ON A GENERALIZATION OF THE
FUNDAMENTAL IDENTITY OF WALD

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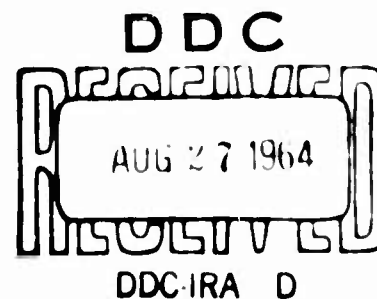
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Summary

In this paper we extend the fundamental identity of Wald
in the theory of sequential analysis to the case where the
variables are Markovian rather than independent.

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§1. Introduction

Let z_1 be a sequence of independent random variables with identical distribution functions, and let S_N denote the partial sum $z_1 + z_2 + \cdots + z_N$. In many interesting situations, we encounter stochastic processes in which a "stop rule" exists, making N itself a random variable which we write as n. Examples of this occur in the study of random walk in the presence of absorbing barriers, in "excluded volume" problems, in sequential analysis, in branching processes, and in many other types of processes. The problem of determining the behavior of the random variable is, in many of these cases, a matter of difficulty, and in other cases, as yet unresolved.

A technique much more powerful in a number of applications than the traditional technique based upon recurrence relations was developed by Wald in 1944, or earlier, [1]. Under light restrictions upon the variables z_i and the nature of the stop rule, he showed that

$$E(\phi(t)^{-n} e^{S_n t}) = 1 \tag{1.1}$$

for $t \geq 0$, where $\phi(t)$ is the characteristic function of the random variable z_1. This was called by Wald the "fundamental

identity", and is closely connected with martingale theory, cf. [2].

The extension to the multi-dimensional case, where the z_1 are independent random k-dimensional vectors was given by Blackwell and Girshick[3], together with a discussion of the determination of S_n in certain cases.

In this paper we wish to indicate how analogues of the Wald identity may be obtained for the case where the z_1 are no longer independent, but possess certain Markovian properties, in the sense that the distribution function of the z_{k+1} depends only upon the value of z_k. More generally, results can be obtained for processes in which z_{k+1} depends upon z_k, z_{k-1}, $\cdots$, z_{k-R}, for fixed R, or upon S_k. Furthermore, the results obtained for the commutative scalar case can be extended to the non-commutative case where we have matrix random variables[4].

We are primarily interested in indicating the nature of the extension and postpone any rigorous discussion and treatment of various special cases until a later date.

§2. Markovian Case

Let $dG(z,x)$ be the distribution function of $z = z_{k+1}$, given that $z_k = x$. Then $\phi_1(x, t) = E(e^{tS_1}) = \int_{-\infty}^{\infty} e^{tz} dG(z, x)$, given $z_0 = x$, and

$$\phi_N(x, t) = E(e^{tS_N} | z_0 = x) =$$

$$\int_{-\infty}^{\infty} e^{tz} \phi_{N-1}(z, t) dG(z, x), \quad N \geq 2, \tag{2.1}$$

Under various assumptions concerning $dG(z, x)$, e.g., a finite range for z, we have, asymptotically as $N \to \infty$,

$$\phi_N(x, t) \simeq \lambda(t)^N \Psi(x, t). \tag{2.2}$$

Note that $\lambda(0) = 1$, and that $\Psi(x, 0)$ is independent of x. Following the Wald proof of (1.1), as in [1], we have

$$\phi_N(x, t) = P_N E(e^{tS_n + t(S_N - S_n)} \mid z_0 = x) + r_N, \tag{2.3}$$

where P_N is the probability that $n \leq N$, and n is, as above, the stochastic variable determined by the stop rule.

Hence

$$P_N(x, t) = P_N E(e^{tS_n} \phi_{N-n}(z_n, t) \mid z_0 = x) + r_N \tag{2.4}$$

Letting $N \to \infty$, we obtain

$$E(e^{tS_n} \lim_{N \to \infty} \phi_{N-n}(z_n, t) / \phi_N(x, t) \mid z_0 = x) = 1 \tag{2.5}$$

Using (2.2), we see that the analogue of Wald's identity is

$$E(e^{tS_n} \Psi(z_n, t) / \lambda(t)^n \Psi(x, t) \mid z_0 = x) = 1. \tag{2.6}$$

Still proceeding formally, let us differentiate (2.6) with respect to t, and set $t = 0$. A rigorous discussion of differentiation of Wald's identity is given by Wald in [5]. The result is

$$E(S_n) = \lambda'(0)E(n) + \phi(x) - E(\phi(z_n)), \tag{2.7}$$

where $\phi(x) = \partial\Psi/\partial t\big|_{t=0}$.

In order to use this to compute $E(n)$, if $\lambda'(0) \neq 0$, we must have some information concerning the values of z_n and S_n at the stage where the stop rule is enforced. In many cases, e.g., random walk with absorbing barriers, this information is readily derivable from the process itself.

§3. Other Types of Identities

The advantage of using the exponential function resides in its simple functional equation. In place of considering the function $e^{S_N t}$, consider $\prod_{k=1}^{N} (z_k + c)$, where c is a constant. Using the same arguments as above, we have

$$E\Big(\prod_{k=1}^{n} (z_k + c) \big/ (a + c)^n\Big) = 1, \tag{1}$$

where $a = E(z_k)$, for the case where the z_k are independent variables, and $|a + c| > 1$.

§4. Matrix Variables

Let us also point out that analogous results may be obtained for the non-commutative case where we have matrix random variables, cf. [4] for a discussion of some problems in this field.

There are several possible generalizations of the scalar case. We may consider

$$E\left(\left[\prod_{k=1}^{n}(Z_k + C)\right](A + C)^{-n}\right), \tag{1}$$

or

$$E\left(\left[\prod_{k=1}^{n}|Z_k + C|\right]|A + C|^{-n}\right), \tag{2}$$

where $|\quad|$ represents the determinant of the random matrix variable $Z_k + C$, with C a fixed matrix.

In turn, we may consider

$$E\left(e^{tr(T(Z_1+Z_2+\cdots+Z_n)} \phi(T)^{-n}\right), \tag{3}$$

where $\phi(T) = E(e^{tr(TZ)})$.

Formulas corresponding to those derived in the one-dimensional case may be obtained for these.

[1] Wald, A., On Cumulative Sums of Random Variables, "Annals of Math. Stat.", Vol XV (1944), pp. 283-296.

[2] Doob, J. L., Stochastic Processes, Wiley, 1953.

[3] Blackwell, D., and Girshick, A., On Functions of Sequences of Independent Chance Vectors with Applications to the Problem of the "Random Walk" in k Dimensions, "Annals of Math. Stat.", Vol. XVII (1946), pp. 310-317.

[4] Bellman, R., Limit Theorems for Non-Commutative Operations—I, "Duke Math. Jour.", pp. 491-500.

[5] Wald, A., Differentiation Under the Expectation Sign in the Fundamental Identity of Sequential Analysis, "Annals of Math. Stat.," Vol. XVII (1946), pp. 493-499.