NOTES ON THE THEORY OF
DYNAMIC PROGRAMMING—VI
THE WAREHOUSING MODEL

By

Richard Bellman

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The RAND Corporation
1700 MAIN ST. • SANTA MONICA • CALIFORNIA

Summary

The purpose of this paper is to show how the functional equation technique of the theory of dynamic programming yields a very simple computational algorithm for the solution of mathematical models arising in stock level studies.

A numerical solution of these problems relying upon linear programming techniques had previously been given by Charnes and Cooper.

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§1. Introduction

In a recent report, [2], Charnes and Cooper present a solution by means of linear programming techniques of one version of what is called the "warehouse problem". As formulated by A. Cahn, [1], it reads

> "Given a warehouse with fixed capacity and an initial stock of a certain product, which is subject to known seasonal price and cost variations, what is the optimal pattern of purchasing (or production), storage and sales?"

The purpose of this note is to indicate how problems of this general nature may be approached by means of the functional equation technique of the theory of dynamic programming, and thereby reduced to a very simple and straightforward computational problem.

In order to compare the two approaches more readily, we shall use the formulation and notation of Charnes and Cooper, [2].

§2. Analytic Formulation

Following the report of Charnes and Cooper, let

(1) B = the fixed warehouse capacity

A = the initial stock in the warehouse

Consider a seasonal product to be bought (or produced) and sold for each of $i = 1, 2, \ldots, n$ periods. For the i^{th} period, let

(2) c_i = cost per unit

p_i = selling price per unit

x_i = amount bought (or produced)

y_i = amount sold

The constraints are as follows:

(3) (a) Buying constraints: The stock on hand at the end of the i^{th} period cannot exceed the warehouse capacity

(b) Selling constraints: The amount sold in the i^{th} period cannot exceed the amount available at the end of the $(i-1)^{st}$ period

(c) Non-negativity constraints: Amounts purchased or sold in any period are non-negative

Analytically

(4) Buying constraint: $A + \sum_{j=1}^{i} (x_j - y_j) \leq B, \ i = 1, 2, \cdots, n,$

Selling constraint: $y_i \leq A + \sum_{j=1}^{(i-1)} (x_j - y_j), \ i = 1, 2, \cdots, n,$

for $i = 1$, this is $y_1 \leq A$

Non-negativity: $x_i, y_i \geq 0.$

The problem is to determine the quantities x_i and y_i so as to maximize the over-all profit

$$(5) \qquad P = \sum_{j=1}^{n} (p_j y_j - c_j x_j).$$

§3. Dynamic Programming Treatment

It is clear that the maximum profit will be a function of the original quantity of stock, A, and the duration of the process n. Define

$$(1) \qquad f_n(A) = \text{Max } P,$$

where the maximum is taken over all admissible values of the x_i and y_i. We have

$$(2) \qquad f_1(A) = \text{Max } (p_1 y_1 - c_1 x_1),$$

over all x_1, y_1 satisfying

$$(3) \qquad \begin{aligned} &(a) \quad y_1 \leq A \\ &(b) \quad A + (x_1 - y_1) \leq B, \end{aligned}$$

or

$$(4) \qquad f_1(A) = p_1 A.$$

We now wish to derive a recurrence relation connecting $f_n(A)$ and $f_{n+1}(A)$. If x_1, y_1 are chosen, the constraints on the remaining variables are

(5) (a) $\sum_{j=2}^{i} (x_j - y_j) \leq B - (A + (x_1 - y_1))$,

(b) $y_i \leq (A + (x_1 - y_1)) + \sum_{j=2}^{i-1} (x_j - y_j)$.

Hence, for $n \geq 2$,

(6) $$f_n(A) = \max_{x_1, y_1} \left[p_1 y_1 - c_1 x_1 + f_{n-1}(A + x_1 - y_1) \right],$$

where the maximum is taken over the region

(7) (a) $y_1 \leq A$

(b) $x_1 - y_1 \leq B - A$, $x_1, y_1 \geq 0$.

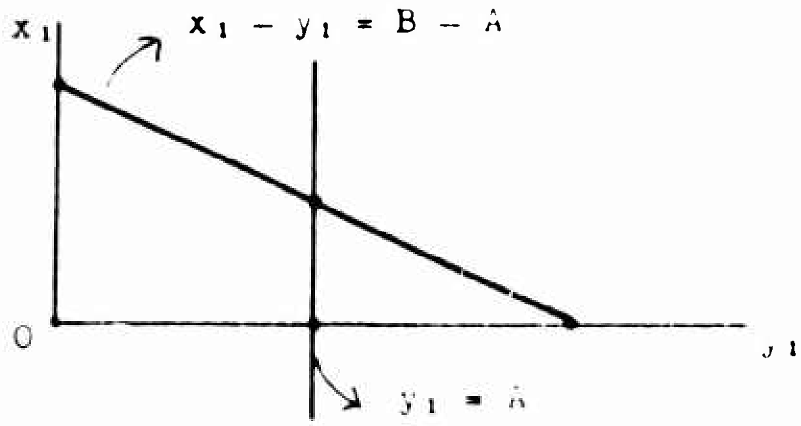
The variable A assumes all values in the interval $[0, B]$.

§4. Discussion

Let us now discuss the actual computation of the solution. As far as the memory and tabulation problems are concerned, we are dealing with a sequence of functions of one variable. Consequently, no difficulties arise from this direction.

The maximization, however, is over a two dimensional region, and a variable region at that. Hence, we might expect that the computation would be slowed down by this fact. Fortunately, we are rescued by the linearity of the process.

Consider the region defined by the equations in (3.7)



We suspect that the maximum will occur at one of the vertices, and this may be established rigorously in several ways, either directly from linear programming, or in an inductive fashion. In the figure above, we have assumed that $A > B - A$ or $2A > B$. In this case, the vertices are

(1) $\qquad P_1(0, 0),\ P_2(0, B - A),\ P_3(A, 0),\ P_4(A, B).$

If $A < B - A$, there are only three vertices

(2) $\qquad P_1(0, 0),\ P_2(0, B - A),\ P_5(B - A, 0).$

Taking all five vertices as possible maximizing points, which takes care of the two cases $B - A \gtrless A$ at one time, we can reduce (3.6) to

(3)
$$f_n(A) = \text{Max} \begin{bmatrix} 1. & f_{n-1}(A), \\ 2. & -c_1(B - A) + f_{n-1}(B), \\ 3. & p_1 A \\ 4. & p_1 A - c_1 B + f_{n-1}(B) \\ 5. & p_1(B - A) + f_{n-1}(2A - B) \end{bmatrix}$$

for $A \geq 0$, with $f_n(A) = 0$ for $A \leq 0$.

This computation is now a very simple one. The quantity B is taken as fixed, and A assumes all values in the interval $[0, B]$.

Bibliography

1. Cahn, A. S., "The Warehouse Problem", Bull. Amer. Math. Soc., Vol. 54 (1948), p. 1073.

2. Charnes, A., and Cooper, W. W., "Generalizations of the Warehousing Model", O. N. R. Research Memorandum, No. 34, 1955, Graduate School of Industrial Administration, Carnegie Institute of Technology.