# ON CONVERSES OF SCHWARZ'S INEQUALITY

By

Richard Bellman

P-770

22 November 1955

Approved for OTS release



The RAND Corporation

1700 MAIN ST. • SANTA MONICA • CALIFORNIA

Summary

The purpose of this paper is to present a general method for treating questions involving the converse of the Schwarz and Hölder inequalities. ( )

# ON CONVERSES OF SCHWARZ'S INEQUALITY

Richard Bellman

## §1. Introduction

The well-known inequality of Schwarz states that

$$(1) \qquad \left(\int_0^1 u(x)v(x)dx\right)^2 \leq \left(\int_0^1 u^2 dx\right) \left(\int_0^1 v^2 dx\right),$$

for any two functions u and v belonging to $L^2(0, 1)$. Without additional restrictions upon u and v, there is no non-trivial inequality going in the other direction; i.e., one of the form

$$(2) \qquad \left(\int_0^1 u(x)v(x)dx\right)^2 \geq k\left(\int_0^1 u^2 dx\right) \left(\int_0^1 v^2(x)dx\right),$$

where $k > 0$.

If, however, u and v are restricted to lie within certain function classes, there do exist inequalities of the above form with k a positive constant dependent upon the classes chosen. The first discussion of problems of this variety occurs in the papers of Blaschke and Pick [1], for the case where u(x) and v(x) are concave.

The purpose of this paper is to present a general method for attacking these problems which is equally applicable to other function classes and to multi-dimensional versions. Although it does not solve any particular problem completely, it reduces each problem to a particular investigation which in some cases can be carried through completely.

We shall begin with the one-dimensional case, demonstrating

<u>Theorem 1.</u> <u>Let</u> $u(x)$ <u>and</u> $v(x)$ <u>be concave functions of</u> $x$ <u>for</u> $0 \leq x \leq 1$, <u>normalized by the conditions</u>

(3) (a) $\int_0^1 u^2 dx = 1, \int_0^1 v^2 dx = 1,$

(b) $u(0) = u(1) = 0, \; v(0) = v(1) = 0.$

<u>Then</u>

(4) $$\int_0^1 u(x)v(x)dx \geq \frac{1}{2}.$$

This result is certainly contained in the paper of Blaschke and Pick cited above, although apparently not explicitly stated in the above form. The minimum is attained for

(5) $u(x) = 0 \leq x < 1, \; u(1) = 0,$

$v(x) = (1 - x)\sqrt{3}, \; 0 < x \leq 1, \; v(0) = 0.$

We shall mention the generalizati n, involving the Hölder inequality, and then turn to the multidimensional case where concavity is defined by $\nabla^2 u \leq 0$. Finally we shall mention the corresponding problems for more general operators. A number of interesting problems arise in this way, which we make no attempt to resolve here.

## §2. <u>Concavity</u>

Let $u(x)$ belong to the class of functions over $[0, 1]$ which are concave and zero at the endpoints. Consider the more restrictive class of functions possessing non-positive second derivatives

$$(1) \qquad u''(x) = -f(x), \; f(x) \geq 0,$$

$$u(0) = u(1) = 0.$$

Minimization over all concave functions in the original class is equivalent to determination of the infimum over all concave functions satisfying (1).

Using the Green's function for the operator u", with the above boundary conditions, namely

$$(2) \qquad K(x,y) = x(1 - y), \; 0 \leq x \leq y \leq 1,$$

$$= (1 - x)y, \; 1 \geq x \geq y \geq 0,$$

we may write u(x), as defined by (1), in the form

$$(3) \qquad u(x) = \int_0^1 K(x, y)f(y)dy.$$

§3. Auxiliary Problem

Given a non-negative function h(x), belonging to $L^2(0, 1)$, let us determine the minimum over all concave u(x), normalized as in (1.3), of the linear functional

$$(1) \qquad L(u) = \int_0^1 u(x)h(x)dx$$

Using the restricted class defined by (2.1), let us determine the infimum of

$$(2) \qquad \Lambda(f) = \int_0^1 h(x)\left[\int_0^1 K(x, y)f(y)dy\right]dx$$

$$= \int_0^1 f(y)\left[\int_0^1 h(x)K(x, y)dx\right]dy$$

over all $f \geq 0$ satisfying

$$(3) \qquad \int_0^1 u^2dx = \int_0^1 \int_0^1 K_2(y, z)f(y)f(z)dydz = 1,$$

where

$$(4) \qquad K_2(y, z) = \int_0^1 K(x, y)K(x, z)dx.$$

To solve this problem we can use the following

Lemma. *Let* $\sum_{i,j=1}^{N} a_{ij}x_ix_j$ *be a positive definite quadratic form, and let* $b_i \geq 0$ *in* $L(x) = \sum_{i=1}^{N} b_ix_i$. *Then the minimum of* $L(x)$ *subject to the constraints*

$$(5) \qquad \text{(a)} \quad \sum_{i,j=1}^{N} a_{ij}x_ix_j = 1,$$

$$\text{(b)} \quad x_i \geq 0,$$

*is*

$$(6) \qquad \underset{1 \leq i \leq N}{\mathrm{Min}} \frac{b_i}{\sqrt{a_{ii}}},$$

*attained at a point of the form*

$$(7) \qquad x_i = \frac{1}{\sqrt{a_{ii}}}, \quad x_j = 0, \; j \neq i.$$

Geometrically, the result is easily obtained if we think of moving the plane $L(x) = k$ parallel to itself until it intersects the surface defined by (5a) in a point of the type given in (7). This will be a point where the plane is at minimum distance from the origin, subject to the conditions of (5).

Analytically, it is a simple consequence of the convexity of the function $\sum_{i,j=1}^{N} a_{ij}x_i x_j$ as a function of the vector $x = (x_1, x_2, \cdots, x_N)$. This method is important since it readily generalizes to general problems involving Hölder's inequality.

Taking the limiting form of the above lemma, we see that

$$(8) \qquad \operatorname*{Min}_u L(u) = \operatorname*{Inf}_f \Lambda(f) = \operatorname*{Min}_y \left[ \frac{\int_0^1 K(x, y)h(x)dx}{\sqrt{K_2(y, y)}} \right]$$

§4. <u>Proof of Theorem 1.</u>

The above result leads to a proof of Theorem 1. We have

$$(1) \qquad \int_0^1 u(x)v(x)dx \geq \operatorname*{Min}_y \left[ \frac{\int_0^1 K(x, y)v(x)dx}{\sqrt{K_2(y, y)}} \right]$$

Inserting the representation of (2.3) for the concave function $v(x)$, $v(x) = \int_0^1 K(x, y)g(y)dy$, $g(y) \geq 0$, we have

$$(2) \qquad \int_0^1 K(x,y)v(x)dx = \int_0^1 g(z)\, K_2(y, z)dz.$$

Using (2.8) in the integral in the right-hand side of (4.1), we have

$$(3)\qquad \int_0^1 K(x,\, y)v(x)dx \geq \underset{z}{\text{Min}} \left[\frac{K_2(y,\, z)}{\sqrt{K_2(z,\, z)}}\right].$$

Combining results, we obtain

$$(4)\qquad \int_0^1 u(x)v(x)dx \geq \underset{0\leq y,z\leq 1}{\text{Min}} \left[\frac{K_2(y,\, z)}{\sqrt{K_2(y,\, y)}\sqrt{K_2(z,\, z)}}\right]$$

It remains to show that this inequality is non-trivial. A direct calculation, carried out by Oliver Gross, shows that this minimum is attained at $y = 1$, $z = 0$, and at the symmetric point $y = 0$, $z = 1$. The value of the function at this point is determined by continuity. The limiting functions are $u(x) = \sqrt{3}\, x$, $0 \leq x < 1$, $u(1) = 0$, $v(x) = \sqrt{3}\,(1 - x)$ $0 < x \leq 1$, $v(0) = 0$, yielding the minimum value $1/2$.

## §3. $L^p$-Case

It is easy to show, using the above methods that

$$(1)\qquad \int_0^1 u(x)v(x)dx \geq \underset{0\leq y,z\leq 1}{\text{Min}} \frac{K_2(y,\, z)}{\left[\int_0^1 K(x,y)^p dx\right]^{1/p}\left[\int_0^1 K(x,z)^{p'} dx\right]^{1/p}}$$

provided that

$$(2)\qquad (a)\ \int_0^1 u(x)^p dx = 1,\ \int_0^1 v(x)^{p'} dx = 1,$$

$$(b)\ \ u(0) = v(0) = 0,\ u(1) = v(1) = 1.$$

A similar calculation to that carried out above for the case $p = 2$ shows that the minimum is attained by the same functions as above, suitably normalized. Hence

$$(3) \qquad \int_0^1 u(x)v(x)dx \geq \frac{(p+1)^{1/p}(p'+1)^{1/p'}}{6} .$$

The maximum of the right-hand side is at $p = p' = 2$.

§6. Multi-Dimensional Case

The same argumentation as above yields the following result

Theorem 2. Let $u(P)$, $v(P)$ be defined for $P$ in some region $R$, and satisfy the conditions

(1) (a) $\nabla^2 u, \nabla^2 v \leq 0$, $P \epsilon R$,

(b) $u, v = 0$ on $B$, the boundary of $R$.

(c) $\int_R u^2 dV, \int_R v^2 dV = 1$.

Let $K(P, Q)$ be the Green's function for the region $R$, with the above boundary condition. Let $K_2(P, Q) = \int_R K(P, P')K(P', Q)dV'$. Then

$$(2) \qquad \int_R uv dV \geq \underset{P,Q \epsilon R}{\text{Min}} \frac{K_2(P, Q)}{\sqrt{K_2(P, P)}\sqrt{K_2(Q, Q)}}$$

The difficulty of the problem is now shifted to the question of determining when the right-hand side is non-zero. For special regions—circular, spherical, cylindrical, rectangular—it should not be difficult to determine the precise value. For general regions we have no method for establishing the positivity.

## §7. Discussion

Results analogous to the above can be obtained for finite sums, and for other types of function classes. One way of obtaining these classes is to start with a linear differential operator $L(u)$, and consider the class of functions defined parametrically by the equation

$$(1) \qquad L(u) = f,$$

together with appropriate boundary conditions. The enlarged class will consist of the original class together with the limit functions derived from improper $f$.

One interesting class is the class of generalized concave functions, satisfying

$$(2) \qquad u'' + q(x)u = -f(x), \quad f(x) \geq 0,$$

$$u(0) = u(1) = 0,$$

where $q(x) \geq 0$.

## Bibliography

1. Blaschke, W., and G. Pick, Distanzabschätzungen im Funktionerraum—II, Math. Annalen, 77 (1916), pp. 277-302.