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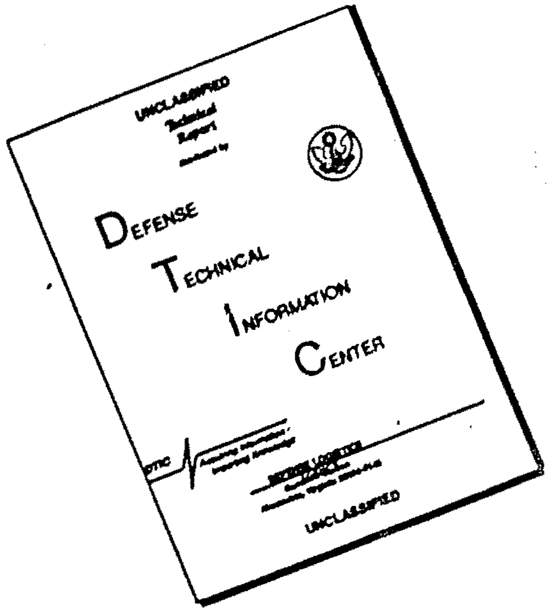

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THE FIXED CHARGE PROBLEM

by

Warren M. Hirsch
George B. Dantzig

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SUMMARY

A fundamental unsolved problem in the programming area is one in which various activities have fixed charges (e.g., set-up time charges) if operating at a positive level. Properties of a general solution to this type problem are discussed in this paper. Under special circumstances it is shown that a fixed charge problem can be reduced to an ordinary linear programming problem.

THE FIXED CHARGE PROBLEM

by

Warren M. Hirsch
George B. Dantzig

STATEMENT OF PROBLEM

Many problems of economics and logistics involve the planning of a large number of interdependent activities in as economical a way as is possible. Mathematically, such problems often can be reduced to the following form: Let $Q, P_1, \ldots, P_n$ be vectors in m-dimensional Euclidean space, and let X denote the set of all vectors $x = (x_1, x_2, \ldots, x_n)$ in n-dimensional Euclidean space which satisfy the relations

$$(1) \qquad \sum_{j=1}^{n} x_j P_j = Q \qquad\qquad m < n$$

$$(2) \qquad x_j \geq 0 \qquad\qquad j=1,2,\ldots,n \; .$$

Suppose $\phi = \phi(x)$ is a real-valued function. Find a vector $\hat{x} \in X$ such that

$$\phi(\hat{x}) \leq \phi(x)$$

for all $x \in X$. In problems of linear programming, ϕ is of the form

$$(3) \qquad \phi(x) = \sum_{i=1}^{n} \phi_i(x_i) \quad ,$$

where ϕ_i is a linear function, and techniques for its solution are available. [Ref. 1, 2, 3, and 6] (It may be assumed without loss of generality that each of the functions ϕ_i vanish at the origin, for it is clear that the problem of minimizing $\sum_{i=1}^{n} a_i x_i$ is equivalent to that of minimizing $\sum_{i=1}^{n} a_i x_i + {}_i$.)

There are, however, numerous applications in which the functions ϕ_i appearing in (3) are linear only for positive values of the argument and have a jump discontinuity of magnitude β_i at the origin. In this case, putting

$$\delta(u) = \begin{cases} 1 \; , & u > 0 \\ 0 \; , & u = 0 \; , \end{cases}$$

we can write ϕ in the form

$$(4) \qquad \phi = \sum_{i=1}^{n} a_i x_i + \beta_i \delta(x_i) \quad ,$$

where $a_1, \ldots, a_n$, $\beta_1, \ldots, \beta_n$ are constants. The fact that ϕ is piecewise linear (and a fortiori non-linear) makes it difficult to apply directly the methods of linear programming. We shall refer to the problem of minimizing (4) under the constraints (1) and (2) as the fixed charge problem, and to the constant β_i as the fixed charge associated with the i^{th} activity.

A practical situation in which this problem may arise is the following: Consider a factory in which n distinct operations are performed by each of m machines. Let h_{ij} and c_{ij}, $i=1,2,\ldots,m$, $j=1,2,\ldots,n$ denote, respectively, the time consumed and cost incurred by the j^{th} machine in performing the i^{th} operation on a unit amount of goods. Suppose that there is incurred a fixed cost, say β_{ij}, in setting up the j^{th} machine to perform the i^{th} operation. (We ignore the time involved in this activity.) Let x_{ij} denote the quantity of goods on which the i^{th} operation is performed by the j^{th} machine, and b_i the total quantity of goods on which it is required to perform the i^{th} operation. Evidently the quantities x_{ij} must satisfy the conditions

(5) $$\sum_{j=1}^{n} x_{ij} = b_i \,, \qquad i=1,2,\ldots,m$$

and

(6) $$x_{ij} \geq 0 \qquad i=1,2,\ldots,m \quad j=1,2,\ldots,n \,.$$

Moreover, if c_j denotes the total machine hours available for the j^{th} machine, we have the additional constraints

(7) $$\sum_{i=1}^{m} h_{ij}x_{ij} = c_j , \qquad j=1,2,\ldots,n .$$

The total cost of operation is given by the function

(8) $$\sum_{i=1}^{m} \sum_{j=1}^{n} [c_{ij}x_{ij} + \beta_{ij}\delta(x_{ij})] ,$$

which has the form (4). Thus the problem of minimizing the operating cost (8) under the restrictions (5), (6), and (7) is just the fixed charge problem.

SOLUTION IN SPECIAL CASE

Under special circumstances it is possible to reduce the fixed charge problem to that of linear programming. In particular, if the fixed charges are all equal and non-negative, we have the following theorem:

Theorem 1: Let $\phi = \sum_{i=1}^{n} a_i x_i + \beta \sum_{i=1}^{n} \delta(x_i)$, $\beta \geq 0$, and $\phi_1 = \sum_{i=1}^{n} a_i x_i$. Suppose that the vector Q appearing in (1) does not lie in any sub-space of Euclidean m-space spanned by fewer than m of the vectors $P_1,\ldots,P_n$ (non-degeneracy assumption). Then there exists an extreme point $\hat{x} \in X$, having exactly m positive components, at which both ϕ_1 and ϕ are minimized. Moreover, any point $\hat{x}$ with m positive components which minimizes ϕ_1 also minimizes ϕ.*

Proof: Under the non-degeneracy assumption, it is clear that any vector $x \in X$ satisfying (1) must have at least m non-zero components. Since the sum, $\sum_{i=1}^{n} \delta(x_i)$, is equal to the number of non-zero components of $x=(x_1,\ldots,x_n)$, we have

$$(9) \quad \min_{x \in X} \phi(x) \geq \min_{x \in X} \phi_1(x) + \beta \min_{x \in X} \sum_{i=1}^{n} \delta(x_i) \geq \min_{x \in X} \phi_1(x) + m\beta .$$

It is well known from the theory of linear programming that, under the assumption of non-degeneracy, there is an extreme point $\hat{x} \in X$ having exactly m positive components at which ϕ_1 is a minimum. Let $\hat{x}$ be any such point. At this point the function ϕ assumes the value

$$\phi(\hat{x}) = \phi_1(\hat{x}) + m\beta .$$

Hence, from (9),

$$\phi(\hat{x}) = \phi_1(\hat{x}) + m\beta \geq \min_{x \in X} \phi(x) \geq \min_{x \in X} \phi_1(x) + m\beta = \phi_1(\hat{x}) + m\beta = \phi(\hat{x}) .$$

Thus it is clear that

---

*Since the simplex method leads to a point $\hat{x}$ with m positive components at which ϕ_1 is a minimum, this method can also be used to solve the fixed charge problem with a constant positive fixed charge.

$$g(\hat{x}) = \min_{x \in X} g(x) .$$

EXISTENCE THEOREM

In the general case when the fixed charges β_i vary with i, although we have no expedient computational algorithm, we shall show that the form (4) is minimized at an extreme point of the convex X. More precisely, we have the following theorem:

Theorem 2. Assume that [illegible] $\sum [\alpha_j x_j + \beta_j \delta(x_j)] \to \infty$ [illegible]. Then if [illegible] ≥ 0, there exists an extreme point $\hat{x} \in X$ such that $\sum_{j=1}^{n} [\alpha_j \hat{x}_j + \beta_j \delta(\hat{x}_j)] = \hat{g}$. (Note that no assumption of non-degeneracy is made here.)

Proof (1). [illegible] proof is as follows: [illegible] point [illegible] on the convex X. Since the convex [illegible] defined by [illegible] (1), (2) is the intersection of a hyperplane in n-space with the positive orthant, through the point x, there exists a line segment L which lies entirely in the convex and which intersects a coordinate plane of the positive orthant in one direction, although it may extend to infinity in the other direction. Put

---

(1) We wish to acknowledge the helpful suggestions of R. S. Lehman in formulating this proof.

$$\phi_1(x) = \sum_{j=1}^{n} \alpha_j x_j$$

and

$$\phi_2(x) = \sum_{j=1}^{n} \beta_j \delta(x_j)$$

Then

$$\phi = \phi_1 + \phi_2 \ .$$

As we proceed along L from x to W the number of non-zero components of points on L (and hence the function ϕ_2) remains constant. Accordingly, the restriction of ϕ to these points is linear, and in one direction ϕ increases (more precisely, does not decrease), while in the other direction it decreases. If the line is finite in extent, we take W to be the plane in the direction of which ϕ is decreasing. If the line extends to infinity in one direction, ϕ must decrease on L in the other direction (i.e., toward W) for otherwise $\phi \to \infty$ as $x \to \infty$ on L, contrary to assumption. Now at the point x' of intersection of L and W an additional zero component is added to the argument of ϕ_2, and since the coefficients β_j are non-negative, this implies that

$$\phi_2(x') < \phi_2(x) \ .$$

We thus see that

$$\phi(x') < \phi(x) .$$

An iteration of the argument starting with x' leads in a finite number of steps to an extreme point at which ϕ is minimized.

We now fill in the gaps in the preceding discussion by supplying an analytic proof which, of course, is nothing but a translation of the geometric argument.

A point $x \in X$ is an extreme point of X if and only if the vectors P_j associated with its non-zero components in (1) are linearly independent. Hence the set $Z \subset X$ of extreme points of X is finite. We shall prove Theorem 2 by showing that to each point $x \in X$ there corresponds an extreme point $x^* \in Z$ such that

(10) $\phi(x^*) \leq \phi(x)$,

and, accordingly, that

(11) $\inf_{x \in X} \phi(x) = \min_{x \in Z} \phi(x)$.

Let x be an arbitrary point, not an extreme point, of X. There exist distinct points $\xi \in X$, $\eta \in X$ such that

(12) $x = \lambda_0 \xi + (1 - \lambda_0)\eta$ $0 < \lambda_0 < 1,\ \xi \neq \eta$.

We put

(13) $x(\lambda) = \eta + \lambda(\xi - \eta)$

and observe that

(14) $x(\lambda_0) = x$.

Denoting the i^{th} component of $x(\lambda)$ by $x_i(\lambda)$, we have

$$\sum_{j=1}^{n} x_j(\lambda)P_j = \sum_{j=1}^{n} [\eta_j P_j + \lambda(\sum_{j=1}^{n} \xi_j P_j - \sum_{j=1}^{n} \eta_j P_j)] = Q + \lambda(Q - Q) = Q \ .$$

Hence $x(\lambda) \in X$ for all λ such that $x_i(\lambda) \geq 0$, $i=1,2,\ldots,n$. Accordingly, it follows that $x(\lambda) \in X$ for all $\lambda > 0$, if and only if, for all indices i, $\xi_i \geq \eta_i$. Similarly $x(\lambda) \in X$ for all $\lambda < 0$ if and only if $\xi_i \leq \eta_i$ for all i. Thus, the set Λ of all λ's for which $x(\lambda) \in X$ is bounded either above or below. We assume without loss of generality that

$$-\infty < \inf_{\lambda \in \Lambda} \lambda \ .$$

Let

$$\underline{\lambda} = \inf_{\lambda \in \Lambda} \leq 0$$

and

$$x_1 = [illegible]$$

for λ [illegible] therefore

$$x_1(\lambda) [illegible] \lambda < [illegible]$$

We shall show that for $\underline{\lambda} < \lambda < \bar{\lambda}$, $x_1(\lambda)$ is zero or positive according as $x_1(\lambda_0)$ is zero or positive. We begin with [illegible] suppose that $x_1(\lambda_0) = 0$; that is,

(16) $$\lambda_0 \xi_1 + (1 - \lambda_0)\eta_1 = 0 .$$

Since $0 < \lambda_0 < 1$, $\xi_1 \geq 0$, and $\eta_1 \geq 0$, equation (16) holds only if $\xi_1 = 0$ and $\eta_1 = 0$. Hence,

$$x_1(\lambda) = \eta_1 + \lambda(\xi_1 - \eta_1) = 0 , \quad \underline{\lambda} < \lambda < \bar{\lambda} .$$

Suppose now that $x_1(\lambda_0) > 0$. Then, if $\xi_1 \neq \eta_1$, we see from (13) that $x_1(\lambda)$ is a strictly monotonic function of λ. Consequently, since $\underline{\lambda} < \lambda_0 < \bar{\lambda}$, (15) can hold only if $x_1(\lambda) > 0$ for $\underline{\lambda} < \lambda < \bar{\lambda}$. On the other hand, if $\xi_1 = \eta_1$, we see from (13) that

$$x_1(\lambda) = [illegible]) > 0 .$$

From the linearity of the functions $x_i(\lambda)$ and the fact that $x(\lambda) \notin X$ for $\lambda < \underline{\lambda}$, it follows that there is an integer i_0 such that $x_{i_0}(\lambda_0) > 0$ and $x_{i_0}(\lambda) < 0$ for $\lambda < \underline{\lambda}$. The continuity of the function $x_{i_0}(\lambda)$, together with (15), then insures that

$$x_{i_0}(\underline{\lambda}) = 0 .$$

Summarizing the results of the preceding paragraphs, we have seen that any zero component of $x(\lambda_0)$ is a zero component of $x(\underline{\lambda})$, while $x(\underline{\lambda})$ has at least one zero component for which the corresponding component in $x(\lambda_0)$ is positive. Evidently $x(\underline{\lambda}) \in X$.

If $\bar{\lambda} < \infty$, similar considerations show that the vector $x(\bar{\lambda})$ has at least one more zero component than $x(\lambda_0)$, and $x(\bar{\lambda}) \in X$.

Recalling that $\phi_1(x) = \sum_{j=1}^{n} \alpha_j x_j$ and $\phi_2(x) = \sum_{j=1}^{n} \beta_j \theta(x_j)$, it is now clear that

(18) $$\phi_2(x(\underline{\lambda})) < \phi_2(x(\lambda_0)) = \phi_2(x)$$

and, if $\bar{\lambda} < \infty$,

(19) $$\phi_2(x(\bar{\lambda})) < \phi_2(x(\lambda_0)) = \phi_2(x) .$$

If $\bar{\lambda} = \infty$,

(20) $\quad \phi_2(x(\lambda)) = \phi_2(x(\lambda_0)) = \phi_2(x)$

for all $\lambda \geq \lambda_0$, since, for $\underline{\lambda} < \lambda < \overline{\lambda}$, $x_1(\lambda)$ is zero or positive according as $x_1(\lambda_0)$ is zero or positive.

Observing now that $\phi_1(x(\lambda))$ is a linear function of λ, it follows that if $\overline{\lambda}$ is finite, either

(21) $\quad \phi_1(x(\underline{\lambda})) \leq \phi_1(x(\lambda_0)) = \phi_1(x)$

or

(22) $\quad \phi_1(x(\overline{\lambda})) \leq \phi_1(x(\lambda_0)) = \phi_1(x)$.

If $\overline{\lambda} = \infty$, then (21) must hold; for, in this case, the relation $\phi_1(x(\underline{\lambda})) > \phi_1(x(\lambda_0))$ implies that $\phi_1(x(\lambda)) \rightarrow -\infty$ as $\lambda \rightarrow \infty$, since ϕ_1 is linear. This contradicts the assumption $\inf_{x \in X} \phi(x) > -\infty$.

From (18) - (22) we see that there is a point $x' \in X$ for which

$$\phi(x') = \phi_1(x') + \phi_2(x') \leq \phi_1(x) + \phi_2(x) = \phi(x) ,$$

where x' has at least one more zero component than x. (If $\overline{\lambda} = \infty$, then $x' = x(\underline{\lambda})$; if $\overline{\lambda} < \infty$, then x' is one of the two vectors $x(\underline{\lambda})$, $x(\overline{\lambda})$.) If x' is an extreme point, (10) is proved. Otherwise, an iteration of the argument replacing x by x' yields a new point x'' such that

$$\phi(x'') \leq \phi(x') ,$$

where x'' has at least one more zero component than x'. It is clear that after at most n such constructions there is produced an extreme point x^* satisfying (10).

A MORE GENERAL FIXED CHARGE PROBLEM

In certain logistical problems, in particular in connection with the question of transshipment, one is led to consider a generalization of the fixed charge problem. Suppose that k is a divisor of n, say,

$$n = k \cdot p ,$$

and put

$$(23) \qquad \psi(x) = \sum_{j=1}^{n} a_j x_j + \sum_{r=1}^{p} \beta_r \delta\Big(\sum_{j=(r-1)k+1}^{rk} x_j \Big) .$$

Clearly ψ reduces to the function ϕ defined in (4) when k=1 and p=n. We refer to the problem of minimizing ψ under the side conditions (1) and (2) as the generalized fixed charge problem. For this problem there is no easy analogue of theorem 1. However, the statement and proof of theorem 2 carry over without change; so that if $\inf_{x \in X} \psi(x) > -\infty$ there is an extreme point $\hat{x} \in X$ such that $\psi(\hat{x}) \leq \psi(x)$.

for all $x \in X$.

We have encountered this generalization of the fixed charge problem in connection with the following transportation question: Consider a hypothetical network of cities between each pair of which specified quantities of homogeneous cargo are to be delivered. Let b_{ij} denote the cargo to be delivered from city i to city j. Suppose that the cost of shipping u units of cargo from i to j is given by a function of the form

$$\phi_{ij}(u) = \alpha_{ij}u + \beta_{ij}\delta(u) .$$

The problem is to route shipments so that the total cost of shipments is minimized. Let x_{ijk}, $i=1,2,\ldots,n$, $j=1,2,\ldots,n$, $k=1,2,\ldots,n$, $i \neq k$, $i \neq j$, denote the quantity of materiel to be shipped from i to j for ultimate transshipment to k. Observe that $\sum_{\substack{j=1 \\ j \neq i}}^{n} x_{ijk}$ represents the total goods shipped from i for the final destination k. Since $\sum_{\substack{m=1 \\ m \neq i \\ m \neq k}}^{n} x_{mik}$ represents cargo received by i for ultimate transshipment to k, the variables x_{ijk} must satisfy the constraints

(24) $$\sum_{\substack{j=1 \\ j \neq i}}^{n} x_{ijk} - \sum_{\substack{m=1 \\ m \neq i \\ m \neq k}}^{n} x_{mik} = b_{ik} , \qquad \begin{matrix} i=1,2,\ldots,n \\ k=1,2,\ldots,n \\ i \neq k \end{matrix} .$$

The total quantity of goods shipped from i to j ($i \neq j$) is given by $\sum_{k=1, k \neq i}^{n} x_{ijk}$. Hence the cost of shipment from i to j is

$$\alpha_{ij} \sum_{k=1, k \neq i}^{n} x_{ijk} + \beta_{ij} \delta\left(\sum_{k=1, k \neq i}^{n} x_{ijk} \right) ,$$

and the total cost of all shipments is

$$(25) \qquad \sum_{i,j=1, i \neq j}^{n} \left[\alpha_{ij} \sum_{k=1, k \neq i}^{n} x_{ijk} + \beta_{ij} \delta\left(\sum_{k=1, k \neq i}^{n} x_{ijk} \right) \right] .$$

The problem of minimizing (25) under the side conditions (24) and

$$(26) \qquad x_{ijk} \geq 0 ,$$

is evidently of the generalized fixed charge type.

RELATION TO QUADRATIC PROBLEMS

It is of some interest to note that we may replace the problem of minimizing the non-linear function Φ defined in (23) under linear side conditions by one of minimizing a linear function under linear side conditions plus an additional quadratic constraint. Specifically let $(x,y) = (x_1,\ldots,x_n,y_1,\ldots,y_p)$ represent a point in n+p dimensional Euclidean space. Suppose that $(\hat{x},\hat{y}) = (\hat{x}_1,\ldots,\hat{x}_n,\hat{y}_1,\ldots,\hat{y}_p)$ minimizes the function

$$(27) \qquad \tilde{\Psi}(x,y) = \sum_{i=1}^{n} \alpha_i x_i + \sum_{r=1}^{p} \beta_r y_r$$

under the constraints (1), (2), and

$$(28) \qquad 0 \leq y_r \leq 1 , \qquad r=1,2,\ldots,p$$

$$(29) \qquad \sum_{r=1}^{p} (1 - y_r) \cdot \sum_{i=(r-1)k+1}^{rk} x_i = 0 .$$

The constants β_r, $r=1,2,\ldots,p$ are assumed to be non-negative. We shall prove that $\hat{x} = (\hat{x}_1,\ldots,\hat{x}_n)$ minimizes the function Ψ in (26) under the side conditions (1) and (2). In the first place,

$$(30) \qquad \inf \tilde{\Psi}(x,y) \leq \inf \Psi(x)$$

(where the infinium is taken over the appropriate side conditions), since to each point x satisfying (1) and (2) there corresponds a point y such that (x,y) satisfies (1), (2), (28), and (29), and

$$\tilde{\Psi}(x,y) = \Psi(x) .$$

Indeed, we need only define

$$y_r = \delta\Big(\sum_{i=(r-1)k+1} x_i\Big)$$

Observe next that if $\sum_{i=(r-1)k+1} \hat{x}_i > 0$, then in order to satisfy (29) we must have $\hat{y}_r = 1$; if $\sum_{i=(r-1)k+1} \hat{x}_i = 0$, we must have $\hat{y}_r = 0$; for otherwise we could achieve a lower value for [illegible] by setting $\hat{y}_r = 0$. It is thus clear that

$$\hat{y}_r = \delta\Big(\sum_{i=(r-1)k+1} x_i\Big)$$

and, accordingly, that

(31) $$\inf \Psi(x) \geq \inf \tilde{\Psi}(x,y) = \tilde{\Psi}(\hat{x},\hat{y}) = \Psi(\hat{x}) \geq \inf \Psi(x) ,$$

where again the infinium is taken over the appropriate set of constraints. We see from (31) that

$$\inf \Psi(x) = \inf \tilde{\Psi}(x,y) ,$$

which completes the proof.

Another form of the problem can be obtained by putting $z_r = 1 -$ [illegible]. The function $\tilde{\Psi}(x,y)$ to be minimized can then be replaced by

(32) $$V^*(x,z) = \sum_{j=1}^{n} \alpha_j x_j - \sum_{r=1}^{p} \beta_r z_r \quad ,$$

omitting the constant, $\sum_{r=1}^{p} \beta_r$, which is inessential. The constraints then become

(33) $$0 \leq z_r \leq 1$$

and

(34) $$\sum_{r=1}^{p} z_r \sum_{i=(r-1)k+1}^{rk} x_i = 0 \ .$$

This form suggests a possible connection between the generalized fixed charge problem and other programming problems involving the optimization of a linear form where, in addition to being non-negative, the variables are constrained by a condition of the form

(35) $$x_i = 0 \text{ or } 1 \qquad\qquad i=1,2,\ldots,n$$

or of the form

(36) $$x_i = 0 \quad \text{or} \quad x_i \geq a_i \ .$$

Both (35) and (36) can be recast as quadratic constraints formally similar to (34). For example, (35) becomes

$$(37) \qquad \sum_{i=1}^{n} x_i(1-x_i) = 0 .$$

Feasible computational methods for solving [illegible] involving (37) have been developed for special cases, for [illegible]. In the [illegible] of the traveling salesman problem. Whether or not these methods can be carried over to the fixed charge problem is still an open question.

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