THE PROBLEM OF AIMING AND EVASION

Rufus Isaacs

P-642

14 March 1955

45 p

COPY	/ OF	/	p
HARD COPY	$.2.00		
MICROFICHE	$.0.50		

The RAND *Corporation*

1700 MAIN ST. • SANTA MONICA • CALIFORNIA

43)

SUMMARY

The general problem of a marksman versus a mobile
target, with a time lag in the gunner's information
as to the target's position, ~~appears in many guises~~
~~in many situations. It is a classic military problem.~~
Formulated in terms of game theory, the desiderata
are: How should the target best maneuver to confound
prediction of his position? How and when should the
marksman make this prediction? What hit probability
is to be expected when both participants behave
optimally?

This paper discusses this general class of problems
and then settles on one which seems to be the simplest
possible example that is not trivial. Nevertheless
it is difficult. In two previous papers devoted to
it, the evader's best strategy and value of the game
~~were given. Here~~ the emphasis is on the marksman.
He has no optimal strategy, but does have an _ideal_
strategy with the property that every near optimal
strategy is close to it. He also has a class of
passive ~~e~~ strategies such that if and only if he
obeys their dictates will he either come within ε
of the best hit probability or else always remain
in a position where it is possible to do so.

THE PROBLEM OF AIMING AND EVASION

Rufus Isaacs

1. Introduction

One of the most classic of military problems is: how best
to aim at a mobile target which is deliberately maneuvering so
as to confound prediction of his position. The answer must be
sought in the theory of games, whence we consider simultaneously
the apposite question: how best should the target maneuver.

Such antagonists appear in a great variety of situations.
They may be sniper against infantryman, antiaircraft gun against
plane, bomber against ship. Whatever be their nature, the crucial
feature these situations have in common is a time lag between the
detection of the target and the arrival of the projectile. This
lag may be composed of a number of summands such as the delay
between detection of the target and aiming of the firing device,
and the flight time of the projectile itself. But this decomposi-
tion does not concern us here; it suffices to consider the time
lag as a whole.

The theory of games warns us to expect mixed strategies
from both participants and a modicum of common sense confirms
the warning. When a player of a game employs a mixed strategy,
it means that he does not make his decisions in accordance with
any predetermined, certain plan, but invokes a certain amount of
randomness. A game theoretic solution prescribes not the dictates

of behavior but their exact probabilities so as respectively to
minimize or maximize the probability of a hit. It is clear that
this will be the case in the present type of problem. For if the
target were to follow any proscribed, certain plan, it would
plainly be a ruinous policy as soon as the gunner became aware
of it. Likewise any fixed policy of the gunner would enable the
target always to escape once he learned it. Then our goal is
optimal mixed strategies or policies of best regulated randomness
for each player.

So far as we know, this entire field is virtually virginal.
We do not claim any deep inroads here. We deal with a single
problem, described below, which is the simplest nontrivial one
we could devise, yet which embodies the features discussed above.
It is but the first rung of the ladder.

The circumstances that led to this problem, we think, are
instructive. Originally this was its guise.

A battleship in midocean is aware of an enemy bomber's
presence, but the plane is too high for precise detection. The
ship is interested only in not being hit; it has no offensive
means. The plane has one bomb and we suppose — to avoid extra-
neous factors — that the bomber's aim is excellent. The battle-
ship knows this, but knows nothing about when or where the bomb
will be dropped until after detonation. It is to maneuver so as
to minimize the hit probability. We suppose that its only kinematic
restriction is that it travels with a fixed speed v. There is a
time lag T between the bomber's last sighting of the ship and

detonation. Thus the bomber must aim at an anticipated position of the ship.

Game theory attempts to answer the three intersticial questions:

How best should the battleship maneuver? (Optimal strategy of player I)

When and where should the bomber strike? (Optimal strategy of player II)

What is the hit probability when both players use best tactics? (Value of the game)

If at a certain time the ship is sighted at a certain position, then when the bomb strikes he may be located anywhere in a disk of radius vT. To minimize the chance of an immediate hit, the ship should be at all points of the disk with equal probability.[*] For if he favored one portion of the disk, by bombing thereat the plane scores a disparately high hit probability. But there is only one path — a straight one — by which the ship can reach a peripheral point and many by which he can reach a given interior one. Thus to achieve equiprobability, the ship's mixed strategy must attach an unduly high probability to straight paths. But plainly such a course is detrimental to future positions. For if the bomber waits a little and observes this straight path tactic, nothing could be easier than an extrapolation and a certain hit. In other words, if the ship attempts equiprobability

---

[*] For peripheral positions we should make a correction if we take any inaccuracy of bomber into account. We will not do so here.

at one instant he renders his later distributions extremely
unequal. The battleship must compromise between present and
future and seek a probability distribution which, although it
is as near uniform as possible, can be maintained indefinitely.

As simple as this problem sounds circumstantially, it is
difficult technically. To gain a foothold, we simplified it
further. We made the ocean one-dimensional and discrete. That
is, we supposed the battleship to be located on one of a long
row of points and at each unit of times he hops to one adjoining
one, enjoying the sole choice of a right or left jump. The time
lag was to be an integral number n of time units, or — the
same thing — of jumps. This is tantamount to saying that the
bomber knows all positions of the battleship which precede his
present one by n jumps or more. If n = 1, the bomber knows all
but the most recent of the ship's positions and there are but two
possibilities for that: one space to right or left of the last
observed one.

This case — n=1 — is trivial. The ship makes each decision —
left or right — by the toss of a coin. The bomber can bomb at any
time and when he does he also decides between the two possibilities
with a coin.[*] Then the value of the game (hit probability) is 1/2.

_____

[*] (For the game theory tyro only.) If at some time, the ship
elected, say, the probabilities: Left: .6; Right: .4, the
bomber need only wait for this time and bomb on the left;
then hit probability = .6. Similar considerations hold
vice versa. Thus the unique optimal strategies require
50-50 decisions on the parts of both players.

Our intention was now to take up n = 2, 3, 4, ⋯ and, from
the knowledge gained, proceed to the continuous case. Thence, we
hoped to restore planarity to the ocean and approach practicality
by more realistic assumptions about the ship's kinematics, accuracy
of the bomber, number of bombs, etc.

But the case of n = 2 proved to be an incubus. A considerable
amount of effort by several people was expended before its shell
began to crack. This paper will be the third one devoted to it;
see [1, 2]. We can expect the general class of aiming-and-evasion
problems to be more difficult than anticipated, but by no means
hopeless.

We have been occupied with a subject we call differential
games, with pursuit games as one of its more cogent applications.
A drawback is the difficulty of handling cases where the information
of the players is incomplete. It is our hope that the present
problem will adumbrate techniques in this field also, and we are
thus guided in our nomenclature of the players:

P, the pursuer, bomber, or marksman

E, the evader, battleship, or target.

We cite one innovation of technique that appears to be of some
generality in games like the present one which admit of a "stationary"
or steady state character. By this we mean that after a full cycle
of moves (usually one by each player) the game is either terminated
cr a situation recurs which resembles a fresh start of the game.
An initial move of one player is replaced by a chance move with a

preassigned probability x. Then for each x we have a new game
and the value of this game we denote by $\phi(x)$. Often the
resemblance just mentioned becomes an identity except for
the value of x, which circumstance enables us to write a
functional equation satisfied by $\phi(x)$. For a simple example
of this method — the initial one for us — see our paper [3]
dealing with a recreational game.

In the previous two papers [1, 2], dealing with the current
game, E was accorded this treatment of having his initial move
"chancified." Here we shall do the same for P. These alter-
native possibilities present a curious duality of techniques
whose interrelationships may bear interesting fruit.

But methods later! Let us now return to case of n = 2 —
our subject proper. The course of E can be shown conveniently
on the diagram of Figure 1. His starting point
is O and on his first move he travels to either
d or e. If he went left to d, on his second
move he may go to a or b and so forth.
Always P knows E to be at one of three
positions; if he was last observed, say,
at d, and P wishes to fire he will do
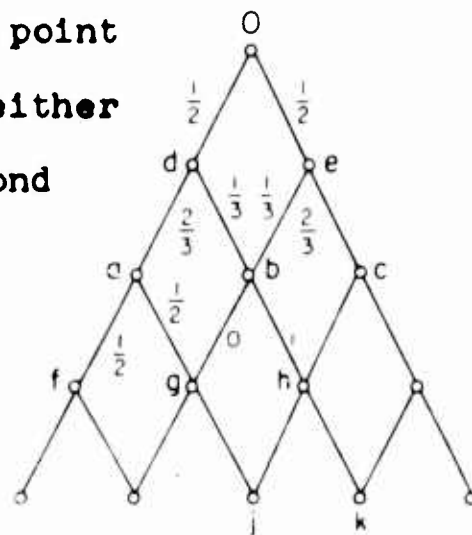so at one of f, g, or h.

Fig. 1

The same dilemma of present or future
benefits that beset the battleship also confronts E, but we are
now in position to examine matters more succinctly. If E is
concerned only with a single instant, his best and safest course

is to make the three probabilities of where he will be when under fire each equal to 1/3. For then P spots him with probability 1/3 and this is clearly the lowest value E can can hope for. Let us suppose E is guided by this consideration alone.

On the grounds of symmetry, we suppose E to make his first choice (d or e) with probabilities 1/2.[*] To cause the probabilities of being at a, b, or c to be 1/3, E must make his second move with probability 2/3, 1/3, 1/3, 2/3 as marked on Figure 1. If E is at d he must similarly equalize his chances of arriving f, g, or h. This determines some of his third move probabilities; they are also marked on the figure. But the probability b—to—h is 1! Thus, should E reach b via d, P can fire then and, by splitting his target choice between j and k, score with probability 1/2 or more.

Let us endeavor to find a less ambitious but more enduring strategy for E. We may expect that in such a strategy each decision will depend on prior moves. As E's course more than two moves ago is known to his opponent, it is reasonable to suppose that this dependence will not reach very far back. Let us suppose the choice depends on the previous move only. Precisely let E move in the same direction as his last move with probability $1 - x$ and let him make a turn with probability x. This strategy is certainly stationary; it is expounded in Figure 2, which diagram applys to any position except the

_____

[*] The symmetrization is not necessary. The reader can verify that our reasoning holds whatever the initial probability.

one at the very outset. Then the probabilities
of E's reaching 1, 2, or 3 are respectively

$$\left.\begin{array}{l} (1-x)^2 \\[1em] x \\[1em] x(1-x) \end{array}\right\} \quad (1)$$

Fig 2
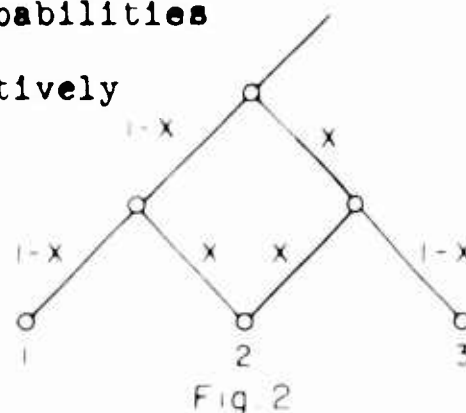
In accordance with the tenets of game
theory, we presume that P will elect the largest of these three
quantities. The best possible x for E is then that value which
renders the maximum of the three polynomials (1) a minimum.
Plots of (1) are sketched in Figure 3 with maximum overscored.
It is minimum at V, a root of

Fig. 3

$$x = (1-x)^2 . \qquad (2)$$
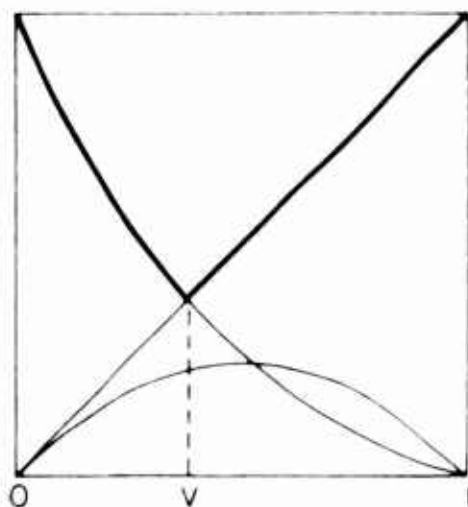
Then

$$V = \frac{3-\sqrt{5}}{2} = .382 \cdots . \qquad (3)$$

This number is also the probability
E's arriving at points 1 or 2 and
so is the payoff when E plays as
just described (we grant P sense
enough not to fire at 3).

It turns out that V is actually the value of the game and
the strategy just described, which we henceforth denote by M,

is the optimal strategy for E and indeed the only such.* On the
other hand, it turns out that P does not possess an optimal
strategy, the situation being thus: for any $\varepsilon > 0$, there is a
(mixed) strategy for P which assures him of a hit with proba-
bility $\geq V - \varepsilon$, but no strategy insures V. A strategy of this
type will be called a near optimal strategy or an ε-strategy.

These results are not easy to prove. They are the subjects
of papers [1, 2]. Dubins deserves the honor of priority. His
paper came to our attention some months after its publication.
By this time a RAND version was ready for the press; the work
was done independently and the methods differed enough to warrant
a second treatment.

Neither paper gave a near optimal strategy for P in the
sense of furnishing him explicit playing instructions. As this
facet of the problem is of obvious importance in more realistic
versions, we present a third approach which stresses this aspect.

On this topic we shall later obtain the following results.
In the next section our game will be imbedded in a family of
games. For these games P has what we term an ideal strategy.
It is for most of the family not an ε-strategy, but it is true
that every ε-strategy is nearly the ideal strategy, the nearness
increasing with the smallness of ε. We also delineate a class
called passive ε-strategies. For each ε and each play of a

_____

* We complete the definition of M. On his very first move
E may elect any probability p such that $V \leq p \leq 1 - V$.
It is easy to verify that P can still attain at most V.

game these impose well-defined restrictions on each move of P such that:

If he conforms he will either attain a hit probability exceeding the value (the best possible) $-\xi$ or he will always be in position where, with proper subsequent play, it will be possible for him to do so. But if ever he violates the restrictions, E can prevent him from coming with ξ of the value.

The ideal strategy is a passive ξ-strategy for every ξ.

2. The "Chancified" Games

We say P plays an α-strategy when the following holds:

Let $\alpha = 1/\sqrt{5}$. Whenever P decides to fire he will aim at the leftmost [rightmost] of the three points where E may be with probability α if E's last observed move was left [right] and he aims at the center point with probability $1 - \alpha$.

Fig 4
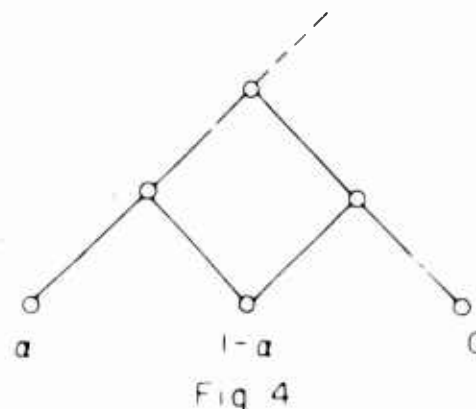
To act at the very opening of the game, P must supply E with a fictitious preceding move. See Figure 4, where the dotted line is the (possibly fictitious) preceding move. We will motivate this concept in Section 7.

We coin two families of new games. For the game F_r, we amend our original rules:

There is a fictitious minus-first move, say, from the left, and

Fig.5
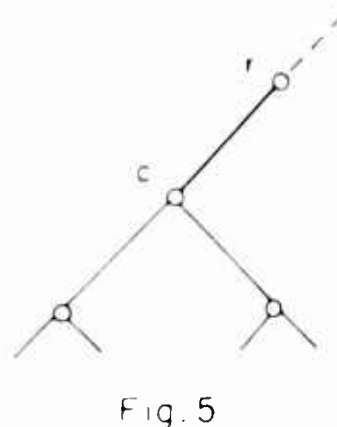
P is constrained to play an α-strategy. The opening move is chancified; P is impelled to fire with a preassigned probability r. Also E is obliged to make his first move to the left. (See Figure 5.)

The game H_r is the same except that E makes his first move to the right.[*]

For F_r, let

$$f(r) = \sup \inf \text{(payoff)}$$

Fig 6

which inf extends over all strategies of E and sup over those of P. Or, to put it otherwise, $f(r)$ is the upper bound of all hit probabilities that P can attain in the game F_r, no matter how skillfully he is opposed.

Let $h(r)$ be defined analogously for H_r.

We shall obtain a pair of functional equations for $f(r)$, $h(r)$. Here we shall do it heuristically.

Let P elect the firing probability c, to be fixed later, for the second move of F_r. If E's second move is leftwards, P fires at him with probability r and then hits with probability a. If P does not fire, the situation is tantamount to the commencement of the game F_c. Assuming that P also strives toward his upper bound

_____

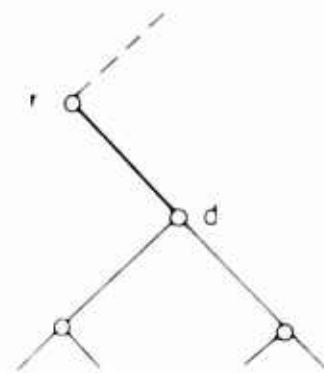[*] The reader may ask: Why does the chancifying process lead to two families of games and consequently two functional equations? It need not; see Section 9.

when playing this latter game, the hit probability under these circumstances is

$$r\alpha + (1-r)f(c) \ . \tag{4}$$

If E chooses rightwards for his second move, the hit probability is $1 - \alpha$ if P fires immediately. If P does not, he is faced with the game H_c. Thus the chance of a hit is

$$r(1-\alpha) + (1-r)h(c) \ . \tag{5}$$

Now we suppose E adroit enough so that his left—right choice selects the minimum of (4) and (5). Then P, to play well, should pick c with the intent of making this minimum as large as possible. Thus

$$f(r) = \sup_{0 \le c \le 1} \ \min \begin{cases} r\alpha + (1-r)f(c) \\ r(1-\alpha) + (1-r)h(c) \end{cases} \ . \tag{6}$$

Similar considerations applied to H_r lead to

$$h(r) = \sup_{0 \le d \le 1} \ \min \begin{cases} (1-r)f(d) \\ r(1-\alpha) + (1-r)h(d) \end{cases} \ . \tag{7}$$

The functions $f(r)$ and $h(r)$ appear to be extraordinarily complicated. It seems that the interval $0 \le r \le 1$ is to be divided into infinitely subintervals with the functions possessing distinct analytic expressions on each. Furthermore to ascertain these expressions appears bafflingly difficult.

This amazing complexity is disconcerting when we bear in mind
that we are still dealing with but one of the simplest versions
of our problem.

We have computed plots of f and h which appear at the end
of this paper. These were executed with naive computational
techniques out with enough care so that, if data is take from
the plots, they will fulfill the functional equations to within
the limits of graphical accuracy. There are also plots of the
c and d which furnish the maxima.

One would hardly suspect the involved character of f and h
from their innocent looking graphs. Are we to conclude that there
is some simple but closely approximate method of treating the
present class of problems? We do not know.

RAND Report RM-1385, A Game of Aiming and Evasion: General
Discussion and the Marksman's Strategies is a mathematically more
scrupulous version of the present paper. In it a number of
properties of f and h necessary for our work are rigorously
proved. We list them below. If the reader accepts our plots
as close depictions of the functions, most of these properties
will appear obvious. The above report also contains a rigorous
derivation of the functional equations (6) and (7).

In the report it is shown that all solutions of (6) and (7)
are continuous. Then the sup appearing on their right sides may
be replaced by max.

We introduce the numbers

$$R_1 = 2V, \quad R_2 = V$$

and use R to mean R_1 where we are speaking of F_r or f and R_2 when speaking of H_r or h. When $r \geq R^*$ and only then the functions are linear; in fact

$$f(r) = a, \quad h(r) = a(1-r) .$$

The largest maximizers here are

$$\bar{c}(r) = \min \left[\frac{1-2a}{a} \frac{r}{1-r}, \mathbf{4} \right.$$

$$\bar{d}(r) = \min \left[\frac{1-a}{a} \frac{r}{1-r}, 1 \right]$$

but these are not unique, as shown by the shaded portions of plots c and d. Remark that

$$\bar{c}(R_1) = \bar{d}(R_2) = R_1 .$$

More interesting is the range $r < R$. Here the maximizing c and d are unique for each r and will be denoted by $c(r)$ and $d(r)$. They are continuous and

$$c(r) < R_1, \quad d(r) < R_2 .$$

Further $f(r)$ is decreasing and $h(r)$ increasing. When $c = c(r) [d = d(r)]$ the two lines on the left of (6) $[(7)]$ have equal values.

_____

* The case $r = 1$ corresponds to certain firing and so no significance then attaches to c or d.

At $r = 0$, f and h are differentiable and

$$f'(0) = A = 1 - 2a$$

$$h'(0) = -B = -\frac{3a-1}{2} .$$

Further $c(0) = 0$, $c'(0)$ exists and $= V$. For $0 \leq r \leq r_1 < R_1$, there exists $k = k(r_1)$ such that $c(r) \leq kr$.

We shall not use any of these results until Section 4.

It is clear from (6) and (7) that

$$h(r) \leq f(r) . \qquad (0 \leq r \leq 1) \qquad (8)$$

Also

$$f(0) = h(0) = \sup_{c} \min \begin{cases} f(c) \\ h(c) \end{cases} = \sup_{c} h(c) \qquad (9)$$

and we will denote the common value of these four quantities by U.

Consider the game like F_r or H_r except that the compulsion of E's first move is waived. E will exploit his new liberty in favor of a low payoff; from (8), the sup inf of the new game is $h(r)$. Now put $r = 0$. This means that P can't fire on the first move and so the game virtually starts from the second. It is thus equivalent to our original game in all ways except P's constraint to an a-strategy. Its sup inf is clearly U. Remark that the election of an a-strategy is at P's disposal; thus, in playing the original game, he can always attain a hit probability arbitrarily close to U.

In the next section we prove that $U \geq V$. As we already
know that E, by playing M, can attain a payoff $\leq V$, we conclude
that V is the value of the game.

3. The Value of the Game

We find

$$\alpha = .447 \cdots > V$$

and so $U \geq \alpha$ implies $U > V$, which cannot be. Hence $U \leq \alpha$.

Let ψ be the set of all pairs a, b such that $a > 0$ and the
inequalities

$$f(r) > U + ar \tag{10.1}$$

$$h(r) > U - br \tag{10.2}$$

hold for all sufficiently small positive r. Note $b > 0$ or else
$\sup_r h(r)$ would exceed $h(0)$, contradicting (9).

Lemma 1. ψ is not vacuous.

Proof. It contains the pair $\frac{1}{2}(\alpha-U)$, α. For, if $r > 0$,
we put c = 0 in (6) and then d = 0 in (7):

$$f(r) \geq \min \begin{cases} r\alpha + (1-r)U \\ r(1-\alpha)+(1-r)U \end{cases} = r\alpha + (1-r)U = U + r(\alpha-U) > U + \frac{1}{2}(\alpha-U)r$$

$$h(r) \geq \min \begin{cases} (1-r)U \\ r\alpha+(1-r)U \end{cases} = U - rU > U - \alpha r .$$

<u>Lemma</u> 2.[*] If a, b $\in \Psi$, so does a', b' where

$$a' = 1 - a - U - \frac{b}{a+b} (1-2a) \qquad (11.1)$$

$$b' = - (1-a-U) + \frac{b}{a+b} (1-a) . \qquad (11.2)$$

<u>Proof</u>: Let R be the set of all r for which (10.1) holds. In (6), if we restrict the range of c to R, the sup cannot increase; then we may make replacements from (10).

$$f(r) > \max_{c \in R} \min \begin{cases} ra + (1-r)(U+ac) \\ r(1-a) + (1-r)(U-bc) \end{cases} . \qquad (12)$$

The two lines on the right are equal when c = c_o;

$$c_o = \frac{r}{1-r} \frac{1-2a}{a+b} .$$

As the upper line is an increasing function of c, and the lower one decreasing, c_o furnishes the max providing $c_o \in R$. But such is the case when $\frac{r}{1-r}$, and hence r, is positive and sufficiently small. When c = c_o the common value of the two lines in (12) is

$$U + r \left[1-a-U- \frac{b}{a+b} (1-2a)\right] = U + a'r .$$

Treating h(r) analogously leads to

$$d_o = \frac{r}{1-r} \frac{1-a}{a+b}$$

$$h(r) > U - r \left[-1+a+U+ \frac{b}{a+b} (1-a)\right] = U - b'r$$

---

[*] The underlying idea is due to Oliver Gross.

for r positive and small.

Finally

$$a' > 1 - \alpha - U - (1-2\alpha) = \alpha - U > 0 .$$

Lemma 3. $U \geq V$.

Proof. Suppose $U < V$. We will show that if we start with any member of ψ and construct a sequence of them by repeated applications of Lemma 2, we will be led to one with $b \leq 0$. This absurdity gives our result.

Let a, b, a', b' be as in Lemma 2 and

$$K = \frac{b}{a+b} , \qquad K' = \frac{b'}{a'+b'} .$$

Addition of the equations (11) gives

$$\frac{b'}{K'} = a' + b' = \alpha K \tag{14}$$

while (11.2) yields

$$K' = \frac{b'}{\alpha K} = \frac{-(1-\alpha-U)+K(1-\alpha)}{\alpha K} = \phi(K) .$$

We show that iteration by ϕ will ultimately lead to a $K \leq 0$ and thus a $b \leq 0$.

First, if $K > 0$, then

$$\phi(K) < K . \tag{15}$$

For (15) is true for sufficiently small positive K (then $\phi(K) < 0 < K$) and so a violation of (15) would imply a K_o such that

$$\phi(K_o) = K_o$$

or

$$\alpha K_o^2 - (1-\alpha)K_o + (1-\alpha-U) = 0 . \qquad (16)$$

But the discriminant of this quadratic is

$$(1-\alpha)^2 - 4\alpha(1-\alpha-U) = 2 - 6\alpha + 4\alpha U = 4\alpha(U - \frac{3-5\alpha}{2}) = 4\alpha(U-V) < 0 .$$

Secondly suppose, starting with any value, all the iterates of ϕ were positive. By (15) they are decreasing and so converge. The limit would be a root of (16).

Theorem 1. The value of the game is V.

Proof. As in the last few paragraphs of Section 2.

Corollary 1.1. $U = V$

Corollary 1.2. For each $\varepsilon > 0$, there is an ε—strategy for P which is an α—strategy.

This corollary solves half the problem of the marksman's best strategies. We now know how he is to aim; the remaining question is when is he to fire. For a further discussion of the α—strategies see Section 7.

The work of Scarf and Shapley in [4] tells that E has an optimal strategy for all r for both F_r and H_r and that these games have values. It follows, from the general principles of game theory, that the values must be $f(r)$ and $h(r)$.

4. The Ideal Strategies for P

We deal with the games F_r and H_r. They have the advantage of reducing P's decisions to choices only of when to fire. As discussed earlier, his near optimal strategies for these games suffice to yield at least some such strategies for our subject game. We will see later that this yield is more consummate than at first appears.

Assume $f(r)$ and $h(r)$ have been ascertained. How do they function in determining P's strategy? Consider P's situation in a play of, say, F_r. He has no choice as to his first firing probability, it being r (which we shall also call r_o). The derivation of the functional equation (6) makes it plausible that his next firing probability will be a value of c which furnishes a maximum to the right side. Select such a value and call it r_1. Suppose E's next move is straight. Then, if P has not yet fired, he is now faced with the game H_{r_1}. By the same reasoning as befor , a sensible choice for his next firing probability will be a maximizing d of (7) witn $r = r_1$. Select one such and label is r_2. Proceed thus. we will denote the strategies so generated (for H_r as well as F_r) collectively

by Q. It is clear that if P has an optimal strategy it must belong to Q. If he has not — as seems more likely — what is the role of Q?

At the conclusion of this paper will be found plots of $c(r)$ and $d(r)$, the maximizing c and d of (6) and (7). Playing Q amounts to successively iterating the r_j from these curves using the c or d one according as E's last move was a straight or a turn.

We now examine a typical play.[*] Suppose E moves as shown in Figure 7 and that P plays Q, obtaining the firing probabilities as shown. Put

$$\pi_n = \prod_{j=0}^{n} (1-r_j) ,$$

the probability that P has not fired at the first $n + 1$ opportunities. For convenience, we also define π_{-1} as 1. Then the probability of a hit in this play is
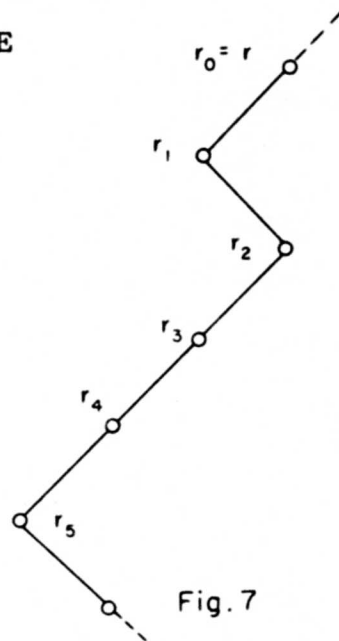

Fig. 7

$$\mathcal{H} = (\pi_{-1})r_0(1-\alpha) + \pi_0 r_1(1-\alpha) + \pi_1 r_2(0) + \pi_2 r_3 \alpha + \pi_3 r_4 (1-\alpha) + \cdots . \tag{17}$$

From the way in which the r_j were selected, we have

---

[*] Here and in later illustrations, E plays a pure strategy. Such suffices, for if P can overcome all pure strategies of E he can overcome a mixture.

$$\min \begin{cases} r_0a+(1-r_0)f(r_1) \\ r_0(1-a)+(1-r_0)h(r_1) \end{cases} = f(r_0)$$

$$\min \begin{cases} (1-r_1)f(r_2) \\ r_1(1-a)+(1-r_1)h(r_2) \end{cases} = h(r_1)$$

$$\min \begin{cases} (1-r_2)f(r_3) \\ r_2(1-a)+(1-r_2)h(r_3) \end{cases} = h(r_2)$$

$$\min \begin{cases} r_3a+(1-r_3)f(r_4) \\ r_3(1-a)+(1-r_3)h(r_4) \end{cases} = f(r_3)$$

$$\vdots$$

(18)

and with none but a Q strategy could these equalities be attained.
By a judicious selection of one of the lines on each left side, we
obtain from (18):

$$r_0(1-a) + (1-r_0)h(r_1) \geq f(r_0)$$

$$r_1(1-a) + (1-r_1)h(r_2) \geq h(r_1)$$

$$(1-r_2)f(r_3) \geq h(r_2)$$

$$r_3a + (1-r_3)f(r_4) \geq f(r_3)$$

$$\vdots$$

(19)

or rearranged and multiplied by the π_j:

$$\pi_{-1} r_0 (1-\alpha) \geq \pi_{-1} f(r_0) - \pi_0 h(r_1)$$

$$\pi_0 r_1 (1-\alpha) \geq \pi_0 h(r_1) - \pi_1 h(r_2)$$

$$0 \geq \pi_1 h(r_2) - \pi_2 f(r_3) \qquad (20)$$

$$\pi_2 r_3 \alpha \geq \pi_2 f(r_3) - \pi_3 f(r_4)$$

$$\vdots$$

The $\mathcal{N}_n$ below are the truncations of (17). From (20)

$$\mathcal{N}_0 = r_0(1-\alpha) \geq f(r) - \pi_0 h(r_1)$$

$$\mathcal{N}_1 \geq f(r) - \pi_1 h(r_2)$$

$$\mathcal{N}_2 \geq f(r) - \pi_2 f(r_3)$$

or in general

$$\mathcal{N}_n \geq f(r) - \pi_n (f \text{ or } h(r_{n+1})) . \qquad (21)$$

We now see at once

<u>Theorem</u> 2. A sufficient condition that a Q strategy attain the best possible value for P in a particular play is

$$\lim_{n \to \infty} \pi_n = 0 . \qquad (22)$$

This condition will be met should any $r_j = 1$ (certain firing). Otherwise — as is well known — it is tantamount to the divergence of $\sum_{j=0}^{\infty} r_j$.

Theorem 3. If P has an optimal strategy at all, it must be a strategy Q.

Proof: Suppose P plays a strategy not Q. Then in at least one of (18) the sign $=$ becomes $<$. Let E move straight or turn on each move according as the upper or lower line on the left side of the corresponding (18) is the smaller (either way in the case of equality). Then in (19), $\geq$ is replaced by $=$ or $<$, with at least one instance of the latter. Thus $\geq$ is replaced by $<$ in (21) for all large n.

Theorem 4. If $r < R$, then P has no optimal strategy.

Proof. Let P play Q. For $r < R$, the signs $\leq$ of (19) become $=$ and the same is true (21). Thus (22) is a necessary as well as a sufficient condition.

Let us take the case of F_r. Let E make all straight moves. Then

$$r_{j+1} = c(r_j) \leq kr_j$$

and so

$$r_j \leq k^j r$$

and so Σr_j converges.

For H_r, let E pick straight for his first free move. Then P is confronted by H_{r_1} with $r_1 < R_1$, and we revert to the preceding case.

Theorem 5. If $r \geq R$, then any strategy Q is optimal.

Proof. Here $r_j \geq R_1$ and so Σr_j diverges.

Remark. The most efficient strategy Q utilizes the functions $\bar{c}(r)$ and $\bar{d}(r)$. If $r > R$, it is not hard to see that finitely many iterations, at each stage by one or the other of these functions, will lead to an $r_j = 1$, thus terminating the play.

If $r = R$, Corollary 9.1 shows that $r_1 = R_1$. If E makes all straight moves henceforth, all the remaining $r_j = R_1$. This is the only instance where Q is optimal, yet cannot be made finite.

The role of Q now emerges. Aside from the uninteresting cases of larger r, Q is unique and not optimal, and indeed there is none such. If P deviates appreciably from Q, the inequalities, which at least some of (18) become, will be severe. Their compensation by (19) can be frustrated by E, because his moves decide which line on the left of (18) is to be effective. The result will be a payoff appreciably defective. We draw the following rough conclusion (these ideas will be dissected with more precision in the next section):

In general P has no optimal strategy, but any ϵ–strategy must be close to Q, the closeness increasing with the smallness of ϵ .

It seems apt to term a strategy with this property an ideal strategy. A precise statement is made by Theorem 7 below.

5. The Passive ϵ–Strategies

The case $r = 0$ is really our desideratum, for as we have seen in Section 2, it is, aside from the restriction to α–strategies, the original subject game. The strategy Q for it leads to the vapid situation: all the $r_j = 0$; P never fires.

We now turn to ϵ–strategies, taking some positive ϵ as given, with P seeking a payoff $> V - \epsilon$. The ϵ gives him license to depart from the sterility of the all zero r_j. Thus it is that we find use for F_r and H_r with $r > 0$.

Our procedure is a recurrent one, somewhat like that of the last section. But not only will P ascertain an r_{j+1} from r_j, but also an ϵ_{j+1} from ϵ_j. This ϵ_j is that circumscription on the jth move permitted him by the preassigned $\epsilon = \epsilon_o$ of the outset.

Let m_j be E's j^{th} move ($j = 0. 1, 2, \cdots$ so taken that m_o is the preassigned move indigenous to the game); $m_j =$ either "Straight" or "Turn." The quantities $r_o = r$, $\epsilon_o = \epsilon > 0$, and m_o (deciding between F_r and H_r) are given at the outset. At a

later stage, these will be known (of course, assuming P has not yet fired):

$$m_0, \; m_1, \; \cdots, \; m_j$$

$$r_0, \; r_1, \; \cdots, \; r_j$$

$$\varepsilon_0, \; \varepsilon_1, \; \cdots, \; \varepsilon_j \; .$$

It is now time for P to select r_{j+1}. We will say he is playing a strategy Q_ε if he does so as follows for each $j \geq 0$:

1) If $r_j \geq R$, he plays a strategy Q from this point on. Here R means R_1 if m_j was Straight and R_2 if m_j was Turn.

2) If $r_j < R$ he picks r_{j+1} so that, if m_j was Straight

$$\min \begin{cases} r_j \alpha + (1-r_j) f(r_{j+1}) \\ r_j(1-\alpha) + (1-r_j) h(r_{j+1}) \end{cases} > f(r_j) - \varepsilon_j \qquad (23)$$

and utilizes a corresponding inequality, which the reader will readily infer, if m_j was Turn.

If P has not yet fired, E now picks m_{j+1}.

If Case I held, P will have no need of further ε_j, but if II held, ε_{j+1} must be defined. Suppose, for example, m_{j+1} was Turn. Then we take

$$\varepsilon_{j+1} = h(r_{j+1}) - h(c(r_j)) + \frac{\varepsilon_j}{1-r_j} \; . \qquad (24)$$

There is an analogous equation for each of the other three possibilities of m_j, m_{j+1}.

The cycle is ready for repetition.

We must show that this process can be effected. First, the ε_j obtained will always be positive. In the instance displayed above ((23) and (24)), we know from the functional equation that

$$r_j(1-\alpha) + (1-r_j)h(c(r_j)) = f(r_j) .\qquad(25)$$

The combination of (25) and (23), using the lower line on the left, will show that the right side of (24) is positive. If $\varepsilon_0 > 0$, by induction, all the ε_j appearing will be positive. We observe that if Case I arises once, it does so thereafter. Finally, we see that an r_{j+1} can always be found satisfying (23)(or its analogue), for $c(r_j)$ is one such.

We now turn to the $\mathcal{H}_n$, the hit probabilities for a finite segment of initial stages, and handle them as we did in the previous section. Like (17) we find

$$\mathcal{H}_{n-1} = \sum_{j=0}^{n-1} \pi_{j-1}\, r_j\, a_j \qquad(26)$$

where a_j is α, $1 - \alpha$, or 0 according to the values of the pair m_j, m_{j+1}. If we suppose that Case II has held thus far and proceed in a manner similar to the derivation of (21), we obtain

$$\mathcal{H}_{n-1} = \left[f(r) \text{ or } h(r)\right] - \varepsilon - \pi_{n-1}\left\{\left[f_n(r_n) \text{ or } h_n(r_n)\right] - \varepsilon_n\right\}. \qquad(27)$$

Recall that P's objective is to obtain a payoff exceeding the first two terms on the right of (27). What (27) states is

this: If the play is interrupted so that P is confronted with F_{r_n} or H_{r_n}, he can attain his objective according as he obtains in the new subgame a payoff exceeding its value less ε_n.

From the initiation the reader had in the last section, he should have no trouble in completing the proof of

Theorem 6. Let P play Q_ε. If at any time Case I arises, then P will attain his objective; if it does not, P, after any finite number of moves, will be in a position such that it is possible for him to attain his objective. On the other hand, if at any time, when faced with Case II, P selects his firing probability in violation of (23), then E can prevent him from attaining his objective.

This theorem exposes the nature of Q_ε. We, of course, suppose $r < R$. Then, as long as Case II persists, we have seen that P is compelled to abide by Q_ε in order to play an ε-strategy. But the latter desideratum is by no means guaranteed. For example, we see that Q is a strategy Q_ε for every $\varepsilon > 0$. But we know Q is not optimal; hence it is not an ε-strategy if ε is sufficiently small. On the other hand, as long as P adheres to Q_ε he is safe, in that he has not forfeited the possibility of exceeding (value $-\varepsilon$) as the payoff.

His situation is much like that of a person asked to select the terms of infinite series one at a time in such a way that the series converges. After each individual selection the possibility

of convergence has not been destroyed. But this does not mean that the entire aggregate of selections will spell convergence.

We term a strategy with the above properties a passive ε-strategy.

We conclude with a sharp statement that Q is an ideal strategy but omit the proof. In case $r \geq R$ the strategies Q (and only they) are actually optimal and nothing more need be said. We assume $r < R$. In this case we know that Q is unique.

We wish to establish a measure of the closeness of two strategies. Suppose E adheres to some pure strategy in a particular play. Any mixed strategy U of P will result in a sequence of firing probabilities $\{r_0 = r, r_1, r_2, \cdots\}$. For a second strategy U', let the sequence be $\{r_0' = r_0 = r, r_1', r_2', \cdots\}$. We define

$$d_n(U, U') = \max_{0 \leq j \leq n} |r_j - r_j'|$$

and $D_n(U, U')$ by $\max d_n$ over all pure strategies of E.[*]

Theorem 7. Given $r < R$ and one of F_r, H_r, as well as an integer $n \geq 0$ and $\delta > 0$, then we can find $\bar{\varepsilon} = \bar{\varepsilon}(\delta, n)$ so that if U is ε-strategy for P with $\varepsilon < \bar{\varepsilon}$ then

$$D_n(Q, U) < \delta .$$

---

[*] Of course, we need consider only E's first n moves.

6. The ϵ-Strategies; The Unsolved Margin

We know that for P to play an ϵ-strategy with a small ϵ he must remain in the vicinity of a strategy Q_ϵ. But exactly how this should be done is still an open question. The following theorem and more particularly its corollary offer a lead.

<u>Theorem</u> 8. An ϵ-strategy can be executed in a finite number of moves.

<u>Proof</u>. If $r > R$, $Q_\epsilon = Q$ and we know the latter terminate in finitely many moves. If $r < R$, the ϵ-strategy begins with Case II and is thus Q_ϵ. If it switches to Case I it can be terminated. If not, we see by (27) that P can exceed $e_o(r) - \epsilon$ only if for some n, $e_n(r_n) - \epsilon_n < 0$. But this means the desired hit probability has been achieved in finitely many moves. Thus the ensuing r_j are irrelevant; we can take one of them $= 1$. (This strange situation can happen. For example, it happens at the very outset if ϵ is taken absurdly large.) The only remaining case — the r_j stay R_1 and E plays all straights — seems unimportant. No doubt, P can increase R_1 a mite and take a small loss $(< \epsilon)$ in payoff.

<u>Corollary</u> 8. Every play in which P employs an ϵ-strategy can be culminated with an $r_j = 1$.

A necessary condition for a strategy Q_ϵ not to stagnate then is that we find some means of avoiding persistenly small r_j.

To see what happens should the r_j (and also the ϵ_j) remain small, we might use linear approximations of f and h. From Section 2 these turn out to be

$$f(r) \cong V + Ar$$

$$h(r) \cong V - Br \ .$$

The expressions on the right actually are formal solutions to the functional equations with the maximizers

$$c: \quad V \frac{r}{1-r}$$

$$d: \quad 2 \frac{r}{1-r} \ .$$

(The first of these is corroborated by Section 2.) However, we will linearize to the full and use

$$c(r) \cong Vr$$

$$d(r) \cong 2r \ .$$

Let γ_j be c if m_j = Straight and d if m_j = Turn.

Now the basic inequality (23) of Q_ϵ condemns r_{j+1} to an interval containing $\gamma_j(r_j)$. We know that use of $\gamma_j(r_j)$ itself for r_{j+1} is Q and not even close to optimal. In fact it fails because the r_j can remain small (Theorem 4 and its proof). Therefore it seems sensible always to take r_{j+1} the part of the prescribed interval lying to the right of $\gamma_j(r_j)$. Let k_j be the fraction of $(r_{j+1}-\gamma_j(r_j))/(\bar{r}_{j+1}-\gamma_j(r_j))$ where $\bar{r}_{j+1}$ is

the right boundary of the interval. The task of P reduces to selecting the k_j. Using the approximations (30) and (32), the governing equations are found to be

$$r_{j+1} = b_j r_j + \frac{k_j}{B} \varepsilon_j$$

$$\varepsilon_{j+1} = (1 + k_j(1 - b_{j+1})) \varepsilon_j$$

where $b_j = V$ if m_j is straight and 2 if m_j is turn. We start with a given small r_0, ε_0 and compute the later ones recurrently by (33), each time selecting $k_j (0 \leq k_j < 1)$. At the same time E is selecting the m_j. When we apply (33) m_0, m_1, $\cdots$, m_j is known, but not m_{j+1}. Thus the value of b_{j+1} is unknown when k_j is chosen. In brief, the order of choices and computations is

$$m_0, \; r_0, \; \varepsilon_0 \; - \; \text{given}$$

k_0 chosen by P

r_1 computed

m_1 chosen by E (deciding b_1)

ε_1 computed

k_1 chosen by P

r_2 computed

etc.

The problem: To devise a scheme for selecting the k_j so
that no matter how E picks the m_j, the r_j will not remain small.

We have been unable to solve it.

At the least, the solution will point the way to ε-strategies.
Very likely, after developing some results that would furnish
bounds to errors arising from (30) and (32), it would actually
supply the ε-strategies. There exists an exact version of (33),
but, due to the complicated nature of f and h, it would be
difficult to write the right sides explicitly. Remark that
our previous results guarantee that the problem has a solution.

7. Motivation for the ω-strategies

Let us consider a miniature game in which P has but one
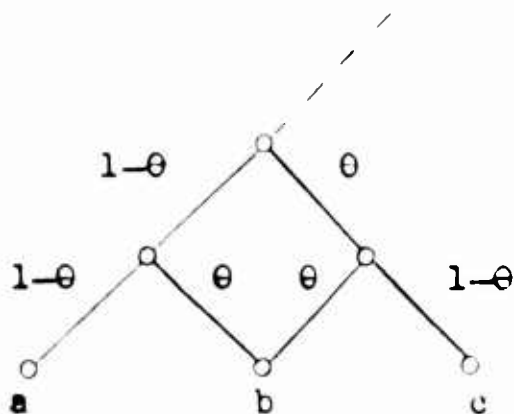chance to fire and three choices of where to aim. He uses a

Figure 8.

mixed strategy taking a, b, c as the probabilities as shown in
Figure 8. As for E, he makes just two moves and each of these

with probability θ for a turn; he does nought but select θ.
There is fictitious move preceding the first so that straight
and turn may be distinguished there.

The payoff is

$$a(1-\theta)^2 + \theta b + \theta(1-\theta)c \ .$$

The computation is omitted, but the solution is

opt. strat. for P: $a = \alpha, \ b = 1 - \alpha, \ c = 0$

opt. strat. for E: $\theta = V$

Value: V .

It follows that if in the original game we wish to fix the
aiming probabilities at all, they can only be fixed at the values
above. For otherwise E could find a θ in the little game render-
ing him a payoff $< V$. In the original game, then, E could play a
strategy like M but using this θ instead of V and attain the same
payoff.

Let us bear in mind that generally the parameters entailed in
an ε-strategy are not sharply delineated. For the player may choose
to play an ε'-strategy with $\varepsilon' < \varepsilon$. He can exploit the margin
in payoff by slight alterations in the quantities he controls.

Applying this principle to our case, we see that P will have
ε-strategies that are not α-strategies but close to α-strategies.
Has he any that are not close? If the answer is no, as seems
likely, the following conjecture seems reasonable.

Conjecture — Q is an ideal strategy not only among the α—strategies but among all the strategies of P.

8. The Truncated Versions

We can apply a technique here that proved useful in [2]; we amend the rules by requiring that P can fire only on the first n moves. The max min functions will be replaced by $f_n(r)$ and $h_n(r)$. The functional equations (6) and (7) become recurrence relations; we affix to each f or h the subscript n + 1 when it appears on the left side and n when it appears on the right. We may take $f_o(r)$ and $h_o(r)$ as 0; all the functions are then determined recurrently.

We may proceed as in [2]. The truncated games fall under the general tenets of game theory and we know at once that both players have optimal strategies. The functions $f_n(r)$, $h_n(r)$ increase with n, for P can apply an optimal strategy for the case with a small n to the case with a larger, filling in the residual moves arbitrarily. The proof of Lemma 5[*] is easily modified to show that the $f_n(r)$ and $h_n(r)$ are equi-continuous. Thus the $f_n(r)$ $[h_n(r)]$ converge uniformly to $f(r)$ $[h(r)]$.

One conclusion is evident:

Theorem 9. An ε—strategy can be found which terminates in n moves, the n depending only on ε.

---

[*] of RM-1385

9. The General "Chancified" Game and Functional Equation

We now relinquish P's constraint to α—strategies. We chancify as follows: P shall fire initially with probability r and his aiming probabilities shall be as shown in Figure (8). As before, a fictitious pre—initial move is requisite. It is required that E make his initial move straight (to the left in Figure 8). The max min (or value) we denote by $\phi(r, a, b)$.
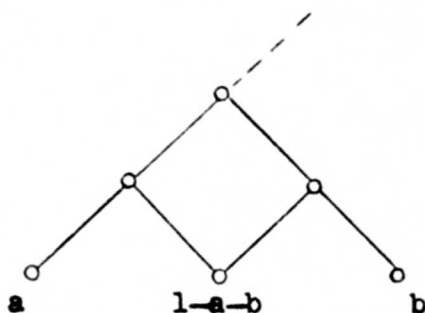

Figure 9

If we demand E make a turn initially, the corresponding function is $\phi(r, b, a)$. The functional equation for ϕ is derived along lines we have seen before; it is

$$\phi(r,a,b) = \max_{c,x,y} \min \begin{cases} ra & +(1-r)\phi(c,x,y) \\ r(1-a-b)+(1-r)\phi(c,y,x) \end{cases} .$$

Here the maximizers are of course subject to

$$0 \leq c \leq 1, \ x \geq 0, \ y \geq 0, \ x + y \leq 1 .$$

The connection with our earlier work lies in

$$f(r) = \phi(r,a,0) , \qquad h(r) = \phi(r,0,a)$$

and in these spec'al cases (34) reduces to (6) or (7).

REFERENCES

1. A Discrete Evasion Game, L. E. Dubins; Inst. Air Weapons Research, Technical Note No. 2.

2. A Game of Aiming and Evasion, R. Isaacs and S. Karlin; RAND Report No. RM-1316.

3. A Card Game with Bluffing, R. Isaacs; an unnumbered RAND report. Also Am. Math. Mo., Vol. 62, No. 2.

4. Games with Information Lag, H. Scarf and L. Shapley, RAND Report No. RM-1320.

The functions f(r) and h(r)
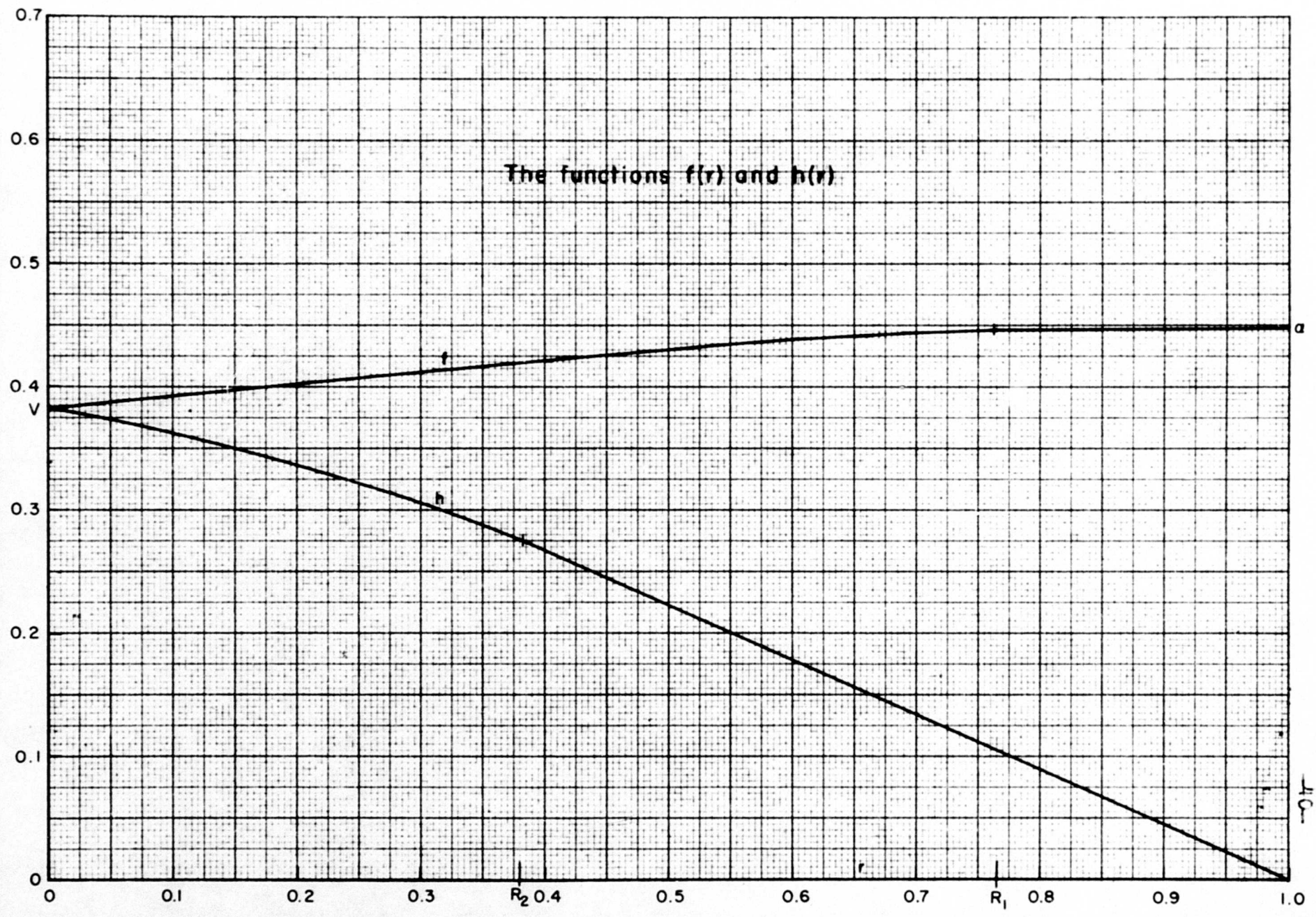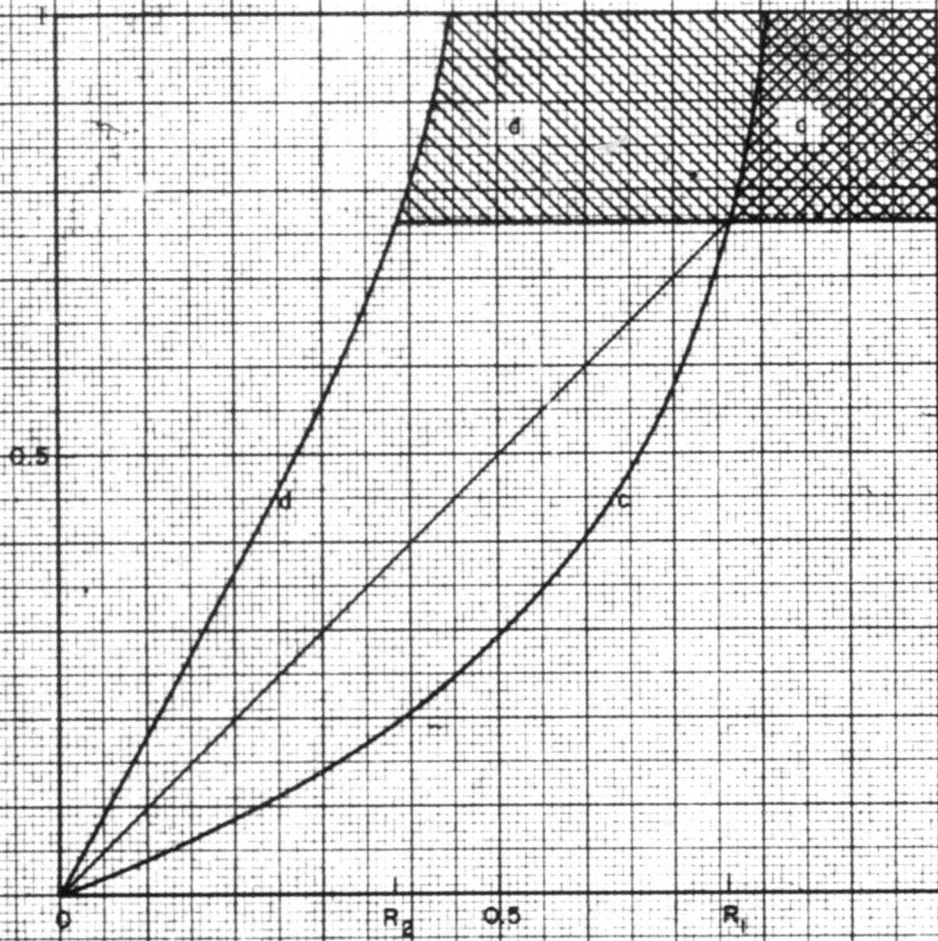
The Maximizing c and d

The shaded areas represent cases where the functions are not single valued. The 45° line is included to facilitate iteration.