CLEARINGHOUSE FOR FEDERAL SCIENTIFIC AND TECHNICAL INFORMATION CFSTI
DOCUMENT MANAGEMENT BRANCH 410.11

LIMITATIONS IN REPRODUCTION QUALITY

ACCESSION # 604 633

- [x] 1. WE REGRET THAT LEGIBILITY OF THIS DOCUMENT IS IN PART UNSATISFACTORY. REPRODUCTION HAS BEEN MADE FROM BEST AVAILABLE COPY.

- [ ] 2. A PORTION OF THE ORIGINAL DOCUMENT CONTAINS FINE DETAIL WHICH MAY MAKE READING OF PHOTOCOPY DIFFICULT.

- [ ] 3. THE ORIGINAL DOCUMENT CONTAINS COLOR, BUT DISTRIBUTION COPIES ARE AVAILABLE IN BLACK-AND-WHITE REPRODUCTION ONLY.

- [ ] 4. THE INITIAL DISTRIBUTION COPIES CONTAIN COLOR WHICH WILL BE SHOWN IN BLACK-AND-WHITE WHEN IT IS NECESSARY TO REPRINT.

- [ ] 5. LIMITED SUPPLY ON HAND: WHEN EXHAUSTED, DOCUMENT WILL BE AVAILABLE IN MICROFICHE ONLY.

- [ ] 6. LIMITED SUPPLY ON HAND: WHEN EXHAUSTED DOCUMENT WILL NOT BE AVAILABLE.

- [ ] 7. DOCUMENT IS AVAILABLE IN MICROFICHE ONLY.

- [ ] 8. DOCUMENT AVAILABLE ON LOAN FROM CFSTI (TT DOCUMENTS ONLY).

- [ ] 9.

NBS 9/64

PROCESSOR: PM

AD-604633

604633

(1)

NOTES IN THE THEORY OF
DYNAMIC PROGRAMMING – II:
A FUNCTIONAL EQUATION
ARISING IN ALLOCATION THEORY

by

Richard Bellman

P-631

25 January 1955

The RAND Corporation
1700 MAIN ST. • SANTA MONICA • CALIFORNIA

9

SUMMARY

~~We derive~~ The structure of the solution of

$$f(x) = \max_{0 \le y \le x} \left[g(y)+h(x-y)+f(ay+b(x-y)) \right] ,$$

in the case where g and h are concave is derived.

NOTES IN THE THEORY OF DYNAMIC PROGRAMMING - II: A FUNCTIONAL EQUATION ARISING IN ALLOCATION THEORY

by

Richard Bellman

§1. Introduction

An equation we have used for illustrative purposes in a number of papers is

$$(1) \qquad f(x) = \max_{0 \le y \le x} \left[g(y)+h(x-y)+f(ay+b(x-y)) \right] .$$

This equation arises in connection with a multi-stage allocation process which we shall describe below.

The purpose of the present note is to complete the description of the solution we have given elsewhere, [1], for the case where g and h are concave, monotone increasing functions. The method of successive approximations that we employ is a very powerful analytic tool for the treatment of functional equations of this general class.

§2. Description of the Process

We are given a resource, x, to divide into two parts, y and x-y. From y we obtain a return of g(y); from x-y a return of h(x-y). In so doing we expend a certain amount of the original quantity and are left with a new quantity ay+b(x-y), where $0 < a, b < 1$. This process is now continued indefinitely. How does one allocate at each stage so as to maximize the overall return from the process?

If we let $f(x)$ denote the overall return obtained employing an optimal policy, it is easy to see that an application of the principle of optimality, cf. [2], [3], yields the functional equation in (1.1).

§3. The Character of the Solution

The result we wish to prove is

Theorem. *Let us assume that*

(1) a. $g(x)$ *and* $h(x)$ *are both strictly concave for* $x \geq 0$, *monotone increasing, and* $g(0) = h(0) = 0$.

b. $g'(0)/(1-a) > h'(0)/(1-b)$, $h'(0) > g'(\infty)$, $b > a$.

Then the optimal policy has the following form:

(a) $y = x$ *for* $0 \leq x \leq \bar{x}$, *where* $\bar{x}$ *is the root of*

$$h'(0) = g'(x)+(b-a)g'(ax)+(b-a)ag'(a^2x)+ \dots \tag{2}$$

(b) $y = y(x)$ *for* $x \geq \bar{x}$ *where* $0 < y(x) < x$ *and* $y(x)$ *is the solution of*

$$g'(y)-h'(x-y) + (a-b)f'(ay+b(x-y)) = 0 \quad . \tag{3}$$

Remark: We have given the result only for one set of the possible inequalities connecting $g'(0)$ and $h'(0)$. It is easy to see that the same methods suffice to cover the other cases.

Proof: Let us proceed by successive approximations. For the results we shall tacitly assume concerning the existence and uniqueness of the solution of (1.1) and the convergence of the successive approximants, we refer to [1], and [3]. The criterion above in (3) for the determination of $y(x)$ is constructive since we can always obtain $f(x)$ directly by means of an iterative process. Here, however, we are primarily interested in the structure of the solution.

Set

$$f_1(x) = \max_{0 \le y \le x} \left[g(y) + h(x-y) \right] . \tag{4}$$

Since, by assumption, $g'(0) > h'(0)$, for small x we have $g'(y) - h'(x-y) > 0$ for $0 \le y \le x$. Hence $g(y) + h(x-y)$ is monotone increasing in $0 \le y \le x$ and the maximum occurs at $y = x$. As x increases, the equation $g'(y) - h'(x-y) = 0$ will ultimately have a root at $y = x$, and then a root inside the interval $[0,x]$. The critical value of x is given as the solution of $g'(x) - h'(0) = 0$. This equation has precisely one solution, which we call x_1. For $x \ge x_1$ let $y_1 = y_1(x)$ be the unique solution of $g'(y) = h'(x-y)$. The uniqueness is a consequence of the concavity assumptions concerning g and h.

Thus we have

$$\begin{aligned} f_1(x) &= g(x) , && 0 \le x \le x_1 , \\ &= g(y_1) + h(x-y_1) , && x \ge x_1 . \end{aligned} \tag{5}$$

and

$$(6)\qquad f_1'(x) = g'(x) \;, \qquad 0 \leq x < x_1$$

$$= [g'(y_1)-h'(x-y_1)]\frac{dy_1}{dx} + h'(x-y_1) = h'(x-y_1),$$

for $x > x_1$.

Since $y_1(x_1) = x_1$, we see that $f_1'(x)$ is continuous at $x = x_1$, and hence, for all values of $x \geq 0$.

Now let us turn to the second approximation

$$(7)\qquad f_2(x) = \max_{0 \leq y \leq x} [g(y)+h(x-y)+f_1(ay+b(x-y))] \;.$$

The critical function is now $D(y) = g'(y)-h'(x-y)+f_1'(ay+b(x-y))(a-b)$. Since $g'(0)-h'(0)+f_1'(0)(a-b) = g'(0)-h'(0)+g'(0)(a-b) >$ $h'(0)[(1-a)(1+a-b)/(1-b)-1] > 0$, we see that $D(y)$ is again positive for all y in $[0,x]$ for small x. Hence the maximum occurs in (7) at $y = x$ for small x. As x increases, there will be a first value of x where $D(x) = 0$. This value, x_2, is determined by the equation $g'(x) = h'(0)+(b-a)f_1'(ax)$. Comparing the two equations

$$(8)\qquad \begin{aligned} g'(x) &= h'(0) \\ g'(x) &= h'(0) + (b-a)f_1'(ax) \;, \end{aligned}$$

we see that $0 < x_2 < x_1$.

Hence the equation for x_2 has the simple form

$$(9)\qquad g'(x) = h'(0) + (b-a)g'(ax) \;.$$

Thus $y = x$ for $0 \leq x \leq x_2$ in (7) and $y = y_2(x)$ for $x \geq x_2$, where $y_2(x)$ is the unique solution of

(10) $$g'(y) = h'(x-y) + (b-a)f_1'(ay+b(x-y)) \quad .$$

Furthermore

(11) $$f_2'(x) = g'(x) \quad , \qquad 0 \leq x \leq x_2$$
$$= h'(x-y_2) + b\, f_1'(ay_2+b(x-y_2)) \quad , \qquad x \geq x_2 \quad ,$$

and $f_2'(x)$ is continuous at $x = x_2$.

Comparing (10) with the equation $g'(y) = h'(x-y)$ defining y_1, we see that $y_2(x) < y_1(x)$. In order to carry out the induction and obtain the corresponding results for all members of the sequence $\{f_n\}$, defined recurrently by the relation $f_{n+1} = \underset{0\leq y\leq x}{\text{Max}} [\, g(y)+h(x-y)+f_n(ay+b(x-y)) \,]$, we require the essential inequality $f_2'(x) \geq f_1'(x)$. There are three intervals $[0,x_2]$, $[x_2,x_1]$, $[x_1,\infty]$, to examine, each one requiring a separate argument. Using (10) and (11) we have

(12) $$f_2'(x) = \frac{bg'(y_2) - ah'(x-y_2)}{b-a}$$

for $x \geq x_2$. Combining (6) and the equation for y_1 we have

(13) $$f_1'(x) = \frac{bg'(y_1) - ah'(x-y_1)}{b-a}$$

The function $[\, bg'(y)-ah'(x-y) \,]/(b-a)$ is monotone decreasing in y for $0 \leq y \leq x$. Since $y_2 < y_1$ we see that $f_2'(x) > f_1'(x)$. This completes the proof for the interval $[x_1,\infty]$. The interval $[0,x_2]$ yields equality. The remaining interval is $[x_2,x_1]$. In this interval, we have

(14) $$f_1'(x) = g'(x)$$
$$f_2'(x) = \frac{bg'(y_2) - ah'(x-y_2)}{b-a} .$$

Hence in this interval, since $0 \leq y_2 \leq x$,

(15) $$f_2'(x) \geq \frac{bg'(x) - ah'(0)}{b - a} \geq g'(x) ,$$

since $g'(x) \geq h'(0)$ is a consequence of $g'(y) \geq h'(x-y)$ for $0 \leq y \leq x$ and $0 \leq x \leq x_1$. This completes the proof that $f_2'(x) \geq f_1'(x)$.

We now have all the ingredients of an inductive proof which shows that

(16) a. $x_1 > x_2 > \dots x_n > \dots > 0$

b. $f_1'(x) < f_2'(x) < f_n'(x) < \dots$

c. $y_1(x) > y_2(x) > \dots$

Since $f_n(x)$ converges to $f(x)$, $f_n'(x)$ to $f'(x)$, $y_n(x)$ to $y(x)$ and x_n to $\bar{x}$, we see that the solution has the indicated form.

BIBLIOGRAPHY

1. R. Bellman, "A Problem in the Theory of Dynamic Programming", Econometrica, January 1954, pp. 37-48.

2. ———————, "The Theory of Dynamic Programming", Bull. Amer. Math. Soc., Vol. 60 (1954), pp. 503-516.

3. ———————, "An Introduction to the Theory of Dynamic Programming", RAND Report No. R-245, 1953.

/md