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DYNAMIC PROGRAMMING AND A NEW FORMALISM IN THE THEORY OF INTEGRAL EQUATIONS

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Summary

It is shown that the functional equation technique of the theory of dynamic programming may be used to derive functional differential equations for the characteristic values of a certain integral equation $\lambda u(s) = \int K(s,t)u(t)dt$, similar to those obtained for the eigenvalues of differential equations.

Dynamic Programming and a New Formalism in the Theory of Integral Equations

By

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1. Introduction.

In a series of papers [1],[2],[3], and in two monographs,[4],[5], we have treated various aspects of the mathematical theory of multi-stage decision processes, a rather imposing title which we have shortened to 'dynamic programming'. In several of these, notably [6],[7], we have shown that the calculus of variations may be viewed as a multi-stage decision process of continuous type, and that the functional equation technique which we have employed in connection with various other continuous processes may be utilized in this discipline to yield new results and new derivations of old results in the one-dimensional case.

In the process of extending these methods to include multi-dimensional variational problems, the theory of functional analysis enters in a very natural way since boundary and initial values are now functions rather than numbers.

It turns out that an excellent testing ground for these new tools is the theory of integral equations which combines the new aspects of functional analysis with greater analytic simplicity than afforded by partial differential equations. In addition, the results are of independent interest.

We shall sketch below an application of the theory of dynamic programming to the problem of obtaining a functional equation for

the characteristic values of the integral equation

$$\lambda u(t) = \int_0^T K(s,t)u(s)ds \tag{1.1}$$

Following this line of thought, we shall in a subsequent work develop corresponding results for some multidimensional problems in the calculus of variations.

2. Causality and Optimality.

As is discussed and illustrated by means of a wealth of examples in Hille's treatise, [8], a large part of analysis may be very properly, profitably and naturally subsumed under the general theory of semi-groups of operations. Here the basic functional equation for a multi-stage process is

$$f(P;\ s + t\ ) = f(\ P(s);\ t), \tag{2.1}$$

which, as remarked by Hadamard, expresses the principle of causality, cf. Hille, [9].

If we consider a multi-stage decision process, where there are many alternative routes to be pursued, the analogue of the above equation is

$$f(P;\ s + t\ ) = \max_Q f(\ P(s,Q);\ t) \tag{2.2}$$

which expresses the principle of optimality, cf [3], [6].

As we have shown elsewhere, [5], [6], various results in the calculus of variations and in eigenvalue problems connected with differential equations may be obtained from (2.2) and its

limiting form as $s \to 0$, the "infinitesmal generator" of (2.2).

We now turn to the application of (2.2) to the theory of integral equations.

3. Integral Equations.

Let $K(s,t)$ be a symmetric kernel belonging to L^2 over the square $0 \leq s,t \leq T$, which is assumed to be positive definite. The characteristic values of (1.1) are then equivalent to the relative minima of $\int_0^T u^2 dt$ subject to the constraint that $\int\int_0^T K(s,t)u(s)u(t)dsdt = 1$. In order to convert this minimization problem into one with the proper invariant qualities, we employ the device used in [6] and [7] and imbed the problem within the larger problem of determining the relative minima of the functional

$$J(a;\ r(t),u(t)) = \int_a^T u^2(t)dt, \tag{3.1}$$

subject to the constraint

$$\int_a^T \int_a^T K(s,t)u(s)u(t)dsdt + 2\int_a^T r(t)u(t)dt = 1 \tag{3.2}$$

where $r(t) \in L^2(0,T)$. Let us for the sake of simplicity consider only the absolute minimum and define the function $f(a;r(t))$ as follows:

$$f(a;\ r(t)) = \operatorname{Min}_u J(a;r(t),u(t)) \tag{3.3}$$

The function f is "une fonction des lignes" in the sense of Volterra, [11].

Let us now proceed, using the principle of optimality embodied in (2.2) to obtain an approximate functional equation for f, under the assumptions of continuity for $K(s,t)$ and $r(t)$ as functions of s and t.

Let h be a small positive quantity. Then, setting for typographical convenience $v = u(a)$, we have

$$\text{(a)} \quad f(a;r(t)) = hv^2 + \int_{a+h}^{T} u^2 dt + o(h),$$

$$\text{(b)} \quad \int_{a+h}^{T}\int_{a+h}^{T} K(s,t)u(s)u(t)\,ds\,dt + 2\int_{a+h}^{T}[r(t) + hK(a,t)]u(t)\,dt = 1 - hr(a)v + o(h), \tag{3.4}$$

Introducing the change of variable, $u(t) = (1-hr(a)v/2)w(t)$, in order to normalize the relation of (3.4b), we see that the problem of choosing $u(t)$ in $[a+h,T]$ is a problem of precisely the same form as the original. Hence we obtain the approximate functional equation

$$f(a;r(t)) = \min_{v}\left[hv^2 + (1-hr(a)v)f(a+h;(r(t)+hK(a,t)(1+hr(a)v/2))\right] + o(h). \tag{3.5}$$

To obtain a limiting functional equation of differential form, we must employ functional derivatives.

4. A Gateaux Differential Equation.

The appropriate functional derivative to employ here is the Gateaux derivative, rather than the Volterra. We shall employ essentially the notation of Hille, [10], and write $\delta_h(f)$ for the Gateaux derivative of f with increment h,

$$\delta_h(f) = \lim_{\epsilon \to 0} \frac{f(a;r(t)+\epsilon h(t)) - f(a;r(t))}{\epsilon} \tag{4.1}$$

Using this notation, the limiting form of (3.5) is

$$f_a = - \min_v \left[v^2 + vr(a)\, \delta_{r(t)}(f) + \delta_{K(a,t)}(f) \right] , \qquad (4.2)$$

where f_a is the usual partial derivative.

This functional differential equation may be used to provide a sequence of successive approximations to f, which can be made to converge montonically if we choose an initial approximation in policy rather than function space, cf. [7].

Furthermore, due to the simple dependence of the expression in (4.2) upon v, the function f may be eliminated and an equation for $v = v(a; r(t))$, also involving Gateaux differentials, derived. This equation will have a "characteristic theory", and, as in the case of the calculus of variations, the characteristics will be associated with the Euler equation obtained by classical variational techniques. These topics will be discussed in subsequent papers.

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