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CONSTANT STRAIN WAVES IN STRINGS

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SUMMARY

A non-linear theory is developed for constant-strain waves in elastic strings. The speed of longitudinal and transverse waves is related to the strain and tension. The results can be used to calculate tension due to impact and thus breaking loads. Some generalizations are suggested.

INTRODUCTION: EQUATIONS OF MOTION

If it is assumed that the tension T in a string is given by

$$T = T_o + F(\sigma) \qquad (1)$$

where

T_o = initial tension = constant

σ = strain and $F(\sigma) = E\,\sigma$

then solutions to the equations of motion can be given for the propagation of waves of constant strain and tension. Such solutions are not restricted to small displacements or slopes and are hence of special interest. The waves which propagate consist of straight line segments which move with constant velocity. The accelerations are thus zero except at points where the velocity jumps. At these points the acceleration is infinite. A convenient way to discuss the speed of propagation is by considering the differential equations of motion. The differential equations can be considered to contain the impulse-momentum laws. Another approach is to apply the impulse-momentum laws directly.

Newton's laws of motion applied to an infinitesimal element of string, are

$$\rho u_{tt} = (T \cos \theta)_x \qquad (2)$$

$$\rho v_{tt} = (T \sin \theta)_x \qquad (3)$$

where (see Figure 1)

u, v = horizontal and vertical displacements respectively of point located at x in unstrained string.

θ = inclination of string

ρ = mass per length = constant

No external forces are considered. The horizontal projection of an element originally of length Δx is $(1 + u_x)\ \Delta x$; the vertical projection $v_x\ \Delta x$; and its length $\Delta s = \sqrt{(1 + u_x)^2 + v_x^2}\ \Delta x$. Therefore

$$\sin\theta = \frac{v_x}{\sqrt{(1 + u_x)^2 + v_x^2}} \tag{4}$$

$$\cos\theta = \frac{(1 + u_x)}{\sqrt{(1 + u_x)^2 + v_x^2}} \tag{5}$$

$$\sigma = \frac{\Delta s - \Delta x}{\Delta x} = \sqrt{(1 + u_x)^2 + v_x^2} - 1 \tag{6}$$

The characteristic lines, which represent the locus of possible discontinuities or wave fronts, can be found for the system of equations (1) - (6). Two sets of characteristics appear representing different wave velocities. These speeds are

$$C_1 = \sqrt{\frac{1}{\rho}\frac{dF}{d\sigma}} = \sqrt{\frac{E}{\rho}} \tag{7}$$

$$C_2 = \sqrt{\frac{1}{\rho}\,\frac{T_o + F(\sigma)}{1 + \sigma}} = \sqrt{\frac{1}{\rho}\,\frac{T_o + E\sigma}{1 + \sigma}} \tag{8}$$

C_1 is the speed of longitudinal waves in a solid bar of the same material.

C_2, for small strains, becomes the speed of transverse waves in a string.

For most applications $C_1 > C_2$. It should be noted that one speed of propagation is variable and depends on the strain σ. The details of the characteristics are not shown here since the same result about propagation speeds is derived by physical considerations in paragraph 2.

The general solution for constant strain waves can be written in the form

$$v\ (x,\ t) = \text{linear function of } (x,\ t)$$

$$u\ (x,\ t) = \text{linear function of } (x,\ t)$$

Then since v_x, u_x, v_t, u_t are constants it is easily seen from (1) - (6) that θ, σ, T are constants and that the differential equations (2) and (3) are satisfied trivially, except where the velocity jumps. The simplest example leading to waves of this type is discussed below.

RADIATION PROBLEM: SEMI-INFINITE STRING

Consider the string at rest initially with zero displacement

$$u = u_t = v = v_t = 0 \qquad t = 0 \tag{9}$$

Also, consider at first an infinite string $-\infty < x < \infty$. Let a constant vertical velocity be imported at the point where x = 0 when t = 0. Then by symmetry u = 0 at that point for all time and hence x = 0 there. The boundary conditions are thus, for t > 0.

$$v\ (0,\ t) = V = \text{constant} \tag{10}$$

$$u\ (0,\ t) = 0$$

It is now sufficient to consider x > 0. The solution of the correct form is

$$v = \begin{cases} V\ (t - \frac{x}{c_v}) & t > \frac{x}{c_v} \\ 0 & t < \frac{x}{c_v} \end{cases} \tag{11}$$

$$u = \begin{cases} \dfrac{Ux}{c_v} & t > \dfrac{x}{c_v} \\ \dfrac{U\left(t - \dfrac{x}{c_u}\right)}{1 - \dfrac{c_v}{c_u}} & \dfrac{x}{c_u} < t < \dfrac{x}{c_v} \\ 0 & t < \dfrac{x}{c_u} \end{cases} \qquad (12)$$

where

c_u = velocity u waves, and c_u = velocity of v waves.

The various regions in the (x, t) diagram and the shape of the waves are shown in Figure 2. The shape of the waves was determined by the fact that $c_v < c_u$ and that $u = 0$ at $x = 0$ and for $x > c_u t$. It can be shown, and is verified below, that the momentum equations cannot be satisfied if c_v is assumed greater than c_u. The undetermined constants in (11) and (12) are U, c_v, c_u. These are computed by applying the horizontal impulse-momentum law along $x = c_u t$ and the horizontal and vertical laws along $x = c_v t$.

Before doing this it is convenient to compute the slope and the strains which are constant in the various regions. We have

$$u_{1_x} = \frac{U}{c_v} \qquad v_{1_x} = -\frac{V}{c_v}$$

$$u_{2_x} = \frac{-\dfrac{U}{c_u}}{1 - \dfrac{c_v}{c_u}} \qquad v_{2_x} = 0 \qquad (13)$$

$$\sigma_1 = \sqrt{\left(1 + \frac{U}{c_v}\right)^2 + \frac{V^2}{c_v^2}} - 1 \qquad \sin\theta_1 = \frac{-\dfrac{V}{c_v}}{\sqrt{\left(1 + \dfrac{U}{c_v}\right)^2 + \dfrac{V^2}{c_v^2}}}$$

$$\sigma_2 = -\frac{\dfrac{U}{c_u}}{1 - \dfrac{c_v}{c_u}}$$

$$\theta_2 = \theta_3 = \sigma_3 = 0 \qquad \cos\theta_1 = \frac{1 + \frac{U}{c_v}}{\sqrt{(1 + \frac{U}{c_v})^2 + \frac{V^2}{c_v^2}}}$$

The subscripts denote the regions shown in Figure 2.

Consider now the vertical motion. Formally, we may write

$$T \sin\theta = T_1 \sin\theta_1 \, H\,(t - \frac{x}{c_v}) \tag{14}$$

where

$$H\,(z) = \text{unit step function} = \begin{matrix} 1 & z > 0 \\ 0 & z < 0 \end{matrix}$$

Thus (3) becomes

$$\rho v_{tt} = - \frac{1}{c_v} T_1 \sin\theta_1 \, \delta\,(t - \frac{x}{c_v}) \tag{15}$$

where δ = delta function. Integration of (1ʳ with respect to t gives the velocity jump across $x = c_v t$

$$- \frac{1}{c_v} T_1 \sin\theta_1 = \rho v_{1_t} \quad \text{which by (11)} = \rho V \tag{16}$$

Similarly

$$T \cos\theta = T_1 \cos\theta_1 \, H\,(t - \frac{x}{c_v}) + T_2 \left\{ H\,(t - \frac{x}{c_v}) - H\,(t - \frac{x}{c_v}) \right\} + T_3 \left\{ 1 - H\,(t - \frac{x}{c_u}) \right\} \tag{17}$$

and

$$\frac{1}{c_u}(T_3 - T_2) = \rho\,(u_{2_t} - u_{3_t}) \text{ which by (14)} = \frac{\rho U}{1 - \frac{c_v}{c_u}} \tag{18}$$

$$\frac{1}{c_v} T_2 - \frac{1}{c_v} T_1 \cos\theta_1 = \rho\,(u_{1_t} - u_{2_t}) \text{ which by (14)} = \frac{-\rho U}{1 - \frac{c_v}{c_u}} \tag{19}$$

Now since σ_1, σ_2 are known in terms of (U, c_v, c_u), T_1 and T_2 are known functions of U, c_v, c_u and the Eqs. (16), (18) and (19) determine U, c_v, c_u. That is, T_1 and T_2 can be expressed as

$$T_1 = T_o + E\,\sigma_1 = T_o + E\left\{\sqrt{\left(1 + \frac{U}{c_v}\right)^2 + \frac{V^2}{c_v^2}} - 1\right\}$$

$$T_2 = T_o + E\,\sigma_2 = T_o - E\,\frac{\frac{U}{c_u}}{1 - \frac{c_v}{c_u}} \tag{20}$$

$$T_3 = T_o$$

Thus (18) becomes

$$\frac{\rho U}{1 - \frac{c_v}{c_u}} = \frac{1}{c_u}\,E\,\frac{\frac{U}{c_u}}{1 - \frac{c_v}{c_u}}$$

from which c_u may be found as

$$c_u = \sqrt{\frac{E}{\rho}} = C_1 \tag{21}$$

It follows that (16) and (19) are two equations for U and c_v, and these become

$$\begin{aligned} T_1 \sin\theta_1 &= -\rho\,Vc_v \\ T_1 \cos\theta_1 &= T_o - \rho\,Uc_u \end{aligned} \tag{22}$$

or

$$\tan\theta_1 = \frac{-\rho\,Vc_v}{T_o - \rho\,Uc_u} \quad \text{which by (13)} = \frac{-\frac{V}{c_v}}{1 + \frac{U}{c_v}} \tag{23}$$

$$T_1^2 = (\rho\,Vc_v)^2 + (T_o - \rho\,Uc_u)^2 = (T_o + E\,\sigma_1)^2 \tag{24}$$

(23) can be solved for U to yield

$$U = \frac{c^2 - c_v^2}{c_u + c_v} \tag{25}$$

where $c^2 = \frac{T_o}{\rho}$ = speed of v waves in a small displacement theory.

Using (25), (24) is the equation for c_v.

Before solving for c_v it is useful to find expressions for the strain. Using (25)

$$\sigma_1 = \sqrt{\left(1 + \frac{U}{c_v}\right)^2 + \frac{V^2}{c_v^2}} - 1 = \frac{1}{c_v}\sqrt{\left[\frac{c^2 + c_u c_v}{c_u + c_v}\right]^2 + V^2} - 1 \qquad (26)$$

also

$$T_o - \rho U c_u = \rho c_v \left[\frac{c^2 + c_u c_v}{c_u + c_v}\right] \qquad (27)$$

Hence (24) becomes

$$T_1^2 = \rho^2 c_v^2 \left\{c_v^2 (1 + \sigma_1)^2\right\} = (T_o + E\,\sigma_1)^2 \qquad (28)$$

which can be solved for σ_1

$$\sigma_1 = \frac{c_v^2 - c^2}{c_u^2 - c_v^2} = \frac{-U}{c_u - c_v}, \text{ or } c_v = C_2 = \sqrt{\frac{c^2 + \sigma_1 c_u^2}{1 + \sigma_1}} \qquad (29)$$

A comparison of (29) and (20) yields the interesting result that

$$\sigma_1 = \sigma_2 \text{ and } T_1 = T_2 \qquad (30)$$

The strain and tension have the same constant value over the entire disturbed part of the string.

Now using (27) and (29), equation (24) yields the following quartic equation for c_v (V, c_u, c).

$$(c_u - c_v)^2 (c^2 + c_u c_v)^2 + V^2 (c_u^2 - c_v^2)^2 = c_v^2 (c_u^2 - c^2)^2 \qquad (31)$$

When c_v is found from this equation the strain is given by (29). Two limiting cases are of interest

(i) V is small; (31) shows that as $V^2 \to 0$ a solution exists independent of V, namely $c_v = c$. The strain is zero and this theory goes over to the usual linearized theory.

(ii) $c^2 \to 0$; waves are sent out in a string whose initial tension is zero. Writing

$$\left\{\begin{array}{l} V = \alpha\, c_u \\ c_v = \lambda\, c_u \end{array}\right\} \qquad (32)$$

(33) becomes

$$\lambda^4 (1 + \alpha^2) - 2 \lambda^3 - 2 \alpha^2 \lambda^2 + \alpha^2 = 0 \qquad (33)$$

For $\alpha, \lambda << 1$ an approximate solution to (33) can be obtained by neglecting the λ^4 term in (33). Solving the resulting cubic equation yields

$$\lambda = \frac{\alpha^{2/3}}{2^{1/3}} + \Theta (\alpha^2) \qquad (34)$$

Hence, approximately

$$c_v = (\frac{c_u}{2})^{\frac{1}{3}} V^{\frac{2}{3}} \qquad (35)$$

The last result clearly shows the non-linearity dependence of transverse wave speed on incoming velocity. The corresponding expression for strain is

$$\sigma_1 = \frac{1}{(\frac{c_u}{V})^{\frac{4}{3}} \cdot 2^{\frac{2}{3}} - 1} \doteq \frac{1}{2^{\frac{2}{3}}} (\frac{V}{c_u})^{\frac{4}{3}} \qquad (36)$$

Now $T_1 = T_2 = E \sigma_1$ so that (36) can be used to compute the velocity of motion which would break a given string.

It can be noted that the stress and strain are independent of the area of string. The inclination of the string is given by

$$\sin \theta_1 = - \frac{1}{2^{1/3}} (\frac{V}{c_u})^{5/3} \left\{ \frac{1}{2^{2/3}} (\frac{c_u}{V})^{4/3} - 1 \right\} \doteq - \frac{1}{2} (\frac{V}{c_u})^{1/3} \qquad (37)$$

and the constant force required to produce the wave is

$$F_1 = - T_1 \sin \theta_1 = \frac{E}{2^{1/3}} (\frac{V}{c_u})^{5/3} \qquad (38)$$

Some typical numerical values are those for 3/16'' carbon steel cable where $E = 4.7 \times 10^5$ lbs, $c_u = 17{,}050$ ft/sec; if $V = 690$ ft/sec

$$T_1 = 3900 \text{ lbs} \qquad (39)$$

Reflection problems can be treated in the same way. It can be shown that the reflections of the longitudinal wave at an end travels back with the speed C_1, relative to the original coordinate x. Upon reflection from a fixed end $(u = 0)$ the strain and hence that part of the tension due to strain is doubled. Upon reflection from a free end the strain becomes zero and the tension is then T_o.

The reflection of the returning longitudinal wave with the on-coming transverse wave can also be worked out in the same way.

CONCLUDING REMARKS

It has been shown how the problem of propagation and reflection of constant-strain waves in an idealized elastic string can be treated. The results are especially interesting since they are not limited to small deflections. Longitudinal and transverse waves occur which travel with different speeds relative to the string. One limiting case shows how a wave propagates because of elasticity in a string with zero initial tension.

The results have a practical application in estimating the suddenly applied constant force which will break a rope. An example of this was presented for zero initial tension which clearly shows the non-linear dependence of tension on force (Eqs. (38) - (40)). It should be remarked that the results for the idealized string are independent of the cross-section area, so long as the string behaves in a similar manner.

The case of varying velocity applied to a string can be treated by replacing the velocity curve by segments of constant velocity. Then the actual motion can be approximated by various constant-strain waves. Similar problems occur where a load moves as a string.