A NOTE ON PRIMITIVE MATRICES*

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Suppose that A is a square matrix consisting of nonnegative elements. In certain considerations it is important to know when all the elements of some power of A are strictly positive. Frobenius [2][2] gave a very simple necessary and sufficient condition for this to happen. In this note we give a simple proof of this result. Our proof is algebraic in nature and avoids the use of the convergence of powers of a matrix.

All matrices considered here will have real elements. For two such matrices (not necessarily square) $B = (b_{ij})$, $C = (c_{ij})$ we define

$$B \geqq C \quad \text{if} \quad b_{ij} \geqq c_{ij} \quad \text{for each } i, j.$$

$$B \geq C \quad \text{if} \quad B \geqq C \text{ but } B \neq C$$

$$B > C \quad \text{if} \quad b_{ij} > c_{ij} \quad \text{for each } i, j.$$

A square matrix $A \geqq 0$ (A is then called nonnegative) is said to be <u>indecomposable</u> if for no permutation matrix P does

$$PAP^{-1} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \quad \text{where the } A_{ii} \text{ are square submatrices.}$$

The fundamental result about nonnegative, indecomposable matrices is due to Frobenius [2]; this, and other, results have recently been rederived and extended in a greatly simplified manner by Wielandt [3] and Debreu and Herstein [1]. It is

THEOREM. *Let $A \geqq 0$ be an indecomposable matrix. Then A has a positive characteristic root r such that*

1. *r is a simple root.*

2. to r can be associated a characteristic vector $x > 0$.

3. if α is any other characteristic root of A, $|\alpha| \leq r$.

If $A > 0$ then 3. can be sharpened to $|\alpha| < r$ for all characteristic roots $\alpha \neq r$ of A.

If $A \geq 0$ is indecomposable and if A has no characteristic root other than r of maximal absolute value then A is said to be primitive.

In this paper we prove the

THEOREM* (Frobenius). Let $A \geq 0$. Then $A^m > 0$ for some integer $m > 0$ if and only if A is primitive.

Suppose that $A^m > 0$. Then A must be indecomposable; for if

$$PAP^{-1} = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix} \text{ then } PA^mP^{-1} = \begin{pmatrix} B^m & C_m \\ 0 & D^m \end{pmatrix}$$

contradicting $A^m > 0$.

Now suppose that r and $re^{i\varphi} \neq r$ are characteristic roots of A of maximal absolute value. Then A^m, A^{m+1} are both positive and have r^m, $r^m e^{im\varphi}$, and r^{m+1}, $r^{m+1}e^{i(m+1)\varphi}$ respectively as roots of maximal absolute value. Since the largest root of a positive matrix is simple and is actually greater than any other root in absolute value. We must have $r^m e^{im\varphi} = r^m$, $r^{m+1}e^{i(m+1)\varphi} = r^{m+1}$, whence $e^{i\varphi} = 1$, a contradiction.

There remains but to show that if A is primitive then $A^m > 0$ for a suitable integer $m > 0$. This will be proved as a consequence of the following few lemmas, which by themselves are of some interest.

Lemma 1. If A is primitive then A^m is primitive for every positive integer m.

Proof. Since r is a simple root of A and is the only root of A of absolute value r, r^m is a simple root of A^m and is the only root of A^m of absolute value r^m. So we need but show that A^m is indecomposable for every integer

$m > 0$. Suppose that for some s A^s is not indecomposable; we can then assume that $A^s = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$. Now $Ax = rx$ for $x > 0$, so $A^s x = r^s x$; partition x according to the partitioning of A^s and we have $\begin{pmatrix} B & C \\ 0 & D \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = r^s \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

That is $Dx_2 = r^s x_2$, and since x_2 is positive, r^s is a characteristic root of D. Since the transpose, A', of A is also indecomposable, we have $A'Y = rY$ for $Y > 0$. Partitioning as above we obtain that r^s is a characteristic root of B', and so of B. Being a characteristic root of both B and D r^s must be a multiple root of A^s, which is a contradiction. The lemma is thereby proved.

Lemma 2. (Wielandt). <u>Let ϵ be any positive number. Suppose $A \geq 0$ is an nxn indecomposable matrix. Then $(\epsilon I+A)^{n-1} > 0$ where I is the identity matrix.</u>

<u>Proof</u>. It clearly suffices to show that for any vector x, $x \geq 0$, $(\epsilon I+A)^{n-1} x > 0$. Let

$$x_\nu = (\epsilon I+A)^{\nu-1} x. \quad \text{Then} \quad x_{\nu+1} = \epsilon x_\nu + Ax_\nu .$$

Hence a zero component can occur in $x_{\nu+1}$ only where a zero component already occured in x_ν. However, not every such zero component can be preserved in $x_{\nu+1}$. For if so, by a suitable reordering of the coordinates,

$$x_\nu = \begin{pmatrix} p \\ 0 \end{pmatrix},\ p > 0, \text{ whence } x_{\nu+1} = \epsilon \begin{pmatrix} p \\ 0 \end{pmatrix} + \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} p \\ 0 \end{pmatrix} = \begin{pmatrix} q \\ 0 \end{pmatrix},$$

from which it follows that $A_{21}p = 0$. This together with $p > 0$ forces $A_{21} = 0$, violating the indecomposability of A. So each application of $\epsilon I + A$ to x decreases the number of zero coordinates by at least one. Hence $(\epsilon I+A)^{n-1}x > 0$.

As an easy consequence of Lemma 2 we obtain

Lemma 3. *If* $A = (a_{ij})$ *is indecomposable and* $a_{ii} > 0$ *for each* i *then* $A^{n-1} > 0$.

For let ϵ be chosen satisfying $0 < \epsilon < \min_i a_{ii}$. Then $A = \epsilon I + B$ where $B \geq 0$ is indecomposable. Lemma 2 then yields $A^{n-1} > 0$.

Let $A^m = (a_{ij}^{(m)})$. Then we have

Lemma 4. *Let* $A \geq 0$ *be indecomposable. Then for any* i,j *we can find an* $m = m(i,j) > 0$ *so that* $a_{ij}^{(m)} > 0$.

Proof. Consider first the case $i \neq j$. Since $(I+A)^{n-1} = A^{n-1} + \binom{n-1}{1} A^{n-2} + \ldots + I > 0$ by Lemma 2, $a_{ij}^{(m)} > 0$ for some $m \leq n-1$. Now suppose $i = j$. Since A is indecomposable, no column of zeros can occur in A. So there is a k with $a_{ki} > 0$. If $k = i$ then $a_{ii}^{(m)} > 0$ for all m trivially. If, on the other hand, $k \neq i$, then $a_{ik}^{(m)} > 0$ for some m, and since $a_{ii}^{(m+1)} = \sum_r a_{ir}^{(m)} a_{ri} \geq a_{ik}^{(m)} a_{ki} > 0$ the lemma is proved.

We are now in position to complete the proof of Theorem*. Let A be primitive. Pick m_1 so that in A^{m_1}, $a_{11}^{(m_1)} > 0$. Let $A_1 = A^{m_1} = (a_{ij}(1))$. By Lemma 1 A_1 is primitive, so there is an m_2 such that in $A_1^{m_2}$, $a_{22}^{(m_2)}(1) > 0$. Since $a_{11}(1) = a_{11}^{(m_1)} > 0$, $a_{11}^{(m_2)}(1) > 0$. Let $A_2 = A_1^{m_2}$. Continuing in this way we arrive at an $A_n = A^{m_1 m_2 \ldots m_n}$ which is primitive and whose diagonal elements are all positive. By Lemma 3 $A_n^t > 0$ for some t, hence $A^m > 0$ for some suitably chosen integer m.

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FOOTNOTES

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2. Numbers in square brackets refer to the bibliography at the end of this paper.

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