MULTIDIMENSIONAL UTILITIES

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**Summary**: A generalization of the von Neumann and Morgenstern theory of utility, by omission of the Archimedean postulate, and an extension to the infinite dimensional case are given. ( )

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## §1. Introduction.

This paper generalizes the von Neumann and Morgenstern theory of utility by omitting the Archimedean postulate. The work was done originally at The RAND Corporation in the summer of 1951 by N. Dalkey and R. M. Thrall. Refined methods were introduced by J. G. Wendel and the author in order to simplify the work and extend it to the infinite dimensional case.

Two distinct concepts enter into a formulation of utility theory: the set of prospects, which we shall call the mixture space and the ordering or utility on this set. For convenience we shall treat the mixture space first (axioms M1 – M5) and then introduce the ordering axioms (O1 – O3). The two sets of axioms, taken together, characterize a utility space. As an intermediate step, we shall consider a weak utility space, where axioms O1 – O3 are weakened. The weakened axioms are still strong enough to permit identification of "indifferent" elements and still preserve the operations of mixture and order.

The main result is that a mixture space may be embedded in a vector space; a utility space may be embedded in an ordered vector space. The last section characterizes ordered vector spaces.

## § 2. Algebraic Preliminaries.

We now introduce the following notation, to be useful in what follows. Let S be any subset of a vector space V over the real numbers. We define:

$C(S)$ = the convex closure of S

= the set of elements $X \in V$ which are of the form $X = \sum x_i S_i$ where $x_i > 0$, $\sum x_i = 1$, $S_i \in S$.

$P(S)$ = the cone generated by S

= the set of $X \in V$ which are of the form $X = \sum x_i S_i$ where $x_i > 0$, $S_i \in S$.

$H(S)$ = the hyperplane generated by S

= the set of $X \in V$ which are of the form $X = \sum x_i S_i$ where $\sum x_i = 1$, $S_i \in S$.

$V(S)$ = the vectorspace generated by S

= the set of elements X of the form $X = \sum x_i S_i$ where $S_i \in S$.

The above sums are taken to be finite. We note that

$C(S) \subseteq P(S)$, $H(S) \subseteq V(S)$.

I. The set $P(S)$ is closed under addition and scalar multiplication by positive reals. Such a set is called a cone. In the course of various embeddings to be done, it happens that it is frequently more natural to embed in a cone rather than in a vector space. A cone C may be characterized by the following properties (see §3, P1 — P8 for the complete set of axioms):

1. C is a commutative semigroup with cancellation under the operation +.

2. C is closed under scalar multiplication by positive reals. The usual associative and distributive laws are satisfied.

If C is a cone, it is possible to embed C in a vector space V so that addition and scalar multiplication in C may be extended to V and so that C generates V. This embedding is unique and may be accomplished by the familiar method of first embedding C in a group C' under + (cf. Van der Waerden, Moderne Algebra, p. 43, where a ring is embedded in a field). The group C' may be made into a vector space by defining

$$x(A-B) = xA - xB; \; x > 0$$
$$0(A-B) = 0 \qquad ;$$
$$-x(A-B) = xB - xA; \; x > 0, \; A, B \in C.$$

We may then verify that C' is a vector space V and $C = P(C)$,

$V = V(C)$.

II. If $S$ is a convex subset of $V(C(S) = S)$, then $P(S)$ consists of elements of the form $xA$ where $x > 0$ and $A \in S$. We seek a condition (given that $S$ is convex) that this representation is unique. First, suppose the representation is not unique. Then $x_1A_1 = x_2A_2$ where $x_1 \neq x_2$ or $x_1 = x_2$ and $A_1 \neq A_2$. In the first case, we may assume (by dividing by $x_1 - x_2$) that $x_1 - x_2 = 1$. Then $0 = x_1A_1 - x_2A_2 \in H(S)$. The second case $x_1 = x_2 > 0$ is impossible. Conversely, suppose $0 \in H(S)$. Then $0 = \sum x_iA_i$ where $\sum x_i = 1$. By transposing, if necessary, we obtain $\sum y_iA_i = \sum z_iA_i$ where $y_i > 0$, $z_I > 0$ and $\sum y_i \neq \sum z_i$. This gives two different representations of the element $\sum y_iA_i$ in the form $xA$, $A \in S$, since the $x$ may be taken to be $\sum y_i$ and $\sum z_i$. Thus, a necessary and sufficient condition that any element in $P(S)$ be represented uniquely in the form $xA$ is that $0 \notin H(S)$.

III. We set down some definitions concerning orderings for future reference.

If $S$ is any set, an ordering $\prec$ is a binary relation on $S$. Thus, for any elements $X, Y \in S$ it is determined whether $X \prec Y$. The relation $\prec$ is transitive if $X \prec Y$, $Y \prec Z$ imply $X \prec Z$. It is a total ordering if it is transitive and if for any distinct elements X, Y either $X < Y$ or $Y < X$. It is reflexive if $X < X$, and irreflexive if we never have $X < X$.

## §3. Mixture Spaces.

A mixture space is a set $M = \{A, B, \ldots\}$ which satisfies the following axioms:

**M1.** For any $A, B \in M$ and for any real p, $0 \leq p \leq 1$, the p-mixture of A and B, denoted by ApB, is a uniquely defined element of M.

M2. $ApB = B(1-p)A$

M3. $Ap(BrC) = (A\frac{p}{p+r-pr}B)(p+r-pr)C$

M4. $ApA = A$

M5. If $ApC = BpC$ for some $p > 0$, then $A = B$.

Taking $r = 0$ and $B = C$ in M3, we have with the help of M4

$$ApB = Ap(B_0B) = (A_1B)pB.$$

By the cancellation law (M5) we obtain

M6. $A_1B = A$.

If V is a vector space, we may define $ApB = pA + (1-p)B$. It is then easily shown that V is a mixture space. We shall show that this is the most general mixture space: any mixture space is isomorphic to a convex subset of a vector space.

The embedding of M will be accomplished by first embedding M in a cone P and then, as indicated in §2, embedding P in a vector space V. We shall be concerned only with the embedding in a cone. For convenience, we set down the axioms for a cone which were indicated in §2. A cone P is a set $\{A, B, \ldots\}$ satisfying the following axioms:

P1. There is an operation + in P such that $A + B$ is a uniquely determined element of P, where $A, B \in P$. Scalar multiplication by positive real numbers x is defined so that xA is a uniquely defined element of P.

P2. $A + B = B + A$

P3. $A + (B+C) = (A+B)+C$

P4. If $A + B = A + C$ then $B = C$

P5. $x(A+B) = xA + xB$, $x > 0$

P6. $(x+y)A = xA + yA$

P7. $x(yA) = (xy)A$

P8. $1 \cdot A = A$

Given the mixture space M, we define the set P to be the space of all ordered couples $(x, A)$ where $x > 0$, $A \in M$, and we formally write $(x, A) = xA$. Thus $x \cdot A = yB$ if and only if $x = y$ and $A = B$. (Later, A will be identified with $1 \cdot A$ and this definition of equality simply places M in V in such a way that $0 \notin H(M)$. An extraneous dimension is thus introduced.)

The set P being defined, we now define addition and scalar multiplication in P.

D1. $r(sA) = rs(A)$

D2. $rA + sB = (r+s)A_{\frac{r}{r+s}}B.$

And we now verify that P is a cone.

P1 is trivial.

P2. $rA + sB = (r+s)A_{\frac{r}{r+s}}B$ (D2)

$= (s+r)B_{\frac{s}{r+s}}A$ (M2)

$= sB + rA$ (D2).

P3. $rA + (sB+tC) = rA + (s+t)B_{\frac{s}{s+t}}C$

$= (r+s+t)\left[A_{\frac{r}{r+s+t}}(B_{\frac{s}{s+t}}C)\right].$

Similarly,

$(rA+sB) + tC = (r+s+t)\left[(A_{\frac{r}{r+s}}B)_{\frac{r+s}{r+s+t}}C\right].$

With the help of M3 it may be verified that

$A_{\frac{r}{r+s+t}}(B_{\frac{s}{s+t}}C) = (A_{\frac{r}{r+s}}B)_{\frac{r+s}{r+s+t}}C,$

which proves P3.

P4. If $rA + sB = rA + tC$, then

$(r+s)A_{\frac{r}{r+s}}B = (r+t)A_{\frac{r}{r+t}}C.$

Hence, $s = t$ and by M5, $B = C$. Thus $sB = tC$ proving P4.

P5. $x(rA+sB) = x(r+s)A_{\frac{r}{r+s}}B)$

$= (xr+xs)(A_{\frac{xrs}{xr+xs}}B)$

$= (xr)A + (xs)B$

$= x(rA) + x(sB)$

P6. $(x+y)rA = (xr+yr)A$

$$= (xr+yr)\left(A \frac{xr}{xr+yr} A\right) \qquad (M4)$$

$$= (xr)A + (yr)A$$

$$= x(rA) + y(rA).$$

P7. $x(y(rA)) = x(yrA) = (xyr)A = (xy)(rA)$

P8. $1\cdot(rA) = (1\cdot r)A = rA.$

P is thus a cone. If $A \in M$, we define $f(A) = 1\cdot A \in P$ to obtain the following lemma.

Lemma 1. There is a function f mapping M into a cone P in such a manner that

1) f is 1–1 into P.

2) $f(ApB) = pf(A) + (1-p)f(B)$ for $0 < p < 1$.

Proof. Define $f(A) = 1\cdot A$. Then 1) is a consequence of definition of equality in P and 2) is verified by the following computation:

$$\begin{aligned} f(ApB) &= 1\cdot ApB \\ &= (p+(1-p))ApB \\ &= pA + (1-p)B \\ &= p(1\cdot A) + (1-p)(1\cdot B) \\ &= pf(A) + (1-p)f(B). \end{aligned}$$

As in §1, the cone P may be taken to be a cone in some vector space V which is generated by P. The statement of the embedding theorem follows.

__Theorem 3.1__. __Let M be a mixture space. Then there exists a vector space V over the real numbers and a function f mapping M into V such that__

1. f is one-one into V.
2. $f(ApB) = pf(A) + (1-p)f(B), \quad 0 \leq p \leq 1$.
3. $f(M) = C(f(M))$, i.e., $f(M)$ is convex.
4. $V = V(f(M))$.
5. $0 \notin H(f(M))$.

__Proof__: Let V be generated by the cone P of Lemma 1. Then 1. and 2. are true by Lemma 1. (The cases $p = 0$ and $p = 1$ are trivial by M6.) To prove that $f(M)$ is convex, observe that if $A', B' \in f(M)$, so that $A' = f(A)$, $B' = f(B)$, then $pA' + (1-p)B' = f(ApB) \in f(M)$ by 2. Hence $f(M)$ is convex. Since $f(M)$ generates P (because the elements $1 \cdot A$ in the proof of Lemma 1 generate P) and P generates V we have 4. As for 5., we need to prove that if $xf(A) = yf(B)$ for $x,y > 0$ then $x = y$ and $f(A) = f(B)$. In terms of the cone P of Lemma 1, we have $xA = yB$ so that $x = y$ and $A = B$, proving the result.

We now prove that the embedding is unique. To do this we first prove the following algebraic lemma.

__Lemma 3.1__. Let V and V' be vector spaces with elements $x, y, z \ldots$ and $x', y', z' \ldots$ respectively. Let C and C' be convex subsets of V and V' respectively and let $0 \notin H(C)$,

$0 \in H(C')$, $V = V(C)$, $V' = V'(C)$. Let g map C onto C' in such a way that

a. g is one-one from C onto C'.

b. $g(pA+(1-p)B) = pg(A) + (1-p)g(B)$ for $0 < p < 1$ and $A, B \in C$.

Then g may be extended in one and only one way to all of V such that

a'. g is one-one from V into V'.

b'. g is linear.

(Thus, if $X = \sum_i x_i A_i$, $A_i \in C$, then $g(X) = \sum x_i g(A_i)$, and this extension of g is uniquely defined and satisfies a'. and b'.)

Proof: Obviously there is at most one extension. Let us first extend g to P(C), by defining $g(xA) = xg(A)$ where $x > 0$ and $A \in C$. (Observe that since C is convex, and $0 \notin H(C)$, xA is the most general element of P(C).) There is no question of whether g is properly defined, since the representation xA is unique.

$\alpha$. g is one-one onto P(C'). g is one-one since if $xg(A) = yg(B)$ with $x, y > 0$, $A, B \in M$, we have $g(A), g(B) \in C'$ and two representations of an element in P(C'). Hence $x = y$ and $g(A) = g(B)$, and therefore $A = B$. That g is onto P(C') is clear.

$\beta$. g is additive on P(C). For we have

$$g(xA+yB) = g(x+y)\left[\frac{x}{x+y} A + \frac{y}{x+y} B\right]$$

$$= (x+y)g(A \frac{x}{x+y} B)$$

$$= (x+y)\left[g(A) \frac{x}{x+y} g(B)\right] \qquad \text{(by b)}$$

$$= xg(A) + yg(B)$$

$$= g(xA) + g(yB).$$

$\gamma$. g is homogeneous. For,

$$g(x(yA)) = g(xyA)$$

$$= xyg(A)$$

$$= xg(yA).$$

We now extend g to V. Let R, S, T, V,... denote elements of P(C). The most general element X of V is of the form X = R − S. Define g(X) = g(R) − g(S). g is properly defined, since if R − S = T − U then R + U = S + T, g(R) + g(U) = g(S) + g(T) (by $\beta$) and hence G(R) − g(S) = g(T) − g(U). We now prove a' and b'.

It is easily verified, with the help of $\beta$ and $\gamma$ that g is linear. To prove that it is one-one, assume g(X) = 0. Letting X = R − S, we have g(R−S) = 0, g(R) = g(S). Hence, by $\alpha$, $R = S$ and $X = 0$. Finally, $g$ clearly maps $V$ onto $V'$ and this proves the lemma.

__Theorem 3.2__ (__Uniqueness__). __Let M be a mixture space and let V and V' be vector spaces. Suppose f maps M into V and f' maps M into V' as in Theorem 3.1. Then V and V' are isomorphic under a mapping g which sends an element__ $X = \sum x_i f(A_i)$ $(A_i \in M)$ __into__ $g(X) = \sum x_i f'(A_i)$.

__Proof__: By Lemma 3.1 applied to the function $g = f'f^{-1}$.

The above discussion has not depended on the dimension of V. If the (essentially unique) V of Theorem 1 is (n+1)-dimensional, then we say that the mixture space M is n-dimensional. In this case, M is simply an n-dimensional convex set.

## §4. Utility Spaces.

A utility space is a mixture space M with an order relation $<$ imposed on its structure. In view of §3, we assume that M is a convex subset of a vector space V with $V = V(M)$, $0 \notin H(M)$. The relation $<$ is required to satisfy the following axioms:

O1. The relation $<$ is a total ordering on M, which is irreflexive and transitive.

O2. If $A < B$ and $0 < p < 1$, then $pA + (1-p)C < pB + (1-p)C$.

Observe that the converse of O2 holds since $<$ is a total, irreflexive ordering.

Before considering utility spaces, we shall consider weak utility spaces. These are mixture spaces with an ordering $\leq$ satisfying

W1. The relation $\leq$ is a binary, reflexive and transitive relation of M. It is a total ordering.

W2. If $A \leq B$ then $pA + (1-p)C \leq pB + (1-p)C$ for $0 < p < 1$.

W3. If $pA + (1-p)C \leq pB + (1-p)C$ for some p such that $0 < p < 1$, then $A \leq B$.

<u>Definition 4.1</u>. $A \sim B$ if and only if $A \leq B$ and $B \leq A$.

It is easily verified that this is an equivalence relation. Let $[A]$ be the equivalence class containing A, i.e., $X \in [A]$ if and only if $X \sim A$.

<u>Definition 4.2</u>. $[A] < [B]$ if and only if $A \leq B$ and $[A] \neq [B]$.

It is then seen that the relation $<$ on the equivalence classes is uniquely defined and is transitive, irreflexive. It is a total ordering. Then O1 is satisfied. Let M' be the set of equivalence classes. We make M' into a mixture space by defining

$$[A]p[B] = [ApB].$$

With the help of W2, it is easily verified that this definition is unique. That M' is a mixture space follows from the fact that M is one.

To prove O2, assume $0 < p < 1$ and $[A] < [B]$. Since $A \lesssim B$ we have $pA + (1-p)C \lesssim pB + (1-p)C$. We need only prove that $pB + (1-p)C \not\lesssim pA + (1-p)C$. But assume that $pB + (1-p)C \lesssim pA + (1-p)C$. By W3, we would have $B \lesssim A$, hence $A \sim B$, $[A] = [B]$ which contradicts $[A] < [B]$. Hence O2 is verified.

We see that W2 was needed to define mixtures, W3 to define order.

Given a weak utility space M, we identify elements by the above process to obtain a utility space M'. Any information obtained on M' will then be reflected in M. For this reason we consider only utility spaces.

An ordered vector space V is one which satisfies the following axioms:

V1. There is a binary, transitive and irreflexive relation $<$ on V'. The relation $<$ is a total ordering.

V2. If $A > B$ then $A + C > B + C$ for $A, B, C \in V$.

V3. If $A > B$ and $x > 0$ then $xA > xB$.

An ordered vector space is thus a utility space. Axioms O1 and O2 are easily seen to hold. The purpose of this section is to embed a utility space in an ordered vector space. Since the utility space M is embedded in the vector space V as a mixture space, it suffices to extend the order relationship from M to V. We assume that $V = V(M)$ and $0 \notin H(M)$. The order relation will first be extended to $P = P(M)$ and then to $V = V(P)$.

To extend the order on M to $P = P(M)$, we define

$$xA > yB \text{ if and only if } x > y$$

or

$$x = y \text{ and } A > B \text{ for } x,y > 0;\ A,B \in M.$$

We now verify V1, V2 and V3.

V1 is trivially verifiable.

V2. Assume $xA > yB$. We must prove that $xA + zC > yB + zC$. But $xA + zC = (x+z)\, A \frac{x}{x+z} C$ and $yB + zC = (y+z)\, B \frac{y}{y+z} C$. If $x > y$, we have the result, since $x + z > y + z$. If $x = y$, then $x + z = y + z$ and since $A > B$, we have $A \frac{x}{x+z} C > B \frac{y}{y+z} C$ by O2. Hence $xA + zC > yA + zC$ in any case.

V3. Let $xA > yB$. Then the condition $rxA > ryB$ is equivalent to the condition $xA > yB$. This proves V3.

Finally, the order on P will be extended to V by defining

$$A - B > C - D \text{ if and only if } A + D > B + C$$

for $A,B,C,D \in p$.

It is easily seen that the definition is unique and we omit this proof. We now verify the axioms V1, V2 and V3.

V1 is trivially true.

V2. Let $A - B > C - D$. We must prove that $(A-B) + (E-F) > (C-D) + (E-F)$ for $A,B,\ldots F \in p$. But we have $A + D > B + C$. Hence $A + D + E + F > B + C + E + F$. By definition, we have $(A+E) - (B+F) > (C+E) - (D+F)$. Hence, by rearranging, we have the result.

V3. If $A - B > C - D$ and $x > 0$, we have

$A + D > B + C$,

$xA + xD > xB + xC$,

$xA - xB > xC - xD$,

$x(A-B) > x(C-D)$.

Thus, the utility space M is embedded in an ordered vector space V. As we have mentioned before, an extraneous dimension has been introduced since $O \notin H(M)$. The dimension of M has been defined as the dimension of $H(M)$. Once the above embedding has been done it is an easy matter to embed M in a vector space of its own dimension. To do this we simply embed $H(M) \supset M$ in a vector space by selecting any point $A \in H(M)$ and defining $f(X) = X - A$ for $X \in H(M)$. The image of $H(M)$ is then a vector space whose dimension is $H(M)$. Moreover, f is one-one and f preserves convex combinations and order. This seemingly roundabout method of introducing an extra dimension greatly simplifies the embedding procedure since all of our embeddings seem to be accomplished easily by first extending definitions to $P(M)$ where $O \notin H(M)$. Lastly, it should be pointed out that the "proper" embedding should be thought of as an embedding in an affine space, since the relations of p-mixtures and order are preserved under the affine group. We use a vector space for convenience in manipulation.

## §5. Ordered Vector Spaces.*

We now proceed to characterize ordered vector spaces. The definition has been given in §3 by axioms V1, V2 and V3. It is convenient to introduce equivalent substitutes by V2 and V3 as follows:

V2' If $A > 0$, $B > 0$ then $A + B > 0$

V3' If $A > 0$, $x > 0$ then $xA > 0$.

We have

$$A > B \text{ if and only if } A - B > 0.$$

Thus the order is determined by the positive elements which we denote by $V^+$. V2' and V3' imply that $V^+$ is a cone. Since the order is a total order and is irreflexive, $V^+$ is a maximal cone not containing 0.

We now introduce a definition which reflects the failure of the Archimedean property. Let $A, B \in V^+$ $(A, B > 0)$. Then we say that A dominates B if $A > xB$ for all real $x > 0$. We write $A >> B$ or $B << A$. This is defined only for positive elements of V. The relation $<<$ is irreflexive and transitive; and $A << B$ implies $A < B$. Given A and B, if neither $A << B$ nor $B << A$ we write $A \sim B$, A is equivalent to B. Equivalently, $A \sim B$ if and only if $xA < B < yA$ for some positive reals x and y. Again this is defined for $A, B \in V^+$. The relation $\sim$ is seen to be an equivalence relation. We denote by $[A]$ the

*This section has appeared, in essence, in the Proceedings of A.M.S., December, 1952. It is included for the sake of completeness.

equivalence class containing A. We may then define [A] > [B] to mean A < < B. We then have a total order on the equivalence classes, since this definition may be seen to depend only on the equivalence classes and not on their representatives. The seemingly irrelevant reversal of order is for later convenience.

Lemma 5.1. If $A >> B$ and $C > 0$ then $A + C >> B$.

Proof: If $x > 0$, then $A > xB$. Hence $A + C > xB$.

Lemma 5.2. If $A >> A_1, A_2, \ldots, A_n$ and $x_1, \ldots, x_n > 0$, then $A >> x_1A_1 + \cdots + x_nA_n$.

Proof: If $x > 0$ we have

$$A > nx\ x_iA_i$$

Summing and dividing by n, we have

$$A > x(x_1A_1 + \cdots x_nA_n),$$

proving the lemma.

Lemma 5.3. If $A >> A_1, \ldots, A_n$ and $x, x_1, \ldots, x_n > 0$, then

$$xA + x_1A_1 + \cdots + x_kA_k >> x_{k+1}A_{k+1} + \cdots + x_nA_n$$

Proof: By Lemmas 5.1 and 5.2.

Corollary 5.1. If $\{A_t\}$ is a set of elements of $V^+$ no two of which are equivalent, then the $A_t$ are linearly independent.

Proof: If there is a linear relation among the $A_t$, then we will obtain a relation of the form

$$xA + x_1A_1 + \cdots + x_kA_k = x_{k+1}A_{k+1} + \cdots + x_nA_n,$$

where the x's are positive, the A's belong to the given set, and $A >> A_1, \ldots, A_n$. But this contradicts Lemma 5.3.

As usual, we define $|A| = \pm A$ according as A is non-negative or negative. The usual laws apply:

$$|A + B| \leq |A| + |B|$$

$$|xA| = |x||A|.$$

With this notation we may state the following important lemma:

Lemma 5.4. Let $A \sim |B|$. Then there is a unique real number x such that $xA = B$ or $|xA - B| << A$.

Proof: The uniqueness of x is immediate, for if $x_1 \neq x_2$ we have

$$|(x_1 - x_2)A| \leq |x_1A - y| + |x_2A - y|,$$

and if the terms on the right were zero or dominated by A we should have $A << A$.

To prove that such an x exists, we assume that $B > 0$ with no loss in generality. Since $A \sim B$, we have $yA < B < zA$

for some positive real numbers y and z. Let x be the supremum of the numbers y for which $yA < B$. Let $\varepsilon > 0$ be an arbitrary positive real number. Then we have

$$(x-\varepsilon)A < B < (x+\varepsilon)A,$$

$$-\varepsilon A < B - xA < \varepsilon A,$$

$$|B - xA| < \varepsilon A.$$

Then either $B - xA = 0$ or $|B - xA| << A$ by definition.

We may now easily characterize finite dimensional ordered vector space. The result is known (cf. Garrett Birkhoff, Lattice Theory, p. 240), but we give it here as an illustration of the method used for the general (infinite-dimensional) case.

<u>Theorem 5.1.</u> <u>Let</u> V <u>be a finite dimensional ordered vector space</u>. <u>A basis</u> $A_1, A_2, \ldots, A_n$ <u>may be chosen so that the ordering in</u> V <u>is lexicographic</u>, <u>i.e.</u>,

$$A = \sum_{i=1}^{n} x_i A_i > 0$$

<u>if and only if the first non-vanishing</u> $x_i$ <u>is positive</u>.

<u>Proof</u>: Let $V^+$ be decomposed into equivalence classes as above. For each equivalence class t, let $A_t$ be an arbitrary element of it. By Corollary 5.1, the set $\{A_t\}$ is finite. We choose the notation so that $A_1, \ldots, A_k$ are the

representatives and $A_1 >> A_2 >> \cdots >> A_k$. Moreover, $k \leq n$ = dimension of V.

Let $A \in V$. If $A \neq 0$ then $|A|$ belongs to some equivalence class, say $|A| \sim A_{t_1}$. By Lemma 5.4, either $A = x_1 A_{t_1}$ or $|A - x_1 A_{t_1}| << A_{t_1}$. We repeat the process on $A - x_1 A_{t_1}$ if it is not zero and continue until the zero element is reached. (It must be reached in $\leq k$ steps.) Thus A is a linear combination of the $A_i$'s; and since A was arbitrary it follows that the A's constitute a basis for V. Thus $k = n$.

Now let

$$A = \sum x_i A_i .$$

By Lemma 5.3, $A > 0$ if and only if the first non-vanishing coefficient $x_i$ is positive. This completes the proof. Observe that a basis $A_1, \ldots, A_n$ is a lexicographic basis if and only if $A_i > 0$ and $A_1 >> \ldots >> A_n$.

<u>Theorem 5.2</u>. <u>Let</u> $A_1, \ldots, A_n$ <u>and</u> $A_1', \ldots, A_n'$ <u>be basis elements in the sense of Theorem</u> 5.1. <u>Then</u> $A_i' = TA_i$ <u>where</u> T <u>is a lower triangular matrix with positive diagonal elements. Conversely, if</u> T <u>is such a matrix and</u> $A_1, \ldots, A_n$ <u>constitute a lexicographic basis then so do</u> $TA_1, \ldots, TA_n$.

<u>Proof</u>: We have $A_1 >> \ldots >> A_n$ and $A_1' >> \ldots >> A_n'$.

Hence $A_i \sim A_i'$. Thus $A_i' = x_i A_i$ + terms dominated by $A_i$ by Lemma 5.4. $A_i' = x_i A_i + x_{i+1} A_{i+1} + \ldots$ where $x_i > 0$. This proves the first part of the theorem. For the second part, we observe that $TA_i \sim A_i$. Hence $TA_1 >> \ldots >> TA_n$ proving the second part.

Im terms of coordinates, we may state Theorem 5.1 as follows: with respect to some basis, the vector $X = (x_1, \ldots, x_n) > 0$ if and only if $x_1 > 0$ or $x_1 = 0$, $x_2 > 0, \ldots,$ or $x_1 = \ldots x_{n-1} = 0$, $x_n > 0$.

In order to consider the infinite dimensional case, we define a lexicographic ordered vector space as follows: Let T be a totally ordered set. Let $V_T$ be the set of all real valued functions on T which vanish except on some well ordered subset of T. We define $f > 0$ if $f \neq 0$ and if $f(t_0) > 0$ where $t_0$ is the first t for which $f(t) \neq 0$. It is easily verified that $V_T$ is an ordered vector space. For T finite, we get the finite-dimension ordered vector spaces. It may then be shown that the ordered set T is order-isomorphic to the ordered set of equivalence classes in $V_T$. If V is any ordered vector space, we let T be the set of equivalence classes with the previous definition of order. Then the result is that V is embeddable in a subspace of $V_T$ which contains the characteristic functions of points. The proof is by transfinite induction. The details appear in a paper by the author and J. G. Wendel in the December, 1952, Proceedings of the A.M.S. (Vol. 3, No. 6).