POLYNOMIAL-LIFE APPROXIMATION

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SUMMARY

This is a short note on approximate representation for computing
purposes of continuous real-valued functions of several real
variables by finite sums of functions separable in the individual
variables. ( )

POLYNOMIAL-LIKE APPROXIMATION

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Frequently, in machine computations, one may substitute for a
function of relatively complicated analytic structure a function of
simpler form without appreciable error. Moreover, if the function is
given tabularly, a considerable reduction of tabular input data (load
on the memory) may be achieved. In what follows, we deal with real-
valued functions of continuous real variables x, y, ... Computationally,
it is usually better to work with variables assuming finitely many discrete
values. In the formulas which follow, however, the discrete analogues are
apparent, integration to be replaced by summation, etc. We have in mind
multivariate approximations by polynomial-like forms, in particular, by
functions M of the form

$$(1) \qquad M(x, y) = \sum_{j=1}^{n} \phi_j(x)\psi_j(y) ,$$

in the bivariate case. Some of the following results are generalizable to
functions of more than two variables.

Of special interest is the so-called slide rule form, $f(x) + g(y)$.
We shall give formulas for approximating to a given real-valued continuous

function, defined over the unit square $0 \leq {x \atop y} \leq 1$, to minimize, respectively, each of the following measures of goodness of fit:

(a) $\int_0^1 \int_0^1 \left[z(x, y) - f(x) - g(y) \right]^2 dxdy$, and

(b) $\max\limits_{0 \leq {x \atop y} \leq 1} \left| z(x, y) - f(x) - g(y) \right|$.

A simple variational technique yields for (a):

$f(x) = \int_0^1 z(x, y)dy + c_0$, and

$g(y) = \int_0^1 z(x, y)dx + c_1$, where

c_0 and c_1 are any real numbers such that

$$c_0 + c_1 = -\int_0^1 \int_0^1 z(x, y)dxdy .$$

Recursion formulas for (b) are based heuristically on the fact that given two numbers a and b, one can approximate to them by a single number c; taking c to be the arithmetic mean of a and b minimizes the maximum absolute error. The recursion formulas for (b) are as follows:

$$f_{n+1}(x) = \frac{1}{2} \left\{ \max_y \left[z(x, y) \cdot g_n(y) \right] + \min_y \left[z(x, y) - g_n(y) \right] \right\} ,$$

$$g_n(y) = \frac{1}{2} \left\{ \max_x \left[z(x, y) - f_n(x) \right] + \min_x \left[z(x, y) - f_n(x) \right] \right\} ,$$

and f_o arbitrary but continuous, $n = 0, 1, \ldots$. DILIBERTO & STRAUS[1] have shown quite generally that the above process converges to a pair of continuous functions (f, g) which has the required property. Included in their discussion is a precise method, discovered independently by the writer, for estimating the value of the minimum. In this connection, the reader is referred to their paper.

We now return to the more general form (1). Let $n > 0$ be a prescribed integer. We assume that z (continuous over the unit square) is not of the form (1) for this particular n. We shall give two sets of formulas for approximating to z by continuous functions of the form (1), which satisfy respectively:

(c) The error vanishes on some rectangular grid of lines $x = x_1$, $x_2, \ldots, x_n$, $y = y_1, y_2, \ldots, y_n$ ($x_i \neq x_j$ and $y_i \neq y_j$ if $i \neq j$)

(d) $\int_0^1 \int_0^1 \left[z(x, y) - \sum_{j=1}^{n} \phi_j(x)\psi_j(y) \right]^2 dxdy$ is minimized.

For (c), functions ϕ_j, ψ_j can be computed successively as follows. Choose a point (x_1, y_1) such that $z(x_1, y_1) \neq 0$ and two constants c_1, c_1' such that $c_1 c_1' = \frac{1}{z(x_1, y_1)}$ and set

$$\phi_1(x) = c_1 z(x, y_1) \quad \text{and}$$

$$\psi_1(y) = c_1' z(x_1, y) .$$

We compute ϕ_2 and ψ_2 in precisely the same manner, except that z is now replaced by a new function z_2 given by

$$z_2(x, y) = z(x, y) - \phi_1(x)\psi_1(y) .$$

Thus, we choose a point (x_2, y_2) such that $z_2(x_2, y_2) \neq 0$, and two constants c_2, c_2' such that $c_2 c_2' = \dfrac{1}{z_2(x_2, y_2)}$ and set

$$\phi_2(x) = c_2 z_2(x, y_2) \quad \text{and}$$

$$\psi_2(y) = c_2' z_2(x_2, y) .$$

We continue in this manner, with

$$z_{k+1}(x, y) = z_k(x, y) - \phi_k(x)\psi_k(y)$$

until we have the required number of functions. A simple inductive argument shows that condition (c) is satisfied.

That the solution of problem (d), i.e. finding a least squares fit to z of the form (1), bears a direct relationship to the theory of integral operators in Hilbert space was pointed out by George Brown at

RAND a few years ago. His observations are essentially these:

Let λ_1, λ_2, ... , λ_n be the <u>largest</u> n eigenvalues of the symmetric kernel

$$K(x, y) = \int_0^1 z(x, t)z(y, t)dt$$

and ϕ_1, ϕ_2, ... , ϕ_n be corresponding eigenfunctions (normalized so that $\int_0^1 \phi_j^2(x)dx = 1$) and set

$$\psi_j(y) = \int_0^1 z(x, y)\phi_j(x)dx \quad , \quad j = 1, 2, \ldots , n,$$

then the ϕ's and ψ's defined in this way form a solution to (d), the ϕ's being mutually orthogonal and the ψ's being mutually orthogonal. The computations can be carried on sequentially. Let λ_1 be the largest eigenvalue of K, ϕ_1 a corresponding normalized eigenfunction and ψ_1 defined as above. Having found ϕ_1 and ψ_1 we form the symmetric Kernel, K_1 associated with the residual function $z(x, y) - \phi_1(x)\psi_1(y)$. Then, the largest eigenvalue of K_1 is the next largest eigenvalue of K and we take ϕ_2 to be the corresponding normalized eigenfunction with ψ_2 defined as above. We continue in this manner until we have the required number of functions. Incidentally, for n = 1, a variational technique yields the necessary conditions for an extremum:

$$\phi(x) = \frac{\int_0^1 z(x, y)\psi(y)dy}{\int_0^1 \psi^2(y)dy}$$

$$\psi(y) = \frac{\int_0^1 z(x, y)\phi(x)dx}{\int_0^1 \phi^2(x)dx}$$

These can be used to generate an iterative computation for ϕ and ψ. However, questions of convergence, proper initiating functions to achieve the largest eigenvalue, etc. seem to be difficult. One final remark — the minimum value of $\int_0^1 \int_0^1 \left[z(x, y) - \sum_{j=1}^n \phi_j(x)\psi_j(y) \right]^2 dxdy$ is precisely the sum of the remaining eigenvalues of K, i.e. the sum of the eigenvalues minus the sum of the largest n eigenvalues, the former sum being equal to $\iint z^2 dxdy$.

S. P. DILIBERTO + E. G. STRAUS, "On the approximation of a function of several variables by the sum of functions of fewer variables," Pacific J. of Math. 1, p. 195-210, 1951.