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APPLICATIONS OF THE KAC-SIEGERT METHOD FOR FINDING  
OUTPUT PROBABILITY DENSITIES FOR RECEIVERS  
WITH SQUARE LAW DETECTORS

R. C. Emerson

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Applications of the Kac-Siebert Method for Finding Output Probability

Densities for Receivers with Square Law Detectors

R. C. Emerson

The RAND Corporation  
Santa Monica, California

Abstract



The method of Kac and Siebert for finding the output probability density characteristic function for receivers with square law envelope detections is discussed and a parallel development is given for the square law rectifier. Procedures are then outlined for determining the probability density functions directly, i.e., without solving the eigenvalue problem or inverting the characteristic function. The method depends on expanding the density function in an orthonormal series the coefficients of which are expressed in terms of cumulants which in turn are obtained from the system kernel by straightforward quadratures.

As an example to illustrate the procedure, a receiver with Gaussian I.F. and Gaussian audio frequency pass characteristics is treated in detail, and the output probability density functions are found for various sinusoidal input signal strengths and I.F. vs. audio bandwidth ratios.

List of Symbols

$E_1(t)$	I.F. input voltage
$E(t)$	I.F. output (detector input) voltage
$E^2(t)$	Detector output (audio input) voltage
$E_0(t)$	Audio output voltage
$F_{if}(\omega)$	I.F. voltage transfer function
$f_{if}(t)$	Fourier transform of $F_{if}(\omega)$
$F_a(\omega)$	Audio voltage transfer function
$f_a(t)$	Fourier transform of $F_a(\omega)$
$g(u,v)$	System kernel defined by equation (5)
$h_j(u)$	$j^{\text{th}}$ normal eigenfunction of $g(u,v)$
$\lambda_j$	Eigenvalue corresponding to $h_j(u)$
$e_j(t)$	Component of $E_1(t)$ along $h_j(x)$ (see equation (10))
$S(t)$	Signal component of $E_1(t)$
$N(t)$	Noise component of $E_1(t)$
$s_j(t)$	Signal component of $e_j(t)$ (see equation (13))
$\eta_j(t)$	Noise component of $e_j(t)$ (see equation (14))
$\phi_0$	I.F. input noise power per unit frequency
$\bar{I}(y,t)$	Characteristic function of the probability density of $E_0(t)$
$S$	Signal Power at input to the detector
$N$	Noise Power at input to the detector

List of Symbols (Cont.)

$X$	Signal-to-noise power ratio at input to the detector
$Y$	Audio output voltage measured in units of $\bar{N}$
$K_n$ or $K_n(t)$	$n^{\text{th}}$ cumulant (defined by equation (33))
$g^n(u, v)$	$n$ times iterated kernel (defined by equation (35))
$\sigma$	Bandwidth measure for Gaussian I.F. amplifier
$\nu$	Bandwidth measure for Gaussian audio amplifier
$\gamma$	$\sigma/\nu$
$\Delta f$	Noise Bandwidth of I.F. amplifier
$H_j(x)$	$j^{\text{th}}$ Hermite polynomial (defined by equation (74))
$\omega_0$	Center frequency of the I.F. amplifier
$K_1^{S+N}$	First output cumulant for signal plus noise
$K_1^N$	First output cumulant for noise alone
$K_2^{S+N}$	Second output cumulant for signal plus noise
$K_2^N$	Second output cumulant for noise alone
$S_0, S_s$	Detectability criteria (see equations (79) and (80))

## Introduction

The upper limit on radio receiver performance is often determined by the ability of the equipment to detect weak signals in the presence of system, or pre-system noise. Since this noise is basically random in its fine structure, the degree of signal contamination and consequent equipment malfunction must be described statistically, i.e., by means of expectations, probability distributions, etc. Considerable attention has been given in recent years to statistical analysis of electronic circuitry and particularly, because of the common requirement for "detectors" in systems, to non-linear circuits. Some of the names in the literature associated with this work are Rice,<sup>(1)</sup> Middleton,<sup>(2)</sup> North,<sup>(3)</sup> Van Vleck,<sup>(2)</sup> Marcum,<sup>(4)</sup> Goudsmit,<sup>(5)</sup> Fubini,<sup>(6)</sup> and Johnson<sup>(6)</sup> as well as Smith<sup>(7)</sup> in England to mention only a few.

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(1) S. O. Rice, "Mathematical Analysis of Random Noise," Bell Syst. Tech. Jour., 23, 282 (1944); 25, 46 (1945).

(2) J. H. Van Vleck and D. Middleton, "Theory of the Visual vs. Aural or Meter Reception of Radar Signals in the Presence of Noise," R.R.L. Report No. 411-86, May 1944.

(3) D. O. North, "Analysis of the Factors Which Determine Signal/Noise Discrimination in Radar," R.C.N. Technical Report PTR 6-C, June 1943.

(4) J. I. Marcum, A Statistical Theory of Target Detection by Pulsed Radar: Mathematical Appendix, The RAND Corporation, Research Memorandum RM-753, July 1, 1948.

(5) S. A. Goudsmit, "Statistics of Circuit Noise," R.R.L. Report No. 43-20, Jan. 1943.

(6) E. G. Fubini and D. C. Johnson, "Signal-to-Noise Ratio in A.M. Modulated Receivers," Proc. I.R.E., Vol. 86, Dec. 1948, pp. 1461-1467.

(7) R. A. Smith, "The Relative Advantages of Coherent and Incoherent Detectors: A Study of Their Output Noise Spectra under Various Conditions," TRE Technical Note No. 75.

The springboard for this paper is a work by M. Kac and A.J.F. Siegert (8) who have investigated the statistical effects of uncorrelated Gaussian (white) noise with and without signal for a system comprising an I.F. amplifier, a square law envelope detector, and an audio amplifier. They have derived an exact formula for the first probability distribution of the output voltage for such a system—an important result because of its generality both with respect to I.F. and audio pass band characteristics, and the unrestricted form of the signal wave assumed. Their expression for the probability distribution is, however, unsuitable for most engineering applications, first, because it depends on inverting a rather complicated characteristic function which may not be possible in closed form, and, secondly, the explicit expression for the characteristic function depends on the solution of a certain eigenvalue problem which none but the most experienced in dealing with integral equations are equipped to solve.

This paper provides a means for using the Kac-Siegert method in its widest generality without the necessity of finding eigenvalues, and without dealing with the characteristic function at all. It depends on the fact that the cumulants of the output distribution are rather simply related to the system operator so that by employing any of the well-known orthonormal systems for expanding density functions, i.e., Gram-Charlier, Laguerre, etc., one may compute the output probability distribution to any desired degree of accuracy by straightforward techniques. The procedure is illustrated in Section IV for a system in which the I.F. and audio amplifiers possess Gaussian band pass characteristics of arbitrary bandwidth.

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(8) M. Kac and A.J.F. Siegert, "On the Theory of Noise in Radio Receivers with Square Law Detectors," J. Applied Physics, Vol. 18, pp. 383-397, April, 1947.

This treatment will depart from that of Kac and Siegert in one important respect. As was previously mentioned, their detector is an "envelope detector," that is, it consists of a square law rectifier followed by an appropriate smoothing circuit to attenuate the high frequency residue. This action is accomplished mathematically by resolving the input voltage into sine and cosine components which are then squared and added to give the output voltage. This approach appears to be a carry-over from the original derivations of the detector output probability density where no separate audio amplifier is considered. Since this treatment is to include audio filtering as a specific function, the smoothing circuitry referred to above will be included in the audio filter. This point of view leads to a great simplification of the mathematics. In addition, the theory will be slightly more flexible since it will be possible by removing the audio amplifier completely to determine the probability density function for the detector alone, i.e., without smoothing circuits, or to obtain the result of Kac and Siegert by modifying the audio filter so as to account for the additional smoothing action.

Section I

The System Operator

Figure (1) illustrates the system under investigation.

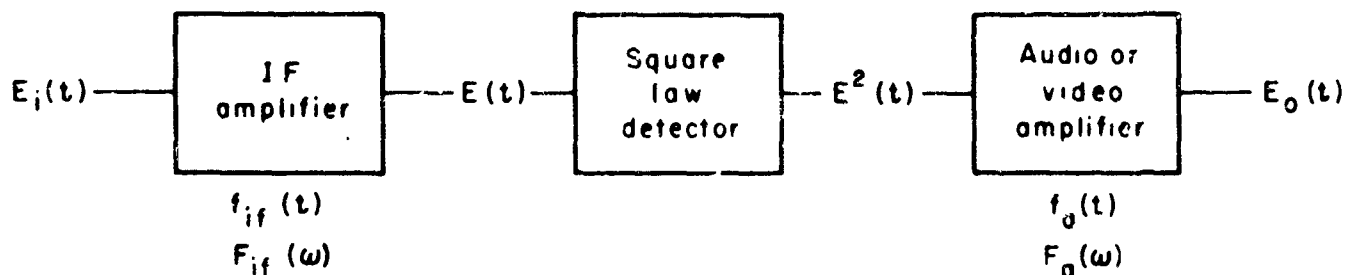


Figure (1)

The I.F. amplifier is characterized by the voltage-frequency function  $F_{if}(\omega)$  and/or its Fourier transform  $f_{if}(t)$ ; similarly for the audio amplifier. The voltage,  $E(t)$ , applied to the square law detector is given by the familiar formula,

$$E(t) = \int_{-\infty}^{\infty} f_{if}(t-x) E_i(x) dx \quad (1)$$

The output,  $E^2(t)$ , of the detector is obtained by squaring the input. It may be expressed by the double integral formula,

$$E^2(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{if}(t-x) E_i(x) E_i(y) f_{if}(t-y) dx dy \quad (2)$$

This voltage is applied to the audio amplifier. The output voltage of this amplifier is given by the linear operation,

$$E_o(t) = \int_{-\infty}^{\infty} f_a(t-x) E^2(x) dx \quad (3)$$



On substituting from equation (2) and after making certain substitutions of variables, one obtains the following formula for the output in terms of the input voltage:

$$E_0(t) = \iint_{-\infty}^{\infty} E_1(t-u) g(u,v) E_1(t-v) dudv \quad (4)$$

where,

$$g(u,v) = \int_{-\infty}^{\infty} f_{if}(u-z) f_a(z) f_{if}(v-z) dz \quad (5)$$

Equation (4) r.h.s. is the system operator, and the function  $g(u,v)$ , the system kernel.

One comment concerning the limits of integration seems in order at this point. The realizability of the I.F. and audio filters implies their inability to predict; consequently, the functions  $f_{if}$  and  $f_a$  vanish for negative arguments and so the integrals may be extended over the entire time domain. Notice also that the so-called "high frequency terms" produced by frequency addition in the detector have not been neglected. They are included in  $E_0(t)$  to the extent they are passed by the audio amplifier. This residue need not be given special attention in the derivation of the characteristic function, but may be discarded later on if desired.

#### Expansion of the System Kernel

The next step in the development is to expand the function  $g(u,v)$  into the uniformly convergent bilinear series,

$$g(u,v) = \sum_j \lambda_j h_j(u) h_j(v) \quad (6)$$

where the  $h_j(x)$  and  $\lambda_j$  are respectively the  $j^{\text{th}}$  normal orthogonal eigenfunction

and corresponding eigenvalue of the integral equation

$$\lambda h(x) = \int_{-\infty}^{\infty} g(x,y) h(y) dy \quad (7)$$

For such an expansion to exist it is sufficient, first, that  $g(u,v)$  be symmetric which can be verified by appealing to equation (5), and secondly, that  $g(u,v)$  be positive semi-definite. On referring to equation (4) it will be verified that  $g(u,v)$  will be positive semi-definite if for all input functions,  $E_1$ , the output is non-negative. Since the detector output is always non-negative, one has as a sufficient condition for positive semi-definiteness on  $g(u,v)$  that,

$$f_a(x) \geq 0 \quad (8)$$

This restriction on the audio amplifier characteristic may actually be too stringent for particular I.F. amplifiers, but it will be carried along in order to guarantee the validity of the general result.

On substituting equation (6) into equation (4) the system operator reduces to

$$E_o(t) = \sum \lambda_j [e_j(t)]^2 \quad (9)$$

where

$$e_j(t) = \int_{-\infty}^{\infty} E_1(t-x) h_j(x) dx \quad (10)$$

The input voltage is now expressed as the sum of signal plus noise, viz.,

$$E_1(t) = S(t) + N(t) \quad (11)$$

so that for the output one gets,

$$E_o(t) = \sum_j \lambda_j [s_j(t) + \eta_j(t)]^2 \quad (12)$$

where

$$s_j(t) = \int_{-\infty}^{\infty} S(t-x) h_j(x) dx \quad (13)$$

and

$$\eta_j(t) = \int_{-\infty}^{\infty} N(t-x) h_j(x) dx \quad (14)$$

We are now in a position to find the characteristic function for  $E_o(t)$  for the case where  $N(t)$  is an uncorrelated Gaussian process.

The Characteristic Function

For any fixed  $t$ , let  $N(t)$  be normally distributed. Further, let

$$\overline{N(t_1) N(t_2)} = \phi_0 \delta(t_1 - t_2) \quad (15)$$

where  $\phi_0$  is the noise power per unit frequency and  $\delta(t_1 - t_2)$  is a unit impulse function at  $t_1 = t_2$ . Then, by equation (14), for any fixed  $t$ , the  $\eta_j(t)$  will be normally distributed. and, furthermore, since the functions  $h_j$  are orthogonal,

$$\overline{\eta_j(t) \eta_k(t)} = \phi_0 \delta_{jk} \quad (16)$$

where  $\delta_{jk}$  is the Kronecher delta. Consequently, for any fixed time,  $t$ , the vector  $\eta(t)$  defined by,

$$\eta(t) = [\eta_1(t), \eta_2(t), \dots] \quad (17)$$

has a multidimensional probability distribution given by,

$$dP[\eta(t)] = \prod_j \frac{d\eta_j(t)}{\sqrt{2\pi\phi_0}} e^{-\frac{\eta_j(t)^2}{2\phi_0}} \quad (18)$$

The characteristic function  $\bar{\Phi}(\xi, t)$  of  $E_0(t)$  is now given by,

$$\bar{\Phi}(\xi, t) = \text{ave } e^{i\xi E_0(t)} = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} e^{i\xi \sum_j \lambda_j [s_j(t) + \eta_j]^2} \prod_j \frac{d\eta_j}{\sqrt{2\pi\beta_0}} \cdot e^{-\frac{\eta_j^2}{2\beta_0}} \quad (19)$$

which may be integrated by completing the squares, giving

$$\bar{\Phi}(\xi, t) = \prod_j [1 - 2i\xi \lambda_j \beta_0]^{-1/2} \cdot \frac{s_j(t)^2}{2\beta_0} \cdot \frac{2i\xi \lambda_j \beta_0}{1 - 2i\xi \lambda_j \beta_0} \quad (20)$$

This is the result for the square law rectifier corresponding to that of Kac and Siegert for the square law envelope detector. Their result can be obtained directly from this by observing that in the case of the envelope detector for each variate of noise present in the input an extra independent variate of noise corresponding to the "out of phase" voltage component is added to the output. And, since the  $\eta_j$  are independent, this has the effect of multiplying the characteristic function, equation (20), by its value for noise alone. Under these conditions the noise power per cycle must be split equally between the two components. For Kac and Siegert the noise power per cycle is taken to be unity, so letting  $\beta_0 = 1/2$ ,  $s_j(t)^2 = p^2(t) + q^2(t) = \alpha^2 + \beta^2$  (in their notation), and changing the exponent of the term in brackets from  $-1/2$  to  $-1$ , yields their formula.

#### The Case of No Audio Filter

The output probability density function,  $P'(E_0)$ , can be derived in this case by inverting the characteristic function. Here we have,

$$F_s(\omega) = 1 \quad (21)$$

$$f_s(x) = \delta(x - 0) \quad (22)$$

From equation (5) we get,

$$g(u,v) = f_{if}(u) f_{if}(v) \quad (23)$$

and from equation (6), we see that there is but one eigenvalue,  $\lambda$ , and eigenfunction,  $h(u)$ . These are found to be,

$$\begin{aligned} \lambda &= \int_{-\infty}^{\infty} f_{if}(x)^2 dx \\ &= \int_{-\infty}^{\infty} |F_{if}(\omega)|^2 d\omega \end{aligned} \quad (24)$$

and,

$$h(u) = \frac{f_{if}(u)}{\sqrt{\lambda}} \quad (25)$$

The inversion of the characteristic function is then found to be:

$$\begin{aligned} P'(E_0) &= \int_0^{\infty} e^{-i\zeta E_0} \cdot \frac{s(t)^2 \frac{2\pi\lambda i\zeta}{1-2\pi\lambda i\zeta}}{\sqrt{1-2\pi\lambda i\zeta}} d\zeta \\ &= \frac{e^{-\frac{1}{2\pi\lambda} \left[ \frac{E_0}{\lambda} \cdot s(t)^2 \right]} \cosh \left[ \frac{s(t)}{\lambda} \sqrt{\frac{E_0}{\lambda}} \right]}{\sqrt{2\pi\lambda E_0}} \end{aligned} \quad (26)$$

Now,  $\pi_0 \lambda$  and  $s(t)^2 \lambda$  are respectively the r.m.s. noise power and signal power entering the detector. Then, normalizing these variables as follows:

$$\frac{s(t)^2 \lambda}{\pi_0 \lambda} = X \quad (27)$$

$$\frac{E_0}{\pi_0 \lambda} = Y \quad (28)$$

we get,

$$P'(Y) = \frac{e^{-\frac{1}{2}(X+Y)}}{\sqrt{2\pi Y} \cosh \sqrt{XY}} \quad (27)$$

This result may be verified easily by taking the distribution for the noise voltage,  $\eta$ , into the detector to be,

$$dP(\eta) = \frac{d\eta}{\sqrt{2\pi}} e^{-\frac{\eta^2}{2}} \quad (30)$$

For rectification in the detector, make the substitution,

$$Y = (\eta + \sqrt{X})^2 \quad (31)$$

This gives,

$$dP(Y) = \frac{dY}{2\sqrt{2\pi Y}} \left\{ e^{-\frac{1}{2}(\sqrt{Y} + \sqrt{X})^2} + e^{-\frac{1}{2}(\sqrt{Y} - \sqrt{X})^2} \right\} \quad (32)$$

which reduces immediately to equation (29).

## Section II

### The Cumulants for the Output Probability Density

In principle, the probability density function for the output voltage can be obtained from the characteristic function, equation (20), by a Fourier inversion. However, except in very special cases, i.e., infinitely wide audio pass band, it is a very difficult problem to find the  $\lambda_j$  and  $h_j$  required to determine  $\bar{\Phi}(\xi, t)$  explicitly, let alone to accomplish the inversion. The alternative is to find suitable methods for approximating the density function directly.

There are several well-known orthonormal systems suitable for approximating probability density functions. Two of these systems, the Gram-Charlier and the Laguerre, are discussed in Section III.

Particularly simple coefficients for these approximating series arise when they are expressed in terms of the cumulants of the probability density function. The cumulants,  $K_n$ , are defined by the following identity:

$$\exp\left(\sum_{n=1}^{\infty} K_n \frac{(i\xi)^n}{n!}\right) = \bar{\Phi}(\xi) \quad (33)$$

where  $\bar{\Phi}(\xi)$  is the characteristic function.\* So, taking the logarithm of  $\bar{\Phi}(\xi, t)$ , equation (20), and expanding in powers of  $i\xi$ , we get,

$$K_n(t) = (2\phi_0)^n \frac{(n-1)!}{2} \sum_j \lambda_j^n + (2\phi_0)^{n-1} \frac{1}{n!} \sum_j \lambda_j^n s_j(t)^2 \quad (34)$$

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\* See M. G. Kendall, "The Advanced Theory of Statistics," Vol. 1, for a complete discussion of cumulants and their relation to statistical approximation methods.

The n times iterated kernel,  $g^n(u,v)$ , is defined by,

$$g^n(u,v) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} g(u,x_1) g(x_1,x_2) \dots g(x_{n-1},v) dx_1 \dots dx_{n-1} \quad (35)$$

By virtue of equation (6) it is easily verified that,

$$\sum_j \lambda_j^n = \int_{-\infty}^{\infty} g^n(u,u) du \quad (36)$$

and

$$\sum_j \lambda_j^n s_j(t)^2 = \iint_{-\infty}^{\infty} S(t-u) g^n(u,v) S(t-v) du dv \quad (37)$$

Thus, the formula for the  $n^{\text{th}}$  cumulant becomes,

$$K_n(t) = (2\theta_0)^n \frac{n!}{2} \left\{ \frac{1}{n} \int_{-\infty}^{\infty} g^n(u,u) du + \frac{1}{\theta_0} \iint_{-\infty}^{\infty} S(t-u) g^n(u,v) S(t-v) du dv \right\} \quad (38)$$

which may be evaluated by straightforward quadratures.

Now, equation (38) can be derived by a more direct process. This is accomplished by noting that  $k_n$  is equal to a certain algebraic expression involving the first n moments, each of which can be evaluated by raising  $E_0(t)$  as expressed in equation (4) to the appropriate power and then averaging.

The formulas for the higher moments become progressively more complicated because of the large number of ways noise can be paired with noise, but on forming the required algebraic combination, equation (38) results. Since this procedure is extremely tedious, a derivation of equation (38) along these lines will not be included. It is mentioned, however, because it illustrates that the condition of positive semi-definiteness imposed on the kernel  $g(u,v)$  via equation (8) to insure uniform convergence of the Mercer series, equation (7), is



mathematically sufficient but not necessary, and does not constitute a restriction on the validity of equation (38). In fact, equation (38) is valid for any kernel if the corresponding system output voltage has moments up to and including the  $n^{\text{th}}$ . This will be the case in any practical receiver.

(38)

d

n

Section III

Two Orthonormal Systems

For approximating the output probability density functions two different orthonormal systems are especially useful. The first, the Weber-Hermite system which gives rise to the Gram-Charlier series, type A, is particularly suited to those functions which approximate the normal. In terms of this series,  $P'(E_0)$  is given by,\*

$$P'(E_0) \approx \frac{1}{\sqrt{K_2}} \sum_{j=0}^{\infty} \alpha_j \phi^{(j)} \left( \frac{E_0 - K_1}{\sqrt{K_2}} \right) \quad (39)$$

where

$$\phi^{(j)}(x) = \frac{d^j}{dx^j} \left\{ \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \right\} \quad (40)$$

and

$$\left. \begin{aligned} \alpha_0 &= 1 \\ \alpha_1 &= \alpha_2 = 0 \\ \alpha_3 &= -\frac{K_3}{6K_2^{3/2}} \end{aligned} \right\} \quad (41)$$

For density functions which approximate the Rayleigh distribution, the following series\*\* which derives from the orthonormal system of Laguerre is useful:

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\* See Kendall, *ibid*, and Marcum (4) for discussions of convergence, methods of grouping terms, evaluation of additional  $\alpha_j$  and further details.

\*\* In this form, the series seems to be due to Marcum (4), which see for evaluation of additional  $\alpha_j$  and details concerning the derivation.

$$P^1(E_0) \approx \frac{K_1}{K_2} \sum_{j=0}^{\infty} \beta_j \psi^{(j)} \left( \frac{K_1 E_0}{K_2} \right) \quad (42)$$

where

$$\psi^{(j)}(x) = \frac{d^j}{dx^j} \left\{ x^{\frac{K_1^2}{K_2} - 1} e^{-x} \right\} \quad (43)$$

and

$$\left. \begin{aligned} \beta_0 &= \frac{1}{\frac{K_1^2}{K_2} \Gamma\left(\frac{K_1^2}{K_2}\right)} \\ \beta_1 &= \beta_2 = 0 \\ \beta_3 &= \frac{K_1^2}{6K_2} \frac{2 - \frac{K_1 K_3}{K_2^2}}{\Gamma\left(\frac{K_1^2}{K_2} + 3\right)} \end{aligned} \right\} \quad (44)$$

Section IV

The Case of Gaussian I.F. and audio with Sinusoidal Signal

To illustrate the method consider the case in which the I.F. has a Gaussian pass band of width  $\sigma$ , centered at the very high frequency,  $f_0$ . Similarly let the audio have a Gaussian pass band of width  $\nu$ , centered at zero frequency (see Figure (2)).

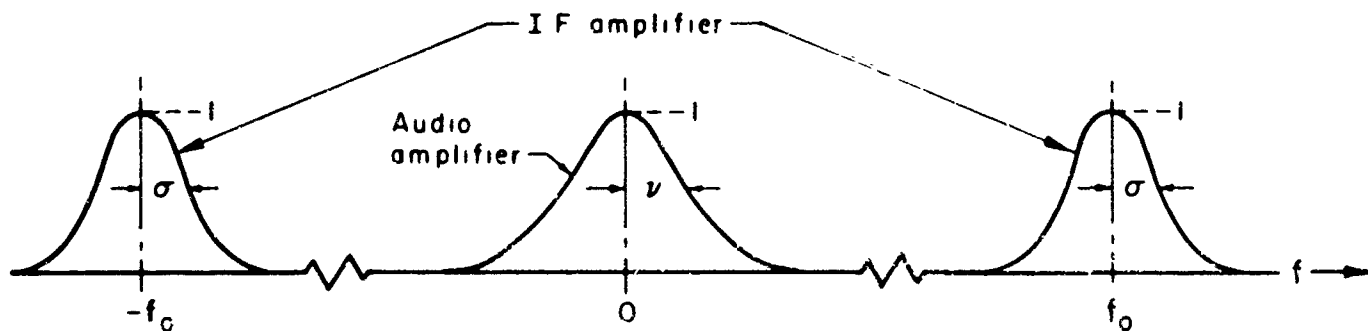


Figure (2)

Thus,

$$F_{if}(f) = e^{-\frac{(f+f_0)^2}{2\sigma^2}} + e^{-\frac{(f-f_0)^2}{2\sigma^2}} \quad (45)$$

$$F_a(f) = e^{-\frac{f^2}{2\nu^2}} \quad (46)$$

and,

$$f_{if}(x) = 2\sqrt{2\pi\sigma^2} e^{-\frac{(2\pi\sigma)^2}{2} \frac{x^2}{2}} \cos \omega_0 x \quad (47)$$

$$f_a(x) = \sqrt{2\pi\nu^2} e^{-\frac{(2\pi\nu)^2}{2} \frac{x^2}{2}} \quad (48)$$

By means of equations (5) and (35) we calculate,\*

$$g(u,v) = \frac{4\pi\sigma^2 \cos \omega_0(u-v)}{\sqrt{1+2\gamma^2}} \exp - \frac{(2\pi\sigma)^2}{4} \left\{ (u-v)^2 \cdot \frac{1}{1+2\gamma^2} (u+v)^2 \right\} \quad (47)$$

$$g^2(u,v) = \frac{4\pi^{3/2}\sigma^3 \cos \omega_0(u-v)}{\sqrt{(1+2\gamma^2)(1+\gamma^2)}} \exp - \frac{(2\pi\sigma)^2}{4} \left\{ \frac{1+\gamma^2}{1+2\gamma^2} (u-v)^2 + \frac{1}{1+\gamma^2} (u+v)^2 \right\} \quad (50)$$

$$g^3(u,v) = \frac{8\pi^2\sigma^4 \cos \omega_0(u-v)}{\sqrt{(1+2\gamma^2)(2+\gamma^2)(2+3\gamma^2)}} \exp - \frac{(2\pi\sigma)^2}{4} \left\{ \frac{2+\gamma^2}{2+3\gamma^2} (u-v)^2 + \frac{2+3\gamma^2}{(2+\gamma^2)(1+2\gamma^2)} (u+v)^2 \right\} \quad (51)$$

where,

$$\gamma = \frac{\sigma}{v} \quad (52')$$

From these equations we get,

$$\int_{-\infty}^{\infty} g(u,u) du = 2\sigma\sqrt{\pi} = \Delta f \quad (53)$$

$$\int_{-\infty}^{\infty} g^2(u,u) du = \frac{2\sigma^2\pi}{\sqrt{1+2\gamma^2}} = \frac{1}{2} \frac{\Delta f^2}{\sqrt{1+2\gamma^2}} \quad (54)$$

$$\int_{-\infty}^{\infty} g^3(u,u) du = \frac{4\sigma^3\pi^{3/2}}{2+3\gamma^2} = \frac{1}{2} \frac{\Delta f^3}{2+3\gamma^2} \quad (55)$$

\* These formulas have been obtained by neglecting the terms in  $\cos \omega_0(u+v)$  which represent the high frequency residue. They are vanishingly small for sufficiently large  $\omega_0$ .

Where the noise bandwidth of the I.F. is defined to be,

$$\Delta f = \frac{\int_{-\infty}^{\infty} |F_{if}(f)|^2 df}{\frac{1}{2} [F_{if}(f_0) + F_{if}(-f_0)]} = 2\sigma\sqrt{\pi} \quad (56)$$

Let the input signal voltage be,

$$S(t) = \sqrt{2S} \cos \omega_0 t \quad (57)$$

where S is the average power. (This is also the average signal power applied to the detector because of equation (45)). We then calculate,\*

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S(t-u) g(u,v) S(t-v) du dv = S \quad (58)$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S(t-u) g^2(u,v) S(t-v) du dv = \frac{1}{2} \sqrt{\frac{\Delta f}{1+\gamma^2}} S \quad (59)$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S(t-u) g^3(u,v) S(t-v) du dv = \frac{1}{2} \frac{\Delta f^2}{\sqrt{(2+\gamma^2)(2+3\gamma^2)}} S \quad (60)$$

Thus, by means of equation (38) we get for the first three cumulants,

$$K_1 = N [1+X] \quad (61)$$

$$K_2 = \frac{N^2}{\sqrt{1+\gamma^2}} \left[ 1 + 2X \sqrt{\frac{1-2\gamma^2}{1+\gamma^2}} \right] \quad (62)$$

$$K_3 = \frac{4N^3}{2+3\gamma^2} \left[ 1 + 3X \sqrt{\frac{2+3\gamma^2}{2+\gamma^2}} \right] \quad (63)$$

\* Again, omitting the high frequency terms in  $\cos \omega_0(2t-u-v)$ .

where

$$N = \bar{\rho}_0 \Delta f \quad (64)$$

is the average noise power at the input to the detector, and

$$X = \frac{S}{N} \quad (65)$$

is the signal-to-noise power ratio at the input to the detector.

The output probability density functions are now expressed in terms of the orthonormal series discussed in Section III. They have been plotted in Figures (3), (4) and (5) for selected values of the input signal-to-noise ratio  $X$  and bandwidth ratio  $\gamma$ . As was pointed out in the preceding footnotes the treatment in this section neglects the high frequency detector residue terms. Consequently, on letting  $\gamma$  pass to zero we do not obtain the case of infinite audio bandwidth discussed in Section I. Rather we obtain in the limit the well-known density function for the square law envelope detector with no additional audio filtering. These curves are labeled  $\gamma = \epsilon$  and the notation  $\gamma = 0$  is reserved for the curves of equation (29).

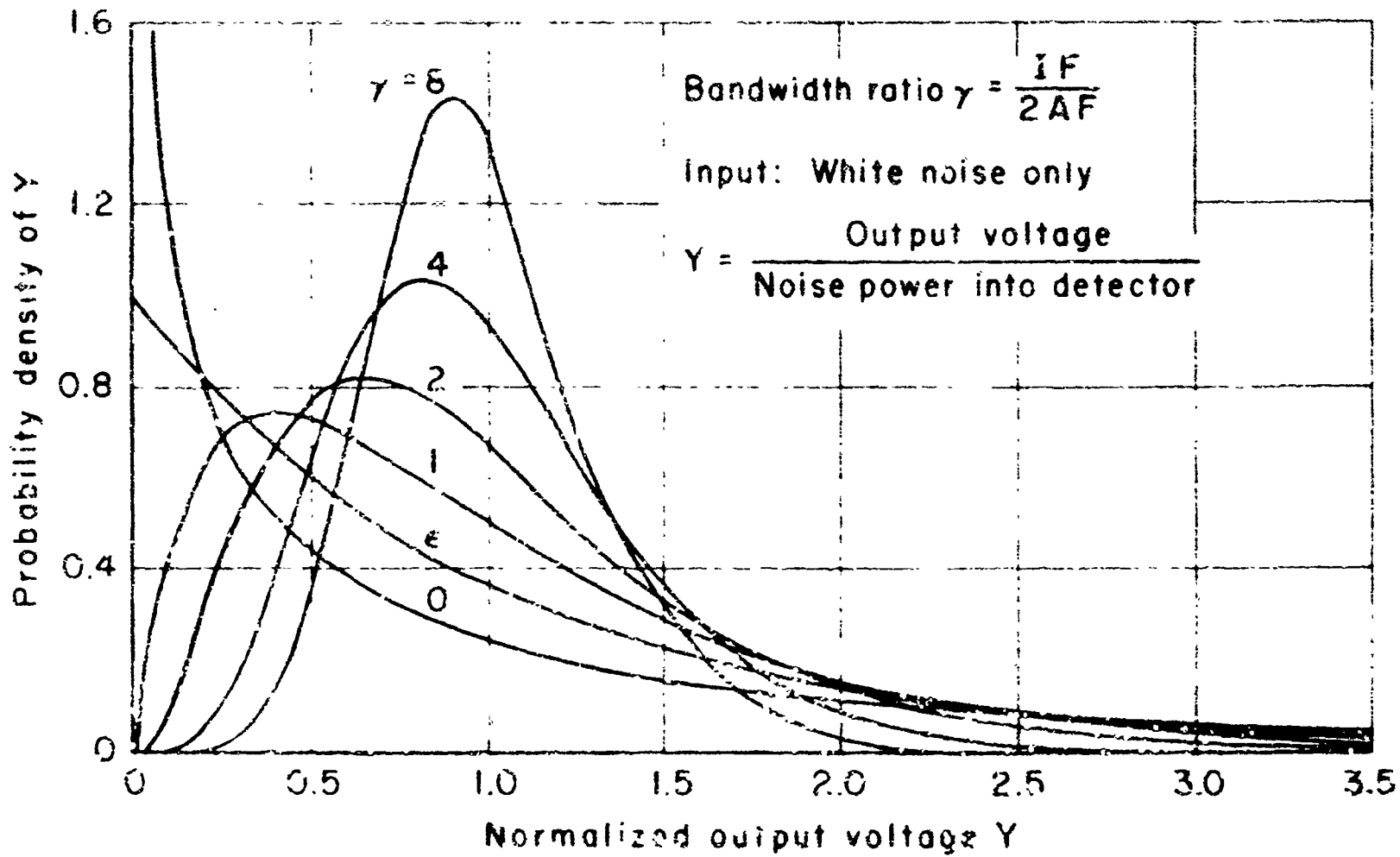


Fig. 3 — Normalized output probability density curves  
Gaussian IF — Gaussian Audio



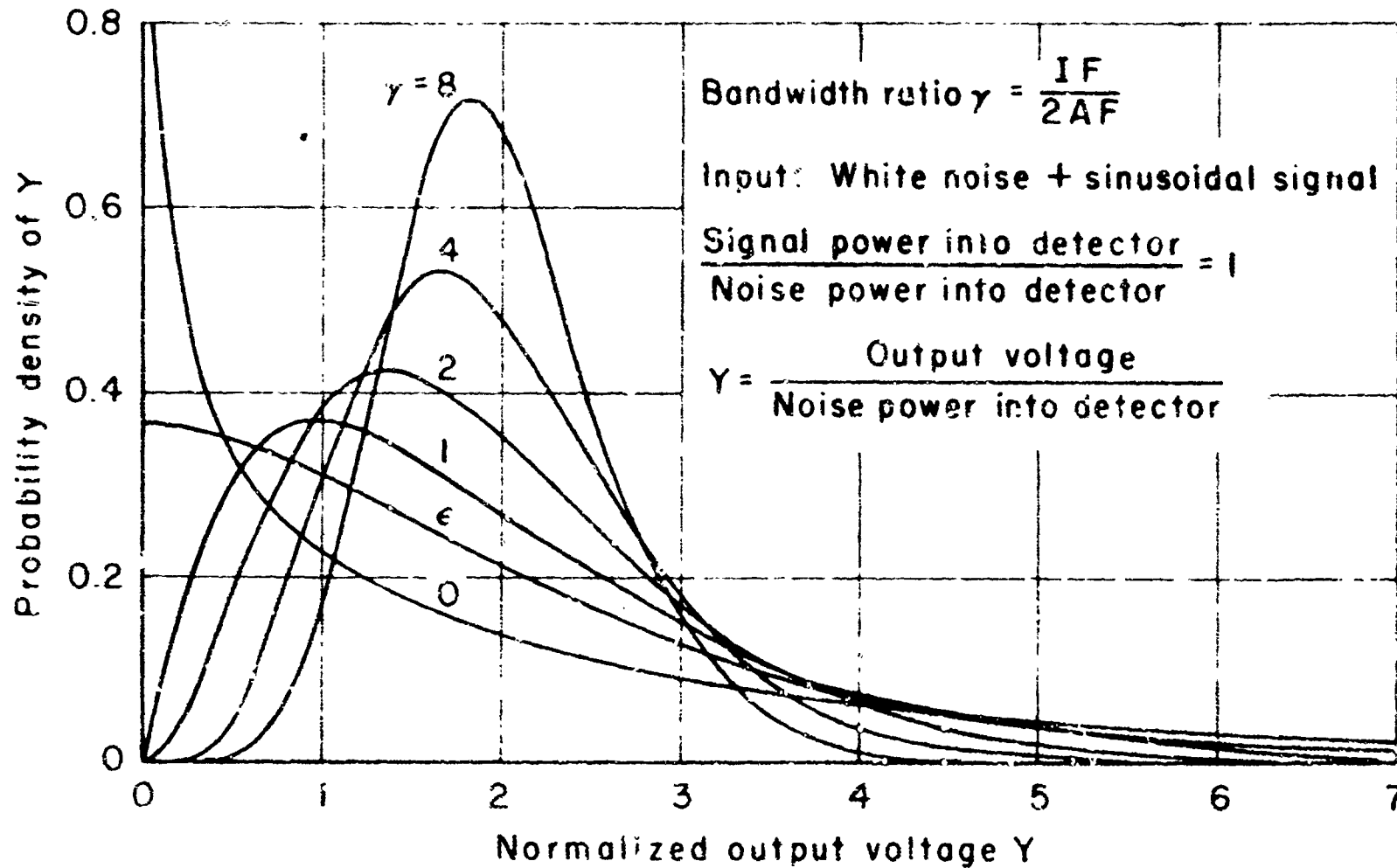


Fig. 4 — Normalized output probability density curves  
 Gaussian IF — Gaussian Audio

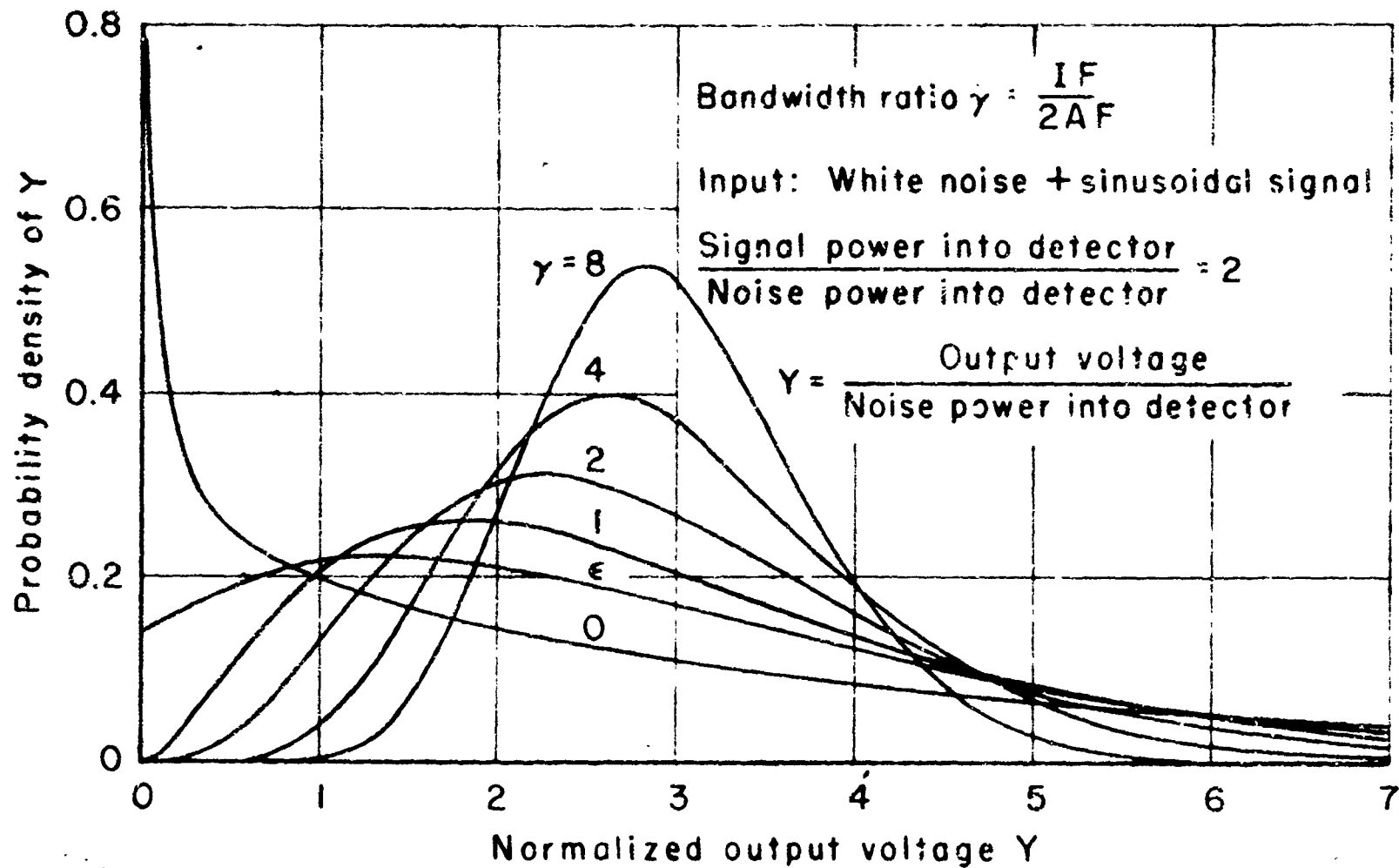


Fig. 5 — Normalized output probability density curves  
 Gaussian IF — Gaussian Audio

Section V

General Formula for  $K_n$  for the Gaussian System

We will now derive the Mercer series, equation (6), for  $g(u,v)$  and obtain the general expression for the  $n^{\text{th}}$  cumulant. This will serve as a check on formulas (61) through (63).

To start with, write,

$$g(u,v) = \frac{4\pi\sigma^2 \cos \omega_0(u-v)}{\sqrt{1+2\gamma^2}} \cdot \frac{-(2\pi\sigma)^2}{2} \left\{ \frac{1+\gamma^2}{(1+2\gamma^2)^2} (u^2+v^2) - \frac{2\gamma^2}{1+2\gamma^2} uv \right\} \quad (66)$$

and make the substitutions,

$$u = \frac{\sqrt{1+2\gamma^2}}{2\pi\sigma} x \quad (67)$$

$$v = \frac{\sqrt{1+2\gamma^2}}{2\pi\sigma} y \quad (68)$$

$$t = \frac{\sqrt{1+2\gamma^2} - 1}{\sqrt{1+2\gamma^2} + 1} \quad (69)$$

Then equation (66) becomes,

$$g(u,v) = \frac{4\pi\sigma^2 \cos \omega_0(u-v)}{\sqrt{1+2\gamma^2}} \cdot \frac{-x^2+y^2}{2} \cdot \left[ \frac{t^2(x^2+y^2) - 2txy}{1-t^2} \right] \quad (70)$$

We now make use of Mehler's formula,\* i.e.:

$$\left[ \frac{t^2(x^2+y^2) - 2txy}{1-t^2} \right] = \sqrt{1-t^2} \sum_{j=0}^{\infty} H_j(x) H_j(y) \frac{t^j}{j! 2^j} \quad (71)$$

\* See Watson, Journal of the London Math. Soc., Vol. 8 (1933), 189.

and after expanding  $\cos \omega_0(u-v)$ , we obtain the eigenvalue pairs,

$$\lambda_j^S = \lambda_j^C = \frac{2\sigma\sqrt{\pi}}{1 + \sqrt{1+2\gamma^2}} \left[ \frac{\sqrt{1+2\gamma^2}-1}{\sqrt{1+2\gamma^2}+1} \right]^j \quad (72)$$

and the corresponding pairs of eigenfunctions,

$$h_j^{S,C}(u) = \sqrt{4\pi\sigma} \frac{e^{-\frac{(2\pi\sigma)^2}{\sqrt{1+2\gamma^2}} \frac{u^2}{2}} H_j \left( \frac{2\pi\sigma u}{\sqrt{1+2\gamma^2}} \right)}{\sqrt{2^j j! \sqrt{\pi}} \sqrt{1+2\gamma^2}} \begin{cases} \sin \omega_0 u \\ \cos \omega_0 u \end{cases} \quad (73)$$

Here  $H_j(x)$  is the Hermite polynomial defined by,

$$H_j(x) = (-1)^j e^{x^2} \frac{d^j}{dx^j} e^{-x^2} \quad (74)$$

On summing with equation (72), we get

$$\sum_{j=0}^{\infty} \lambda_j^n = \sum_{j=0}^{\infty} (\lambda_j^S)^n + \sum_{j=0}^{\infty} (\lambda_j^C)^n = \frac{2(2\sigma\sqrt{\pi})^n}{(\sqrt{1+2\gamma^2}+1)^n - (\sqrt{1+2\gamma^2}-1)^n} \quad (75)$$

For the sinusoidal input signal defined by equation (57) we get

$$s_j^{S,C}(t) = \sqrt{2s} \int_{-\infty}^{\infty} \cos \omega_0(t-x) h_j^{S,C}(x) dx$$

$$= \sqrt{\frac{2s}{2\sigma\sqrt{\pi}}} \sqrt{1+2\gamma^2} \frac{\sqrt{j!}}{2^{j/2} (j/2)!} \begin{cases} \sin \omega_0 t \\ \cos \omega_0 t \end{cases} \text{ for } j \text{ even} \quad (76)$$

$$= 0 \text{ for } j \text{ odd}$$

This result is valid for  $\omega_0$  sufficiently high to justify neglecting the high frequency residue.

On summing with equations (72) and (76) we get,

$$\sum_{j=0}^{\infty} \lambda_j^n s_j(t)^2 = \sum_{j=0}^{\infty} (\lambda_j^S)^n s_j^S(t)^2 + \sum_{j=0}^{\infty} (\lambda_j^C)^n s_j^C(t)^2$$

$$= \frac{2S(2\sigma\sqrt{\pi})^{n-1} \sqrt[4]{1+2\gamma^2}}{\sqrt{(\sqrt{1+2\gamma^2+1})^{2n} - (\sqrt{1+2\gamma^2-1})^{2n}}} \quad (77)$$

Finally, by means of equations 34, 56, 64, and 65 we get for the  $n^{\text{th}}$  cumulant,

$$K_n = \frac{(2N)^n (n-1)!}{(\sqrt{1+2\gamma^2+1})^n - (\sqrt{1+2\gamma^2-1})^n} \left\{ 1 + \sqrt{\frac{(\sqrt{1+2\gamma^2+1})^n - (\sqrt{1+2\gamma^2-1})^n}{(\sqrt{1+2\gamma^2+1})^n + (\sqrt{1+2\gamma^2-1})^n}} \sqrt[4]{1+2\gamma^2} \right\} \quad (78)$$

which reduces for  $n=1, 2,$  and  $3$  to equations (61), (62) and (63), respectively.

## Section VI

Signal Detectability for the Gaussian System

Lawson and Uhlenbeck<sup>(9)</sup> in discussing various detection criteria suggest a "deflection criterion" which is based on measuring the change in the average output brought about by the signal. It is suggested that this change should be comparable to the standard deviation due to noise alone in order that the signal be "detectable." In terms of the cumulants, then, we have for detectability,

$$S_o = \frac{K_1^{S+N} - K_1^N}{\sqrt{K_2^N}} \sim 1 \quad (79)$$

related to this criterion is one obtained by using the standard deviation of the signal plus noise, i.e.,

$$S_s = \frac{K_1^{S+N} - K_1^N}{\sqrt{K_2^{S+N}}} \sim 1 \quad (80)$$

These quantities have been investigated by Smith<sup>(7)</sup> for a variety of filter shapes and detector characteristics. By direct substitution from equations (61) and (62) we get for the Gaussian-square law system,

$$S_o = X \sqrt{1+2Y^2} \quad (81)$$

$$S_s = \frac{X \sqrt{1+2Y^2}}{\sqrt{1+2X \sqrt{\frac{1+2Y^2}{1+Y^2}}}} \quad (82)$$

(9)

J. L. Lawson and G. E. Uhlenbeck, "Threshold Signals," Rad. Lab. Series No. 24, p. 161.

For relatively narrow audio bandwidth, i.e.,  $\gamma$  large, which is the region of interest for the reference, these expressions reduce to,

$$S_o \sim \frac{X}{2^{1/4}} \sqrt{2\gamma} \quad (83)$$

$$S_s \sim \frac{X\sqrt{2\gamma}}{2^{1/4} \sqrt{1+2\sqrt{2} X}} \quad (84)$$

These are in agreement with the formulas given by Smith.

The "Ideal or Threshold Detector" provides another means for determining signal detectability. With this device a "detection" is called if the output voltage at any particular time exceeds a certain threshold. The level of this threshold is then adjusted so that "false alarms", i.e., detections in the absence of signal, are relatively infrequent. The probability of a detection at any instant can then be written,

$$P_d(X, \tau) = \int_{\tau}^{\infty} P'(E_o, X) dE_o \quad (85)$$

and is necessarily an increasing function of signal strength.

Marcum<sup>(4)</sup> has considered the case of a square law envelope detector with video integration, i.e., successive samples of the detector output are numerically averaged rather than applied to a filter. The probability density function for this averaged output is given (in our notation) by the formula,

$$P'(E_o, X, \alpha) = \frac{\alpha}{N} \left(\frac{E_o}{NX}\right)^{\frac{\alpha-1}{2}} \cdot e^{-\alpha\left(\frac{E_o}{N} + X\right)} I_{\alpha-1} \left(2\alpha\sqrt{\frac{E_o X}{N}}\right) \quad (86)$$

where  $\alpha$  is the number of samples averaged. The cumulants for this distribution are given by the formula,

$$K_n^* = \frac{N^n (E-1)!}{\alpha^{n-1}} (1+nX) \quad (87)$$

the first few of which are,

$$K_1^* = N(1+X) \quad (88)$$

$$K_2^* = \frac{N^2}{\alpha} (1+2X) \quad (89)$$

$$K_3^* = \frac{2N^3}{\alpha^2} (1+3X) \quad (90)$$

We now set

$$\alpha = \sqrt{1+2\gamma^2} \quad (91)$$

and notice that fairly good agreement exists between these cumulants and those for the Gaussian audio system. Consequently,  $\sqrt{1+2\gamma^2}$  can be interpreted as the approximate number of samples averaged by the audio filter.\*

It is now possible to obtain a fairly good first approximation to the probability of detection,  $P_d(X, \bar{t})$ , by referring to the appropriate curves in reference (4).

RCE:l.s.e

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\* This is not an exact treatment because the probability density function for the Gaussian system cannot be put precisely into the form of equation (86) unless  $\gamma = 0$ . However, the mean values will correspond exactly, the variance will be correct to within a factor of less than  $\sqrt{2}$  in  $X$ , and the asymmetry will be in error by a factor less than  $4/3$  in  $K_3$  and  $\sqrt{3}$  in  $X$ .