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DRESHER'S INEQUALITY

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Melvin Dresher has proved by an ingenious method, using moment-space theory¹, the inequality which I shall call Dresher's inequality:

If $p \geq 1 \geq r \geq 0$, $f(x) \geq 0$, $g(x) \geq 0$, and $\phi(x)$ is a distribution function², then³

$$(1) \quad \left[\frac{\int [f(x) + g(x)]^p d\phi(x)}{\int [f(x) + g(x)]^r d\phi(x)} \right]^{\frac{1}{p-r}} \leq \left[\frac{\int [f(x)]^p d\phi(x)}{\int [f(x)]^r d\phi(x)} \right]^{\frac{1}{p-r}} + \left[\frac{\int [g(x)]^p d\phi(x)}{\int [g(x)]^r d\phi(x)} \right]^{\frac{1}{p-r}}$$

Dresher's ~~In the purpose of~~ this note ~~we~~ give^s an elementary proof of ~~the~~ *the* inequality, based on the Minkowski inequality and an inequality due to Radon⁴. The Radon inequality is gotten easily by transforming the Hölder

¹ Forthcoming.

² In particular, we may take $\phi(x) = x$, and read each $d\phi(x)$ as dx .

³ We assume that neither f nor g vanishes identically.

⁴ J. Radon, Über die absolut additiven Mengenfunktionen, Wiener Sitzungsber. (II a), 122 (1913), p. 1351. See also Hardy, Littlewood, and Polya, Inequalities Cambridge (1934) p. 61 problem 65, and C. B. Morrey, A class of representations of manifolds, Amer. J. Math. 55 (1933), p. 602.

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inequality, thus Drescher's inequality is seen to be a mélange of the Hölder and Minkowski inequalities. It is appropriate to note here that Drescher's inequality is a generalization of an inequality due to Beckenbach.

The Radon inequality is as follows:

If $\lambda > 0$, $\alpha_i \geq 0$, $\beta_i \geq 0$, $i = 1 \dots, n$, then

$$(2) \quad \frac{(\sum \alpha_i)^{\lambda+1}}{(\sum \beta_i)^\lambda} \leq \sum \frac{\alpha_i^{\lambda+1}}{\beta_i^\lambda}$$

In using this to prove Drescher's inequality, we shall need only two terms.

The inequality is then stated as follows:

If $\lambda > 0$, $\alpha_1 \geq 0$, $\alpha_2 \geq 0$, $\beta_1 \geq 0$, and $\beta_2 \geq 0$, then

$$(3) \quad \frac{(\alpha_1 + \alpha_2)^{\lambda+1}}{(\beta_1 + \beta_2)^\lambda} \leq \frac{\alpha_1^{\lambda+1}}{\beta_1^\lambda} + \frac{\alpha_2^{\lambda+1}}{\beta_2^\lambda}$$

^s Hardy, Littlewood, Polya, loc. cit.

• E. F. Beckenbach, A class of mean value functions, Amer. Math. Monthly 57 (1950), pp. 1-6.

We make in (3) the substitutions

$$\begin{aligned} \alpha_1 &= \left[\int r^p d\phi \right]^{\frac{1}{p}} & \beta_1 &= \left[\int r^r d\phi \right]^{\frac{1}{r}} \\ \alpha_2 &= \left[\int g^p d\phi \right]^{\frac{1}{p}} & \beta_2 &= \left[\int g^r d\phi \right]^{\frac{1}{r}} \end{aligned} \quad \lambda = \frac{r}{p-r}$$

Here, for the moment, we have assumed $p > r > 0$

Then the right side of (3) becomes the right side of (1), and the left side of (3) becomes

$$(4) \quad \frac{\left\{ \left[\int r^p d\phi \right]^{\frac{1}{p}} + \left[\int g^p d\phi \right]^{\frac{1}{p}} \right\}^{\frac{p}{p-r}}}{\left\{ \left[\int r^r d\phi \right]^{\frac{1}{r}} + \left[\int g^r d\phi \right]^{\frac{1}{r}} \right\}^{\frac{r}{p-r}}}$$

But by Minkowski's inequality with $p \geq 1$ and $0 < r \leq 1$ respectively, both

$$(5) \quad \left\{ \left[\int r^p d\phi \right]^{\frac{1}{p}} + \left[\int g^p d\phi \right]^{\frac{1}{p}} \right\}^p \geq \int (r+g)^p d\phi$$

and

$$(5) \quad \left\{ \left[\int r^r d\phi \right]^{\frac{1}{r}} + \left[\int g^r d\phi \right]^{\frac{1}{r}} \right\}^r \leq \int (r + g)^r d\phi ,$$

so that the quantity (4) is not less than the expression on the left side of (1), and Drosher's inequality is proved for $p \geq 1$ and $p > r > 0$.

Extension to the remaining cases is trivial, and thus the proof is complete.

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