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METHODS OF SOLUTION IN GAME THEORY

Melvin Dresher

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The RAND Corporation
1700 MAIN ST. • SANTA MONICA • CALIFORNIA

METHODS OF SOLUTION IN GAME THEORY

Melvin Dresher, The RAND Corporation

Consider the two-person zero-sum game, with a finite number of strategies, described by the payoff matrix $A = \|a_{ij}\|$ where a_{ij} represents the payment to player I if he chooses strategy $i = 1, 2, \cdots, m$ and player II chooses strategy $j = 1, 2, \cdots, n$. Let the column matrices X, Y represent mixed strategies of player I and II respectively, where the components x_i, y_j of the respective matrices are the frequencies of the corresponding strategies. Then there exists a pair of mixed strategies X^*, Y^* such that

$$\min_{Y} X^{*'} AY = \max_{X} X' AY^* = v,$$

where X' is the transpose of X. X^*, Y^* is said to be a <u>solution</u> of the game having a <u>value</u>, v.

If there exists a solution X^*, Y^* such that $AY^* = v1$ and $A'X^* = v1$, where 1 is the column matrix all whose components are unity, then X^*, Y^* is a <u>simple solution</u> of the game. A necessary and sufficient condition that a game described by a square, non-singular matrix A have a simple solution is that all the components of

$$X^* = \frac{(A')^{-1}1}{1'(A')^{-1}1}, \quad Y^* = \frac{A^{-1}1}{1'A^{-1}1}$$

be non-negative. Thus, if a game has a simple solution, it is readily obtained.

If a game does not have a simple solution, it can nevertheless be solved by examining square submatrices for simple solutions. Every solution of a game A is a convex linear combination of a finite number of <u>basic solutions</u> every basic solution is derived from a simple solution of some square non-singular submatrix of A. Thus to obtain all basic solutions we need to examine each square non-singular submatrix of A for a simple solution. Having obtained a simple solution of the submatrix, we introduce zero components for the remaining rows and columns of A, and test whether the full vectors X, Y solve A, i.e., whether

$$\max_i \sum_{j=1}^{n} a_{ij} y_j = \min_j \sum_{i=1}^{m} a_{ij} x_i .$$

If X, Y solves A then it is a solution of A and the value of the game is $\max_i \sum_{j=1}^{n} a_{ij} y_j$.

It is frequently possible to reduce the size of the game matrix by simple dominance considerations. If a row is dominated by some other row or by some convex linear combination of rows, then the dominated row may be eliminated from the matrix. If a row dominates some convex linear combination of other rows, then some one of them may be eliminated. Similarly, we may eliminate columns. In general, reduction by dominance may lose some solutions. However, reduction by strict dominance retains all solutions.

If a game matrix is m x 2, it can be solved graphically in two dimensions. Using trilinear coordinates, we can solve an m x 3 game matrix graphically, also in two dimensions. If the game matrix is very large it is possible to solve it by an iterative process which requires only addition and the location of maxima and minima - each step the method chooses for each player a strategy which is best against the opponent's mixture cumulated to date.

The solution of games having a continuum of strategies is very difficult and methods exist only for a few special classes of payoff functions. If the payoff, M(x, y), is continuous and convex in y for each x, then the value of the game is $v = \min_y \max_x M(x, y)$. Player II has pure strategies which are optimal - every y which minimizes $\max_x M(x, y)$. Player I has optimal strategies which are generally mixed - they make use of all X such that M(X, Y) = v, where Y is any pure strategy which is optimal for player II.

If the payoff M(x, y) is a polynomial in each variable then it is possible to reduce the solution of the game problem to the solution of certain systems of equations - linear in some cases, non-linear in the remaining.