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NOTES ON THE SOLUTION OF

LINEAR SYSTEMS INVOLVING INEQUALITIES

George W. Brown

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NOTES ON THE SOLUTION OF LINEAR SYSTEMS INVOLVING INEQUALITIES

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Consider the problem of minimizing a linear function $\sum b_j x_j$ subject to the conditions[1]

$$\left.\begin{array}{r} \sum_j A_{ij}x_j \geq c_i \\ x_j \geq 0 \end{array}\right\} \begin{array}{l} i = 1, 2, \ldots, m_1 \\ j = 1, 2, \ldots, m_2 . \end{array}$$

Notice at the outset that equalities may be admitted in this form by writing each equality as two inequalities with reversal of signs. Furthermore, the problem may be reformulated so that only inequalities of the form $x_j \geq 0$ are present, by defining appropriate new variables. Thus it is evident that the above form is simply one standard version of a general problem involving both inequalities and equalities.

In principle the solution of the problem stated is trivial. Observe that the set of inequalities defines in m_2-space a convex polyhedron (possibly empty) with at most $m_1 + m_2$ faces of dimension $m_2 - 1$, and that the minimum problem is that of finding an extreme point of the polyhedron in some direction. In general the extremum will be taken on at a vertex, so the problem is that of evaluating $\sum b_j x_j$ at the vertices and choosing that vertex which yields the smallest value. A vertex is of course a point at which a subsystem (of rank m_2) of the inequalities is satisfied exactly as equalities, with the remaining inequalities satisfied. In principle, then, one could invert all subsystems of rank m_2, throwing out those whose solutions fail to satisfy the remaining inequalities, and then evaluate $\sum b_j x_j$. It is clear that this is not a practical method beyond the

(1) Theoretical background of this paper is based on work of H. Weyl, v. Neumann, Ville, Tucker, G. Dantzig, and others, on convex polyhedra and on theory of games.

smallest values of m_1 and m_2. The practical difficulties stem from the fact that the convex polyhedron is specified by its faces, whereas the vertices are at the root of the problem.

In passing, it should be noted that the problem stated above has a very simple dual problem, obtained by transposing the matrix A, and making a few other obvious changes. The dual is the problem of maximizing $\sum c_i y_i$ subject to

$$\left.\begin{array}{r} \sum_i y_i A_{ij} \le b_j \\ y_i \ge 0 \end{array}\right\} \quad \begin{array}{l} i = 1, 2, \ldots, m_1 \\ j = 1, 2, \ldots, m_2 \end{array}$$

The two dual problems have the property that if either problem has a solution so has the other, and the minimum value in one is the maximum value in the other. In certain economic applications the solutions of both problems are required.

Consider now the problem of maximizing $\min_j \sum \xi_i A_{ij}$ subject to $\xi_i \ge 0, \sum \xi_i = 1$; $i = 1, 2, \ldots, m_1$ and the dual problem of minimizing $\max_i \sum A_{ij} \eta_j$ subject to $\eta_j \ge 0, \sum \eta_j = 1$, $j = 1, \ldots, m_2$. This problem provides optimum mixed strategies for the zero-sum game with matrix A, where A_{ij} represents the payment from player 1 to player 2, if player 1 plays his i^{th} strategy and player 2 plays his j^{th} strategy. The celebrated minimax theorem of von Neumann says that under the conditions stated

$$\operatorname*{Max}_{\xi} \operatorname*{Min}_{j} \sum \xi_i A_{ij} = \operatorname*{Min}_{\eta} \operatorname*{Max}_{i} \sum A_{ij} \eta_j .$$

The common value is referred to as the value of the game and the $\{\xi_i\}$ and $\{\eta_j\}$ of the solutions are the optimum mixtures for players 1 and 2, respectively. As in the first problem stated in this paper, geometrical considerations of convex bodies contribute to understanding the problem, and it turns out that in general the problem is practically solved if it is known which submatrix of A to invert.

There is of course an intimate relation between the theory of games problem and the problem first stated, although they are not quite identical problems, since the game problem always has a solution, while the first problem does not necessarily. Briefly summarized, the game problem is directly a special case of the first problem, while the first problem can always be embedded in a game problem, whose solutions yield solutions to the original problem if it has a solution. Thus, if problems of one type can be solved, so can problems of the other type.

Various iterative methods for solution of one or the other of these problems have been given by von Neumann, Dantzig, and others. While some of these methods may be practical over a certain range of problems, all of them have an apparent dependence, in required number of steps, of higher order than the first power of the linear dimensions of the problem. For very large matrices not possessing simplifying special properties, such a dependence can be a very serious obstacle in the way of getting numerical solutions. We will describe briefly, for the game solution, an iterative scheme which is quite different from those previously suggested, in that the amount of calculation required at each iterative step is directly proportional to the linear dimensions of the problem, so that the method has, a priori, some chance of beating the high order dependence.

The procedure to be described can most easily be comprehended by considering the psychology of, let us say, a statistician unfamiliar with theory of games. Such a person, faced with repeated choices of play of a certain game, might reasonably be expected to play, at each opportunity, that one of his strategies which is best against past history, that is, against the mixture constituted by his opponent's plays to date. Such a decision utilizes information of the past in the most obvious manner. The iterative scheme referred to here is based on a picture of two such statisticians playing repeatedly together. For calculation purposes a slight modification is introduced which has the effect that the two players choose alternately, rather than simultaneously.

Restating the method algebraically, let A be the game matrix, let i_n and j_n be the n^{th} choices of strategy for the two sides, let $\xi_i^{(n)}$ and $\eta_j^{(n)}$ be the relative frequencies of strategies i and j in $(i_1, i_2, \ldots, i_n)$ and $(j_1, j_2, \ldots, j_n)$ respectively, then j_n minimizes $\sum_i \xi_i^{(n)} A_{ij}$ and i_{n+1} maximizes $\sum_j A_{ij} \eta_j^{(n)}$. This process defines a sequence i_1, j_1, i_2, j_2, ..., once i_1 is chosen (perhaps arbitrarily), except for possible ambiguities of the extrema. Any convenient rule will do for handling ambiguities. If $\underline{V}_n = \min_j \sum \xi_i^{(n)} A_{ij}$ and $\overline{V}_n = \max_i \sum A_{ij} \eta_j^{(n)}$, it is easily seen that $\underline{V}_n \leq V \leq \overline{V}_n$, where V is the value of the game. The mixtures $\{\xi_i^{(n)}\}$ and $\{\eta_j^{(n)}\}$ are mixed strategies, and the corresponding $\underline{V}_n$ and $\overline{V}_n$ are the most favorable outcomes ensured to each player if he uses the corresponding mixture.

At this moment not much is rigorously established about the properties of this iteration, except that if it converges at all it converges to a solution of the game for each side. Of course it would be sufficient if $\lim\sup \underline{V}_n = \lim\inf \overline{V}_n$. There is considerable support, however, based on experience with the method, and also on the study of a related system of differential equations, for the conjecture that convergence is of the order of $1/n$ and does not depend essentially on the size of the matrix. If this is so, it is extremely important for the solution of large matrices, by virtue of the fact that each iterative steps requires only a number of operations proportional to the linear size of the matrix. Convergence of order $1/n$ is of course painful if high accuracy is needed. In such cases it may be possible, however, to use a method like this to get close to the solution, finishing with one step of another iteration.

The accompanying worksheet shows 25 steps carried out for the 4×3 matrix

i \ j	1	2	3
1	3	1.1	1.2
2	1.3	2	0
3	0	1	3.1
4	2	1.5	1.1

Note that each line is obtained by adding to the previous line, component by component, the corresponding row or column of the matrix, without troubling to divide by n. The $\underline{V}_n$ and $\overline{V}_n$ were calculated at each step, by division of the extrema by n, to show the progress of the calculation. In case of ties the lowest index was taken. Note particularly that $\overline{V}_n - \underline{V}_n$ is decreasing just about like $1/n$, in spite of the excursions which $\overline{V}_n$ and $\underline{V}_n$ make. The initial choice of $i_1 = 2$ was made deliberately as an unfavorable choice, with respect to minimum guaranteed payoff.

It is appropriate to report to this Symposium that preliminary discussions with Messrs. Haar and Singer of the staff of the Harvard Computation Laboratory indicate that Mark III could carry out 1000 of these iterative lines for a 40×40 matrix in comfortably under an hour. Of course the problem has not been completely programmed, but the estimate is believed to be conservative.

WORKSHEET

FORM R-18 2 10 48

Cumulative Payoffs

n	i_n	j=1	j=2	j=3		$\underline{v}_n$	$\overline{v}_n$		j_n	i=1	i=2	i=3	i=4
1	2	1.3	2	0		0	3.1		3	1.2	0	3.1	1.1
2	3	1.3	3	3.1		.65	2.1		1	4.2	1.3	3.1	3.1
3	1	4.3	4.1	4.3		1.37	1.77		2	5.3	3.3	4.1	4.6
4	1	7.3	5.2	5.5		1.30	1.60		2	6.4	5.3	5.1	6.1
5	1	10.3	6.3	6.7		1.26	1.52		2	7.5	7.3	6.1	7.6
6	4	12.3	7.8	7.8		1.30	1.55		2	8.6	9.3	7.1	9.1
7	2	13.6	9.3	7.8		1.11	1.46		3	9.8	9.3	10.2	10.2
8	3	13.6	10.8	10.9		1.35	1.46		2	10.9	11.3	11.2	11.7
9	4	15.6	12.3	12.0		1.33	1.59		3	12.1	11.3	14.3	12.8
10	3	15.6	13.3	15.1		1.33	1.53		2	13.2	13.3	15.3	14.3
11	3	15.6	14.3	18.2		1.30	1.48		2	14.3	15.3	16.3	15.8
12	3	15.6	15.3	21.3		1.28	1.44		2	15.4	17.3	17.3	17.3
13	2	16.9	17.3	21.3		1.30	1.48		1	18.4	18.6	17.3	19.3
14	4	18.9	18.8	22.4		1.34	1.49		2	19.5	20.6	18.3	20.8
15	4	20.9	20.3	23.5		1.35	1.51		2	20.6	22.6	19.3	22.3
16	2	22.2	22.3	23.5		1.39	1.52		1	23.6	23.9	19.3	24.3
17	4	24.2	23.8	24.6		1.40	1.52		2	24.7	25.9	20.3	25.8
18	2	25.5	25.8	24.6		1.37	1.49		3	25.9	25.9	23.4	26.9
19	4	27.5	27.3	25.7		1.35	1.47		3	27.1	25.9	26.5	28.0
20	4	29.5	28.8	26.8		1.34	1.48		3	28.3	25.9	29.6	29.1
21	3	29.5	29.8	29.9		1.40	1.49		1	31.3	27.2	29.6	31.1
22	1	32.5	30.9	31.1		1.40	1.48		2	32.4	29.2	30.6	32.6
23	4	34.5	32.4	32.2		1.40	1.47		3	33.6	29.2	33.7	33.7
24	3	34.5	33.4	35.3		1.39	1.47		2	34.7	31.2	34.7	35.2
25	4	36.5	34.9	36.4		1.40	1.47		2	35.8	33.2	35.7	36.7