SOLUTIONS OF DISCRETE, TWO-PERSON GAMES

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INTRODUCTION

In this paper [redacted] proposes to investigate the structure of solutions of discrete, zero-sum, two-person games. For a finite game-matrix it is well known that a solution (i.e., a pair of frequency distributions describing the optimal mixed strategies of the two players) always exists[redacted] Moreover, the set of solutions is known to be a convex polyhedron, each of whose vertices corresponds to a submatrix with special properties [3].

In Part I of the [redacted] paper [redacted] a fundamental relationship between the dimensions of the sets of optimal strategies is proven, and devote particular attention to the set of games whose solutions are unique. Part II solves the problem of constructing a game-matrix with a given solution. A number of examples and geometrical arguments are interspersed to illustrate the theory, and Part III describes the solutions of some matrices with special diagonal properties.

PART I: STRUCTURE OF SOLUTIONS

§1. Introduction and definitions.

Let Γ be the game described by the matrix $A = (a_{ij})$, with rows a_i and columns a_j $(i = 1,2,\ldots,m;\ j = 1,2,\ldots,n)$. Let X be the set of all optimal

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* Numbers in square brackets refer to the bibliography at the end of the paper.

mixed strategies $x = (x_i)$ $(x_i \geq 0,\ 1 \cdot x = 1,\ \min_y xAy = v)$ of the maximizing player; Y that of the other player. Let $I_1(x)$ be the set of indices i for which $x_i > 0$, and $I_2(y)$ those for which $a_i \cdot y = v$; and similarly $J_1(y)$, $J_2(x)$. Then define

$$I_1 = \sum_{x \in X} I_1(x), \qquad J_1 = \sum_{y \in Y} J_1(y)$$

$$I_2 = \prod_{y \in Y} I_2(y), \qquad J_2 = \prod_{x \in X} J_2(x)$$

$$X_\ell = \text{the set of } x \text{ with } I_1(x) \subseteq I_\ell, \qquad \ell = 1, 2;$$

$$Y_\ell = \text{the set of } y \text{ with } J_1(y) \subseteq J_\ell, \qquad \ell = 1, 2.$$

Thus X_1 is the smallest face of the fundamental simplex of mixed strategies containing X, etc.

The purpose of Part I is to prove the relations:

$$(1) \qquad I_1 = I_2, \qquad J_1 = J_2 \qquad \text{(Theorem 1)}$$

$$(2) \qquad \dim X_1 - \dim X = \dim Y_1 - \dim Y \qquad \text{(Theorem 2)}$$

for all games with finite sets of pure strategies. We also show that the set of m x n game-matrices with unique solutions is dense and open in (mn)-space (Theorem 3). Under (2) we may subsume the corollary that a unique solution must be "square" —that is, involve the same number of pure strategies on each side. This is of especial interest since, by Theorem 3, games "in general position" have unique solutions.

For infinite matrices these results are not valid, even when I_1 and J_1 are finite. Nor does the analogue of Theorem 1 hold if the matrix is replaced by a continuous function, even though the pure strategies form compact sets. Simple examples supporting these assertions will be found in §7.

§2. Reduction to the essential part of the game.

Lemma 1. There exists x in X such that $I_1(x) = I_1$ and $J_2(x) = J_2$ [y in Y such that $J_1(y) = J_1$ and $I_2(y) = I_2$].

Proof: To each $i \in I_1$ and $j \in J_2$ there corresponds an $x^{'(i)}$ with $x_i^{'(i)} > 0$ and an $x^{''(j)}$ with $a_j \cdot x^{''(j)} = v$. The center of gravity of these $x^{'(i)}$ and $x^{''(j)}$ can be taken to be the x of the lemma.

Lemma 2. $I_1 \subseteq I_2$ [$J_1 \subseteq J_2$].

Proof: Choose x and y as in the preceding lemma. Then $\sum x_i(a_i \cdot y) = v$ implies that $x_i > 0$ only if $a_i \cdot y = v$. Hence $I_1 = I_1(x) \subseteq I_2(y) = I_2$.

Let Γ' be the game deduced from Γ by taking only the indices of I_2 and J_2. Any solution of Γ is a solution of Γ'; hence $v(\Gamma) = v(\Gamma')$ and

$$(3) \qquad X(\Gamma) \subseteq X(\Gamma'), \qquad Y(\Gamma) \subseteq Y(\Gamma')$$

Lemma 3. With Γ' derived from Γ as above,

$$(4) \qquad \dim X(\Gamma) = \dim X(\Gamma') \qquad \dim Y(\Gamma) = \dim Y(\Gamma') \ .$$

Proof: Pick $x \in X(\Gamma)$ by Lemma 1 and pick any $x' \in X(\Gamma')$. Then there exists an x'' interior to the segment xx' which is in $X(\Gamma)$!

As simple consequences of (3) and (4), we observe that

$$(5) \qquad I_\ell, J_\ell, X_\ell, Y_\ell(\Gamma) = I_\ell, J_\ell, X_\ell, Y_\ell(\Gamma'), \qquad \ell = 1, 2.$$

In view of (4) and (5), we need to prove the assertions (1) and (2) only for the smaller game Γ'. Or, what is the same thing, we may henceforth assume for Γ itself the properties:

(6) for every $x \in X$, $a_j \cdot x = v$, $j = 1,2,\ldots,n$;

(7) for every $y \in Y$, $a_i \cdot y = v$, $i = 1,2,\ldots,m$.

Under these assumptions we shall verify (in § 4) that each player has an optimal mixed strategy to which every pure strategy contributes positive weight (Theorem 1). In § 3 we describe the geometrical motivation for the algebraic argument.

§ 3. Geometric Analogue.

The game-matrix A may be plotted in n-space as the convex U of the m points a_i. U is then the image under the linear transformation represented by A, of the fundamental simplex of mixed strategies of the x-player.

Suppose for simplicity that $v = 0$, and let Q be the " positive quadrant" in n-space. Then U has no interior point in common with Q, but these two convex polyhedra touch in some non-empty set T, the image of X. Under the reduction assumption (6), T will be precisely the origin. The optimal mixed strategies for the y-player will correspond to the hyperplanes through the origin which separate Q and U.* Reduction assumption (7) means that every separating plane actually contains the entire set U.

The parameter α of § 4 has the effect of shrinking U about an interior point $\bar{a}$. Lemma 5 states that U may be so shrunk and still maintain contact with Q. This leads easily to Theorem 1, which states that the contact point is the center of gravity of a set of positive masses $[x_i > 0]$ at the vertices $[i \in I_2]$ of U.

* The proof that a game has a solution (the minimax theorem) may be reduced to the proof that two convex sets with no interior points in common can always be so separated.

§4. Study of the reduced game.

Let Γ fulfill conditions (6) and (7). Put

$$\bar{a} = \frac{1}{m} \sum a_i ,$$

then for any $y \in Y$, $\bar{a} \cdot y = v$. Form the new game $\Gamma_\alpha = (b_{ij})$ with the rows

$$b_i = (1 - \alpha) a_i + \alpha \bar{a} , \qquad 0 < \alpha < 1.$$

For any $y \in Y$, $b_i \cdot y = v$ for each i, hence $v(\Gamma_\alpha) \leq v$.

Lemma 4. If $Y_1(\Gamma_\alpha)$ and Y have a common point, then $v(\Gamma_\alpha) = v$.

Proof: Take any y in both $Y_1(\Gamma_\alpha)$ and Y, and choose $y' \in Y(\Gamma_\alpha)$ by Lemma 1. The segment yy' may be extended beyond y' to $y'' = (1 + \epsilon) y' - \epsilon y$, $\epsilon > 0$. Then

$$v(\Gamma_\alpha) \leq \max_i b_i \cdot y'' = (1 + \epsilon) v(\Gamma_\alpha) - \epsilon v$$

and thus $v \leq v(\Gamma_\alpha)$. But in any case $v(\Gamma_\alpha) \leq v$!

Lemma 5. There exists $\alpha > 0$ such that $v(\Gamma_\alpha) = v$.

Proof: By compactness there must exist a sequence $\{y^{(n)}\}$, $y^{(n)} \in Y(\Gamma_{\alpha_n})$, $\alpha_n \to 0$, which converges to some mixed strategy y. Since the number of possible sets $Y_1(\Gamma_{\alpha_n})$ is finite, we may further stipulate that they all be the same (closed) set and thus all contain y. But because $b_i \cdot y^{(n)} \leq v$ for each i, n, we find that y is in Y. By Lemma 4 the desired α therefore exists.

Theorem 1. $I_1 = I_2$ $\left[J_1 = J_2 \right]$.

Proof: Choose α by Lemma 5 and $x \in X(\Gamma_\alpha)$. Then the mixed strategy x', all of whose components

$$x_i' = (1 - \alpha) x_i + \alpha / m$$

are positive, satisfies

$$a_j \cdot x' = b_j \cdot x \geq v, \quad \text{all } j,$$

and is therefore in X. Hence I_1 contains every index i in the reduced game.

§5. The fundamental theorem.

Theorem 2. $\dim Y_1 - \dim Y = \dim X_1 - \dim X$.

Proof: We may suppose $v = 0$ without impairing the final result. The conditions defining X may be set forth

(a) $a_j \cdot x = 0$ for all j,

(b) $1 \cdot x = 1$,

(c) $x_i \geq 0$ for all i.

(a) and (b) together define a set C containing X; (a) alone defines a larger set C'. C' is in fact the null space of A, and therefore

(8) $$\dim C' = m - \text{rank } A.$$

Since the origin is in C' but not C, condition (b) actually lowers the dimension, i.e.,

(9) $$\dim C = \dim C' - 1.$$

By Theorem 1 the inequalities (c) hold strictly for some x in X (and thus for a neighborhood); hence

(10) $$\dim X = \dim C.$$

Finally, and obviously,

(11) $$\dim X_1 = m - 1.$$

Therefore, by (8)-(11):

(12) $$\text{rank } A = \dim X_1 - \dim X.$$

This, with the symmetrical expression for A^T, suffices to prove the theorem.

We observe for future use that in games whose value is not zero, the analogue of (12) is

$$\text{rank } A = \dim X_1 - \dim X + 1. \tag{12'}$$

A, in these expressions, is of course the essential part (in the sense of § 2) of the total game-matrix.

The theorem may be interpreted also as a relationship between the zeros and the range of an operator and its adjoint. Let T and T* denote the operator corresponding to A and its transpose, and let η, η^* and R, R* denote the manifold of zeros and the range of T, T*.

Then it is easily shown that

(13)
$$\dim \eta^* = \dim X$$
$$\dim \eta = \dim Y.$$

Now η^* is isomorphic to $R^{\perp}$, the orthogonal manifold to R; thus

$$\dim \eta^* = \dim X_1 - \dim R.$$

But $\dim R = \dim Y_1 - \dim \eta$, hence Theorem 2 is confirmed.

This point of view will be helpful in explaining the method of Part II.

§ 6. Unique solutions.

Let U be the set of m x n game-matrices which have unique solutions (m and n are fixed throughout the discussion). Formally $A \in U$ if and only if

$$\dim X(A) = \dim Y(A) = 0. \tag{14}$$

We shall prove

Theorem 3. U is open and everywhere dense in (mn)-space.

Proof that U is open:

If $\zeta = (\zeta_\ell)$ is a matrix or a vector, define

$$|\zeta| = \max_\ell |\zeta_\ell| .$$

Lemma 6. v, X, and Y are continuous functions of A in the following sense: Given A and any $\delta > 0$, there exists ε such that $|B - A| < \varepsilon$ implies

(a) $|v(B) - v(A)| < \delta$.

(b) $\underset{\substack{x \in X(A) \\ x' \in X(B)}}{\text{minimum}} |x' - x| < \delta$.

(c) $\underset{\substack{y \in Y(A) \\ y' \in Y(B)}}{\text{minimum}} |y' - y| < \delta$.

Proof: Any $\varepsilon < \delta$ is suitable for (a). For (b) and (c), take any sequence $\{B^\mu\} \to A$ and choose for each μ some $x^\mu \in X(B^\mu)$. By compactness, every x^μ after some x^{μ_0} will be within δ of some accumulation point of $\{x^\mu\}$. But since

$$b_j^\mu \cdot x^\mu \geq v(B^\mu), \qquad \text{all } j, \mu,$$

every accumulation point is in X(A). An argument by contradiction now shows that the desired $\varepsilon > 0$ exists.

We now prove that U is open. Let A_1 be the essential part of A:

$$A_1 = (a_{ij}), \ i \in I_1 = I_2, \ j \in J_1 = J_2 .$$

Take $A \in U$. If $|B - A|$ is small, B has a solution x', y' near the unique solution x, y of A, by Lemma 6. (Two-sided uniqueness here makes an essential appearance in the proof. Thus the set U_X of games with unique optimal strategy for the first player is not open.) The components positive in x will be positive in x', hence

$$I_1(A) \leq I_1(B).$$

Also Ay will be near By', hence

$$I_2(B) \leq I_2(A).$$

It follows that A_1 and B_1 correspond and that

(15) $$X_1(A) = X_1(B).$$

Now for any matrix C_o,

$$\lim_{C \to C_o} \text{rank } C \geq \text{rank } C_o;$$

we may therefore write

(16) $$\text{rank } B_1 \geq \text{rank } A_1.$$

Assume $v(A) \neq 0$, then by (L, L') of Part I,

(17) $$\dim X_1(A) - \dim X(A) = \text{rank } A_1 - 1,$$

$$\dim X_1(B) - \dim X(B) \geq \text{rank } B_1 - 1.$$

Equations (15), (16), (17) give us

$$\dim X(B) \leq \dim X(A).$$

After the same argument on Y, we conclude $B \in U$. Since the restriction $v \neq 0$ cannot be relevant in this context, every $A \in U$ has a neighborhood in U.

(A parallel proof could be given in operator terminology. Thus, the companion to (16) is the fact that the dimension of the zeros of an operator can only decrease for small perturbations.)

Proof that U is everywhere dense:

We shall call a matrix A "general" if no r x r + 1 submatrix of A or A^T, with a row of 1's subjoined, has a vanishing determinant. The set G of general m x n matrices is evidently not empty for $mn > 1$.

Lemma 7. G is everywhere dense.

Proof: Take any A; and $C_\epsilon = A + \epsilon B$, $B \in G$. It suffices to find ϵ_o such that $C_\epsilon \in G$ for all positive $\epsilon < \epsilon_o$. Every determinant obtained from C_ϵ as in the preceding paragraph is a polynomial of r^{th} degree in ϵ. Take $\epsilon_o > 0$ smaller than any positive root of these polynomials.

Lemma 8. $G \subseteq U$.

Proof: Take any $A \notin U$ with $y, y' \in Y$, $y \neq y'$; and let B be the submatrix A_1 with a row of 1's subjoined. Then for any $x \in X$,

$$(x,-v)^T B = 0^T. \tag{18}$$

Also

$$B(y - y') = 0. \tag{19}$$

Every submatrix of B having as many rows as B (by(18)), or as many columns (by(19)), must be singular. Therefore at least one of the determinants obtained from A as above vanishes, and A is not general.

The proof of Theorem 3 is completed by direct application of Lemm 7 and 8.

§ 7. Some games with infinite sets of strategies.

The manner in which infinite game-matrices deviate from the theory so far developed will be illustrated here by a number of examples. It is the lack of compactness, rather than the infinite number of strategies, that appears to be chiefly responsible for these deviations. However, we must observe that the non-discrete game with the pay-off function $f(x,y) = y^2 - xy$ defined on the unit

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§7. Some games with infinite sets of strategies.

Virtually none of the foregoing theory applies without drastic modification to games with infinite payoff matrices. We submit here some examples* to justify this assertion. We shall not, however, enter into a systematic study at this time either of infinite game-matrices or of games with continua of strategies.

First we must observe that many infinite matrices do not possess values or optimal strategies, even approximately, and hence should perhaps not be called games at all. As a case in point, consider the unbounded matrix

$$A: \quad a_{ij} = i - j \qquad (i = 1, 2, \cdots, j = 1, 2, \cdots).$$

This game has no value and hence no solution. For consider the mixed strategy x in which x_i is $1/i$ if $i = 2, 4, 8, \cdots$, and zero otherwise. Then $a_j \cdot x = \infty$. By symmetry we are led to the curious result:

$$\inf_y \sup_i a_i \cdot y - \sup_x \inf_j a_j \cdot x = -\infty .$$

(For bounded matrices the corresponding expression is always non-negative. But

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* For helpful comment and suggestions concerning these examples the authors are indebted to H. Kuhn of Princeton University.

with $|A| = \infty$, multiplication of matrices is no longer associative: $x^T(Ay) \neq (x^T A)y$.)

The matrix

$$B: \quad b_{ij} = a_{ij} / \sqrt{1 + a_{ij}^2} \qquad (a_{ij} \text{ as above}),$$

is bounded, but likewise has no value. For, given any y and $\varepsilon > 0$, we may choose n so that $y_1 + \cdots + y_n \geq 1 - \varepsilon$. Then for any $i > n/\varepsilon$, we have $1 \geq b_i \cdot y > 1 - 3\varepsilon$. By symmetry:

$$\inf_y \sup_i b_i \cdot y - \sup_x \inf_j b_j \cdot x = 2.$$

Confining ourselves now to infinite game-matrices which do have solutions, we still find violations of all of our chief theorems. The following two games each have unique solutions.

$$C: \begin{pmatrix} 0 & c_1-1 & c_2 & c_3 & c_4 & \cdots \\ 0 & c_1 & c_2-1 & c_3 & c_4 & \\ 0 & c_1 & c_2 & c_3-1 & c_4 & \\ 0 & c_1 & c_2 & c_3 & c_4-1 & \\ \vdots & & & & & \ddots \end{pmatrix} \qquad c_k \geq c_{k+1} > 0, \quad \sum c_k = 1.$$

$$v(C) = 0; \; X(C) = \{(c_1, c_2, \cdots)\}; \; Y(C) = \{(1, 0, 0, \cdots)\}.$$

(To prove the last, we observe that any optimal y must play all strategies after the first with equal frequency—hence with zero frequency.)

D:

$$\begin{array}{cccccc} d & 2d & 1/2 & 2d & 1/4 & \cdots \\ d & 1 & 2d & 1/3 & 2d & \cdots \end{array} \qquad d \neq 0.$$

$v(D) = d$; $X(D) = \{(1/2, 1/2)\}$; $Y(D) = \{(1, 0, 0, \cdots)\}$.

Theorem 1 fails in the former game, since $J_2(C)$ is infinite, $J_1(C)$ finite. The dimensionality relation of Theorem 2 leads to $\infty = 0$ and $1 = 0$ in the two cases. Thus it breaks down even when the essential part of the game is a finite submatrix.

The continuity of solutions, as set forth in Lemma 6, is violated in the neighborhoods of both games. For example, given any $\varepsilon > 0$ we can move $Y(D)$ a distance of 1 by subtracting ε from columns n and $n + 1$ of D, where $n > 1/\varepsilon$, $n \neq 1$.

The failure of Theorem 3 is illustrated again by both games. It is easy to verify that all x within ε of $X(C)$ or within $\varepsilon/2d$ of $X(D)$ become optimal if any $\varepsilon > 0$ is subtracted from the first column of C or D. Thus U is not open for infinite matrices. U continues to be everywhere dense in the subset of matrices which have solutions.

square $0 \leq x, y \leq 1$ has the saddle point at $x = y = 0$ for its unique solution; hence the analogues of I_1 and J_1 are single points. Yet the counterpart of I_2 is clearly the entire unit interval, contrary to what Theorem 1 would seem to imply.

Consider the infinite matrix:

$$A = \begin{array}{|cccccc} 0 & a_1 - 1 & a_2 & a_3 & a_4 & \cdots \\ 0 & a_1 & a_2 - 1 & a_3 & a_4 & \\ 0 & a_1 & a_2 & a_3 - 1 & a_4 & \\ 0 & a_1 & a_2 & a_3 & a_4 - 1 & \\ \vdots & \vdots & & & & \ddots \end{array} \qquad \begin{array}{l} a_k > 0 \\ \Sigma\, a_k = 1 \end{array}$$

and the two $2 \times \infty$ matrices:

$$B = \begin{array}{|cccccc} b_1 & b_1 + b_2 & 1/2 & b_1 + b_2 & 1/4 & \cdots \\ b_2 & 1 & b_1 + b_2 & 1/3 & b_1 + b_2 & \cdots \end{array} \qquad b_1 b_2 > 0$$

$$C = \begin{array}{|cccccc} c_1 & 1 & -2 & 3 & -4 & --- \\ c_2 & -1 & 2 & -3 & 4 & --- \end{array} \qquad c_1 + c_2 < 0$$

All three games have unique solutions, to wit:

$$v(A) = 0, \quad v(B) = \tfrac{1}{2}(b_1 + b_2), \quad v(C) = \tfrac{1}{2}(c_1 + c_2);$$

$$X(A) = (a_1, a_2, \ldots), \quad X(B) = X(C) = (\tfrac{1}{2}, \tfrac{1}{2});$$

$$Y(A) = Y(B) = Y(C) = (1, 0, 0, \ldots).$$

Only the uniqueness of $Y(A)$ need be justified here. Suppose $X(A)$ to be already known and take

$$y = (y_0, y_1, y_2, \ldots) \in Y(A).$$

Its inner product with the i^{th} row, $i = 1,2,\ldots$, must be $v(A) = 0$:

$$\sum_{j=1}^{\infty} a_j y_j - y_i = 0.$$

Thus $$y_1 = y_2 = y_3 = \ldots,$$

and $$y = (1,0,0,\ldots).$$

Theorem 1 is violated in game A, since $J_2(A)$ is infinite, $J_1(A)$ finite. However when both sets are finite the theorem apparently retains its force.

The dimensionality relation of Theorem 2 leads respectively to $\infty = 0$, $1 = 0$, and $1 = 0$ in the three cases. Thus we see it breaks down even when the essential part of the game is a finite matrix.

In game B we observe that the set J_2 of essential columns is not invariant under certain infinitesimal perturbations; mixed strategies can be found outside of $J_2(B)$ which come arbitrarily close to being optimal. This is obviously impossible in a finite game. To put it another way, this means that the first column may be excluded from B without decreasing the value of the game (defined now using "inf", not "min"). Game C is included to show that this effect may be avoided (the value of C without the first column is zero), though only at the price of an unbounded matrix: $|C| = \infty$.

The failure of Theorem 3 is illustrated by game A, and by game B if $b_1 = b_2$. In each case a level decrease in the first column by any amount $\varepsilon > 0$ destroys uniqueness. It is easy to verify that all mixed strategies within ε of X(A) or within $\varepsilon/(b_1 + b_2)$ of X(B) become optimal. Thus U is not open for infinite matrices. U is indeed everywhere dense in the set of matrices which have solutions, but not every infinite matrix has a solution, or even a value. For example, take the matrix

$$D = (d_{ij}), \; d_{ij} = i-j, \; i = 1,2,\ldots, \quad j = 1,2,\ldots;$$

or the bounded matrix

$$E = (e_{ij}), \; e_{ij} = d_{ij}/\sqrt{d_{ij}^2 + 1}\,, \qquad |E| = 1.$$

The continuity of solutions, as set forth in Lemma 6, apparently holds for infinite matrices whenever their solutions exist.

PART II: CONSTRUCTION OF GAMES WITH GIVEN SOLUTIONS

§8. The problem; Canonical forms.

We suppose that we have been given a pair of (convex) polyhedra X and Y, each situated in a simplex, and that we wish to find a game whose sets of optimal strategies correspond exactly to X and Y.

The problem is made no more difficult by prescribing the value the game is to have.

Let X_1 and Y_1 be the smallest faces of the simplices containing X and Y. From Theorem 2 we know that our problem has no answer unless

$$\text{(20)} \qquad \dim X_1 - \dim X = \dim Y_1 - \dim Y.$$

Our subsequent work will show that condition (20) is sufficient as well (Theorem 4, § 13).

The construction we describe in §§ 10 - 12 produces a specific (except for a certain freedom in ordering the rows and columns) standardised matrix for each X, Y satisfying (20). This gives us in effect a set of canonical forms. However, there is no apparent way of relating an arbitrary matrix to its canonical counterpart short of finding all its solutions and constructing the standard matrix to order

(A finite though tedious process for finding all solutions is described in [3] §6.) These canonical forms, therefore, are not promising as a computational aid. In answering theoretical questions, however, it may sometimes be helpful to have to consider only a small subset of all possible matrices.

A more general classification might lump together games whose sets X, Y are isomorphic under a projective transformation. There would then be only a finite number of types for matrices of each particular size. The canonical games could have their solutions oriented in some natural, simple fashion; for example, the canonical unique solutions could always be of the form:

$$X = x^* = (\frac{1}{r}, \ldots, \frac{1}{r}, 0, \ldots, 0),$$

$$Y = y^* = (\frac{1}{r}, \ldots, \frac{1}{r}, 0, \ldots, 0).$$

The solutions of any game would be identical to the solutions of a game BAC where A is one of a finite set of canonical games and B and C are nonsingular matrices representing the appropriate projective transformations on the two simplices of mixed strategies.

§9. Geometrical description of the construction.

We return to the operator point of view introduced at the end of § 5. Suppose first that the polyhedra given, X and Y, are in fact the intersections of the linear spaces $\bar{X}$ and $\bar{Y}$ with the fundamental simplices ((m - 1)- and (n - 1)-dimensional, respectively) lying in m- and n-dimensional Euclidean space. Suppose further that X and Y contain points interior to their respective simplices. The essential part of any game (see § 2) will have such solutions.

In view of (20), the orthogonal manifolds $\bar{X}^{\perp}$ and $\bar{Y}^{\perp}$ have the same dimension. Take any (non-singular) linear transformation S mapping $\bar{X}^{\perp}$ on $\bar{Y}^{\perp}$, and any projection P of m-space on $X^{\perp}$ which maps X into the origin. Then the game-matrix corresponding to the transformation T = SP has value zero, and sets of optimal strategies X and Y. (See § 5, esp. equations (13).)

In general, the given polyhedra X and Y may have both " natural " faces, caused by the boundary inequalities $x_i \geq 0$, $y_j \geq 0$ of their simplices, and "unnatural" faces defined by arbitrary inequalities. Each unnatural face of X [Y] corresponds to a column [row] outside of the essential part of the game matrix. (See § 12 below.)

Thus our ability to construct a game with given solutions is conditioned not only by the dimensional restriction (20) but also by whether we are provided with enough " dummy" strategies, not involved in any optimal strategy, to take care of the unnatural faces. It is always possible, of course, to handle a surplus of dummy strategies, without disturbing the rest of the construction.

§ 10. Unique solution given.

We proceed now to the algebraic details of the construction. In this section we suppose that the given X and Y are points. The general treatment of § 11 includes this case: we consider it separately only to illustrate the general attack and because of the particular interest of games with unique solutions.

Disregard any dummy strategies, and denote the positive components of the unique optimal strategies by

$$x_o, x_1, \ldots, x_t = x, \qquad t = \dim X_1;$$

$$y_o, y_1, \ldots, y_t = y, \qquad = \dim Y_1.$$

The order may be taken arbitrarily (see Remark 3 below).

Let I^t represent the t x t identity matrix (1 on the main diagonal, zero elsewhere), and let A(c) be the t + 1 x t + 1 matrix as follows:

$$A(c) = \left[\begin{array}{c|cccc} a(c) & \frac{c-x_1}{x_0} & \frac{c-x_2}{x_0} & \cdots & \frac{c-x_t}{x_0} \\ \hline \frac{c-y_1}{y_0} & & & & \\ \frac{c-y_2}{y_0} & & & & \\ \vdots & & I^t & & \\ \frac{c-y_t}{y_0} & & & & \end{array}\right] \qquad a(c) = \frac{c(x_0 + y_0 - 1) + \sum_1^t x_i y_i}{x_0 y_0}$$

Then $v(A(c)) = c$ and, provided $c \neq 1/t$, x,y is the unique solution of A(c).

Proof: One may verify directly that x,y is a solution and that c is the value. To establish uniqueness, we observe that

$$\text{rank } A(c) = \begin{cases} t+1, & c \neq 0, 1/t; \\ t, & c = 0. \end{cases}$$

Then [illegible] (1.1) of [illegible].

$$\dim X(A(c)) = \dim Y(A(c)) = 0.$$

Remark: In the event c = 1/t the mixed strategies x' = y' = (0, 1/t, ..., 1/t) are optimal as well as x and y. $X(A(\frac{1}{t}))$ and $Y(A(\frac{1}{t}))$ are the segments $x'x$ and $y'y$ extended to the edges of the simplices X_1 and Y_1. For a game with unique solution x,y and value 1/t we might use (for example) the matrix $B = \frac{1}{2}(A(\frac{2}{t}))$.

Remark 2: A construction using any non-singular t x t matrix in place of I^t is equally possible. The generalization of the critical value 1/t is $1/\Sigma\Sigma b_{ij}$ where (b_{ij}) is the inverse of the matrix used.

Remark 3: Since the numbering of the rows and columns is purely a matter of nomenclature, the matrix A(c) is not wholly specific. We may remedy this defect by positing

$$x_0 \geq x_1 \geq \cdots \geq x_t, \qquad y_0 \geq y_1 \geq \cdots \geq y_t.$$

§11. Polyhedra of general dimension given.

We shall require in this section only that X [and Y] be completely described in some (r-1)-dimensional [(s-1)-dimensional] plane by the characteristic inequalities of the simplex:

$$x_i \geq 0, \qquad [y_j \geq 0] \;.$$

When a game A has such sets of solutions, its essential part A_1 will have precisely the same sets, i.e:

$$X(A) = X(A_1), \qquad Y(A) = Y(A_1)$$

(compare (3) of §2). It will be most economical, then, to construct a game which is its own essential part. X_1 and Y_1 will comprise the full simplices of mixed strategies.

Define

$$r = 1 + \dim X, \qquad m = 1 + \dim X_1,$$

$$s = 1 + \dim Y, \qquad n = 1 + \dim Y_1,$$

$$t = m - r = n - s.$$

Choose r and s linearly independent points in X and Y respectively

$$x_k = (x_{k1}, \ldots, x_{km}) \qquad k = 1, \ldots, r,$$

$$y_\ell = (y_{\ell 1}, \ldots, y_{\ell n}) \qquad \ell = 1, \ldots, s;$$

and use them as rows in forming the matrices

$$X^r_{rm} = (x_{ki}) \text{ and } Y^s_{sn} = (y_{\ell j}).$$

The superscripts show rank; the double subscripts show size. Our present problem may now be stated algebraically: to find A^ρ_{mn} satisfying

$$(21) \qquad \begin{cases} X^r_{rm} A^\rho_{mn} = v 1_{rn} \\ A^\rho_{mn} (Y^s_{sn})^T = v 1_{ms} \end{cases}$$

with

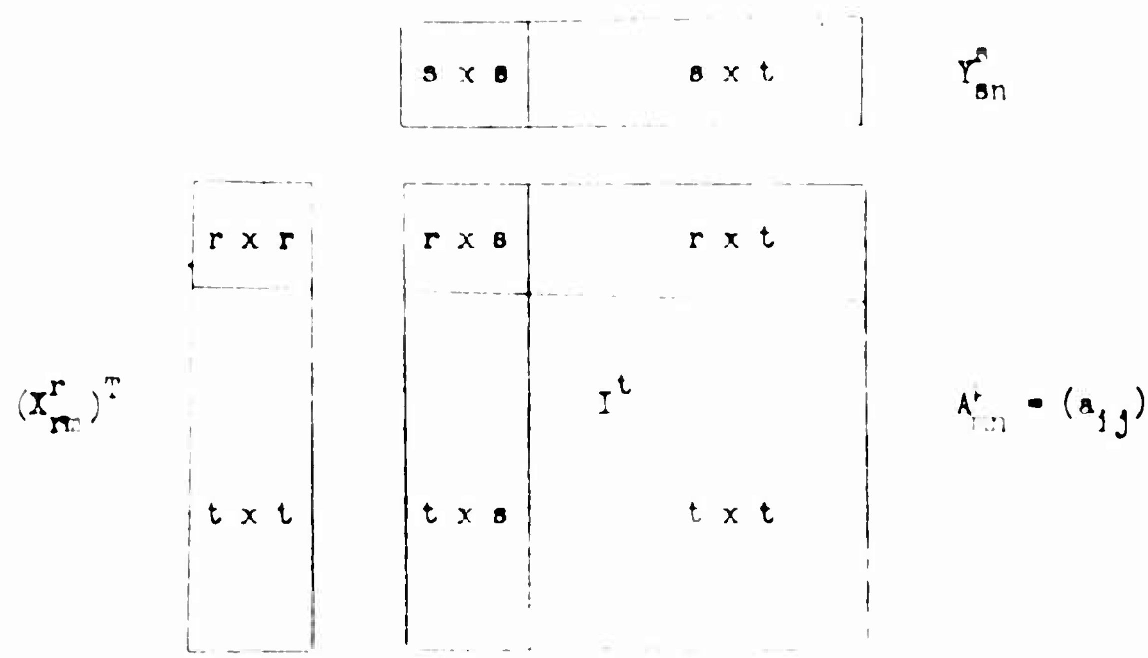
$$(22) \qquad \rho = \begin{cases} t, & v = 0, \\ t + 1, & v \neq 0. \end{cases}$$

The condition on ρ prevents $X(A)$, $Y(A)$ from being of higher dimension than the given X, Y (see (12, 12') in § 5).

System (21) comprises $rn + ms$ equations in the mn unknowns a_{ij}. However, as the following construction reveals, there is just enough interdependence to permit t^2 of the unknowns to be chosen arbitrarily.

Before proceeding with the construction, it will be convenient to rearrange the columns of X^r_{rm} [and Y^s_{sn}] so that the first r [s] columns are linearly independent. Geometrically, this simply means picking the order of coordinates so that the projection of X on the simplicial face defined by the first r coordinates (i = 1, ..., r) will be one-one.

Now we place I^t in the lower right corner of A (see the figure). The remaining elements a_{ij} are then determined uniquely by (11). Specifically,



for the upper right corner (i = 1, .., r; j = r + 1, ..., n) use the systems

$$(23)\qquad \begin{cases} x_{11}a_{1j} + \cdots + x_{1r}a_{rj} = v - x_{1j}, \\ \vdots \\ x_{r1}a_{1j} + \cdots + x_{rr}a_{rj} = v - x_{rj}. \end{cases}$$

The lower left corner is analagous. The upper left may be filled in from either direction: the result must be the same. The finished matrix A is happily independent of our choice of points x_k and y_ℓ.

A simple check reveals that condition (22) on the rank of A is fulfilled except when $v = 1/t$. In that instance we find the extraneous optimal strategy $x' = (0, \ldots, 0, 1/t, \ldots, 1/t)$. The restriction we put on the first r columns of X^r_{rm} tells us that x' is not in X. Therefore, if a value of 1/t is desired, we must use some such device as proposed in Remark 1, § 10.

Remark 2 of § 10 applies with equal force to the present case. As to the point raised in Remark 3, it is not clear that any generalisation of the ordering proposed there would necessarily make the first r columns of X^r_{rm} [s columns of Y^s_{sn}] independent, as required.

§ 12. The most general case.

We must now consider given polyhedra whose boundaries are not entirely described by the natural limits $x_i \geq 0$, $y_j \geq 0$. Each unnatural (r - 2)-face of X [(s - 2)-face of Y] corresponds to a new column a_j [row a_i] outside of the essential submatrix A_1. There is no restriction on the number or arrangement of these unnatural faces, provided of course that X and Y remain convex. Furthermore, there is no interaction between the new columns $j \notin J_1$ and the new rows $i \notin I_1$; elements a_{ij} common to both may be assigned arbitrary values.

It suffices to describe the calculation of a_j for a particular (r - 2)-face F of X. Choose a set of r - 1 independent points $x_2, \ldots, x_r$ on F and another point x_1 in the interior of X. Let the distance of the latter to the plane of F be $\lambda > 0$. Form the matrix $F^r_{rm} = (x_{kl})$. The only condition that a_j must satisfy is

$$F^r_{rm} a_j = (v + \mu, v, \ldots, v), \qquad \mu > 0.$$

To get a definite result we take $a_{ij} = 0$ for $i > r$, and $\mu = \lambda$. The latter makes the result independent of the points chosen. The former is justified because, after our manipulation in § 11, the first r columns of F^r_{rm} constitute a non-singular matrix F^r_{rr}. Thus we have

$$(24) \qquad F^r_{rr} a_j = (v + \lambda, v, \ldots, v),$$

which determines a_j exactly. The similar expression for the y-player involves $v - \lambda$ in place of $v + \lambda$. It is only at this point that the anti-symmetry in the roles of the two players shows up in the construction.

§ 13. Summary.

Sections §§ 11, 12, taken with Theorem 2, § 5, comprise a constructive proof of the following:

Theorem 4. Let X be a convex polyhedron of dimension r - 1 contained in an (m - 1)-dimensional face X_1, but in no smaller face, of an (m' - 1)-dimensional simplex. Let there be just μ (r - 2)-dimensional faces of X not contained in the boundary of X_1. Similarly for Y, s, Y_1, n, n', and ν. Then an m' x n' game-matrix A exists having sets of optimal strategies corresponding exactly to X and Y if and only if

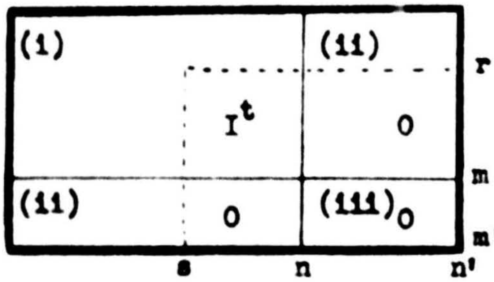
$$m - r = n - s$$

and

$$m' \geq m + \nu, \qquad n' \geq n + \mu.$$

The complete construction may be summed up (see figure):

$A = (a_{ij})$:



(i) Construct the essential submatrix A_1 around the identity matrix I^t, $t = m - r = n - s$, using equations (20) of § 11.

(ii) Compute each additional row or column required as outlined in § 12.

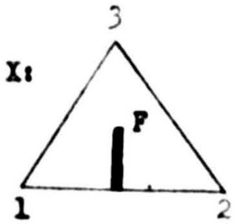
(iii) Square off the matrix by putting $a_{ij} = 0$, $i \notin I_1$, $j \notin J_1$.

§ 14. Example.

Find a game having value 2 and optimal strategies as indicated:

$r = 2$
$m = 3$
$\mu = 1$

X:



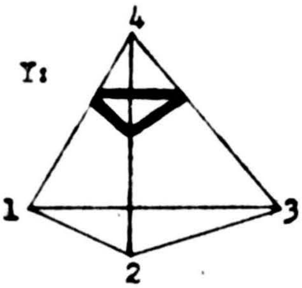
Extreme points (1/2, 1/2, 0)
(1/4, 1/4, 1/2)

In attempting to form X^r_{rm} we find we must reorder the coordinates: 1' = 2, 2' = 3, 3' = 1. Then

$$X^r_{rm} = \begin{pmatrix} 1/2 & 0 & 1/2 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}$$

$s = 3$
$n = r$
$\nu = 0$

Y:



Extreme points (1/3, 0, 0, 2/3)
(0, 1/2, 0, 1/2)
(0, 0, 1/3, 2/3)

$$Y^s_{sn} = \begin{pmatrix} 1/3 & 0 & 0 & 2/3 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1/3 & 2/3 \end{pmatrix}$$

Putting I^t in the lower right corner means setting $a_{3'4} = 1$. Column 4 is found by using X^r_{rm} (as in (23), § 11), and the first three columns may be computed similarly using $(Y^s_{sn})^T$. By taking points with zero components in composing the matrices X^r_{rm} and Y^s_{sn} we were able to make the equations extremely simple.

X has an unnatural face F; we must therefore include a dummy strategy, $j = 5$, for the y-player. To find column 5 we select the points (1/3, 1/3, 1/3) interior to X and (1/4, 1/2, 1/4) in F. The distance between them is $\lambda = \sqrt{6}/12$. Substituting the matrix

$$F^r_{rr} = \begin{bmatrix} 1/3 & 1/3 \\ 1/4 & 1/2 \end{bmatrix}$$

into (24) gives us $a_{1'5}$ and $a_{2'5}$. Finally, we set $a_{3'5} = 0$.

The completed game-matrix:

j:	1	2	3	4	5
i: 1' = 2	0	1	0	3	$4 + \frac{\sqrt{6}}{2}$
2' = 3	2	2	2	2	$2 - \frac{\sqrt{6}}{4}$
3' = 1	4	3	4	1	0

PART III: SOLUTIONS OF SOME SPECIAL GAMES

§ 15. Completely mixed games; basic solutions.

Part III will describe the solutions of three easily recognizable types of square matrices with special diagonal properties. The first two (§§ 16, 17) generalise the " separation of diagonals" criterion used in [2] (Chapter IV, section 18) for solving 2 x 2 matrices, while the third (§ 18) is a special case of the second.

We shall find it convenient to introduce the notions of basic solution and completely mixed game. A solution x, y of A is basic if x is a vertex of X(A) and y a vertex of Y(A). If v(A) is not zero, a solution x, y is basic if and only if there exists a non-singular submatrix of A whose inverse $B = (b_{ji})$ satisfies

$$x_i = \Sigma_j b_{ji} / \Sigma_{i,j} b_{ji}, \qquad y_j = \Sigma_i b_{ji} / \Sigma_{i,j} b_{ji} . \tag{25}$$

The value turns out to be

$$v = 1/ \Sigma \Sigma b_{ji}.$$

(See reference [3] , of which this is the main theorem). The submatrix in question will contain the one defined by $I_1(x)$ and $J_1(y)$, and be contained in the one defined by $I_2(y)$ and $J_2(x)$ (see the definition of § 1).

A game is said to be completely mixed if all of its solutions involve every strategy of both players (reference [1]). It follows that a completely mixed game must have a unique solution and a square matrix, which is non-singular unless the value is zero. The solution may be obtained by inverting the matrix and using (25).

Lemma 9. All four of the game-matrices $\pm A$, $\pm A^T$, or none of them, are completely mixed.

The proof is routine.

§16. Main diagonal separated and dominant.

Consider an n x n matrix $A = (a_{ij})$ which satisfies, for some fixed q,

$$(26) \qquad \begin{cases} a_{ij} > q & \text{if } i = j, \\ a_{ij} < q & \text{if } i \neq j; \end{cases}$$

and either (a):

$$\Sigma_i a_{ij} \geq nq \quad \text{for all } j,$$

or (b):

$$\Sigma_j a_{ij} \geq nq \quad \text{for all } i.$$

Then $v(A) \geq q$ and A is completely mixed. (See Fig. 1 in which q is taken to be zero.)

Proof: (a): Putting $x_1 = \cdots = x_n = \frac{1}{n}$ reveals that $v \geq q$. Should any y in Y have $y_\ell = 0$, then the inequality $\Sigma a_{\ell j} y_j < q \leq v$ tells us that $x_\ell = 0$ for every x in X. But whenever $x_\ell = 0$ the inequality $\Sigma a_{i\ell} x_i < v$ prevents x from being optimal. Thus A is completely mixed.

Proof: (b): Take any y in Y and ℓ such that $y_\ell = \max y_j$. Then

$$v \geq \Sigma a_{\ell j} y_j \geq q.$$

The proof continues as in (a).

Cases (a) and (b) might have been deduced one from the other with the aid of of Lemma 9. Similarly we may reverse the inequalities of the hypothesis. The following example shows that the condition " either (a) ... or (b) ... ", which ensures "uniform" dominance, cannot be replaced by the weaker proviso:

(27) $$\sum_{ij} a_{ij} \geq n^2 q.$$

Example 1:

1	-2	0
-2	1	0
0	0	12

$v = 0$

unique $x \in X$: $(0, 1, 1)$

vertices of Y: $(\frac{1}{3}, \frac{2}{3}, 0)$, $(\frac{2}{3}, \frac{1}{3}, 0)$.

The main diagonal is separated, and dominant in the sense of (27) for every q satisfying (26); yet the game is not completely mixed.

§ 17. Diagonals separated and ordered.

Consider an n x n matrix $A = (a_{ij})$ with

$$a_{ij} \in L_k, \; k \equiv i - j \pmod{n},$$

where the intervals L_k are disjoint and ordered:

(28) $$L_0 < L_1 < \cdots < L_{n-1}.$$

We shall establish that A is completely mixed. (See Fig. 2.)

Proof: Suppose some $x \in X$ has $x_\ell = 0$. Then

$$\sum a_{i\ell} x_i > \sum a_{i,\ell-1} x_i \geq v.$$

Hence every $y \in Y$ has $y_\ell = 0$. Similarly, if some $y \in Y$ has $y_\ell = 0$ then

$$\sum a_{\ell+1,j} y_j < \sum a_{\ell j} y_j \leq v.$$

Hence every x in X has $x_{k+1} = 0$. Repeat these two steps n times, reducing subscripts modulo n when necessary. The resulting absurdity $x = y = 0$ proves that A is completely mixed.

Without the ordering (28), A is not necessarily completely mixed. In any case, use of the mixed strategy $(\frac{1}{n}, \ldots, \frac{1}{n})$ cannot cost either player more than the mean diameter of the sets L_k. The next section solves the case where the L_k are points.

$$\begin{array}{|cccc|c} \hline + & - & \cdots & - & \geq 0 \\ - & + & \cdots & - & \geq 0 \\ \vdots & & & & \vdots \\ - & - & \cdots & + & \geq 0 \\ \hline \geq 0 & \geq 0 & \cdots & \geq 0 & \end{array} \quad (b)$$

(a)

$$\begin{array}{|cccc|} \hline L_0 & L_1 & \cdots & L_{n-1} \\ L_{n-1} & L_0 & \cdots & L_{n-2} \\ \vdots & & & \\ L_1 & L_2 & \cdots & L_0 \\ \hline \end{array}$$

$$\begin{array}{|cccc|} \hline a_0 & a_1 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & \cdots & a_{n-2} \\ \vdots & & & \\ a_1 & a_2 & \cdots & a_0 \\ \hline \end{array}$$

Figure 1. Figure 2. Figure 3.

§ 18. Constant diagonals.

Consider an $n \times n$ matrix $A = (a_{ij})$ with

$$a_{ij} = a_k, \quad k \equiv i - j \pmod{n}.$$

(See Fig. 3.) One immediately observes that $(\frac{1}{n}, \ldots, \frac{1}{n})$, $(\frac{1}{n}, \ldots, \frac{1}{n})$ is a solution and that $v(A) = \frac{1}{n} \sum a_k$. Since any other optimal mixed strategies for either player must be symmetrically disposed about $(\frac{1}{n}, \ldots, \frac{1}{n})$, this solution will be unique if and only if it is basic. Suppose $\sum a_k \neq 0$. Then A is completely mixed if and only if the determinant $|a_{ij}| \neq 0$. But it is easily verified that

$$|a_{ij}| = \prod_{\ell=0}^{n-1} \sum_{k=0}^{n-1} a_k \omega_n^{k\ell}$$

where ω_n is a primitive n^{th} root of unity. If none of the factors vanishes, then A is completely mixed. On the other hand, if $\sum a_k \omega_n^{k\ell} = 0$ for $\ell = \ell_o$, the real part r_{ℓ_o} of the complex vector $(1, \omega_n^{\ell_o}, \omega_n^{2\ell_o}, \ldots, \omega_n^{(n-1)\ell_o})$ is in the null-space of A and A^T. The optimal mixed strategies for either player will be just those of the form

$$\left(\frac{1}{n}, \ldots, \frac{1}{n}\right) + r$$

where r is a vector in the space spanned by all such r_{ℓ_o} and their cyclic permutations.

Example 2:

0	1	3	2
2	0	1	3
3	2	0	1
1	3	2	0

$v = \frac{3}{2}$, $r_{\ell_o} = r_2 = (1, -1, 1, -1)$.

X and Y are the line segments, in their respective tetrahedra, joining the points $(\frac{1}{2}, 0, \frac{1}{2}, 0)$ and $(0, \frac{1}{2}, 0, \frac{1}{2})$.

§19. Conclusion.

Remark 1: Any matrix which is derivable from one of the types here discussed by permutation of the rows or columns is of course not essentially different.

Remark 2: A relaxing of the strict inequalities appearing in §§16, 17 gives rise to a host of special cases, most of them not completely mixed, which are not worth describing in detail.

Remark 3: A 2 x 2 game is completely mixed if and only if its diagonals are separated. Unfortunately, our generalised conditions of §§16, 17 are not even broad enough to cover all 3 x 3 completely mixed games. One of the mavericks is the following:

Example 3:

4	-3	-2
-3	4	-2
0	0	1

$v = \frac{1}{7}$,

unique $x \in X$: $(\frac{1}{7}, \frac{1}{7}, \frac{5}{7})$,

unique $y \in Y$: $(\frac{3}{7}, \frac{3}{7}, \frac{1}{7})$.

Only the main diagonal is separated and it is not dominant.

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