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REPRESENTATION AND ANALYSIS OF SIGNALS

PART XVIII. VECTOR AND TENSOR ALGEBRA OF SIGNALS APPLIED TO SATELLITE NAVIGATION

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by

Dan C. Ross

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VECTOR AND TENSOR ALGEBRA OF SIGNALS APPLIED TO SATELLITE NAVIGATION

by

Dan C. Ross

A dissertation submitted to
The Johns Hopkins University
in conformity with the requirements for
the degree of Doctor of Engineering

Baltimore
The Johns Hopkins University
1964

B

ABSTRACT

The basic concepts of signals and linear systems are formulated in terms of finite-dimensional vector algebra. Important ideas, often confused or omitted in classical signal theory, are clarified by the system of notation and nomenclature presented in the dissertation. Measurement and specification are emphasized in the notation as is appropriate to their importance in engineering practice.

The theory and notation are extended to finite-dimensional tensor product spaces. The extension to multilinear systems of the engineer's intuitive knowledge of linear systems is illustrated. The abstract notions are illustrated by application to the familiar problem of time-domain multiplication.

The utility of the notation and the tensor product concepts is demonstrated by application to satellite navigation signal processing. Descriptions of feasibility tests on the IBM 7094 and excerpts of results are presented. The results confirm the expected simplicity and indicate a surprisingly high accuracy of the processor designed by the tensor product approach.

The dissertation consists of 212 pages including 29 tables and 36 figures.

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I wish to thank Dr. Willis C. Gore who, along with Dr. Huggins, served as referee. I have profited also by many conversations with graduate students and research fellows of the University, particularly A.G. Carlton, D.C. Lai, R.N. McDonough and L. Weiss.

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Last but not least, I am grateful to my wife Deen for her patience and understanding.

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CHAPTER ONE

INTRODUCTION

My main purposes in writing this dissertation are:

(1) To present the theory of signals and linear systems in terms of an appropriate notation and nomenclature based on linear vector algebra, with particular emphasis on finite-dimensional representations.

(2) To extend the theory and notation to multilinear systems through the use of tensor products of finite-dimensional vector spaces.

(3) To demonstrate the utility of the vector and tensor concepts by application to a current problem in systems engineering.

One chapter of the dissertation is devoted to each purpose, and suggestions for further work are given in the closing chapter. The system of notation and nomenclature described in the dissertation has been tested by application to two other problems studied during the research project: (1) optimum filtering and signalling in a communications system, (2) the system identification problem. Future papers on the results of these studies are planned.

point in preserving any difficulties which might have been encountered in such experiences.

The classical literature on signal theory often tends to confuse a physical system and its mathematical model. The principal goal of the notation and nomenclature described in this dissertation is to facilitate precise statements which distinguish between reality and model, between entity and representative, between measured entity and measuring device. The operations of measurement and specification which connect a physical system with its mathematical model, as illustrated in Figure 1, play a major role in the notation system as is appropriate to their importance in engineering practice.

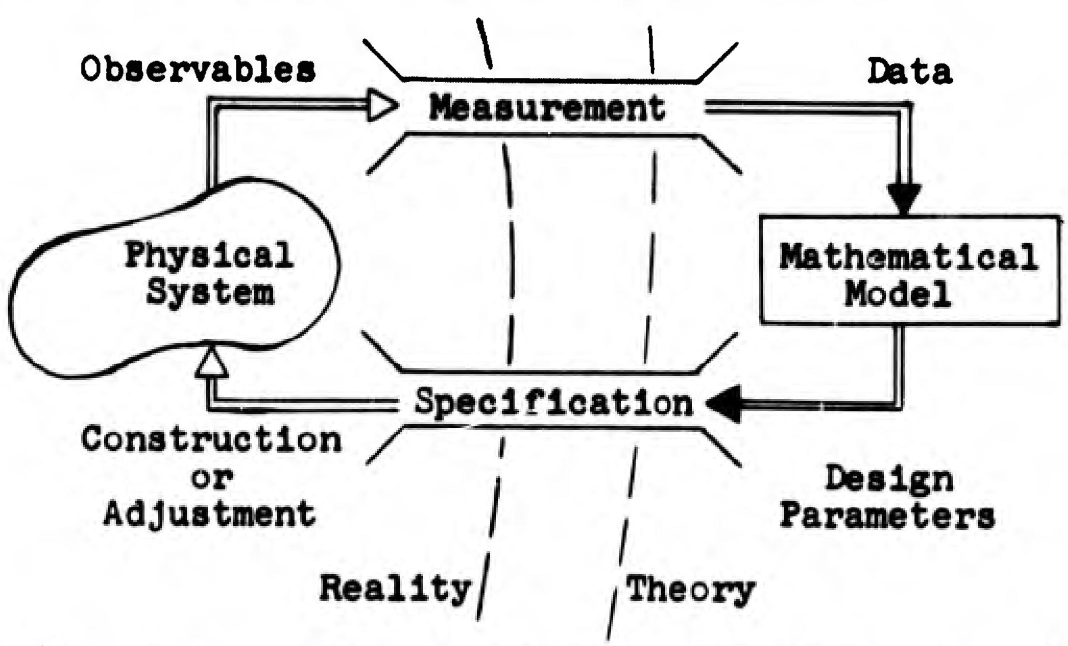


Figure 1 Physical System and Mathematical Model

The notation system presented here was developed for application to practical engineering problems.

Nevertheless, pure mathematicians are invited to evaluate the notation by the criterion of mathematical elegance. The utility of abstract mathematics is beginning to be appreciated by engineers, not merely as a tool for use in calculation but as a language in which to think and communicate about the overall structure of a complex system. Systems engineers will increasingly study algebraic structures and other areas of abstract mathematics in their undergraduate education and will find therein a source of ideas for system structures just as a component engineer is stimulated by his knowledge of physics.

Our system of notation and nomenclature provides for discourse on three levels: (1) physical system, (2) corresponding abstract operator, (3) matrix representative on one of several possible bases. The third level is a generalization of the usual method of description. The first two levels are closely connected, exact isomorphism being prevented only because of the impossibility of exact measurements. The notation employed at the second level is that of abstract vectors and operators, permitting a view of system structure stripped of unnecessary complications related to representations on particular bases. This notation satisfies the objectives of Heaviside in that it provides the simplicity and the overall view needed in the early stages of system design.

Some readers may feel that time and effort spent in developing notation and clarifying nomenclature is wasted. The counter-examples to such opinions are few indeed but are all of major importance: the symbol for zero, the Arabic system of decimal arithmetic, the use of literal symbols for numbers, etc. Thus, the high stakes offset the poor chance for success. Nor is the indiscriminate criticism of new notation free from risk -- witness the judgment of history on the many forgotten authorities who defended the Roman numerals for centuries. Whatever be the merit of the approach described here, there is no doubt that the analytical notation and nomenclature of classical signal theory lacks important features. Thus, the incentive for improvement exists, following the spirit of the following quotations.

"It is a profoundly erroneous truism, repeated by all copy-books and by eminent people when they are making speeches, that we should cultivate the habit of thinking what we are doing. The precise opposite is the case. Civilization advances by extending the number of important operations which we can perform without thinking about them. Operations of thought are like cavalry charges in a battle -- they are strictly limited in number, they require fresh horses, and must only be made at decisive moments."

-- Alfred North Whitehead

"It is hardly possible to believe what economy of thought ... can be effected by a well-chosen term."

-- Henri Poincaré

One of the reasons for emphasizing the use of finite-dimensional algebraic models, as opposed to the more familiar analytic ones, is that engineers need to be rigorous and it is much easier to be rigorous with algebra than with analysis. However, the continuity of physical processes has been an article of faith among many physicists ever since Newton, and has become dogma among many electrical engineers aware of the successes of Maxwell in the last century and of analog computers in the last two decades. These considerations are at the center of the philosophy of science and relate closely to the foundations of mathematics. Many of the most respected scientists, mathematicians, statisticians and communications theorists have written on these fundamental questions. Selected quotations from these works have been collected by the author in a forthcoming paper (Ross 1964). The temptation to present a small sample of these quotations here could not be resisted.

"... we are less interested in continuous variational processes than we are in certain discrete versions of these processes. ...it is often assumed that the discrete version is an approximation to the continuous version. Many times it is far more appropriate to consider the continuous case as a mathematical fiction employed to simplify the analysis of the actual discrete process. This is certainly the case in many control processes."

-- Richard Bellman

"Strictly speaking, there is no need to consider 'continuous' wave forms at all in signal analysis. 'Continuous' functions are the creation of mathematicians, and enable methods of analysis of great elegance to be used. But such analysis may well be done algebraically. Against this, it may be argued that algebraic methods must necessarily introduce approximations; this may be true, but it should be remembered that signal analysis concerns the use of mathematical methods for describing physical signals and their properties. Mathematicians deal with mental constructs, not with descriptions of physical situations. Approximations can be reduced as much as we wish, at the price of increased algebraic labor. A 'continuous' function is not a physical idea but a mathematical one; when solving problems in physics (or applied mathematics), such an idea need not be regarded as holy, as sometimes seems to be the case."

-- Colin Cherry

"It is impossible to prove, by mathematical reasoning, any proposition whatsoever concerning the physical world."

-- G.H. Hardy

"... mathematical existence and physical existence mean basically different things; ... physical existence can never follow from mathematical existence; ... physical existence can in the last analogy only be proved by observation; ... the mathematical difference between rational and irrational forever transcends any possibility of observation. ...

"Be that as it may, there will remain unimpaired the possibility and the grand beauty of a logic and a mathematics of the infinite."

-- Hans Hahn

CHAPTER TWO

VECTOR ALGEBRA OF SIGNALS AND SYSTEMS

The first section of the chapter presents the correspondence between physical signals, linear transducers and measuring processes in the real world and the abstract vectors, linear operators and linear functionals of the mathematician's finite-dimensional vector spaces. In the second section, representation of the abstract entities is identified with the physical process of measurement. The discussion here is limited to concepts essential to the applications discussed in later chapters. The notations and nomenclature which will be used in the dissertation are introduced gradually in the first two sections and are summarized in Appendix 1 for reference. Differences between our notation and that of the existing literature of signal theory are discussed in the third section along with some of the reasons for the differences.

2 - 1 SIGNALS, PATTERNS AND OPERATORS

The traditional point of departure in signal theory is some continuous function of time, usually written f(t). It is usually assumed, perhaps tacitly, that "the signal" is f(t). We do not do this. We consider f(t) to be neither "the signal" nor any signal. What we do consider f(t) to be will be discussed later and what we will mean by "signal" will be presented first.

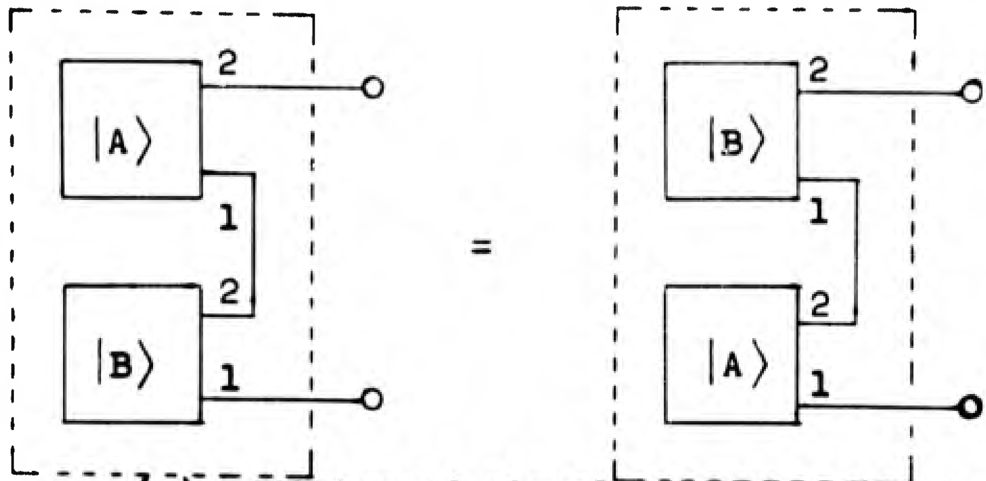
Signal Vectors

A signal is an entity which exists in the physical world. It may be a node voltage, branch current, shaft rotation, pressure, magnetic field intensity, etc. The term "signal" refers to the entire behavior of some particular physical entity throughout one complete operation of the system of which the signal is a part, i.e., throughout one complete operation of the signal generator. From this point of view, an excellent way to describe any particular signal is to specify a generator of that signal, and by "specify" we mean "to identify" some existing generator or "to provide manufacturing, installation and operation instructions" adequate for the production of some desired generator by competent technicians. Indeed, description of a signal by specification of its generator is the only form ultimately useful in engineering for describing the objects of signal analysis or the results of signal synthesis.

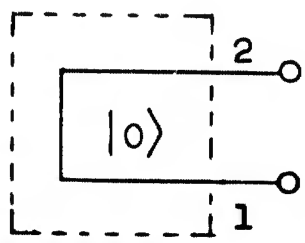
Signals of the same physical type can be combined. Familiar examples are: (1) series connection of voltage generators, (2) superposition of the magnetic fields of two currents, (3) combination of shaft rotations in a mechanical differential, etc. Let us consider the first of these examples in some detail. From experiments on many different voltage generators, we find that our measured results are not affected if the spatial order of the series connection of any two voltage generators is interchanged as in Figure 2(a). In a series connection of three such generators, we find that it makes no difference whether we first connect the first two generators and then the third, or first connect the second and third generators and then connect the combination to the first. In short, we can show experimentally that the composition of signals satisfies all of the axioms of vector addition (H 1958).

Measurement of any of the signals requires some scale of physical magnitude. The set of all possible magnitudes corresponds to the field Ω of scalars which forms a necessary part of the definition of an abstract vector space. If we limit our interest to measurement of existing systems and specification of systems to be constructed, the field of complex rationals is convenient and far more than sufficient. In design problems involving certain optimization steps, a more convenient choice is the field of algebraic numbers. If a valid mathematical model of the system is available, the use of the field of reals or the field of complex numbers may be convenient.

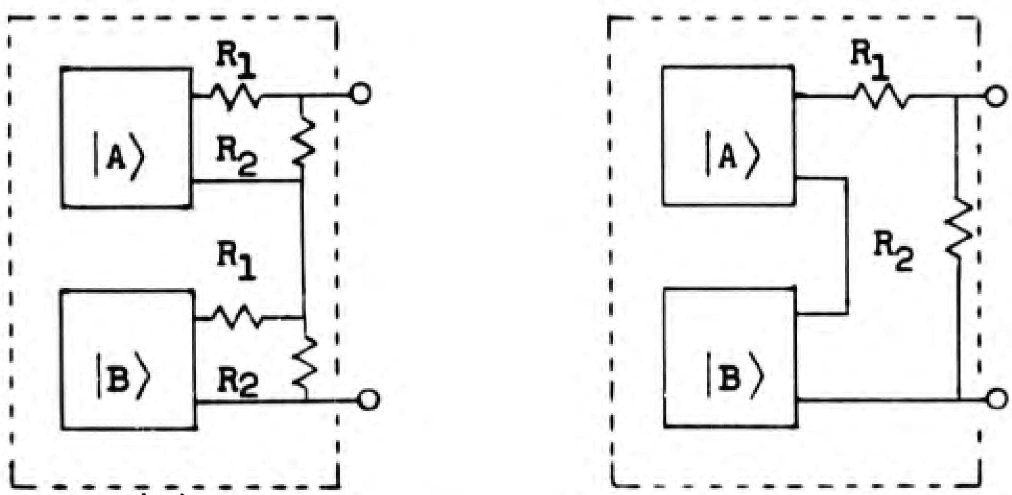
A signal may be multiplied by a magnitude. Familiar examples of scalar multipliers are: (1) resistance networks, (2) electronic amplifiers, and (3) gear trains. Again, from measured results on many sets of signals and



(a) Addition of signals is commutative.



(b) There is a unique signal |0⟩ such that |A⟩ + |0⟩ = |A⟩ for every signal |A⟩.



(c) Multiplication of signals by a magnitude is distributive with respect to signal addition.

Figure 2 Selected Axioms of Signal Vector Algebra magnitude scales, we learn that signal addition and multiplication of signals by magnitudes satisfy (within tolerably small errors) all of the axioms of an abstract vector space $\mathcal{S}$. Thus, we are justified in identifying signals with vectors and magnitudes with scalars. We will denote a signal vector by a symbol of the form $|F\rangle$ in text and by an open arrowhead in system block diagrams.

Finite-dimensional vector spaces are sufficient for signal theory. In some problems, it may be convenient to introduce spaces of infinite dimensionality for approximate calculation or for the purpose of using valid mathematical models. Similarly, when we speak of an ensemble of signals, we will mean an ensemble containing a finite number of signals.

Even though the dimensionality of our signal spaces will seldom be as small as three, it is often convenient to think of a signal as a geometric vector. This follows from the fact that a complex problem can frequently be broken down into a sequence of small steps, each of which involves the consideration of only two or three vectors. Thus, we are able to employ our knowledge of geometry and the mental pictures which it provides to think about complex signal analysis problems in a way which would be impossible if our thinking were confined to strings of mathematical symbols. One must be careful, however, not to confuse the geometric picture of our abstract vector space with either the 3-space of everyday experience or with the 4-space of the theory of relativity. Within limits, thinking and manipulation of geometric vectors is every bit as rigorous as that based on algebraic symbols, though any result obtained from "signal geometry" should be checked by "signal algebra" for the simple reason that checking by an alternate method is always good practice.

In both the real world and the world of mathematics, there are instances where some operation is confused with the result of the operation; in some cases, the same word is used for both ideas. The word "measurement" is an example. Though "signal" and "generator" have not been similarly confused, we have pointed out that a signal may be identified with its generator, and conversely. Since we could describe a generator as a device which converts magnitudes into signals, we might wish to describe the elements of our abstract space as being linear transformations which map scalars into vectors. By an appropriate insertion of unit scalars into our diagrams and equations, it is always possible to adopt this latter point of view.

Pattern Vectors

We will restrict our attention to systems in which signals are not only generated by some transmitting apparatus but are also measured by some physical receiving apparatus. By "measure" we refer to a device or process which produces a set of numbers or magnitudes at its output in response to a signal at its input. Writers on the philosophy of science (Churchman and Ratoosh 1959) customarily insist that the output of a measuring process be a number, and in every application that we will consider the narrower definition suffices. The usual definition seems slightly too restrictive in that the essential function of a measuring process seems to be one of storing, staticizing or recording information in some form suitable for the purposes of reading, interpretation or duplication at some later time and on a time scale having nothing directly to do with the timing associated with the measurement process itself. It appears that such a process of measurement takes place in all systems of interest in signal theory.

A common example of a measuring apparatus is a four-terminal electrical network followed by a sampler and perhaps also by a digitizer. We may sometimes use "siftor" in place of "measuring apparatus" in order to have a term paralleling "generator" and to serve as an abbreviation. Thus, a <u>generator</u> converts magnitudes (or numbers) into signals, and a <u>siftor</u> converts signals into magnitudes (or numbers).

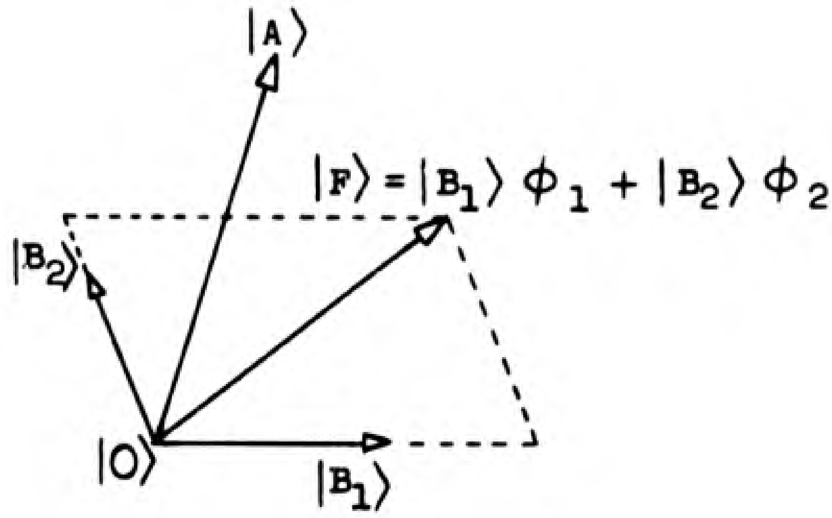
Just as new generators can be formed from old by composition and scalar multiplication, we know from experience that new siftors can be formed from old in quite similar fashion. For example, the composition of two siftors is accomplished by applying the same input signal to both and summing their scalar outputs. Again, we find that the operations of siftor addition and scalar multiplication satisfy all the axioms of an abstract vector space.

In the modern axiomatic treatment of vector algebra, a linear functional is defined as a linear mapping of the vectors in some given vector space into the field of scalars. Furthermore, the set of all such linear functionals is shown to be a vector space. This space of linear functionals is called the dual of the given vector space.

We identify each physical siftor in a signal analysis problem with some linear functional in the world of mathematics. More often than not, we will refer to the linear functional as a vector. To avoid confusion, we use the term pattern vector to distinguish from signal vector. Though patterns and signals are both vectors, they are not at all the same; as we have noted before, they do not even belong to the same space. Most of the existing literature in electrical engineering does not clearly distinguish between signal and pattern, or between impulse response and weighting function. This omission is the source of some confusion and is partially responsible for the lack of any discussion of the essential process of measurement in most introductions to signal analysis. Sharing this responsibility is the universal failure to distinguish between entities in the real world and their corresponding models in the world of mathematics.

We will use a symbol of the form $\langle G|$ to denote a linear functional corresponding to some measuring apparatus. The scalar result obtained by evaluating some linear functional $\langle G|$ at some vector $|F\rangle$, i.e., the result obtained at the output of measuring apparatus $\langle G|$ in response to signal $|F\rangle$, is denoted by $\langle G|F\rangle$ as shown in Figure 3. In flow charts, a siftor is indicated by a box with a signal indicated by an open arrowhead at the input and a scalar magnitude indicated by a solid arrowhead at the output.

Some of the correspondences shown in Figure 3 precede our discussion of them. For example, we will restrict our vector spaces by defining an inner product. Occasionally, we may confuse correspondence with identity, e.g., by writing "operator" when we mean "filter" or vice versa. Such lapses occur frequently in the literature and are to be tolerated but not encouraged.



Linear Algebra	Signal Theory
Vector	Signal
Scalar	Magnitude
Squared length	Energy
Direction	Waveform (or spectrum)
Result of basis change	Transform
Operator	Transducer or filter
Linear functional	Measuring apparatus
Coordinates	Measured results
Inner product	Measurement process

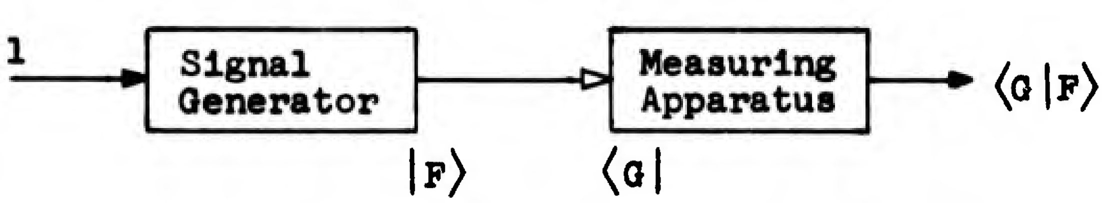


Figure 3
Correspondence Between Signal Theory and
Linear Vector Algebra

Operators

Corresponding to the various linear physical devices and processes which yield an output signal in response to an input signal, e.g., RLC network, electromechanical transducer, linear distortion in a communication channel, etc., we have the mathematical construct variously called linear transformation, linear operator or just operator. Any system that we will consider can be described mathematically as a cascade of signal, operator and pattern. The boundaries between the three elements of the cascade can be located quite arbitrarily. In fact, we can eliminate the middle element entirely by absorbing it as part of the signal or of the pattern. In other words, we are at liberty to consider an operator as operating on a signal or on a pattern, whichever is convenient. This point is illustrated in Figure 4, where we introduce symbols of the form $|H|$ to denote operators.

Normally, we will consider that an operator which operates on signal vectors in some given space will yield an output which is a signal vector in the same space. This is not a necessary restriction and is easily removed. In many problems it is convenient to think of an operator as mapping vectors from one space into vectors in some other space. For example, in the familiar equation

$$e = Ri + L\,di/dt$$

e, Ri and L di/dt represent vectors in the same space, but i represents a vector in another space, and di/dt still another. Ignoring these distinctions, as is frequently done, corresponds to dealing with representatives free of physical units. Such a practice is permitted, provided that the physical units involved in the problem are checked. Similar statements can be made if we regard the pattern space as the domain of the operator.

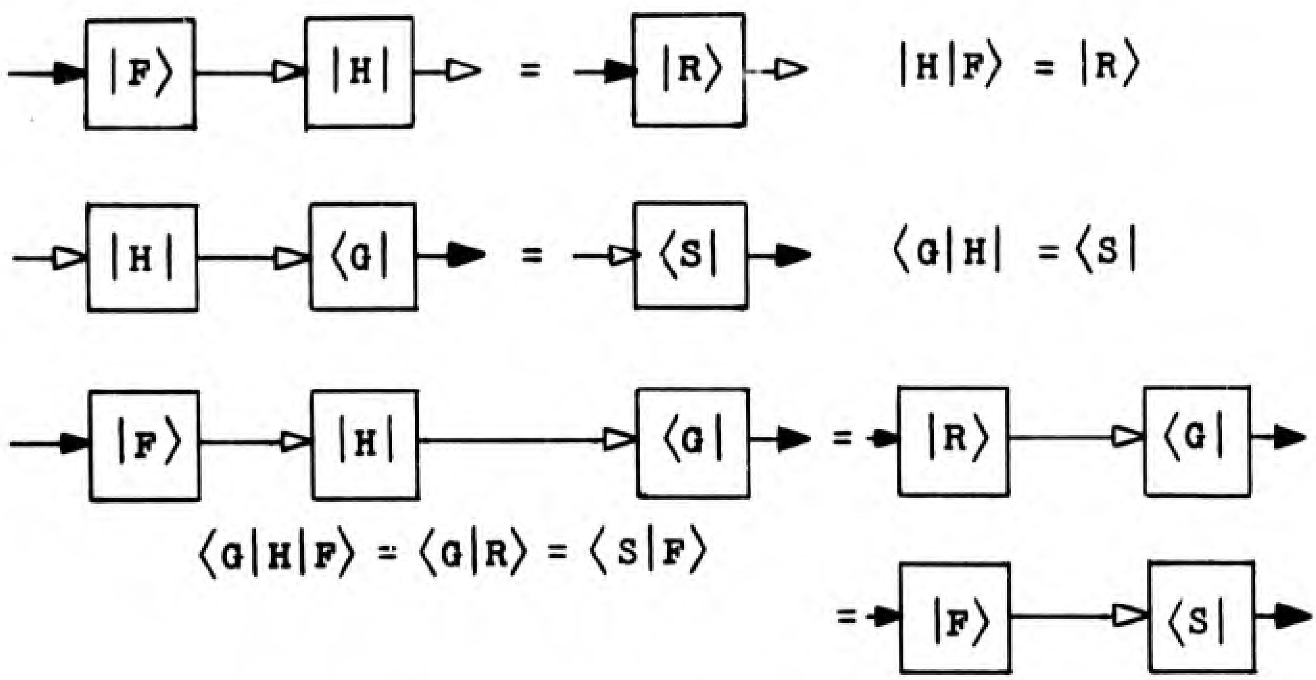


Figure 4 Typical Cascade

Since we are interested here in applying linear algebra to signal theory, it is necessary that we restrict our attention to linear operators, and we will avoid using the adjective "linear" repetitiously.

Extension of our notation to multi-linear systems will be considered in Chapter 3. We may define in physical terms the addition of transducers and their multiplication by magnitudes. From experiments performed on a wide variety of transducers, we find that these operations of addition and multiplication satisfy all the axioms of an abstract vector space. If we wish, we may then regard the idealized mathematical counterpart of the transducer, i.e., the operator, as being not only a transformation defined on some given signal space but also as an element of a vector space itself. If the signal space has n dimensions, then the pattern space has n dimensions and the operator space has n^2 dimensions. Ordinarily, we make no direct use of the fact that operators are vectors in their own right.

Particular operators of general interest are: the null operator $|0|$, the identity operator $|I|$ and the inverse of an invertible operator. The null operator is defined by: $|0|F\rangle = |0\rangle$, and the identity operator by: $|I|F\rangle = |F\rangle$, for all $|F\rangle$. The inverse of an invertible operator $|H|$ is denoted $|H^{-1}|$ and is defined by: $|H|H^{-1}| = |I| = |H^{-1}|H|$. There are several other particular operators of general utility, e.g., the delay operators, which can be defined only by reference to a particular basis.

A <u>subspace</u> $\mathcal{B}$ of an n-dimensional vector space is a vector space satisfying

$$|0\rangle \in \mathcal{B} \subseteq \mathcal{S}$$

$$0 \leq \text{Dim}\,\mathcal{B} \leq n = \text{Dim}\ \mathcal{S}$$

Subspaces of $\mathcal{G}$ are similarly defined. In geometric terms, a subspace is either: (1) the origin $|0\rangle$, (2) a line, plane or hyperplane containing the origin, or (3) the entire space $\mathcal{S}$. The image of the subspace $\mathcal{B}$ under some linear operator $|A|$ is the subspace $|A|\mathcal{B}$ defined as the set of all vectors of the form $|A|X\rangle$ where $|X\rangle \in \mathcal{S}$. In signal theory terms, $|A|$ is a filter, $|X\rangle$ is an input signal, $\mathcal{B}$ is a set of input signals, and the image $|A|\mathcal{B}$ of $\mathcal{B}$ is the corresponding set of output signals. The image of $\mathcal{S}$ under $|A|$ is called the <u>range</u> of $|A|$, and the <u>null-space</u> of $|A|$ is the subspace $\mathcal{N}$ with image $|0\rangle$. The dimensionality of the range of $|A|$ is the <u>rank</u> of $|A|$ and Dim $\mathcal{N}$ is the <u>nullity</u> of $|A|$; the sum of the rank and nullity of any operator must equal n. Thus, for some filter $|A|$, we may regard the range, null-space and rank as analogous to pass-band, stop-band and bandwidth.

We will find uses for a symbol of the form $|F\rangle\langle G|$ which is called a <u>dyad</u> and which is easily seen to be an operator of unit rank. A <u>dyadic</u> is a linear combination of dyads. Any operator can be written as a dyadic; for example,

$$|A| = \sum_{i=1}^{m} |F_i\rangle\langle G_i|$$

provided that m is not less than the rank of $|A|$.

Descriptions of a set of signals in which all of the signals are obtained from only one generator can always be found. Such a description involves identification of each signal in the set with a suitable operator which produces the signal under consideration in response to the output of some "standard" signal generator, as indicated in Figure 5. Similarly, all of the patterns in some given set could be described in terms of a set of operators and some single pattern. Thus, if we wish, we can relocate the two boundaries in the signal-operator-pattern cascade in such a way as to reduce any problem to a study of operators.

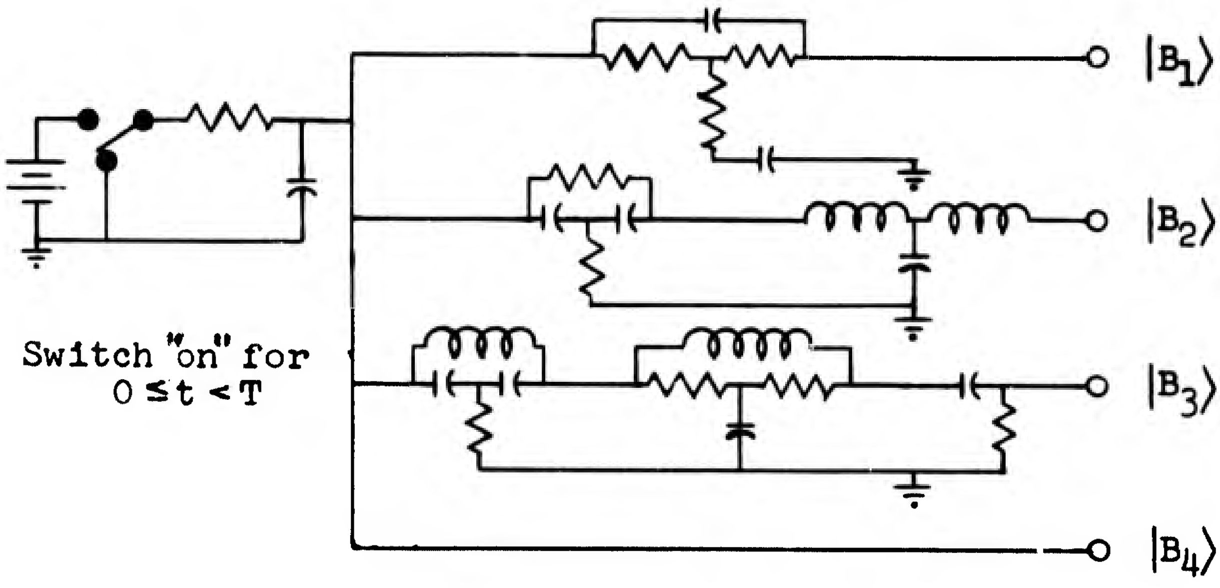


Figure 5 Generation of a Set of Signals

Inner Product

Introduction of such concepts as signal energy, squared error, noise variance, etc., is accomplished by restricting $\mathcal{C}$ and $\mathcal{D}$ to be inner-product spaces. The modern mathematical approach is to define a mapping from the set of ordered pairs of elements of the vector space $\mathcal{D}$ into the field Ω such that the mapping is Hermitian symmetric, linear in the left factor and positive definite. We depart from the usual nomenclature by calling the entity just defined an <u>intra-product</u>. The defining properties of <u>an</u> intra-product $\left[|X\rangle, |Y\rangle\right]$ of vectors $|X\rangle$ and $|Y\rangle$ in $\mathcal{D}$ are given explicitly by

$$\left[|X\rangle, |Y\rangle\right] = \left[|Y\rangle, |X\rangle\right]^* \tag{1}$$

$$\left[|X_1\rangle \beta_1 + |X_2\rangle \beta_2, |Y\rangle\right] = \left[|X_1\rangle \beta_1, |Y\rangle\right] + \left[|X_2\rangle \beta_2, |Y\rangle\right] \tag{2}$$

$$0 \leq \left[|X\rangle, |X\rangle\right] \text{ with equality if and only if } |X\rangle = |0\rangle \tag{3}$$

Intra-product is somewhat inelegant in that it fails to be linear in the right factor; instead, we have

$$\left[|X\rangle, |Y_1\rangle \beta_1 + |Y_2\rangle \beta_2\right] = \left[|X\rangle, |Y_1\rangle\right]\beta_1^* + \left[|X\rangle, |Y_2\rangle\right]\beta_2^* \tag{4}$$

i.e., intra-product is conjugate linear in the right factor. In other words, intra-product is conjugate bilinear and not bilinear. Since Ω is real in the application problem considered in this dissertation, we will not dwell on the details involved with complex fields. The reader is referred to §§ 59-70 (particularly

§§59-61, 67, 69) of Halmos' excellent treatment (H 1958). In his § 67 is the theorem which appears in our notation as follows.

TM To any linear functional $\langle A|$ on a finite-dimensional intra-product space $\mathcal{S}$ there corresponds a unique vector $|Z\rangle$ in $\mathcal{S}$ such that $\langle A|X\rangle = \big[|X\rangle,\ |Z\rangle\big]$ for all $|X\rangle$.

We use this theorem to justify the definition of <u>matching</u> by introducing the notation

$$\langle A| = \widetilde{|Z\rangle} = \langle\tilde{Z}| \tag{5}$$

and calling $\langle A| = \langle\tilde{Z}|$ the <u>match</u> of $|Z\rangle$. The large tilde symbolizes the linear operator "take the match of" which maps a vector space into its dual. The small tilde is a label used to conserve base letters and to recall relationships. We may also write

$$|Z\rangle = \widetilde{\langle A|} = |\tilde{A}\rangle \tag{6}$$

Two applications of "match of" is equivalent to the identity operation. We will return to the subject of matching in our discussion of representation.

Correspondence has been established between: (1) intra-product of two vectors in $\mathcal{S}$, and (2) the interaction of a functional in $\tilde{\mathcal{S}}$ with a vector in $\mathcal{S}$. This correspondence is employed in the modern mathematical literature to justify dropping the linear functional notation in favor of intra-product with the result that attention is focussed on $\mathcal{S}$. Perhaps this choice stems from the bias of the older literature which generally ignored the dual space $\mathcal{G}$.

Since we wish to emphasize the balanced use of both the signal and the pattern spaces, we take the opposite choice by employing the correspondence to drop the intra-product notation. This is accomplished by defining the result of evaluating a linear functional $\langle G|$ in $\mathcal{G}$ at some vector $|F\rangle$ in $\mathcal{S}$, where an intra-product has been defined in $\mathcal{S}$, as the <u>inter-product</u> of $\langle G|$ and $|F\rangle$. Summarizing, we may define inter-product in terms of intra-product by writing

$$\langle G|F\rangle = [\![\, |F\rangle, \ |G\rangle \,]\!] \tag{7}$$

It follows immediately that inter-product is Hermitian symmetric, linear in the left factor, linear (not conjugate linear) in the right factor, positive definite and bilinear (not conjugate bilinear).

If an inter-product has been defined between $\mathcal{G}$ and $\mathcal{S}$, we will call them <u>inter-product spaces</u>. We may also revert to the usual term "inner product" since it will always be clear in context that we mean the inter-product of a pattern vector $\langle G|$ and a signal vector $|F\rangle$. The reader is free to perform the unnecessary steps of replacing the inter-product $\langle G|F\rangle$ with the possibly more familiar intra-product $[\![\, |F\rangle, |G\rangle \,]\!]$ if he finds this convenient.

The proper article to use with inner product is not "the" but "an" as is emphasized in the more recent texts (Hoffman and Kunze 1961, MS&M 1963). Associated with each possible definition of inner product (inter- or intra-) is some Hermitian form or, perhaps more conveniently, some Hermitian matrix $\underline{H}$. Thus, all of the terms associated with inner-product spaces, e.g., orthogonal, normal, orthonormal, unitary, etc., ought to have some such phrase as "with respect to the defined inner product" or "with respect to $\underline{H}$" or just "$\underline{H}$" attached as a suffix or prefix. The real diagonal matrix unitarily similar to $\underline{H}$ is called the *weighting* of the inner product. Ordinarily, we will choose bases and define inner product so that $\underline{H}$ is the unit matrix $\underline{I}$ and the tags regarding $\underline{H}$ may be omitted.

Perhaps some motivation is desirable for the very general definition of inner product given above. In signal theory the procedure of "noise-whitening" is often used. Here, we intend "white" to include not only uniform distribution of noise energy over the n dimensions of $\mathcal{C}$ and $\mathcal{S}$ but also the absence of correlation between the noise on different dimensions. The usual explanation of the whitening procedure is the active or "alibi" interpretation, e.g., "whitening filter" or"whitening operator". An alternative explanation is the passive or "alias" interpretation obtained by considering that noise-whitening involves a change of basis and a change of inner-product definition to one where the noise is white.

We proceed now to define some of the well-known concepts already mentioned which are associated with the inner product. Let $\mathcal{G}$ and $\mathcal{F}$ be n-dimensional inner-product spaces. The inter-product of any element of either space and the match of that element is a positive real scalar called the <u>energy</u> or squared length of the selected element (or matched pair); for example, $\langle\widetilde{F}|F\rangle$, $\langle G|\widetilde{G}\rangle$, etc. The (positive) square root of the energy of some element is called its <u>length</u> or rms value. Any element (or matched pair) of unit length is called <u>normal</u>. If $\langle G|F\rangle = 0$, then each of the pairs: $\langle G|$ and $|F\rangle$, $|F\rangle$ and $|\widetilde{G}\rangle$, $\langle G|$ and $\langle\widetilde{F}|$, $|\widetilde{G}\rangle$ and $\langle\widetilde{F}|$ is said to be <u>orthogonal</u>. An operator $|U|$ which preserves length is called <u>unitary</u> if Ω is complex and <u>orthogonal</u> if Ω is real. For real inner-product spaces, the angle θ between $|F\rangle$ and $|\widetilde{G}\rangle$, or between $\langle\widetilde{F}|$ and $\langle G|$ is defined by

$$\cos\theta = \frac{\langle G|F\rangle}{\sqrt{\langle G|\widetilde{G}\rangle\ \langle\widetilde{F}|F\rangle}} = \frac{\langle\widetilde{F}|\widetilde{G}\rangle}{\sqrt{\langle\widetilde{F}|F\rangle\ \langle G|\widetilde{G}\rangle}} \qquad 0 \le \theta \le \pi \tag{8}$$

This concept of matching may be extended to operators with the aid of Figure 6 where orthonormal bases $\{|B_j\rangle\}$ and $\{\langle\widetilde{B}_i|\}$ in real inner-product spaces $\mathcal{F}$ and $\mathcal{G}$

are assumed given. If and only if the two systems shown in Figure 6 have the same output for every combination of i and j, then $|L|$ is the operator matched to $|K|$ or the match of $|K|$. Thus,

$$|L| = \widetilde{|K|} = |\tilde{K}| \Longleftrightarrow \langle \tilde{B}_i|K|B_j\rangle = \langle \tilde{B}_j|L|B_i\rangle \tag{9}$$

This definition is extended to complex spaces by conjugating the last term in Equation (9). We may also write

$$|K| = \widetilde{|L|} = |\tilde{L}| \tag{10}$$

and, again, two applications of "match of" is the identity operator.

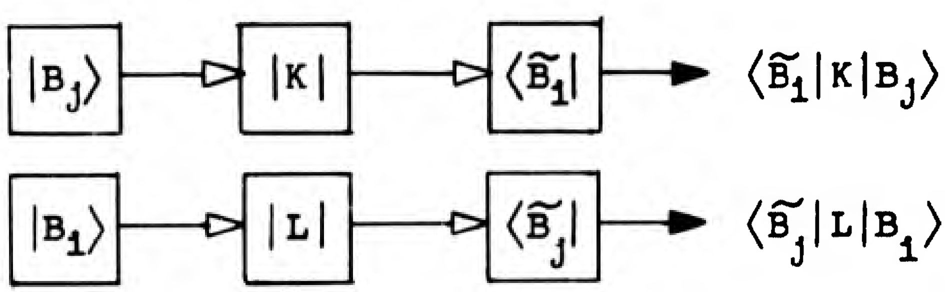


Figure 6 Systems Used in Defining Matching Operators

2 - 2 REPRESENTATION AND MEASUREMENT

Description of signals, transducers and patterns in terms of abstract vectors, operators and functionals provides a powerful way to think about a complex problem, free from the unnecessary complications associated with any particular representation. However, at some point it is necessary to relate the abstract symbols to apparatus and measurements. Thus, a chain of progressively more concrete representations of the abstract entities is required. The first link in this chain is the subject of this section; the remaining links are well-known to engineers for at least the time-domain and frequency-domain representations. This first link involves a choice of basis for representation of the abstract entities. A tremendous range of choice is available here, though the existing literature employs only two or three bases almost exclusively. We adopt the view that we might wish to select a basis for each particular signal analysis problem, rather than the traditional approach of using two or three bases of great generality for all problems. At the beginning of this section, $\mathcal{S}$ and $\mathcal{C}$ are general linear spaces and are later restricted to inner-product spaces.

<u>Basis</u>

Before formally introducing the idea of basis, we need to define linear independence. Following our earlier examples, we will formulate the definition in physical terms. Suppose that we are given two signal generators and that we find, after an adequate series of experiments on the system indicated in Figure 7, that the only scalars which produce the null signal $|0\rangle$ at the output are $\mu_1 = 0$ and $\mu_2 = 0$. This physical situation is described mathematically by saying that $|B_1\rangle$ and $|B_2\rangle$ are linearly independent. That is, the following two statements are equivalent:

$|B_1\rangle$ and $|B_2\rangle$ are <u>linearly independent</u>

$$|B_1\rangle\mu_1 + |B_2\rangle\mu_2 = |0\rangle \Rightarrow \mu_1 = \mu_2 = 0$$

Similarly, n signals $|B_i\rangle$ are linearly independent if and only if no summation of them yields $|0\rangle$ except for the trivial one with all n coefficients μ_i equal to zero.

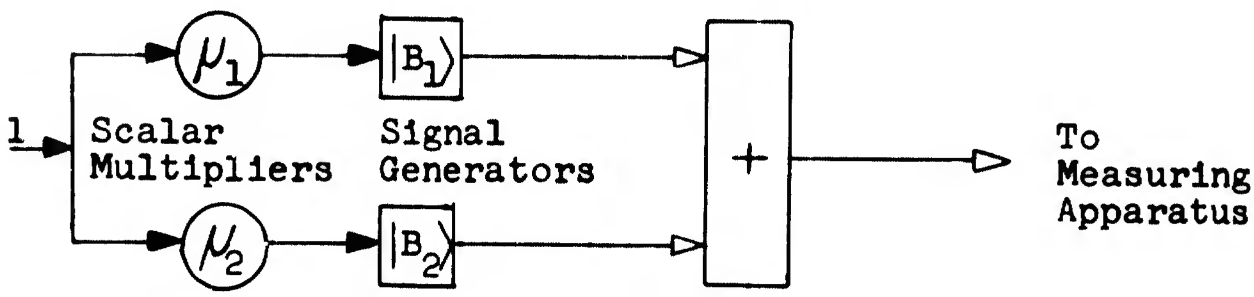


Figure 7 Test for Linear Independence

Now consider the set of signals formed by all possible linear combinations of the n signals $|B_i\rangle$, i.e., the space <u>spanned</u> by $\{|B_i\rangle\}$. This set satisfies all of the axioms of an abstract vector space. The set of vectors $|B_i\rangle$ is a <u>basis</u> for the space and the dimension of the basis, and of the space, is n. We will always be interested in the sequence of labels of the basis elements and will regard $\{|B_1\rangle, |B_2\rangle, \ldots, |B_n\rangle\}$ and $\{|B_2\rangle, |B_1\rangle, \ldots, |B_n\rangle\}$ as different bases.

Conversely, if we are given any finite set of signals we can find a basis for the set by selecting any signal in the set as $|B_1\rangle$, and then selecting $|B_2\rangle$ as the first signal found to be linearly independent of $|B_1\rangle$. The process continues by discarding any signal linearly dependent on the partial basis and adjoining any signal found to be linearly independent of the partial basis. The process stops when every signal in the set has been examined. The residual ordered set of linearly independent signals is a basis for the original set, and the space spanned by the basis is equivalent to the space spanned by the original set of signals.

Representation

The importance of a basis $\{|B_i\rangle\}$ is that it may be used to represent any vector $|F\rangle$ in the space by listing in sequence the unique set of n coefficients ϕ_i in the following equation:

$$|F\rangle = |B_1\rangle\phi_1 + |B_2\rangle\phi_2 \quad \ldots + |B_n\rangle\,\phi_n \tag{11}$$

The ith term on the right side of Equation (11) is the ith component of $|F\rangle$ and consists of the ith basis vector multiplied by the scalar ϕ_i which is called the ith coordinate of $|F\rangle$ on the given basis. Representation of an arbitrary vector on a basis is a mathematical idea corresponding to the engineer's idea of the specification of a generator for an arbitrary signal in terms of some given set of n standard signal generators as indicated in Figure 8.

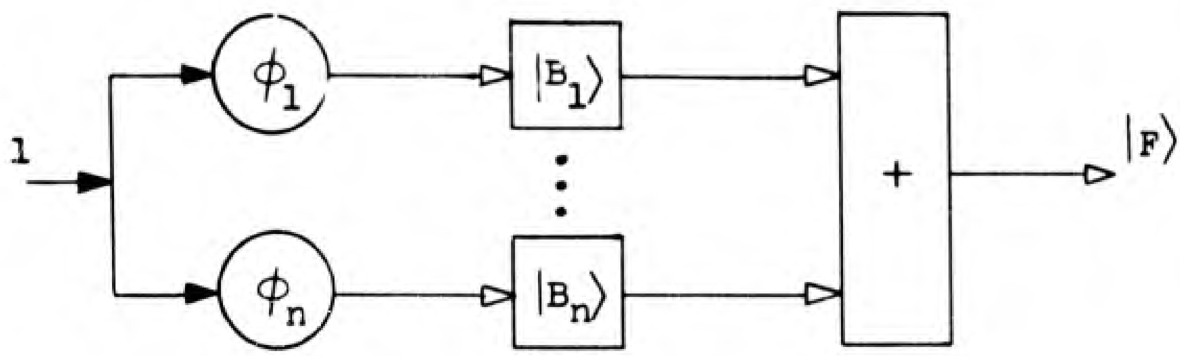


Figure 8 Use of Signal Basis

It is convenient to display the coordinates of a signal vector as a column or $n \times 1$ matrix. Thus, the representative of $|F\rangle$ on $\{|B_i\rangle\}$ is

$$\begin{matrix} 1 \\ 2 \\ \\ n \end{matrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_n \end{bmatrix} = \underline{F}\rangle \qquad (12)$$

Similarly, if $\langle C_i|$ is any basis for the pattern space, we may represent any pattern $\langle G|$ by the n coefficients γ_i in the equation

$$\langle G| = \gamma_1 \langle C_1| + \gamma_2 \langle C_2| \ldots + \gamma_n \langle C_n| \qquad (13)$$

corresponding to a specification of the pattern $\langle G|$ in terms of n standard siftors as indicated in Figure 9. The coordinates of a pattern vector will be displayed as a row or $1 \times n$ matrix. Thus, the representative of

$\langle G|$ on $\{\langle C_i|\}$ is

$$\langle \underline{G} = \boxed{\gamma_1 \mid \gamma_2 \mid \cdots \mid \gamma_n} \tag{14}$$

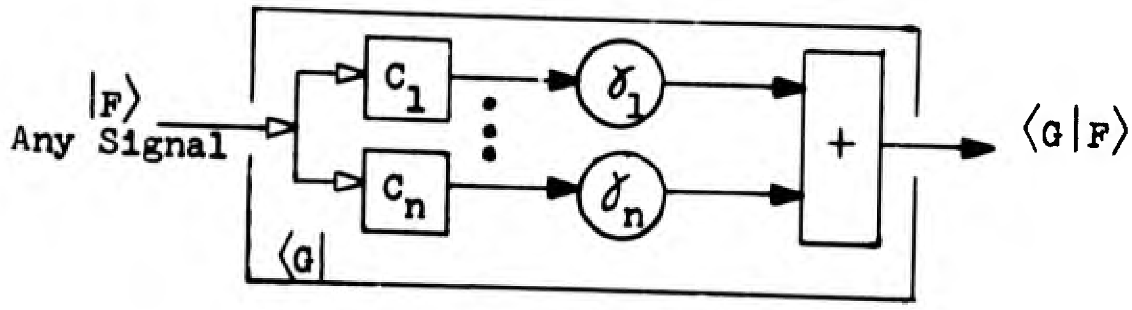


Figure 9 Use of Pattern Basis

Corresponding to any basis $\{|B_i\rangle\}$ for the signal space, there is a basis $\{\langle D_i|\}$ of the pattern space, such that $\langle D_i|B_j\rangle = \delta_{ij}$. Given $\{|B_i\rangle\}$ we say that its dual is $\{\langle D_i|\}$ or that the latter is the <u>dual basis</u>. It is also true that each basis is the dual of the other. There is no need to choose any particular basis for the pattern space imposed by the choice of any basis in the signal space; however, it is usually convenient to employ bases that are mutually dual.

One basis suffices for the representation of vectors, but two bases, one for the signal space and one for the pattern space, are required to represent an operator. For example, an operator $|A|$ is represented on $\{|B_i\rangle\}$ and $\{\langle D_i|\}$ by displaying the n^2 scalars $\langle D_i|A|B_j\rangle$ as an $n \times n$ matrix $\underline{A}$ with row index i and column index j.

Let us now restrict our attention to spaces in which an inner product has been defined and to convenient choices of bases. A basis in which each pair of elements is orthogonal and each element is normal is an orthonormal basis. If $\{\langle D_i|\}$ and $\{|B_i\rangle\}$ are each orthonormal and are also mutually dual, then the bases are mutually matched, i.e., $\langle D_i| = \langle \widetilde{B}_i|$ for all i. An orthonormal basis may be constructed from any basis by means of the Gram-Schmidt orthonormalization procedure described in any of the standard texts on linear algebra or matrix algebra.

We relate our discussion of representation to inner-product spaces by presenting a few theorems which will be needed later. Let $|B_i\rangle$ and $\langle \widetilde{B}_i|$ be orthonormal bases in inner-product spaces $\mathcal{S}$ and $\mathcal{G}$. Let $|F\rangle$ be any signal vector in $\mathcal{S}$ and let $\langle G|$ be any pattern vector in $\mathcal{G}$ defined by

$$|F\rangle = \sum_i |B_i\rangle \phi_i \qquad \langle G| = \sum_i \gamma_i \langle \widetilde{B}_i| \tag{15}$$

The intra-product of $|F\rangle$ with itself is given, in general, by the Hermitian form

$$\big[|F\rangle,\ |F\rangle\big] = \sum_{i,j} \phi_i^* \eta_{ij} \phi_j \qquad \text{where} \qquad \eta_{ij} = \eta_{ji}^*$$

We will be interested in the special case, the Cartesian form, where $\eta_{ij} = \delta_{ij}$.

$$\big[|F\rangle,\ |F\rangle\big] = \sum_{i,j} \phi_i^* \delta_{ij} \phi_j = \sum_i \phi_i^* \phi_i$$

The general element of the column $\underline{F}\rangle$ representing $|F\rangle$

is ϕ_i. Suppose the general element of the row representing $\langle\tilde{F}|$ is ν_i. Then

$$\langle\tilde{F}|F\rangle = \sum_{i,j} \nu_i\phi_j$$

and the definition of inter-product requires that

$$\sum_{i,j} \nu_i\phi_j = \sum_i \phi_i^*\phi_i$$

which must hold for all possible choices of $|F\rangle$. Thus,

$$\nu_i = \phi_i^*$$

and the row representing $\langle\tilde{F}|$ is the adjoint (conjugate transpose) of $\underline{F}\rangle$ which we denote by $\langle\tilde{\underline{F}}$. Note that this notation depends on the assumption that the bases used must be mutually dual and mutually matched. A similar argument shows that $|\tilde{G}\rangle$ is represented by $\tilde{\underline{G}}\rangle$ if and only if $\langle G|$ is represented by $\langle\underline{G}$.

Suppose we are given some operator $|K|$ represented on the given bases by the matrix $\underline{K}$ with general element $\langle\tilde{B}_i|K|B_j\rangle$. Let $|\tilde{K}|$ be represented by the matrix $\underline{J}$ with general element $\langle\tilde{B}_i|\tilde{K}|B_j\rangle$. If Equation (9) is extended to the complex case, we have

$$\langle\tilde{B}_i|K|B_j\rangle = \langle\tilde{B}_j|\tilde{K}|B_i\rangle^* \qquad \underline{K} = \tilde{\underline{J}}$$

Thus, if the large tilde is understood to mean "adjoint of" when applied to representatives, we have the following theorem.

TM Given $|F\rangle$ and $\langle G|$ as in Equation (15), and $|K|$ represented by $\underline{K}$, then (16a)-(16j) follow.

$$\widetilde{|F\rangle} = \langle\tilde{F}| = \sum_i \phi_i^* \langle\tilde{B}_i| \tag{16a}$$

$$\widetilde{\langle G|} = |\tilde{G}\rangle = \sum_i |B_i\rangle \gamma_i^* \tag{16b}$$

$$\widetilde{\underline{F}\rangle} = \langle\underline{\tilde{F}} \tag{16c}$$

$$\widetilde{\langle\underline{G}} = \underline{\tilde{G}}\rangle \tag{16d}$$

$$\widetilde{|K|} = |\tilde{K}| \tag{16e}$$

$$\widetilde{\underline{K}} = \underline{\tilde{K}} \tag{16f}$$

$$\phi_i = \langle\tilde{B}_i|F\rangle \tag{16g}$$

$$\gamma_i = \langle G|B_i\rangle \tag{16h}$$

$$\langle G|F\rangle = \langle\underline{G}\,\underline{F}\rangle = \sum_i \gamma_i\phi_i = \Big(\sum_i \phi_i^*\gamma_i^*\Big)^* = \langle\underline{\tilde{F}}\,\underline{\tilde{G}}\rangle^* = \langle\tilde{F}|\tilde{G}\rangle^* \tag{16i}$$

$$|F\rangle = |0\rangle \Longleftrightarrow \langle\tilde{F}| = \langle 0| \Longleftrightarrow \langle\tilde{F}|F\rangle = 0 \tag{16j}$$

Note that the conjugate isomorphism between $\subsetneq$ and $\supsetneq$ which we call "matching" is quite distinct from the operation of "reversing" (Lai 1960). The "rev" operator |Ƨ| operates on a signal in a real inner-product space to yield a signal with coordinates on some orthonormal basis (e.g., the finite time basis) reversed in sequence from those of the given signal. Thus, |Ƨ| maps from $\supsetneq$ to $\supsetneq$ (or from $\subsetneq$ to $\subsetneq$), while $\widetilde{\ }$ maps from $\supsetneq$ to $\subsetneq$ (or from $\subsetneq$ to $\supsetneq$).

<u>Some Useful Bases</u>

We wish to consider the employment of a basis tailored to the special requirements of each particular signal analysis problem with which we are confronted. This ad hoc approach has received less attention in the classical literature than it deserves. On the other hand, there are a few bases which have been found to be useful in a wide variety of applications and which ought to be given first consideration. Since these bases are well known, our discussion of them will be brief. The only point in describing these familiar bases here at all is to relate them to our notation and nomenclature.

We have emphasized the importance of defining any particular signal $|F\rangle$, operator $|H|$ or pattern $\langle G|$ by identifying or specifying some actual generator, transducer or siftor. Similarly, we define any basis in $\mathcal{S}$ by identifying a bank of generators and any basis in $\mathcal{G}$ by identifying a bank of siftors. Thus we will proceed by defining some of the familiar bases in terms of their correspondents in the real world.

Perhaps the most useful basis of all, for purposes both theoretical and practical, is that defined by apparatus such as that shown in Figure 10. Despite the importance of this *finite time basis*, it is hardly ever discussed in the theoretical literature.

Related to the finite time basis is another basis which *is* discussed in engineering literature of the past decade. This basis is called the sampled-data or *discrete time basis* and is useful in analysis. It is abstracted from the finite time basis by allowing the dimension n to become denumerably infinite. An even more familiar basis, useful in analysis, is the continuous time basis abstracted by imagining that n can become non-denumberably infinite. We can regard a representative of a signal on the discrete time basis as being a column of an infinite number of discrete scalars, and, on the continuous time basis, as an infinite column of densely packed scalars.

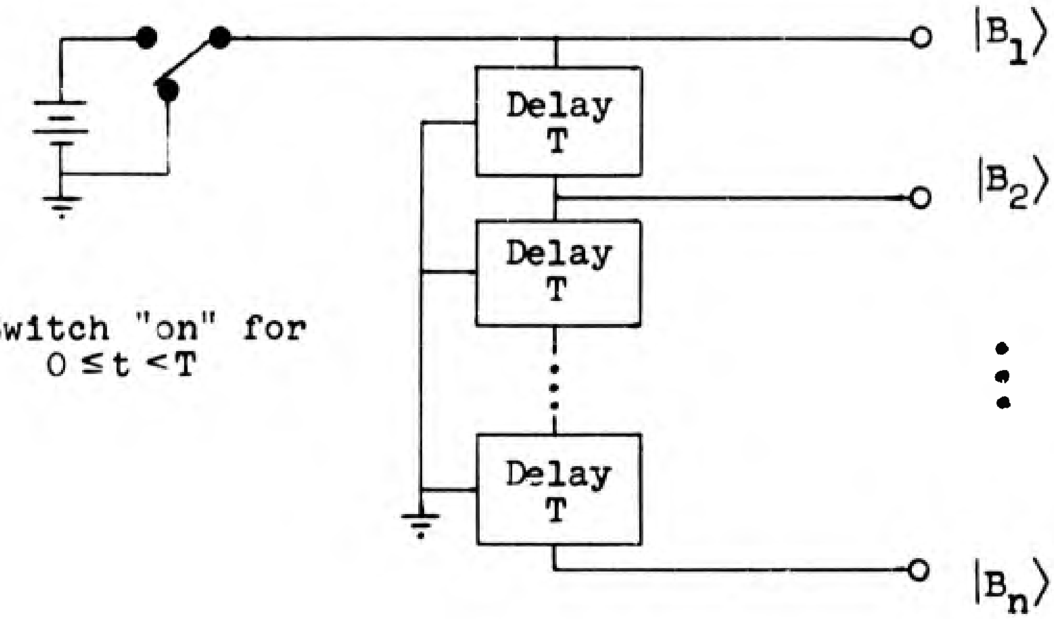


Figure 10 Finite Time Basis

An interesting relationship between our algebraic approach and the familiar continuous time representation may be seen by considering various expressions for the inner product. In Equation (16i), let $\langle G| = \langle \tilde{F}|$. Then

$$\langle \tilde{F}|F\rangle = \langle \underline{\tilde{F}}\ \underline{F}\rangle$$

If we allow n to become uncountably infinite, $\underline{F}\rangle$ becomes a densely packed infinite column of scalars which we indicate by $f\rangle$ and the classical literature denotes by $f(t)$. Similarly, $\langle\underline{\tilde{F}}$ becomes $\langle\tilde{f}$, and Equation (17) becomes

$$\langle\tilde{F}|F\rangle = \langle\tilde{f}\ f\rangle$$

where the latter expression is commonly written in the form

$$\int dt\ f^*(t)\ f(t)$$

Similarly, the continuous time representation of $\langle G|F\rangle$ is $\langle gf\rangle$ or

$$\int dt\ g(t)\ f(t)$$

which rather thoroughly obscures the important fact that inner product is related to measurement which necessarily involves the interaction of two different types of entities, i.e., signal and pattern.

Orthonormal exponential bases (Kautz 1954, Huggins 1956) are useful in both theory and practice. The continuous time representative of each element of such a basis is a linear combination of some finite set of exponential functions of time. The exponents may be either real or complex. A typical orthonormal exponential basis formed on $\{e^{p_i t}\}$ is shown in Figure 11 where we have used the familiar complex frequency basis to describe each of the filters required. The importance of the exponential bases stems from the fact that the eigenfunctions of stationary operators are exponentials.

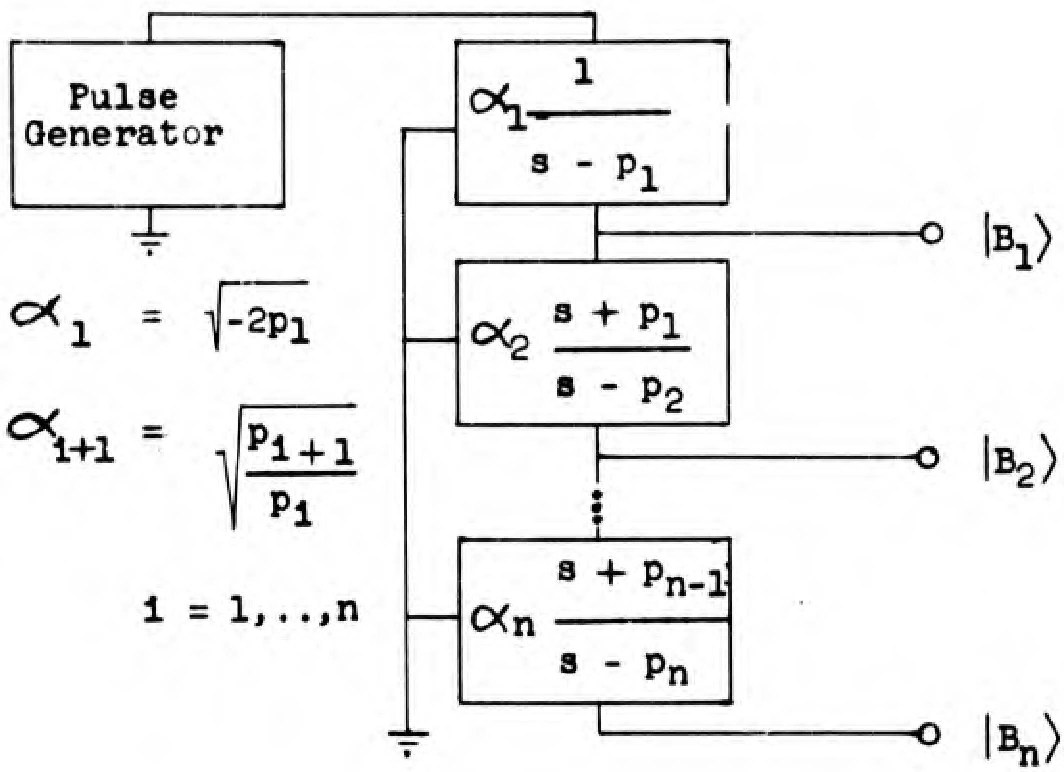


Figure 11 Orthonormal Exponential Basis

An important special type of orthonormal exponential basis is the familiar truncated Fourier series representation, corresponding to a choice of poles at equally-spaced points symmetrically located on the imaginary axis of the s-plane. The truncated Laguerre series representation (Wiener 1949) is an orthonormal exponential basis formed by choosing all the poles at one point on the real axis. The familiar continuous frequency representations (Fourier or Laplace transforms) may be regarded as orthonormal exponential bases with a non-denumerably infinite number of poles densely packed on some simple closed contour in the complex plane.

Extensive development of the experimental techniques has been carried out by investigators at The Johns Hopkins University and elsewhere over the past decade (Huggins 1958, Lai 1958). Orthonormal exponential bases have been successfully applied in analysis of electrocardiograms (Young and Huggins 1962), spoken vowel sounds (Dolansky 1960), non-linear operators (Lory, Lai and Huggins 1959) and nondestructive testing (Litman and Huggins 1963).

Theoretical work on orthonormal exponential bases has been extended to include multi-epoch signals and to exponentials growing in the pre-epoch interval as well as to exponentials decaying in the post-epoch interval (Young 1963). A study of the difficult exponent-selection problem has been completed (McDonough 1963). A generalization of the Kautz procedure using arbitrary zeros and more than one pole at each step has been developed (Ross 1962).

<u>Measurement</u>

In the literature and in everyday usage in engineering, the word "measurement" may mean the process in which a physical entity to be measured interacts with a measuring apparatus to yield a number (or numbers). Alternatively, the word "measurement" may refer to the numerical result (or results) obtained by the process. We will adopt the first meaning and will use the word "coordinate" for the second meaning.

The mathematical process of evaluating some linear functional $\langle G|$ at some vector $|F\rangle$ in a pair of mutually dual inner-product spaces is identified with measurement, the vector $|F\rangle$ with the signal to be measured, and the linear functional or pattern vector $\langle G|$ with the measuring apparatus. This correspondence is illustrated in Figure 12.

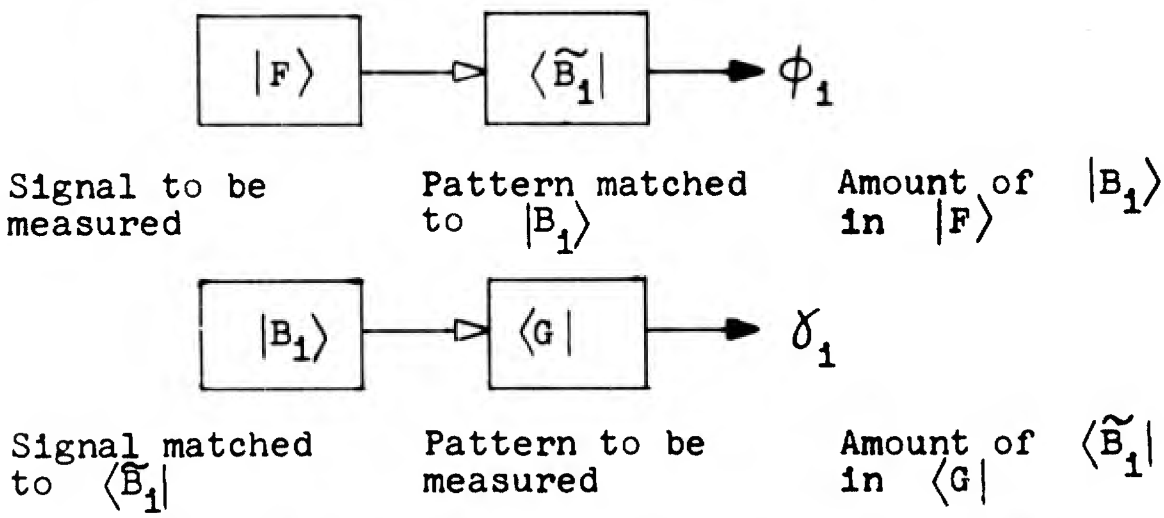


Figure 12 Measurement of Single Coordinates

A convenient way to begin a mathematical treatment of measurement is to discuss the proof of Equation (16g).

TM $\langle \tilde{B}_i | F \rangle = \phi_i$

PF $\{|B_i\rangle\}$ is orthonormal $\Rightarrow \langle \tilde{B}_i | B_j \rangle = \delta_{ij}$

$$|F\rangle \in \mathfrak{S} \Rightarrow |F\rangle = \sum_j |B_j\rangle \phi_j$$

$$\langle \tilde{B}_i | F\rangle = \langle \tilde{B}_i | \sum_j |B_j\rangle \phi_j = \sum_j \langle \tilde{B}_i | B_j\rangle \phi_j$$

$$= \sum_j \delta_{ij} \phi_j = \phi_i \qquad \square$$

Thus, to measure the ith coordinate of a signal in $\mathfrak{S}$ we apply the signal to the measuring apparatus characterized by the pattern matched to the ith element of the basis in $\mathfrak{S}$. If $|F\rangle$ is represented on the continuous time basis by $f(t)$ and $|B_i\rangle$ by $b_i(t)$, the previous statement would be given in classical terms by:

> To measure the ith coordinate of a signal on an orthonormal basis, apply the signal represented by $f(t)$ to a filter matched to $b_i(t)$, i.e., a filter with impulse response $b_i^*(-t)$, and sample the output at $t=0$.

For further discussion of the relationship between the vector space concepts discussed here and the classical approach based on continuous time and/or frequency representations the reader is referred to the doctoral dissertation of D.C. Lai and to the papers by W.H. Huggins and T.Y. Young listed in the Bibliography.

Similarly, Equation (16h) shows that the ith coordinate of a pattern $\langle G|$ on any basis $\{\langle \tilde{B}_i|\}$ may be measured or "calibrated" by applying the signal $|B_i\rangle$ to the pattern

$\langle G|$. Both processes are illustrated in Figure 12 for the case of measurement of a single coordinate.

In Figure 13, the process of measurement of all n coordinates of a signal is illustrated in two ways, the second of which suggests the introduction of an addition to our notation. The entity to be added may be described loosely as a "matrix" in which each "row" is one of the n patterns $\langle\tilde{B}_i|$. However, reflection indicates that this description is unsatisfactory. The entity we are attempting to describe is not a matrix since its elements are vectors, while the elements of a matrix are scalars. We see the entity under consideration operates on a signal vector and yields the representative on $\{|B_i\rangle\}$ of the signal, and that the entity corresponds to the basis of the pattern space. This last remark justifies a redefinition of the term <u>pattern</u> basis to mean precisely the entity which we have just been discussing.

The symbol that we will use for the pattern basis is $\underline{\tilde{B}}|$, incorporating the vertical bar associated with abstract entities and the underlined capitals associated with representatives. This choice is appropriate in view of the mixed nature of $\underline{\tilde{B}}|$ through its role of transformation from signal vectors to representatives of signal vectors.

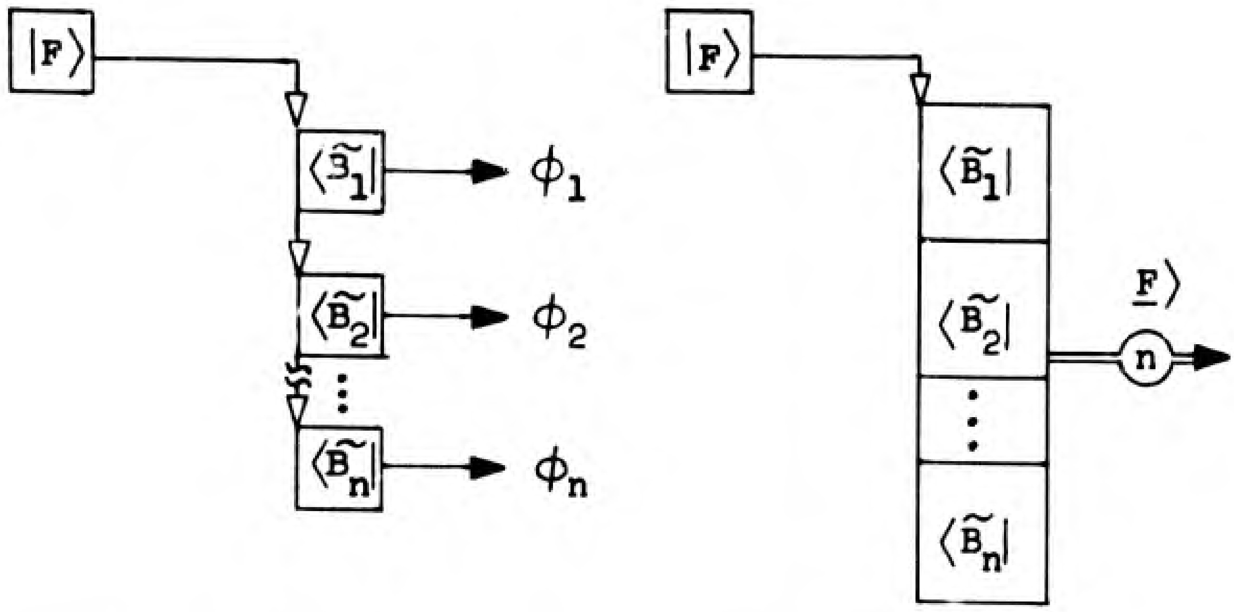
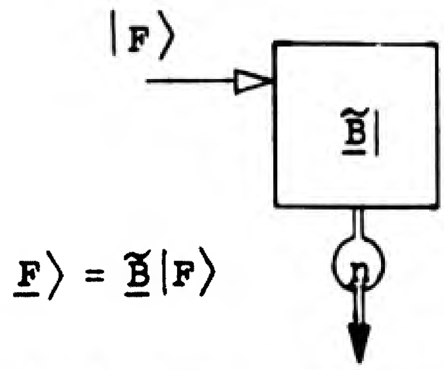


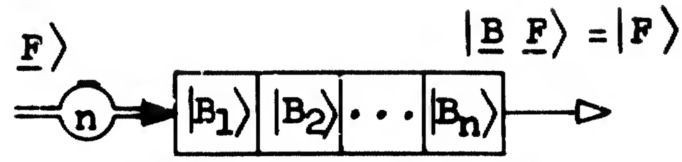
Figure 13 Measurement of All Coordinates of a Signal

Similarly, signal basis is redefined to mean the entity $|\underline{B}$ which may be described loosely as a "matrix" in which each "column" is one of the n signals $|B_i\rangle$. These mixed symbols may be manipulated according to the formal rules of matrix algebra and may be considered as bridges between abstract vector space entities and their representatives. Also, $|\underline{B}$ is identified with the operation of signal generation and $\underline{\tilde{B}}|$ with signal measurement, processes in the real world which bridge between magnitudes and signals. The use of the mixed symbols is illustrated in Figure 14. Note that all of the connections in the system diagram implied by the expression $\langle G|\underline{B}$ cannot be made simultaneously (unless n replicas of $\langle G|$ are available), so it is interpreted as the application of signal generators $|\underline{B}$ one at a time to the pattern $\langle G|$

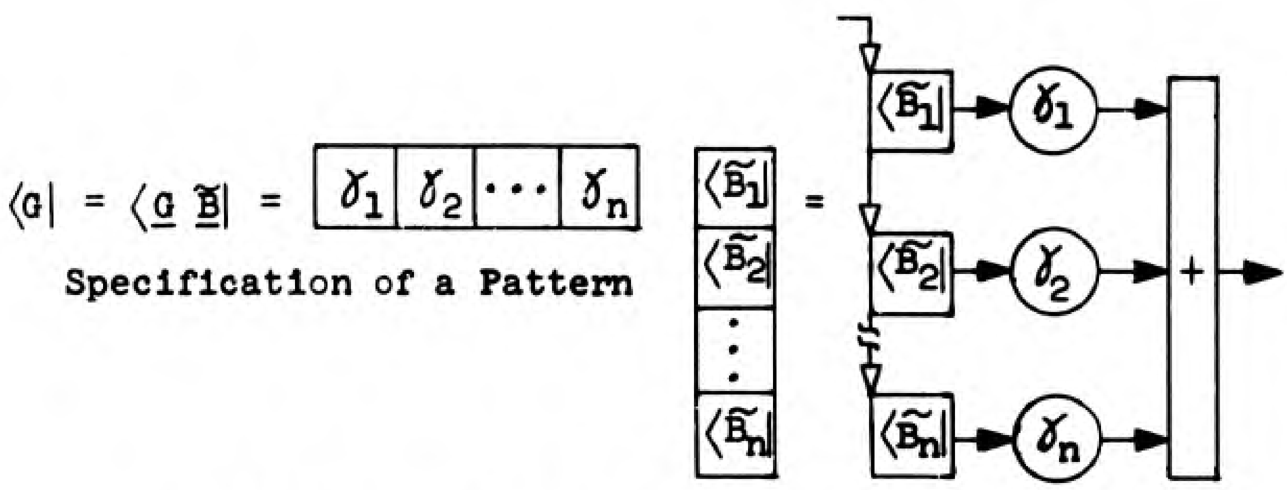
and recording of the results $\langle \underline{G}$.



Measurement of a Signal



Generation of a Signal



Specification of a Pattern

Figure 14 Use of the Mixed Symbols

If we multiply $\underline{\tilde{B}}|$ and $|\underline{B}$ we see than an n x n matrix is obtained with ij<u>th</u> element $\langle \tilde{B}_i | B_j \rangle = \delta_{ij}$ and, therefore,

$$\underline{\tilde{B}}|\underline{B} = \underline{I} \tag{18}$$

The mixed symbols can be generalized. For example, let $|\underline{B}$ indicate some oblique basis in $\mathcal{S}$ and $\underline{D}|$ some oblique basis in $\mathcal{G}$. Then $\underline{D}|\underline{B}$ is the Gram matrix defined by these bases, and

$$\underline{D}|\underline{B} = \underline{I} \tag{19}$$

if and only if the bases are mutually dual.

How shall we interpret the product of $|\underline{B}$ and $\underline{\tilde{B}}|$ in the order given here? Our notation $|\underline{B}\ \underline{\tilde{B}}|$ suggests that the product is an abstract operator which we see to be self-matching and idempotent. Let us see what $|\underline{B}\ \underline{\tilde{B}}|$ does to an arbitrary signal $|F\rangle$.

$$|\underline{B}\ \underline{\tilde{B}}|F\rangle = |\underline{B}\left[\underline{\tilde{B}}|F\rangle\right] = |\underline{B}\ \underline{F}\rangle = |F\rangle = |I|F\rangle$$

Since this is true for all $|F\rangle$, we obtain

$$|\underline{B}\ \underline{\tilde{B}}| = \sum_i |B_i\rangle\langle\tilde{B}_i| = |I| \tag{20}$$

Up to this point we have assumed that the basis $|\underline{B}$ was complete. Now suppose that the operator $|\underline{B}\ \underline{\tilde{B}}|$ formed on the n-dimensional orthonormal basis $|\underline{B}$ and its dual $\underline{\tilde{B}}|$ acts on a signal $|A\rangle$ which lies in some space $\mathcal{S}$ containing the subspace $\mathcal{B}$ spanned by $|\underline{B}$, i.e., Dim $\mathcal{B} \leq$ Dim $\mathcal{S}$ and $\mathcal{B} \subseteq \mathcal{S}$. Clearly, $|\underline{B}\ \underline{\tilde{B}}|A\rangle$ is a signal vector in $\mathcal{B}$, and $|A\rangle - |\underline{B}\ \underline{\tilde{B}}|A\rangle$ lies entirely outside $\mathcal{B}$ in the orthogonal complement of $\mathcal{B}$, i.e., in $\mathcal{B}^\perp$, where $\mathcal{B} \cup \mathcal{B}^\perp = \mathcal{S}$ and every vector in $\mathcal{B}$ is orthogonal to every vector in $\mathcal{B}^\perp$.

Since $|\underline{B}\ \underline{\tilde{B}}|$ is self-matching and idempotent, it is a perpendicular projector (H 1958). That is, if $|A\rangle$ is any signal, then $|\underline{B}\ \underline{\tilde{B}}|A\rangle$ is the perpendicular projection of $|A\rangle$ onto the subspace $\mathcal{B}$ spanned by $|\underline{B}$ as shown in Figure 15.

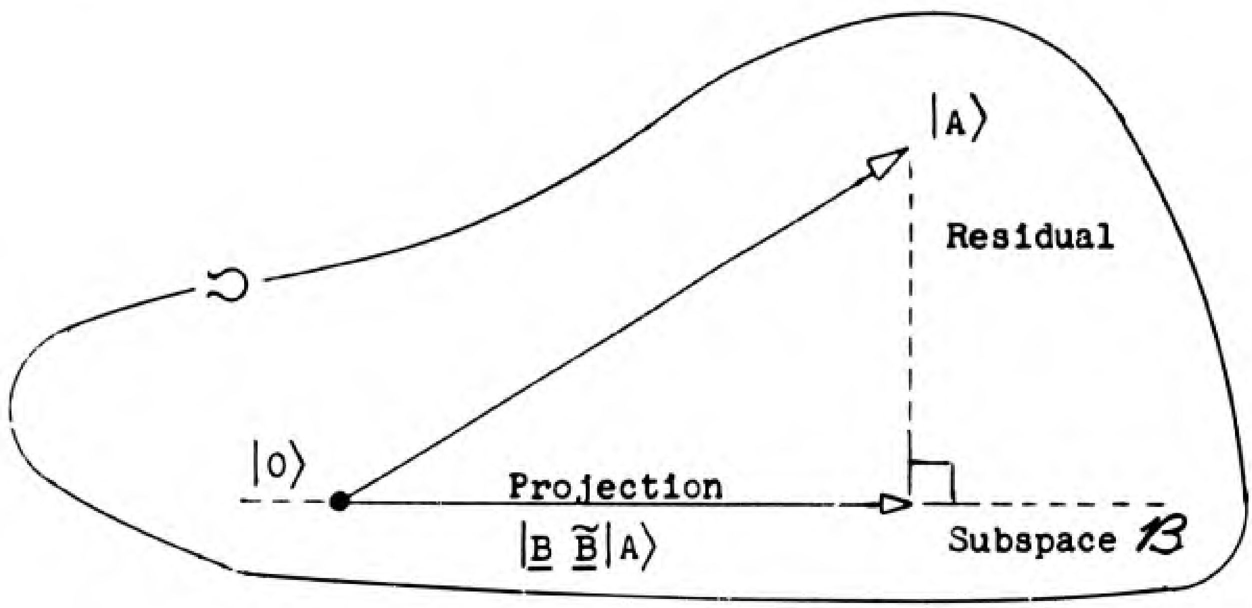


Figure 15 $\quad |\underline{B}\ \tilde{\underline{B}}|$ is a Projector

TM $\quad$ If $|\underline{B}$ is an arbitrary basis and $\underline{D}|$ is its dual, then $|\underline{B}\ \underline{D}|$ is a perpendicular projector.

PF $\quad$ $\underline{D}| = (\tilde{\underline{B}}|\underline{B})^{-1}\tilde{\underline{B}}|$ since this expression satisfies $\underline{D}|\underline{B} = \underline{I}$ and $\underline{D}|$ is uniquely defined by $|\underline{B}$. Then $|\underline{B}\ \underline{D}| = |\underline{B}(\tilde{\underline{B}}|\underline{B})^{-1}\tilde{\underline{B}}|$ is clearly self-matching and idempotent. $\square$

The physical interpretation of $|\underline{B}\ \tilde{\underline{B}}|$ is illustrated in Figure 16, where it is assumed that some signal $|R\rangle$ has two components, a desired signal $|S\rangle$ and some interfering signal $|X\rangle$. Suppose that $|S\rangle$ lies in some subspace $\mathcal{B}$ and that $|X\rangle$, by design or good luck, lies in $\mathcal{B}^{\perp}$. Then, the physical operator corresponding to $|\underline{B}\ \tilde{\underline{B}}|$

will yield $|S\rangle$ in response to $|R\rangle$, and we see that $|\underline{B}\,\underline{\tilde{B}}|$ is a filter. A perpendicular projector could be called a $\mathcal{B}$-pass filter and has been called an "ideal" filter (Zadeh 1952).

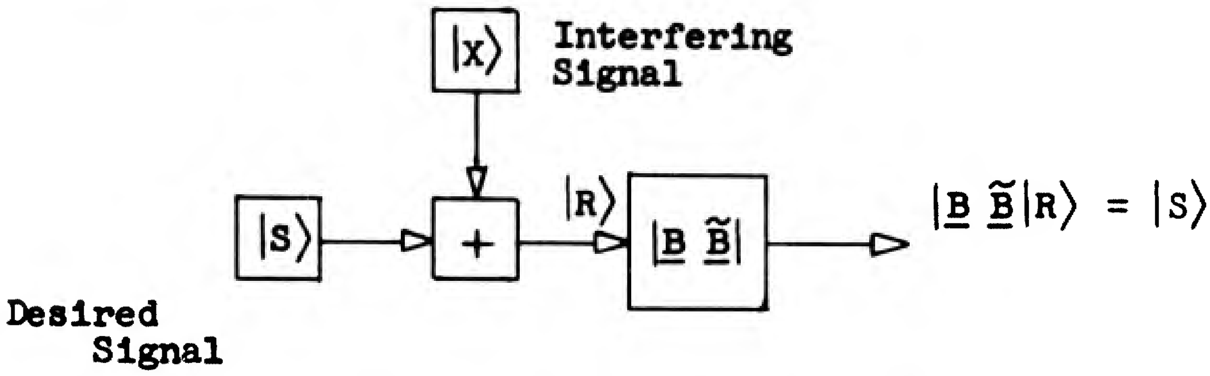


Figure 16 $|\underline{B}\,\underline{\tilde{B}}|$ is a Filter

Measurement of an operator $|A|$ on some orthonormal basis $|\underline{B}$ and its matched dual can be indicated conveniently by the expression $\underline{\tilde{B}}|A|\underline{B}$. If $|\underline{B}$ is oblique and $\underline{D}|$ is its dual, then the measurement of $|A|$ is indicated by $\underline{D}|A|\underline{B}$. In either case, these expressions are equivalent to the matrix $\underline{A}$ representing $|A|$. In summary,

$$\underline{\tilde{B}}|A|\underline{B} = \underline{A} = \left[\langle \tilde{B}_i|A|B_j\rangle\right] \tag{21}$$

for the case of orthonormal bases. This equation indicates that the measurement of an operator $|A|$ is accomplished by connecting each combination of one of the generators $|\underline{B}$ and one of the siftors $\underline{\tilde{B}}|$ to the operator $|A|$ and recording the n^2 measured results as the matrix $\underline{A}$. Conversely, we may specify an operator in terms of its representative on some given pair of complete bases.

$$|A| = |\underline{B}\ \underline{A}\ \underline{\tilde{B}}| \tag{22}$$

Thus, the physical operations of measurement and specification are connected by a sort of duality; one operation maps from the real world to the world of mathematics and the other maps in the opposite direction.

The time scale of the measuring apparatus must be synchronized, at least approximately, with the time scale of the signal generators. Maintenance of synchronism presents measurement problems which are always important and frequently difficult, but we will omit a general discussion of the subject. Suffice it to say that three types of synchronizing techniques are widely used and are well-known: (1) extraction of timing information from the received signals, (2) the use of ancillary signals specially designed to facilitate timing measurements, (3) the transportation of accurately synchronized clocks to the system terminals. The measurement of signal epochs has been discussed in the literature (Young 1963) as well as the design of synchronizing signals (Huffman 1962).

Units of measurement are associated with signals, patterns, operators and bases. Representatives are arrays of scalars without physical units. Consider a case in which some operator $|L|$ maps from a vector space $\mho$ to another vector space Γ. For example, suppose $|L|$ corresponds to a transducer of translational position to voltage. Then the physical units of the various entities are given by

$\vert\underline{B}$ and $\vert F\rangle$ in $\mathcal{B}$	meter
$\vert L\vert$	volt/meter
$\vert\underline{C}$ and $\vert G\rangle = \vert L\vert F\rangle$ in Γ	volt
$\vert\widetilde{L}\vert$	meter/volt
$\widetilde{\underline{B}}\vert$	meter^{-1}
$\widetilde{\underline{C}}\vert$	volt^{-1}

while $\widetilde{\underline{B}}|\underline{B}$, $\widetilde{\underline{C}}|\underline{C}$, $\widetilde{\underline{B}}|F\rangle$, $\widetilde{\underline{C}}|G\rangle$, $\widetilde{\underline{C}}|L|\underline{B}$, $|\widetilde{L}|L|$, $|L|\widetilde{L}|$, $\widetilde{\underline{B}}|\widetilde{L}|\underline{C}$ have no physical units. From this example, we can see a distinction between vectors of one dimension and scalars, i.e., the former have physical units and the latter do not.

In closing our discussion of measurement, we note that several physical and mathematical operations have been shown to be closely related and yet clearly distinguished through the introduction of the mixed symbols $\widetilde{\underline{B}}|$ and $|\underline{B}$. For example, inspection of the expression $|\underline{B}\ \widetilde{\underline{B}}|$ indicates that a perpendicular projector is equivalent to cascading the operations of measurement and generation, as shown in Figure 17. The identity $\widetilde{\underline{B}}|\underline{B}\ \widetilde{\underline{B}}| = \widetilde{\underline{B}}|$ shows that representation is equivalent to cascading perpendicular projection and measurement. Referring to Figure 15, we see that the least-squares approximation by a vector in a given subspace $\mathcal{B}$ to some arbitrarily given vector $|A\rangle$

is obtained by perpendicularly projecting $|A\rangle$ onto $\mathcal{B}$. Throughout this dissertation, the "least-squares" criterion (minimization of squared error or mean squared error) will be employed and the word "best" will refer to this criterion.

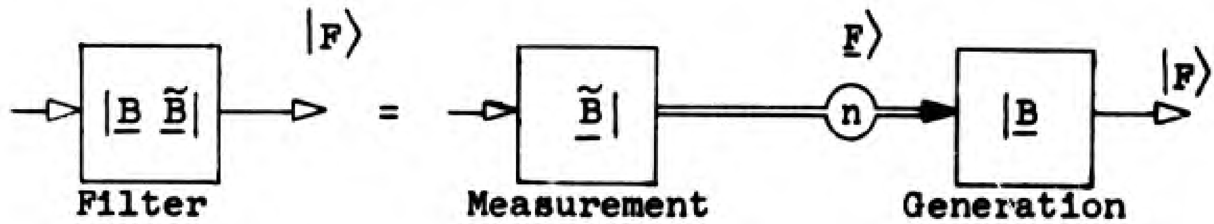


Figure 17
$|\underline{B}\,\tilde{\underline{B}}|$ is a Cascade of Measurement and Generation

<u>Manipulation of Operators and Matrices</u>

A one-to-one correspondence exists between abstract vectors, operators and functionals in a dual pair of n-dimensional linear spaces and their matrix representatives on some pair of dual bases. The mapping in one direction is called "representation"; the inverse mapping is called "abstraction". Our notation and nomenclature are designed to show the isomorphism clearly without allowing it to become confused with identity.

Matrices of three types (1xn, nxn, nx1) arise naturally; general rectangular matrices ($m_1 x m_2$, $m_1 \leq n$, $m_2 \leq n$) arise in representing operators of rank n or less. In summary, all abstract operations symbolized by our vertical bar, i.e., $\langle G|F\rangle$, $\langle G|H|$, $|H|F\rangle$, $|H|K|$ map into matrix multiplications.

As far as multiplication is concerned it is perhaps worth noting that the rule for the matrix product of $m_1 x m_2$ and $m_2 x m_3$ matrices covers all cases. We will not dwell here on the basic rules of matrix algebra, since several excellent works on the subject (Perlis 1952, Marcus 1960) are available. However, some of the more important facts and rules of manipulation will be noted.

We have seen that $|\underline{B}\ \tilde{\underline{B}}|$ is an alternate expression for the unit operator $|I|$ if $|\underline{B}$ is orthonormal and spans $\mathcal{S}$. We are then free to insert or delete $|\underline{B}\ \tilde{\underline{B}}|$ in any chain of abstract symbols. For example,

$$\begin{aligned} \langle G|K|F\rangle &= \langle G|\underline{B}\ \tilde{\underline{B}}|K|\underline{B}\ \tilde{\underline{B}}|F\rangle \\ &= \langle G|\underline{B}\ \tilde{\underline{B}}|K|\underline{B}\ \tilde{\underline{B}}|F\rangle = \langle \underline{G}\ \underline{K}\ \underline{F}\rangle \end{aligned} \tag{23}$$

Thus, we can map conveniently between abstract entities and matrices.

These manipulations can be generalized. Consider the expression$\ldots|J|K|\ldots$and suppose that $|K|\mathcal{S}$ is the subspace $\mathcal{A}$ of $\mathcal{S}$ spanned by $|\underline{A}$. Also, suppose that $\mathcal{G}|J|$ is the subspace $\tilde{\mathcal{Z}}$ of $\mathcal{G}$ spanned by $\tilde{\underline{Z}}|$ and that all bases here are orthonormal. Then

$$\begin{aligned} \ldots|J|K|\ldots &= \ldots|J|\underline{Z}\ \tilde{\underline{Z}}|\underline{A}\ \tilde{\underline{A}}|K|\ldots \\ &= \left[\ldots|J|\underline{Z}\right]\tilde{\underline{Z}}|\underline{A}\left[\tilde{\underline{A}}|K|\ldots\right] \end{aligned} \tag{24}$$

If $\mathcal{A} \subseteq \mathcal{Z}$, then

$$|\underline{Z}\ \tilde{\underline{Z}}|\underline{A}\ \tilde{\underline{A}}| = |\underline{A}\ \tilde{\underline{A}}|\underline{Z}\ \tilde{\underline{Z}}| = |\underline{A}\ \tilde{\underline{A}}| \tag{25}$$

corresponding to a generalization of the cascade of two band-pass filters where one band is completely contained within the other. The generalization of the case of overlapped bands is indicated by

$$|\underline{Z}\ \underline{\tilde{Z}}|\underline{A}\ \underline{\tilde{A}}| = |\underline{A}\ \underline{\tilde{A}}|\underline{Z}\ \underline{\tilde{Z}}| = |\underline{P}\ \underline{\tilde{P}}| \tag{26}$$

which implies that the space spanned by $|\underline{P}$ satisfies

$$\mathcal{P} = \mathcal{A} \cap \mathcal{Z} \tag{27}$$

Generally, however, the matrix $\underline{\tilde{Z}}|\underline{A}$ defined by the two bases must be included between the matrices representing $|J|$ and $|K|$. It is clear from these manipulations how rectangular matrices arise in dealing with operators of less than full rank.

C. Lanczos has recently shown several results concerning rectangular matrices which allow them to be manipulated almost as easily as square matrices of full rank (Lanczos 1958). One result is a generalization of the spectral theorem which will be presented here in operator form.

TM Any operator $|L|$ of rank r on a pair of mutually-dual inner-product spaces can be written in the form

$$|L| = |\underline{B}\ \underline{D}\ \underline{\tilde{A}}| \tag{28}$$

where $\underline{D}$ is a real positive diagonal rxr matrix, $|\underline{B}$ is orthonormal and spans $\mathcal{B} \subseteq \mathcal{D}$, and $\underline{\tilde{A}}|$ is orthonormal and spans $\tilde{\mathcal{A}} \subseteq \mathcal{C}$.

It is easily seen that $|\underline{A}$ consists of the r largest eigenvectors (i.e., corresponding to r largest eigenvalues) of $|\tilde{L}|L|$ and $|\underline{B}$ consists of the r largest eigenvectors of $|L|\tilde{L}|$. Also, $|\tilde{L}|L|$ and $|L|\tilde{L}|$ have r positive real eigenvalues δ_i^2 ; where the diagonal elements of $\underline{D}$ are the δ_i.

Lanczos' theorem provides a convenient introduction to the definition of the pseudo-inverse (Greville 1960). Again, we translate from matrix to operator form.

DN The pseudo-inverse $|L^{\ddagger}|$ of an operator $|L| = |\underline{B}\ \underline{D}\ \tilde{\underline{A}}|$ is given by

$$|L^{\ddagger}| = |\underline{A}\ \underline{D}^{-1}\tilde{\underline{B}}| \tag{29}$$

TM $|L|L^{\ddagger}| = |\underline{B}\ \tilde{\underline{B}}|$

$$|L^{\ddagger}|L| = |\underline{A}\ \tilde{\underline{A}}| \tag{30}$$

The pseudo-inverse is, thus, related to perpendicular projectors and to least-squares approximation and estimation.

An operator of particular importance, along with its matrix representative on any pair of matched dual bases, is the covariance operator $|C|$ of an ensemble $\{|F_k\rangle\}$ of N signals. We assume that the mean signal $\sum_k |F_k\rangle$ is zero, by first subtracting the mean from each $|F_k\rangle$ if necessary. Then, the covariance operator is defined by

$$|C| = \sum_k |F_k\rangle\langle\tilde{F}_k| \tag{31}$$

which is self-matching and non-negative definite. The rank r of $|C|$ is the number of non-zero eigenvalues λ_i of $|C|$, $0 < \lambda_r \leq \ldots \leq \lambda_2 \leq \lambda_1$. Also, the mean energy over the ensemble is

$$\sum_k \langle \tilde{F}_k | F_k \rangle = \sum_i \lambda_i = \mathrm{Tr}|C| = \mathrm{Tr}\ \underline{C} \tag{32}$$

Let $|B$ be the r eigenvectors of $|C|$, i.e.,

$$|C|B_i\rangle = |B_i\rangle \lambda_i \qquad i = 1,2,\ldots,r \tag{33}$$

TM The m normal vectors which best represent an ensemble $\{|F_k\rangle\}$ with covariance $|C|$ are the m largest eigenvectors of $|C|$.

PF Let $m = 1$. $\langle \tilde{B}_1 | B_1 \rangle = 1$. The approximation to $|F_k\rangle$ is $|B_1\rangle\langle \tilde{B}_1 | F_k\rangle$ and contains energy $[\langle \tilde{F}_k | B_1\rangle \langle \tilde{B}_1 |]\ [|B_1\rangle\langle \tilde{B}_1 | F_k\rangle]$ which may be rewritten as $\langle \tilde{B}_1 | F_k\rangle\langle \tilde{F}_k | B_1\rangle$. The mean energy in the approximation is

$$\sum_k \langle \tilde{B}_1 | F_k\rangle\langle F_k | B_1\rangle = \langle \tilde{B}_1 | C | B_1\rangle = \lambda_1$$

Let the tentative choice of $|B_1\rangle$ as the best single basis element be perturbed by adding $|B_i\rangle \phi_i$ and reducing the coefficient of $|B_1\rangle$ from 1 to $\sqrt{1-\phi_i^2}$. The mean energy changes from λ_1 to $\lambda_1(1-\phi_i^2)+\lambda_i\phi_i^2$. Since $\lambda_i \leq \lambda_1$, the change cannot be for the better. Let $m = 2$. Replace $|C|$ with $|C| - |B_1\rangle\lambda_1\langle B_1|$ and repeat, etc. This operator "deflation" procedure also shows that r elements in the basis suffice for exact representation of the given ensemble. □

DN If the covariance operator of an arbitrary ensemble is represented on some matched pair of orthonormal bases by $\underline{C}$, then the correlation matrix of the ensemble is

$$(\mathrm{Dg}\ \underline{C})^{-\frac{1}{2}}\ \underline{C}(\mathrm{Dg}\ \underline{C})^{-\frac{1}{2}} \qquad (34)$$

and the spectrum of the ensemble is the set of eigenvalues of $\underline{C}$. The ensemble is said to be white if $\underline{C}$ is a scalar matrix.

2 - 3 NOTATION AND NOMENCLATURE

In the first two sections of this chapter, we have presented the basic concepts, symbols and terms of our algebraic approach to signal theory, and have said little about comparison with alternate approaches or justification of our approach. Discussion of these comparisons has been reserved primarily for this section. Here we will present some additional details of notation and nomenclature, illustrate the notation and clarify some of the basic concepts by a few simple demonstrations, justify our definition of inner product and related concepts, and discuss some of the advantages of our approach.

Notation Requirements

Reflection on the notation introduced here and comparison with the classical notation and perhaps others are necessary steps prior to listing requirements on a notation system for signal theory. One should also consider the importance of such operations as measurement, least-squares approximation, projection, change of basis, specification, etc., and to the need for distinction between these and other closely related ideas. Whether we are interested solely in analog or in digital instrumentation, or, on the other hand, are concerned with both types as well as hybrids, might affect our thinking here. Some investigators may be influenced by preconceived notions as to whether the world of signals is "really" continuous or not. After due consideration to these matters and others, we conclude that the following is a reasonable list of requirements for a signal theory notation.

The notation and nomenclature ought to:

(1) Distinguish the abstract entities from their representatives.

(2) Distinguish signals, operators and patterns.

(3) Distinguish representatives of signals, operators and patterns.

(4) Facilitate thinking at the systems level in the conceptual phase of problem formulation.

(5) Provide for convenient manipulation in the computational phase of problem solution.

(6) Provide for a variety of representations, some of general utility, others for specific applications.

(7) Provide for convenient description of the important processes of measurement and specification.

(8) Be simple and compact, yet rigorous.

(9) Suggest theorems and facilitate memory of theorems.

(10) Extend conveniently to tensor product spaces.

(11) Relate conveniently to system diagrams.

(12) Preserve the sequence of physical operations.

In short, what is needed is a notation and approach to signal theory which provide an inventor's language suitable for dealing with complex signal-processing problems. Such problems require that analysis and simulation be used in the early stages of development rather than physical experimentation, because of the high cost of the latter. Thus, the notation must be such that it is easy to manipulate rigorously.

The classical notation of linear algebra and analysis fails to meet several of these requirements, and the more common notation of differential and integral equations fails to meet any of them. The only notation in the literature which comes close to being satisfactory is Dirac notation (Dirac 1958). Thus, we follow the suggestion of D.C. Lai and adopt Dirac notation (Lai 1960), with numerous modifications due principally to W.H. Huggins, D.C. Lai and D.C. Ross. The only discrepancy between our modified Dirac notation and one we would presently consider ideal is that it cascades symbols for operator-on-operator from right to left while engineers (at least, in the Occident) draw systems diagrams with the cause-effect stimulus proceeding from left to right. This discrepancy is one which stems from the notation of elementary mathematics. Other difficulties are traceable to the same source (Menger 1955). The discrepancy is maintained here for lack of time to reverse the correspondence: pattern-bra and signal-ket. The more nearly ideal notation has been used in a recent paper (Huggins 1963) and will be employed by the writer in future papers on signal theory and application.

Additional Details on the Notation

The Dirac bra-c-ket $\langle G|F\rangle$ is employed here to indicate the scalar result of a process of measurement involving the interaction of two vectors of different types, i.e., the evaluation of a linear functional $\langle G|$ at some vector $|F\rangle$. As we have noted before, we associate $\langle G|$ with a measuring device and $|F\rangle$ with a device to be measured. In this dissertation, we restrict the spaces to be inner-product spaces. However, the measurement process along with most of our basic concepts apply also to normed linear spaces where the norm might be something other than inner product.

Common type fonts used to distinguish among abstract entities and representatives on various bases are shown in Table 1. Boldface would replace underlines in printed text.

Note that the symbols for representatives do not use the vertical bar (or bars). We see that the bar is connected with the abstract level. The mixed symbols, such as $|\underline{B}$, act as bridges between the levels of abstraction and representation; thus, they are designed so that one side of the symbol suggests abstraction and the other side suggests representation. Script capitals sans bar, bra or ket denote subspaces.

Table 1 Use of Various Type Fonts

Entity	Font	Examples	Classical Counterpart
Scalar	Greek lower case	ρ	ρ
Abstract entity	Roman capital, with vertical bar(s)	$\|R\rangle$	None
Representative on finite basis	Roman capital, underlined	$\underline{R}\rangle$	None
Discrete time representative	Roman lower case, underlined	$\underline{r}\rangle$	$r(t_k)$ or r_k
Continuous time representative	Roman lower case	$r\rangle$	$r(t)$
Cont. frequency representative	Roman capital	$R\rangle$	$R(s)$ or $R(\omega)$

It is rather curious to see that we have selected from mathematical constructs of various levels of abstraction the most abstract entities for identification with the real-world objects with which we are concerned. Therefore, one must be careful to avoid any confusion which might result from the puzzling interchange of roles of the words "abstract" and "concrete" which the correspondence entails. A similar situation occurs in the foundation of mathematics where the term "real number" is used to denote an entity abstracted several levels from the reality experienced by most people.

Classical Notation Systems

Some of the difficulties with the standard notation of linear algebra may be seen by reference to pages 64-68, 78-83, 130-133 of Halmos' book, where he discusses several confusing details pertaining to indices, adjoints and conjugation (H 1958). Three quotations from the places cited are of particular interest:

p 66 "It is a perversity...of nature,.../that/ somewhere in the process of passing from vectors to coordinates the indices turn around."

p 83 "To express this whole tangle of ideas..."

p 132 "One way, for example, of avoiding the unpleasantness of conjugation..."

It would seem that when a pure mathematician of the stature of Halmos uses such words as "perversity", "tangle" and "unpleasantness" that we should be forewarned that the notation contains subtle difficulties which ought to be removed if possible. We will see that our approach eliminates the "perversity" of indices completely, separates the "tangle" of ideas somewhat more clearly, and reduces the "unpleasantness" of conjugation, as compared with the standard notation of modern linear algebra.

It might be argued that the classical approach to signal theory based on integral and differential equations in which $f(t)$ is central is quite useful, adequate, concrete and "natural" and that time and effort spent in developing and studying other notation systems is wasted. It is certainly true that $f(t)$ and its associated paraphernalia has been and will continue to be useful in signal theory. In the sense that $f(t)$ is a representative, perhaps a mathematician would regard it as concrete, but in the sense that it refers to a vector in a space of uncountably infinite dimensionality, it is quite abstract. We have already noted that the $f(t)$ approach is not adequate for our requirements. The feeling that $f(t)$ is a "natural" way to describe signals is related to the fact that physicists and engineers have, following Newton and Maxwell, employed continuous representations of time, frequency, position, etc., almost exclusively. Until the recent advent of adequate digital techniques, there was no other choice.

We might profit by considering just what is meant by the symbol $f(t)$. Is it the value of some function at some particular instant? Or does it refer to the characteristics of the mapping from the domain to the range of the function? Or does it mean the entire set of values of the function corresponding to some given set of instants? The difficulty is that $f(t)$ may denote any of these three ideas, and perhaps others as well (Menger 1955). But $f\rangle$ has only one meaning, i.e., the third meaning given above.

Scientists and engineers are usually impatient with suggested changes in well-established notation, and properly so. However, we are suggesting here not merely a change in notation, but also a revision in the basic mathematical approach, a revision which is needed to clarify and simplify our conceptual basis for signal theory, a revision which is closely related to and will be facilitated by the revolution in the teaching of engineering mathematics which is already started (Stone 1961, CUPM 1962). This revolution will inevitably thrust linear algebra into a dominant role in our thinking and will include the classical approach as an important adjunct. Thus, the notation of signal theory and of engineering generally is going to be overhauled extensively, and we need to be concerned with this problem now.

However, all arguments as to the usefulness of any new notation and approach are to no avail, unless utility can be demonstrated by application to real-world problems. In Chapter 4, the results of an application study using our modified Dirac notation will be presented, and the reader will be able to judge its utility for himself. For the moment, note that expressions of the form H L J are represented classically by double integrals of the form

$$\int H(\tau,\lambda)\ d\lambda\ L(\lambda,\phi)\ d\phi\ J(\phi,\sigma)$$

where the limits are $-\infty$ and $+\infty$ for each of the two dummy variables. An equation with several terms of this complexity can hardly be regarded as a statement in a language suitable for use by systems engineers and inventors.

Duality and Matching

The first statement that needs to be made about duality and matching is that they are not the same. Duality applies to bases and to spaces, while matching applies to pairs of elements, i.e., vectors, functionals and/or operators. For example, $|F\rangle$ has a match but not a dual.

From a mathematical point of view, the "dual of" some given basis or the "match of" some given vector uniquely defines some basis or vector. From an engineering point of view, these terms do not even approach the definition of unique entities, a statement which can be extended to all of our abstract symbols. In order to make this point clear, consider an example in 2-space where we assume that we are given two actual siftors $\langle D_1|$ and $\langle D_2|$ as specific operating pieces of equipment. Now consider some signal generator $|F\rangle$ which produces scalars $\langle D_1|F\rangle$ and $\langle D_2|F\rangle$ at the outputs of the siftors. Consider the waveform of $|F\rangle$, i.e., its representative $f\rangle$ on the continuous time basis. Is there any other waveform which would yield the same two outputs $\langle D_1|F\rangle$ and $\langle D_2|F\rangle$ as the given generator produces? The answer is affirmative; in fact, there are an infinite number of such waveforms. However, there is no way of distinguishing among this set of equivalent generators through the use of $\langle D_1|$ and $\langle D_2|$, so they are all labelled $|F\rangle$ as far as their use in our 2-space is concerned. This set of equivalent signal generators can be progressively distinguished only by considering them as elements in spaces of progressively greater dimension. Similar remarks may also be made about patterns and operators.

We use "adjoint of" differently than do Halmos and other authors of texts in linear algebra. The reason for this difference stems partly from our use of Dirac notation and partly from the fact that engineers may regard the same filter either as operating on an input signal to yield a new signal or as operating on a siftor to yield a new siftor. Note that the definition of adjoint used by Halmos in his §44, i.e., $[Ax,y] = [x,A'y]$, maps over into our notation in the form illustrated in Figure 18. In other words, his definition can be paraphrased in our terms: A system may be divided arbitrarily into two parts, i.e., generator and siftor. This view of the situation is reconciled with the standard mathematical approach by noting that the same equipment cascade appears in the two diagrams (engineer's view) but the vectors and functionals are different (mathematician's view).

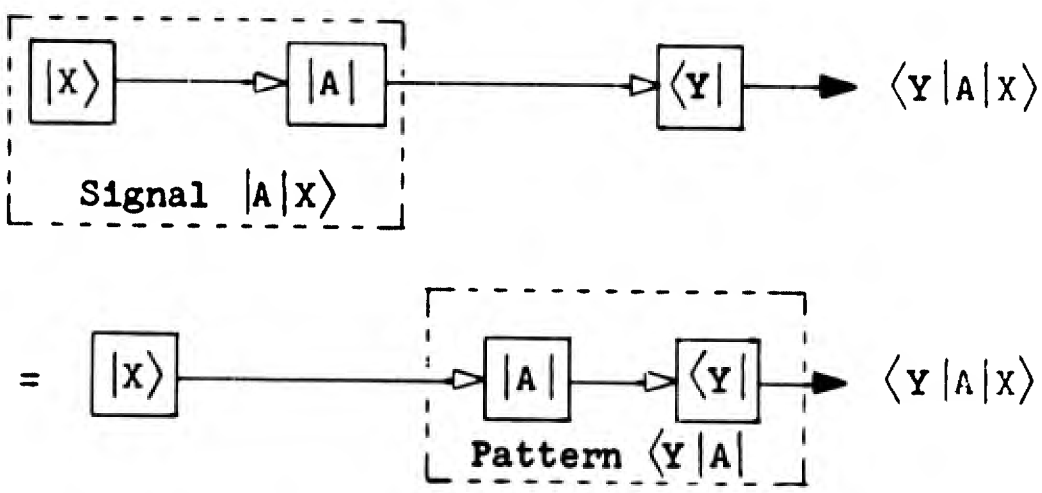


Figure 18 The Signal-Pattern Boundary

<u>Mixed Symbols</u>

The mixed symbols $|\underline{B}$ and $\underline{D}|$, or $|\underline{B}$ and $\tilde{\underline{B}}|$ in the case of orthonormal bases, are useful symbols indeed. They are extremely compact, mnemonic, consistent with the rules of matrix algebra, and closely related to important mathematical and physical operations such as projection and measurement. Their utility derives from their role as bridges between abstract entities and representatives, which we place in correspondence with the roles of measurement and specification in the real world.

Changing from one representation to another involves certain confusing details in the standard notation. The mixed symbols avoid this confusion. Consider a general linear space $\mathcal{S}$ with basis $|\underline{B}$, its dual $\underline{D}|$, a basis $|\underline{A}$ and its dual $\underline{C}|$. Then

$$\underline{D}|\underline{B} = \underline{I} = \underline{C}|\underline{A}$$

$$|\underline{B}\ \underline{D}| = |I| = |\underline{A}\ \underline{C}|$$

We introduce a signal $|F\rangle$ and an operator $|H|$ which are represented by $\underline{F}\rangle$, $\underline{H}$ on $|\underline{B}$ and its dual, and by $\underline{G}\rangle$ and $\underline{K}$ on $|\underline{A}$ and its dual. Then

$$\underline{D}|F\rangle = \underline{F}\rangle$$

$$\underline{C}|F\rangle = \underline{G}\rangle$$

$$\underline{G}\rangle = \underline{C}|I|F\rangle = \underline{C}|\underline{B}\ \underline{D}|F\rangle = \underline{C}|\underline{B}\ \underline{F}\rangle$$

Thus, $\underline{C}|\underline{B}$ is a matrix which premultiplies representatives of signals on $|\underline{B}$ to yield representatives of signals on $|\underline{A}$. A matrix such as $\underline{C}|\underline{B}$ is a <u>basis-change matrix</u>, and the result of a change of basis is a <u>transform</u>.

The representatives of operators are also manipulated easily. For example,

$$\underline{D}|H|\underline{B} = \underline{H}$$

$$\underline{C}|H|\underline{A} = \underline{K}$$

$$\underline{K} = \underline{C}|I|H|I|\underline{A} = \underline{C}|\underline{B}\ \underline{D}|H|\underline{B}\ \underline{D}|\underline{A}$$

$$= \underline{C}|\underline{B}\ \underline{H}\ \underline{D}|\underline{A} = (\underline{C}|\underline{B})\underline{H}(\underline{C}|\underline{B})^{-1}$$

which is, of course, the familiar similarity transformation. The proof that $\underline{D}|\underline{A}$ and $\underline{C}|\underline{B}$ are mutually inverse is seen by multiplying them in either order.

$$\underline{D}|\underline{A}\ \underline{C}|\underline{B} = \underline{D}|I|\underline{B} = \underline{D}|\underline{B} = \underline{I}$$

$$\underline{C}|\underline{B}\ \underline{D}|\underline{A} = \underline{C}|I|\underline{A} = \underline{C}|\underline{A} = \underline{I}$$

The adjoint or conjugate transpose of a matrix combined with our basic notation scheme implies the definition of the match of an operator as shown in Figure 6. This choice agrees with the well-known "matched filter" idea in signal theory and with the "adjoint matrix" of modern algebra. Note that "adjoint" has nothing directly to do with cofactors and inverses as in the older mathematical literature.

We close the discussion of duality and matching with some additional theorems presented in terms of representatives.

TM $\widetilde{\underline{A}\ \underline{Z}} = \tilde{\underline{Z}}\ \tilde{\underline{A}}$

TM $\widetilde{[\alpha_{11}]} = [\alpha^{*}_{11}]$

This last theorem justifies defining the match of a scalar as the conjugate of the scalar.

TM $\underline{\tilde{L}}\,\underline{L}$ and $\underline{L}\,\underline{\tilde{L}}$ are self-adjoint for all $\underline{L}$.

TM $\underline{U}$ represents a unitary operator on orthonormal bases if and only if $\underline{U}^{-1} = \underline{\tilde{U}}$.

Inter-Product and Intra-Product

We have emphasized the interaction of linear functionals and vectors in preference to the classical inner product, i.e., inter-product rather than intra-product. The former was defined in terms of the latter in order to use the standard literature as a background. It is also possible and perhaps desirable to rearrange the order of definitions and theorems so as to develop the inter-product directly.

The system of notation and nomenclature described here works out so well that one is moved to inquire why the concept of intra-product of two vectors in the same space is needed at all. A signal analyst might feel that he needed the idea of intra-product to "correlate" two signals $|F\rangle$ and $|G\rangle$, i.e., to compute the amount of energy in that part of $|F\rangle$ which was "like" $|G\rangle$. But it is just as meaningful to say that one wants to measure one signal with the siftor matched to the other as indicated in Figure 19.

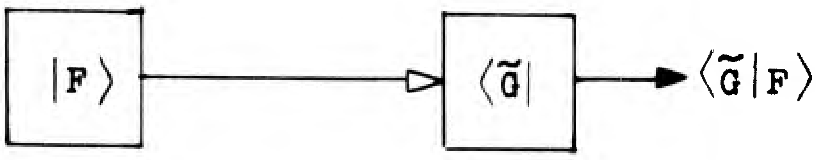


Figure 19 Correlation of $|F\rangle$ and $|G\rangle$

Also, consider a common example from elementary physics which may have been the original motivation for the concept of inner product, i.e., the work done by a force on a body displaced by the force. Here, it seems quite odd to view the situation as the interaction of two vectors of the same kind!

CHAPTER THREE

TENSOR PRODUCT SPACES

Tensor products of finite-dimensional vector spaces are discussed in this chapter. Following the pattern of the preceding chapter, the approach taken is that of modern algebra. Although the older literature (Levi-Civita 1927) may be more familiar to engineers, it is unsatisfactory for our use since it tends to regard tensor algebra as merely an introductory step toward tensor analysis and skips over tensor product spaces completely. In order to make the presentation as simple as possible, we follow Lichnerowicz by defining tensor products in terms of their important properties (Lichnerowicz 1947, 1962). These properties of tensor products appear as theorems rather than as axioms in the more general approach taken in the most recent literature in algebra, some of which may be of interest to mathematically inclined engineers (Chevalley 1956, Jacobson 1953, B 1958, H 1958, MS&M 1963).

The highly abstract approach to tensor products used by the modern algebraist allows for many possible tensor products of two given spaces and then shows that they are naturally isomorphic and that the term "the tensor product" symbolized by "$\otimes$" is justified. Rather than introduce confusion through unnecessary generality, we will always deal with a single tensor product and with a single inner product induced naturally from the two vector spaces on which the tensor product space is erected. In applications, we tie our definitions to some one basis in each space — a restriction which need not be troublesome. Despite all efforts to make our theory independent of choice of bases, there is always one basis which necessarily underlies any particular problem in signal theory — a finite time basis referred to some epoch.

In the interest of brevity, we will depart from the practice of introducing the abstract entities concurrently with their counterparts in the real world. The abstract entities associated with tensor product spaces will be discussed in the first section, representations in the second, and a practical application in the third section of the chapter. Before discussing tensor products in abstract terms, it seems desirable to give a preview of the discussion on repre.entatives so that any reader encountering tensor products for the first time can have some example in mind as the abstract entities are introduced. For the moment, it is sufficient to note two facts. First, the tensor product operation on two abstract entities is represented by the Kronecker product of the corresponding matrices. Second, the Kronecker product of two matrices, for example,

$$
\begin{array}{c|c|c|}
 & 1 & 2 \\ \hline
1 & 2 & -1 \\ \hline
\end{array}
\otimes
\begin{array}{c|c|c|}
 & 1 & 2 \\ \hline
1 & 4 & 3 \\ \hline
2 & -2 & 5 \\ \hline
3 & 0 & -3 \\ \hline
\end{array}
=
\begin{array}{c|c|c|c|c|}
 & (1,1) & (1,2) & (2,1) & (2,2) \\ \hline
(1,1) & 8 & 6 & -4 & -3 \\ \hline
(1,2) & -4 & 10 & 2 & -5 \\ \hline
(1,3) & 0 & -6 & 0 & 3 \\ \hline
\end{array}
$$

is obtained by replacing each element of the first matrix with that element multiplied by the second matrix.

3 - 1 TENSOR PRODUCTS OF VECTORS AND OPERATORS

A tensor product (including tensor as a special case) is an abstract entity and is represented by a matrix. We regard tensor products as <u>invariant</u> under change of basis, just as in the case of vectors. The terms "covariant" and "contravariant" refer to variation directly or inversely with the choice of basis and ought to be applied to representatives and not to the abstract entities. This view agrees with our choice of abstract algebraic structures as appropriate mathematical models of physical generators, siftors and transducers which are unaffected by how we choose to describe their performance.

Tensor Product of Vectors

The formal definition of tensor product space is based on two vector spaces $\backsim$ and Γ over the same field Ω . The real-world entities corresponding to the elements of $\mathcal{S}$ and Γ need not have the same physical units and $n = \text{Dim}\, \mathcal{S}$ need not be equal to $p = \text{Dim}\, \Gamma$. On the other hand, Γ may be quite closely related to $\mathcal{S}$, e.g., it may be identical to $\mathcal{S}$ or to $\mathcal{C}$.

Let $|X\rangle$, $|X_1\rangle$, $|X_2\rangle$ denote arbitrary vectors in $\mathcal{S}$ and let $|Y\rangle$, $|Y_1\rangle$, $|Y_2\rangle$ denote arbitrary vectors in Γ. Let α, β be any scalars in Ω. Let $\{|B_i\rangle : i = 1, \ldots, n\}$ be any basis in $\mathcal{S}$ and let $\{|C_j\rangle : j = 1, \ldots, p\}$ be any basis in Γ. Let $\mathcal{S} \otimes \Gamma$ be a vector space. Let $|X\rangle \otimes |Y\rangle$ denote any element of $\mathcal{S} \otimes \Gamma$. We may then write the formal definition of tensor product as follows.

DN Let every ordered pair of vectors $|X\rangle$ in $\mathcal{S}$ and $|Y\rangle$ in Γ map into some element $|X\rangle \otimes |Y\rangle$ in $\mathcal{S} \otimes \Gamma$ and let the law of correspondence satisfy the following five conditions.

$$|X\rangle \otimes \big[|Y_1\rangle + |Y_2\rangle\big] = |X\rangle \otimes |Y_1\rangle + |X\rangle \otimes |Y_2\rangle \tag{35}$$

$$|X\rangle \otimes \big[|Y\rangle\beta\big] = \big[|X\rangle \otimes |Y\rangle\big]\beta \tag{36}$$

$$\big[|X_1\rangle + |X_2\rangle\big] \otimes |Y\rangle = |X_1\rangle \otimes |Y\rangle + |X_2\rangle \otimes |Y\rangle \tag{37}$$

$$\big[|X\rangle\beta\big] \otimes |Y\rangle = \big[|X\rangle \otimes |Y\rangle\big]\beta \tag{38}$$

The np elements $\{|B_i\rangle \otimes |C_j\rangle\}$ form a basis in $\mathcal{S} \otimes \Gamma$. (39)

Then $\mathcal{S} \otimes \Gamma$ is called the <u>tensor product of the spaces</u> $\mathcal{S}$ and Γ, and $|X\rangle \otimes |Y\rangle$ is called the <u>tensor product of the vectors</u> $|X\rangle$ and $|Y\rangle$ (Licherowicz 1947).

These properties may be paraphrased. Equation (35) states that tensor product is <u>additive</u> in the right factor, (36) states that tensor product is <u>homogeneous</u> in the right factor, and the two properties are equivalent to the single statement that tensor product is <u>linear</u> in the right factor. Similarly, for the left factor and Equations (37) and (38). Note that we need not worry about conjugate linearity in connection with tensor product, even if Ω is complex. The linearity of each of the two factors may be combined into the single property of <u>bilinearity</u>. We may also introduce the symbol $|\underline{B} \otimes |\underline{C}$ to stand for the set of np possible tensor products of the elements of $|\underline{B}$ and $|\underline{C}$. Thus, we may abbreviate the definition as follows.

DN Tensor product maps every ordered pair $|X\rangle$ in $\supset$ and $|Y\rangle$ in Γ into an element $|X\rangle \otimes |Y\rangle$ in $\supset \otimes \Gamma$ such that:

(1) the mapping is bilinear, and

(2) $|\underline{B} \otimes |\underline{C}$ is a basis in $\supset \otimes \Gamma$.

An immediate consequence of the definition is that

$$\text{Dim } \supset \otimes \Gamma = np \qquad (40)$$

Suppose that $|X\rangle$ and $|Y\rangle$ are given by

$$|X\rangle \quad \sum_i |B_i\rangle \beta_i$$

$$|Y\rangle \quad \sum_j |C_j\rangle \gamma_j$$

Then, the tensor product of $|X\rangle$ and $|Y\rangle$ is given by

$$|X\rangle \otimes |Y\rangle = \left[\sum_i |B_i\rangle \beta_i\right] \otimes \left[\sum_j |C_j\rangle \gamma_j\right]$$
$$= \sum_{i,j} \left[|B_i\rangle \otimes |C_j\rangle\right] \beta_i \gamma_j \qquad (41)$$

Thus, the (i,j)th component of $|X\rangle \otimes |Y\rangle$ on the given basis (in $\supset \otimes \Gamma$) is $\left[|B_i\rangle \otimes |C_j\rangle\right] \beta_i \gamma_j$ and the (i,j)th coordinate is $\beta_i \gamma_j$.

Consider the special case $\Gamma = \supset$ and $n = 2 = p$. Then

$$|X\rangle \otimes |Y\rangle = \left[|B_1\rangle \otimes |B_1\rangle\right] \beta_1 \gamma_1 + \left[|B_1\rangle \otimes |B_2\rangle\right] \beta_1 \gamma_2 + \left[|B_2\rangle \otimes |B_1\rangle\right] \beta_2 \gamma_1 + \left[|B_2\rangle \otimes |B_2\rangle\right] \beta_2 \gamma_2 \qquad (42)$$

If we form the tensor product with the factors in reverse order, we have

$$|Y\rangle \otimes |X\rangle = \left[|B_1\rangle \otimes |B_1\rangle\right] \gamma_1 \beta_1 + \left[|B_1\rangle \otimes |B_2\rangle\right] \gamma_1 \beta_2 + \left[|B_2\rangle \otimes |B_1\rangle\right] \gamma_2 \beta_1 + \left[|B_2\rangle \otimes |B_2\rangle\right] \gamma_2 \beta_2 \qquad (43)$$

Inspection of Equations (42) and (43) shows that, in general,

$$|X\rangle \otimes |Y\rangle \neq |Y\rangle \otimes |X\rangle \qquad (44)$$

Note that $|X\rangle \otimes |Y\rangle$ and $|Y\rangle \otimes |X\rangle$ cannot even be compared if $\Gamma \neq \supset$ since $|X\rangle \otimes |Y\rangle$ belongs to $\supset \otimes \Gamma$ and $|Y\rangle \otimes |X\rangle$ belongs to $\Gamma \otimes \supset$.

In the application problem considered in this dissertation, the number of vector spaces over Ω on which the tensor product space is formed will not exceed two. Thus, we need not be concerned here with associativity of the tensor product. It is interesting to note, however, that there is a canonical isomorphism (B 1958) which links any two tensor product spaces based on the same finite set of vector spaces (MS&M 1963). In this sense, the tensor product of spaces is commutative and associative.

Duality and Tensor Products

At several points in the abstract development of the tensor product concepts, we must say "is canonically isomorphic to" rather than "is" or "equals" if we wish to be strictly rigorous. For example, Bourbaki shows that the tensor product of $\mathcal{C}$ and $\urcorner$ (i.e., the duals of Ↄ and $\ulcorner$) is canonically isomorphic to the dual of $\text{Ↄ} \otimes \ulcorner$ and that

$$\left[\langle W| \otimes \langle Z|\right] \; \left[|X\rangle \otimes |Y\rangle\right] = \langle W|X\rangle \langle Z|Y\rangle \tag{45}$$

where $\langle W| \in \mathcal{C}$, $\langle Z| \in \urcorner$, $|X\rangle \in \text{Ↄ}$ and $|Y\rangle \in \ulcorner$ (B 1958 §105). In the interest of convenience, we will follow the example of the mathematicians and write "is" and "equals" even when these expressions are not strictly correct (MS&M 1963 p 525). Thus,

$$\mathcal{C} \otimes \urcorner = \text{the dual of } \text{Ↄ} \otimes \ulcorner \tag{46}$$

and conversely.

Referring to Equations (45) and (46) we see that $\langle W| \otimes \langle Z|$ is to be regarded in two ways: (1) the tensor product of two linear functionals, one in $\mathcal{C}$ and one in

$\daleth$, and (2) a linear functional defined on $\supset \otimes \Gamma$. Note that no apology is needed for the "=" in Equation (45) since the elements on both sides belong in the field Ω.

Tensor Product of Operators

Consider any operator $|K|$ on $\supset$ and $|L|$ on Γ, i.e., $|K|$ maps $|X\rangle$ into $|K|X\rangle$ and $|L|$ maps $|Y\rangle$ into $|L|Y\rangle$. The operator on $\supset \otimes \Gamma$ which maps the tensor product of $|X\rangle$ and $|Y\rangle$ into the tensor product of $|K|X\rangle$ and $|L|Y\rangle$ is denoted $|K| \otimes |L|$, the tensor product of $|K|$ and $|L|$. That is,

$$|K|X\rangle \otimes |L|Y\rangle = \big[|K| \otimes |L|\big] \big[|X\rangle \otimes |Y\rangle\big] \qquad (47)$$

Bourbaki shows the following important result

$$|K_2|K_1| \otimes |L_2|L_1| = \big[|K_2| \otimes |L_2|\big] \big[|K_1| \otimes |L_1|\big] \qquad (48)$$

where the $|K_i|$ are operators on $\supset$ and the $|L_i|$ are operators on Γ and $i = 1, 2$ (B 1958 §1.4). This result extends to operators mapping from one space to another.

Equations (47) and (48) may be described in block diagram form as shown in Figure 20. For the moment, no physical interpretation is intended for $\otimes$; this point will be considered in the closing section of this chapter. The block diagram form of presentation of the basic properties of tensor product is very useful for systems engineers, since such diagrams form the common language of the profession. The fact that the sequence of operators in the diagram is opposite to that in the equations has been noted before. This particular confusion is standard and can easily be purged by making signals correspond to rows and vectors in $\mathcal{C}$ and making patterns correspond to columns and vectors in Ↄ .

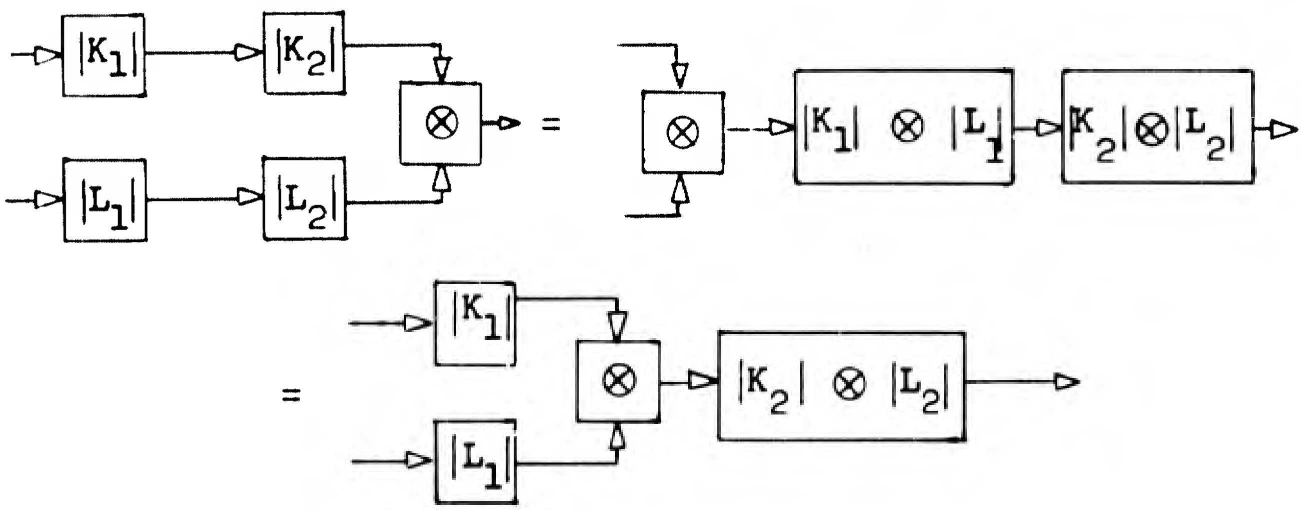


Figure 20 Tensor Product of Operators

The significant fact about the use of tensor product spaces in engineering systems which is made clear by Figure 20 is that operators can be moved through the ⊗ junction by simple rules reminiscent of those used in familiar linear system diagrams. Thus, the engineer is able to extend his intuitive understanding of linear systems in a simple yet rigorous manner. Consideration of diagrams similar to those in Figure 20 leads to the recognition of a class of <u>multi-linear</u> systems intermediate between linear systems and the most general non-linear systems.

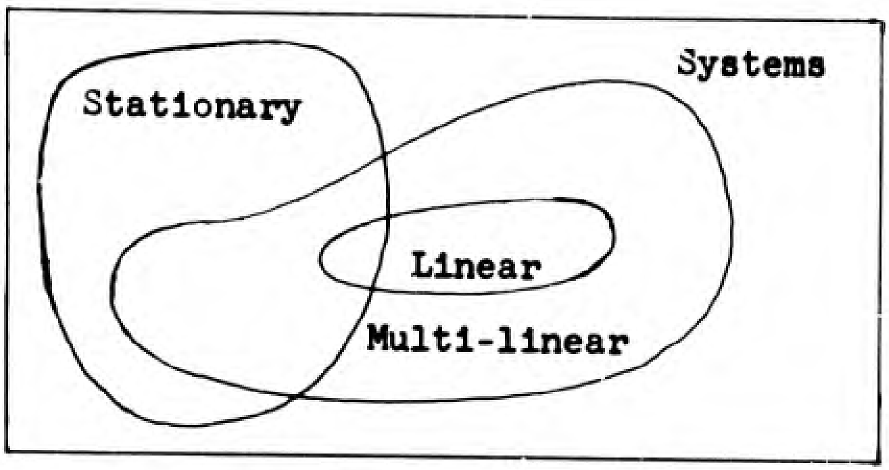


Figure 21 Classification of Systems

Tensor Product of Inner-Product Spaces

If $\subsetneq$ and $\supsetneq$ are inner-product spaces and so are $\urcorner$ and $\ulcorner$, then so are $\subsetneq \otimes \urcorner$ and $\supsetneq \otimes \ulcorner$. This fact follows from the bilinearity (recall that "inner product" is interpreted as "inter-product" here), Hermitian symmetry and non-negativeness of inner product. Bilinearity follows directly from the definition of inter-product between $\subsetneq \otimes \urcorner$ and $\supsetneq \otimes \ulcorner$. The other two properties are established by the following two theorems.

TM If $\langle W|X\rangle = \langle X|W\rangle^*$ and $\langle Z|Y\rangle = \langle Y|Z\rangle^*$, then

$$\left[\langle W| \otimes \langle Z|\right] \left[|X\rangle \otimes |Y\rangle\right] = \left\{\overset{\frown}{\left[|X\rangle \otimes |Y\rangle\right]} \; \overset{\frown}{\left[\langle W| \otimes \langle Z|\right]}\right\}^*$$

PF $\left[\langle W| \otimes \langle Z|\right] \left[|X\rangle \otimes |Y\rangle\right] = \langle W|X\rangle \langle Z|Y\rangle = \langle \tilde{X}|\tilde{W}\rangle^* \langle \tilde{Y}|\tilde{Z}\rangle^*$

$$= \langle \tilde{X}|\tilde{W}\rangle \langle \tilde{Y}|\tilde{Z}\rangle^* = \left[\langle \tilde{X}| \otimes \langle \tilde{Y}|\right] \left[|\tilde{W}\rangle \otimes |\tilde{Z}\rangle\right]^*$$

The last step follows from the theorems

$$\overset{\frown}{|X\rangle \otimes |Y\rangle} = \langle \tilde{X}| \otimes \langle \tilde{Y}| \qquad (49)$$

$$\widetilde{\langle W| \otimes \langle Z|} = |\tilde{W}\rangle \otimes |\tilde{Z}\rangle \tag{50}$$

which may be demonstrated simply via representations which are discussed later. □

TM If $0 \leq \langle\tilde{X}|X\rangle$ and $0 \leq \langle\tilde{Y}|Y\rangle$, then $0 \leq \widetilde{\left[|X\rangle \otimes |Y\rangle\right]} \left[|X\rangle \otimes |Y\rangle\right]$.

PF $\widetilde{\left[|X\rangle \otimes |Y\rangle\right]} \left[|X\rangle \otimes |Y\rangle\right] = \left[\langle\tilde{X}| \otimes \langle\tilde{Y}|\right] \left[|X\rangle \otimes |Y\rangle\right]$

$$= \langle\tilde{X}|X\rangle \langle\tilde{Y}|Y\rangle \geq 0$$

with equality if and only if $|X\rangle \otimes |Y\rangle = |0\rangle$, i.e., if and only if $|X\rangle = |0\rangle$ or $|Y\rangle = |0\rangle$. □

Thus, the inner product defined between $\tilde{\subset}$ and $\tilde{\supset}$ coupled with that defined between $\rceil$ and $\lceil$ induces an inner product between $\tilde{\subset} \otimes \rceil$ and $\tilde{\supset} \otimes \lceil$.

Tensors

The elements of a tensor product space erected on a finite number of vector spaces each consisting of some space $\tilde{\supset}$ or its dual are called tensors. The general example of such a tensor space may be written

$$\Phi^s_r = \left[\bigotimes_i \tilde{\subset}\right] \otimes \left[\bigotimes_j \tilde{\supset}\right] \qquad \begin{array}{l} i = 1, \ldots, r \\ j = 1, \ldots, s \end{array}$$

where $r + s = v$. If $\tilde{\supset}$ is regarded as the starting point as far as choice of basis is concerned, then the elements of Φ^s_r are called tensors of valence v, with representatives which are r times covariant and s times contravariant. Despite the emphasis on invariance in the engineering literature on tensors (Kron 1939), it is somewhat surprising that the word "tensor" has usually been applied to non-invariants and the invariant quantities have been given such names as "geometric object" or no name at all! We regard tensors as invariant under change of basis, although the representatives of the tensors do depend on the basis.

Thus, $\mathcal{C}$ and $\reflectbox{$\mathcal{C}$}$ are spaces of tensors with unit valence. The columns representing elements of $\reflectbox{$\mathcal{C}$}$ are contravariant and the rows representing elements of $\mathcal{C}$ are covariant. The scalars in Ω are called tensors of valence zero. The algebraic structure consisting of all tensor spaces defined on $\reflectbox{$\mathcal{C}$}$ is called the tensor algebra of $\reflectbox{$\mathcal{C}$}$.

Tensor spaces of particular interest are $\reflectbox{$\mathcal{C}$} \otimes \reflectbox{$\mathcal{C}$}$, $\mathcal{C} \otimes \mathcal{C}$ and $\reflectbox{$\mathcal{C}$} \otimes \mathcal{C}$. The first two spaces will be employed in the application considered in this dissertation; the third is the space of operators on $\reflectbox{$\mathcal{C}$}$ (or on $\mathcal{C}$). The really important properties, however, are not those pertaining to the special case of tensor spaces but are those possessed by tensor products in general.

Summary on Tensor Products

The section on tensor products is closed by listing some of the most important theorems relating to abstract entities. Proofs are omitted in most cases. The more basic theorems are proved in the literature (B 1958, MS&M 1963). The remainder may be demonstrated easily through an argument employing representatives on some basis.

TM Scalars α, β in a field Ω and the following entities are introduced as in the previous text.

Signal Vector Spaces	$\supset$	Γ
Dimensionality	n	p
Indices	i, j	μ, ν
Signal Bases	$\vert\underline{B}$	$\vert\underline{C}$
Signal Vectors	$\vert X\rangle$	$\vert Y\rangle$
Pattern Spaces	$\subset$	$\rceil$
Pattern Bases	$\underline{D}\vert$	$\underline{A}\vert$
Pattern Vectors	$\langle W\vert$	$\langle Z\vert$
Operators	$\vert K\vert$	$\vert L\vert$

Then the tensor products of these entities satisfy:

$$\text{Dim } \supset \otimes \Gamma = np = \text{Dim } \subset \otimes \rceil \qquad (51)$$

$$|X\rangle \otimes |Y\rangle \in \supset \otimes \Gamma \qquad (52a)$$

$$\langle W| \otimes \langle Z| \in \subset \otimes \rceil \qquad (52b)$$

$$\left[\langle W| \otimes \langle Z|\right] \left[|X\rangle \otimes |Y\rangle\right] = \langle W|X\rangle \otimes \langle Z|Y\rangle \qquad (53a)$$

$$= \langle W|X\rangle \langle Z|Y\rangle \qquad (53b)$$

$$\otimes \text{ is bilinear} \qquad (54)$$

$$|K|X\rangle \otimes |L|Y\rangle = \left[|K| \otimes |L|\right] \left[|X\rangle \otimes |Y\rangle\right] \qquad (55a)$$

$$\langle W|K| \otimes \langle Z|L| = \left[\langle W| \otimes \langle Z|\right] \left[|K| \otimes |L|\right] \qquad (55b)$$

$$|\underline{B} \otimes |\underline{C} \text{ is a basis in } \supset \otimes \Gamma \qquad (56a)$$

$\underline{D}| \otimes \underline{A}|$ is a basis in $\mathbb{C} \otimes \urcorner$ (56b)

$$\left[\underline{D}| \otimes \underline{A}|\right] \left[|\underline{B} \otimes |\underline{C}\right] = \underline{D}|\underline{B} \otimes \underline{A}|\underline{C} = \left[\delta_{i\mu} \; \delta_{j\nu}\right] \quad (57a)$$

$\delta_{i\mu} \; \delta_{j\nu} = 1$ if $(i,j) = (\mu, \nu)$, 0 otherwise (57b)

$$|K_2|K_1| \otimes |L_2|L_1| = \left[|K_2| \otimes |L_2|\right] \left[|K_1| \otimes |L_1|\right] \quad (58)$$

$$|I| \otimes |I| = |I| \quad (59)$$

$$|0| \otimes |L| = |K| \otimes |0| = |0| \quad (60a)$$

$$|0\rangle \otimes |Y\rangle = |X\rangle \otimes |0\rangle = |0\rangle \quad (60b)$$

$$\langle 0| \otimes \langle Z| = \langle W| \otimes \langle 0| = \langle 0| \quad (60c)$$

Note that the meanings of the unit and null entities vary from one appearance to another in an obvious way. The following theorem includes invertible operators as a special case and follows directly from Lanczos' theorem and the definition of pseudoinverse.

$$\left[|K| \otimes |L|\right]^{\ddagger} = |K^{\ddagger}| \otimes |L^{\ddagger}| \quad (61)$$

$$\widetilde{\left[|X\rangle \otimes |Y\rangle\right]} = \langle\tilde{X}| \otimes \langle\tilde{Y}| \quad (62a)$$

$$\widetilde{\left[\langle W| \otimes \langle Z|\right]} = |\tilde{W}\rangle \otimes |\tilde{Z}\rangle \quad (62b)$$

$$\widetilde{\left[|K| \otimes |L|\right]} = |\tilde{K}| \otimes |\tilde{L}| \quad (62c)$$

$$\left.\begin{array}{l}\text{Spectrum } |K| = \{\lambda_i\} \\ \text{Spectrum } |L| = \{\phi_\mu\}\end{array}\right\} \Rightarrow \text{Spectrum } |K| \otimes |L| = \{\lambda_i \phi_\mu\} \quad (63)$$

$$\left.\begin{array}{l}|K|X\rangle = |X\rangle\lambda \\ |L|Y\rangle = |Y\rangle\phi\end{array}\right\} \Rightarrow \left[|K| \otimes |L|\right] \left[|X\rangle \otimes |Y\rangle\right] = |X\rangle \otimes |Y\rangle \lambda\phi \quad (64a)$$

Similarly for left eigenvectors. (64b)

If $|\underline{B}$ and $|\underline{C}$ are orthonormal, so is $|\underline{B} \otimes |\underline{C}$. (65)

If $|K|$ and $|L|$ are (unitary, self-matching, non-negative, projection, or perpendicular projection) operators, then so is $|K| \otimes |L|$. (66a-e)

3 - 2 REPRESENTATION OF TENSOR PRODUCTS

Since a tensor product space is a vector space, the approach used in Chapter 2 may serve as a guide. For example, a discussion of representation requires the introduction of a basis. In the case of a tensor product space $\mathcal{S} \otimes \Gamma$, it is convenient to choose a basis formed by taking the tensor product of bases in the spaces $\mathcal{S}$ and Γ.

Tensor Product Bases

We have already seen that, if $|\underline{B}$ is a basis in $\mathcal{S}$ and $|\underline{C}$ is a basis in Γ (both usually orthonormal), then $|\underline{B} \otimes |\underline{C} = \{|B_i\rangle \otimes |C_j\rangle : i = 1, \ldots, n;\ j = 1, \ldots, p\}$ is a basis in $\mathcal{S} \otimes \Gamma$. It is convenient to use the ordered pairs (i,j) to index $|\underline{B} \otimes |\underline{C}$ and it is customary to write them in lexicographical sequence, i.e., (1,1), (1,2),..., (1,p); (2,1), (2,2),..., (2,p);...; (n,1), (n,2),..., (n,p). Thus,

$$|\underline{B} \otimes |\underline{C} = \boxed{|B_1\rangle \otimes |C_1\rangle \;\Big|\; |B_1\rangle \otimes |C_2\rangle \;\Big|\; \cdots \;\Big|\; |B_n\rangle \otimes |C_p\rangle} \qquad (67)$$

and if $\underline{D}|$ is the dual of $|\underline{B}$ and $\underline{A}|$ is the dual of $\underline{C}|$ then the dual of $|\underline{B} \otimes |\underline{C}$ is

$$\underline{D}| \otimes \underline{A}| = \begin{bmatrix} \langle D_1| \otimes \langle A_1| \\ \langle D_1| \otimes \langle A_2| \\ \vdots \\ \langle D_n| \otimes \langle A_p| \end{bmatrix} \qquad (68)$$

If $|\underline{B}$ is orthonormal, then $\underline{D}| = \tilde{\underline{B}}|$. Similarly, if $|\underline{C}$ is orthonormal, then $\underline{A}| = \tilde{\underline{C}}|$. If both conditions hold, then $\underline{D}| \otimes \underline{A}| = \tilde{\underline{B}}| \otimes \tilde{\underline{C}}| = \widetilde{|\underline{B} \otimes |\underline{C}}$ and $|\underline{B} \otimes |\underline{C}$ is orthonormal along with its dual.

As we have seen in Chapter 2, the symbol formed by multiplication of a signal basis by a pattern basis is a unit operator on the two spaces under consideration. Thus,

$$\left[|\underline{B} \otimes |\underline{C}\right]\left[\underline{D}| \otimes \underline{A}|\right] = |I|$$

where $|I|$ indicates the unit operator on $\mathcal{S} \otimes \Gamma$. The left side of this equation may be written

$$|\underline{B}\ \underline{D}| \otimes |\underline{C}\ \underline{A}| = |I| \otimes |I|$$

where the first $|I|$ refers to the unit operator on $\mathcal{S}$ and the second $|I|$ refers to the unit operator on Γ. In view of the multiple meanings we assume (often tacitly) for "$+$", "$=$" and juxtaposition, there does not seem to be any reason to balk at writing

$$|I| \otimes |I| = |I| \qquad \text{or} \qquad |0| \otimes |0| = |0|$$

where a single symbol has three meanings in one expression.

Kronecker Product of Matrices

The notation system described in Chapter 2 for linear vector spaces extends naturally to tensor product spaces.

For example, suppose that the signal vectors $|F\rangle$ in $\supset$ and $|G\rangle$ in Γ are given by

$$|F\rangle = |\underline{B}\ \underline{F}\rangle = \sum_{i} |B_i\rangle\, \phi_i$$

$$|G\rangle = |\underline{C}\ \underline{G}\rangle = \sum_{j} |C_j\rangle\, \gamma_j$$

and, thus,

$$|F\rangle \otimes |G\rangle = \sum_{(i,j)} \left[|B_i\rangle \otimes |C_j\rangle\right] \phi_i\, \gamma_j \qquad (69)$$

is the signal vector in $\supset \otimes \Gamma$ called the tensor product of $|F\rangle$ and $|G\rangle$. Note that Equation (69) may be written

$$|F\rangle \otimes |G\rangle = \left[|\underline{B} \otimes |\underline{C}\right]\left[\underline{F}\rangle \otimes \underline{G}\rangle\right] \qquad (70)$$

if it is understood that $\underline{F}\rangle \otimes \underline{G}\rangle$ is the column of scalars $\phi_i\ \gamma_j$ in lexicographic order.

The column $\underline{F}\rangle \otimes \underline{G}\rangle$ can be obtained from the columns $\underline{F}\rangle$ and $\underline{G}\rangle$ by using the rule for Kronecker multiplication of matrices. Thus,

$$\underline{F}\rangle \otimes \underline{G}\rangle = \begin{array}{c} 1 \\ 2 \\ \\ n \end{array}\begin{bmatrix} \phi_1 \\ \phi_2 \\ \cdots \\ \phi_n \end{bmatrix} \otimes \begin{array}{c} 1 \\ 2 \\ \\ p \end{array}\begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \cdots \\ \gamma_p \end{bmatrix} = \begin{array}{c} (1,1) \\ (1,2) \\ \\ (n,p) \end{array}\begin{bmatrix} \phi_1\gamma_1 \\ \phi_1\gamma_2 \\ \cdots \\ \phi_n\gamma_p \end{bmatrix}$$

In general, the <u>Kronecker product</u> of an $m_1 \times n_1$ matrix $\underline{A} = [\alpha_{ij}]$ and an $m_2 \times n_2$ matrix $\underline{B} = [\beta_{\mu\nu}]$ is the $(m_1 m_2) \times (n_1 n_2)$ matrix

$$\underline{A} \otimes \underline{B} = [\alpha_{ij}\, \underline{B}] \qquad (71)$$

with rows indexed by the ordered pairs (i, μ) and columns indexed by the ordered pairs (j, ν) in lexicographic sequence. Note that $\otimes$ means "tensor product" when applied to abstract entities and "Kronecker product" when applied to matrices.

The process of signal measurement in which the dual basis (set of siftors) operates on a signal vector to give the column representing the signal is indicated by writing

$$\underline{F}\rangle \otimes \underline{G}\rangle = \left[\underline{D}| \otimes \underline{A}|\right] \left[|F\rangle \otimes |G\rangle\right] \tag{72}$$

A series of equations similar to (69)-(72) can also be written for patterns and rows.

At this point, the mnemonic advantages of our modified Dirac notation over the usual index and summation notation ought to be apparent. The form of many theorems involving " $\otimes$ " is seen to be

$$[\square \; \triangle] \otimes [\diamond \; \nabla] = [\square \otimes \diamond][\triangle \otimes \nabla] \tag{73}$$

where $\square$, $\diamond$ may be any pair of entities of the same type, e.g., patterns, operators, pattern bases or matrices associated with two vector spaces (not necessarily distinct); similarly for $\triangle$, ∇ with "pattern" replaced by "signal".

Consider the tensor product of operators $|K|$ on $\Im$ and $|L|$ on Γ and their matrix representatives $\underline{K}$ and $\underline{L}$ on the given bases.

$$\underline{K} = \underline{D}|K|\underline{B} \qquad\qquad |K| = |\underline{B}\ \underline{K}\ \underline{D}|$$

$$\underline{L} = \underline{A}|L|\underline{C} \qquad\qquad |L| = |\underline{C}\ \underline{L}\ \underline{A}|$$

The operator $|K| \otimes |L|$ on $\supset \otimes \Gamma$ is then represented by the matrix

$$\left[\underline{D}| \otimes \underline{A}|\right]\left[|K| \otimes |L|\right]\left[|\underline{B} \otimes |\underline{C}\right] = \underline{D}|K|\underline{B} \otimes \underline{A}|L|\underline{C}$$

$$= \underline{K} \otimes \underline{L} \qquad (74)$$

obtained by Kronecker multiplication of $\underline{K}$ and $\underline{L}$.

<u>Adjoints of Kronecker Products</u>

Consider $|F\rangle$ in $\supset$ and its match $\langle\tilde{F}|$ in $\subset$ represented on orthonormal bases $|\underline{B}$ and $\tilde{\underline{B}}|$ by $\underline{F}\rangle = [\phi_i]$ and $\langle\tilde{\underline{F}}$. Similarly for $|G\rangle \in \Gamma$, $\langle\tilde{G}| \in \urcorner$, $|\underline{C}$, $\tilde{\underline{C}}|$, $\underline{G}\rangle = [\gamma_\mu]$ and $\langle\tilde{\underline{G}}$. Then $|F\rangle \otimes |G\rangle$ is represented by $\underline{F}\rangle \otimes \underline{G}\rangle = [\phi_i\gamma_\mu]$. Note that taking the adjoint of $\underline{F}\rangle \otimes \underline{G}\rangle$ yields the row

$$\widetilde{\underline{F}\rangle \otimes \underline{G}\rangle} = \begin{array}{|c|c|c|c|}\hline \phi_1^* \gamma_1^* & \phi_1^* \gamma_2^* & \cdots & \phi_n^* \gamma_p^* \\ \hline\end{array}$$

and that the same result is obtained by Kronecker multiplication of $\langle\tilde{\underline{F}}$ and $\langle\tilde{\underline{G}}$. Thus,

$$\widetilde{\underline{F}\rangle \otimes \underline{G}\rangle} = \langle\tilde{\underline{F}} \otimes \langle\tilde{\underline{G}} \qquad (75)$$

The left side of Equation (75) represents $\widetilde{|F\rangle \otimes |G\rangle}$ which is an element of the dual of $\supset \otimes \Gamma$ while the right side represents $\langle\tilde{F}| \otimes \langle\tilde{G}|$ which is an element of $\subset \otimes \urcorner$. The equality of the two representatives is an illustration of the canonical isomorphism between $\subset \otimes \urcorner$ and the dual of $\supset \otimes \Gamma$.

We have demonstrated Equation (62a) and (62b). It is easy to see that

$$\widetilde{\underline{K} \otimes \underline{L}} = \tilde{\underline{K}} \otimes \tilde{\underline{L}} \tag{76}$$

and, again, by corresponding representatives with entities represented, we can deduce (62c).

<u>Summary on Kronecker Products</u>

The discussion on Kronecker products is completed by listing some of the most important theorems without proofs. Other aspects of matrix algebra have also been summarized in a convenient list of theorems (Marcus 1960).

TM The following matrices are introduced.

n x n matrices $\underline{K}$

p x p matrices $\underline{L}$

n x 1 columns $\underline{X}\rangle$

p x 1 columns $\underline{Y}\rangle$

1 x n rows $\langle\underline{W}$

1 x p rows $\langle\underline{Z}$

Then the Kronecker products of these matrices have the properties:

$$\left[\langle\underline{W} \otimes \langle\underline{Z}\right]\left[\underline{X}\rangle \otimes \underline{Y}\rangle\right] = \langle\underline{W}\ \underline{X}\rangle \otimes \langle\underline{Z}\ \underline{Y}\rangle \tag{77a}$$

$$= \langle\underline{W}\ \underline{X}\rangle \langle\underline{Z}\ \underline{Y}\rangle \tag{77b}$$

$\otimes$ is bilinear (78)

$$\underline{K}\ \underline{X}\rangle \otimes \underline{L}\ \underline{Y}\rangle = \left[\underline{K} \otimes \underline{L}\right]\left[\underline{X}\rangle \otimes \underline{Y}\rangle\right] \tag{79a}$$

$$\langle\underline{W}\ \underline{K} \otimes \langle\underline{Z}\ \underline{L} = \left[\langle\underline{W} \otimes \langle\underline{Z}\right]\left[\underline{K} \otimes \underline{L}\right] \tag{79b}$$

$$\underline{K}_2\,\underline{K}_1 \otimes \underline{L}_2\,\underline{L}_1 = \left[\underline{K}_2 \otimes \underline{L}_2\right]\left[\underline{K}_1 \otimes \underline{L}_1\right] \qquad (80)$$

$$\underline{I} \otimes \underline{I} = \underline{I} \qquad (81)$$

$$\underline{0} \otimes \underline{L} = \underline{K} \otimes \underline{0} = \underline{0} \qquad (82a)$$

$$\underline{0}\rangle \otimes \underline{Y}\rangle = \underline{X}\rangle \otimes \underline{0}\rangle = \underline{0}\rangle \qquad (82b)$$

$$\langle\underline{0} \otimes \langle\underline{Z} = \langle\underline{W} \otimes \langle\underline{0} = \langle\underline{0} \qquad (82c)$$

Note that the meanings of the unit and null matrix symbols vary from one appearance to another in an obvious way. The following theorem includes invertible matrices as a special case and follows directly from Lanczos' theorem and the definition of pseudoinverse.

$$\left[\underline{K} \otimes \underline{L}\right]^{\ddagger} = \underline{K}^{\ddagger} \otimes \underline{L}^{\ddagger} \qquad (83)$$

$$\widetilde{\underline{X}\rangle \otimes \underline{Y}\rangle} = \langle\widetilde{\underline{X}} \otimes \langle\widetilde{\underline{Y}} \qquad (84a)$$

$$\widetilde{\langle\underline{W} \otimes \langle\underline{Z}} = \widetilde{\underline{W}}\rangle \otimes \widetilde{\underline{Z}}\rangle \qquad (84b)$$

$$\widetilde{\underline{K} \otimes \underline{L}} = \widetilde{\underline{K}} \otimes \widetilde{\underline{L}} \qquad (84c)$$

$$\left.\begin{array}{l}\text{Spectrum } \underline{K} = \{\lambda_1\} \\ \text{Spectrum } \underline{L} = \{\phi_\mu\}\end{array}\right\} \Rightarrow \text{Spectrum } \underline{K} \otimes \underline{L} = \{\lambda_1 \phi_\mu\} \qquad (85)$$

$$\left.\begin{array}{l}\underline{K}\,\underline{X}\rangle = \underline{X}\rangle\lambda \\ \underline{L}\,\underline{Y}\rangle = \underline{Y}\rangle\phi\end{array}\right\} \Rightarrow \left[\underline{K} \otimes \underline{L}\right]\left[\underline{X}\rangle \otimes \underline{Y}\rangle\right] = \underline{X}\rangle \otimes \underline{Y}\rangle\lambda\phi \qquad (86a)$$

Similarly for eigenrows. (86b)

If $\underline{K}$ and $\underline{L}$ are (square, scalar, diagonal, column, row, triangular, unitary, self-adjoint, non-negative, idempotent, or normal) matrices, then so is $\underline{K} \otimes \underline{L}$. (87a-k)

The theorems above on Kronecker products are, for the most part, images of those on tensor products. The correspondence is made via mutually dual tensor product bases in $\supset \otimes \ulcorner$ and $\subset \otimes \urcorner$. Other important theorems following from (85) are given as follows.

$$\text{TM} \quad \mathrm{Tr}\left[\underline{K} \otimes \underline{L}\right] = \left[\mathrm{Tr}\ \underline{K}\right]\left[\mathrm{Tr}\ \underline{L}\right] \tag{88}$$

$$\mathrm{Det}\left[\underline{K} \otimes \underline{L}\right] = \left[\mathrm{Det}\ \underline{K}\right]^{n}\left[\mathrm{Det}\ \underline{L}\right]^{p} \tag{89}$$

$$\mathrm{Rank}\left[\underline{K} \otimes \underline{L}\right] = \left[\mathrm{Rank}\ \underline{K}\right]\left[\mathrm{Rank}\ \underline{L}\right] \tag{90}$$

3 - 3 TIME-DOMAIN MULTIPLICATION

We proceed now to apply the abstract concepts of tensor product spaces and the corresponding Kronecker product representatives to a signal-processing problem which arises frequently in communication, computation, measurement and control systems engineering. The problem is that of instantaneous multiplication, i.e., multiplication in the time domain. A general discussion of the problem will be presented in this section. A specific application to satellite navigation will be considered in some detail in Chapter 4.

Examples of real-world devices which may be modeled by the product $f(t)g(t)$ of two functions of time include: (1) an amplitude modulator in a radio communication system, (2) a potentiometer used as a multiplier in an analog computer. In the first example, $f(t)$ might be the value of a modulating signal at the instant t and $g(t)$ the value of a sinusoidal carrier at the same instant. In the second example, $f(t)$ might represent an electrical input and $g(t)$ a mechanical motion of the potentiometer contact. In both examples, the value of the electrical output of the device at the instant t is expressed by the product $f(t)g(t)$.

<u>Algebras</u>

Instead of the usual analytic formulation of time-domain multiplication, we are interested here in a formulation in terms of abstract algebraic structures. We have seen that finite-dimensional vector spaces are appropriate models of signals and related physical entities. Can time-domain multiplication be included in such a model? More specifically, if $|F\rangle \in \mathcal{S}$ and $|G\rangle \in \mathcal{S}$, is there an element of $\mathcal{S}$ corresponding to the output of a time-domain multiplier with $|F\rangle$ and $|G\rangle$ as inputs? Since the definition of a vector space does not provide for a multiplication of vectors which yields a vector in the same space, the answer to both questions is negative. Then is there an extension of the definition of a vector space which will serve as a model of time-domain multiplication?

An n-dimensional vector space $\mathcal{S}$ furnished with a closed multiplication $\boxtimes$ of vectors is called an <u>algebra</u> (H 1958) and the product of any two basis elements $|B_i\rangle$, $|B_j\rangle$ under this multiplication must satisfy

$$|B_i\rangle \boxtimes |B_j\rangle = \sum_k |B_k\rangle \gamma_{ijk} \qquad (91)$$

for some set of n^3 scalars $\gamma_{ijk} \in \Omega$ (Chevalley 1956). The γ_{ijk} are called the "constants of structure" or the "multiplication table" of the algebra.

It is immediately clear that an algebra will not meet our requirements, since the multiplication furnished by the algebra leads to a contradiction in physical units. For example, if the unit of each $|B_i\rangle$ is the volt, then we would want to assign the unit $(\text{volt})^2$ to each $|B_i\rangle \boxtimes |B_j\rangle$. However, Equation (91) indicates that the unit of $|B_i\rangle \boxtimes |B_j\rangle$ is the same as that of each basis element, since the γ_{ijk} have no physical unit. The elements of an algebra must be free of physical units if Equation (91) is to make complete sense. Thus, we must look in other directions for an appropriate extension which will include time-domain multiplication and the vector structure of signal theory discussed in Chapter 2.

Change of Basis

One of the important ideas in the application of the abstract algebra of vectors and tensor products to physical problems is that of invariance, i.e., invariance under change of basis. It is informative to consider the effect of basis change on t-domain multiplication.

We begin by considering the system of Figure 22 involving the instantaneous multiplication of two signals $|C\rangle$ and $|D\rangle$ in the inner-product space $\mathcal{S}$ defined by an N-dimensional finite time basis $|\underline{t}$. Several interpretations of $|\underline{t}$ and $\underline{\tilde{t}}|$ are possible. Each element $|t_k\rangle$ of $|\underline{t}$ may be a signal generator consisting of a battery and a switch such that the output is $\sqrt{1/T}$ for $(k-1)T \leq t < kT$; each element $\langle\tilde{t}_k|$ of $\underline{\tilde{t}}|$ may be an integrator of weight $\sqrt{1/T}$ over the same interval of time. In problems where an appropriate analytic model of the system is available, $|t_k\rangle$ may be represented on $|t$ as a Dirac delta at the mid-point of the k<u>th</u> sampling interval, and similarly for $\langle\tilde{t}_k|$ on $\tilde{t}|$.

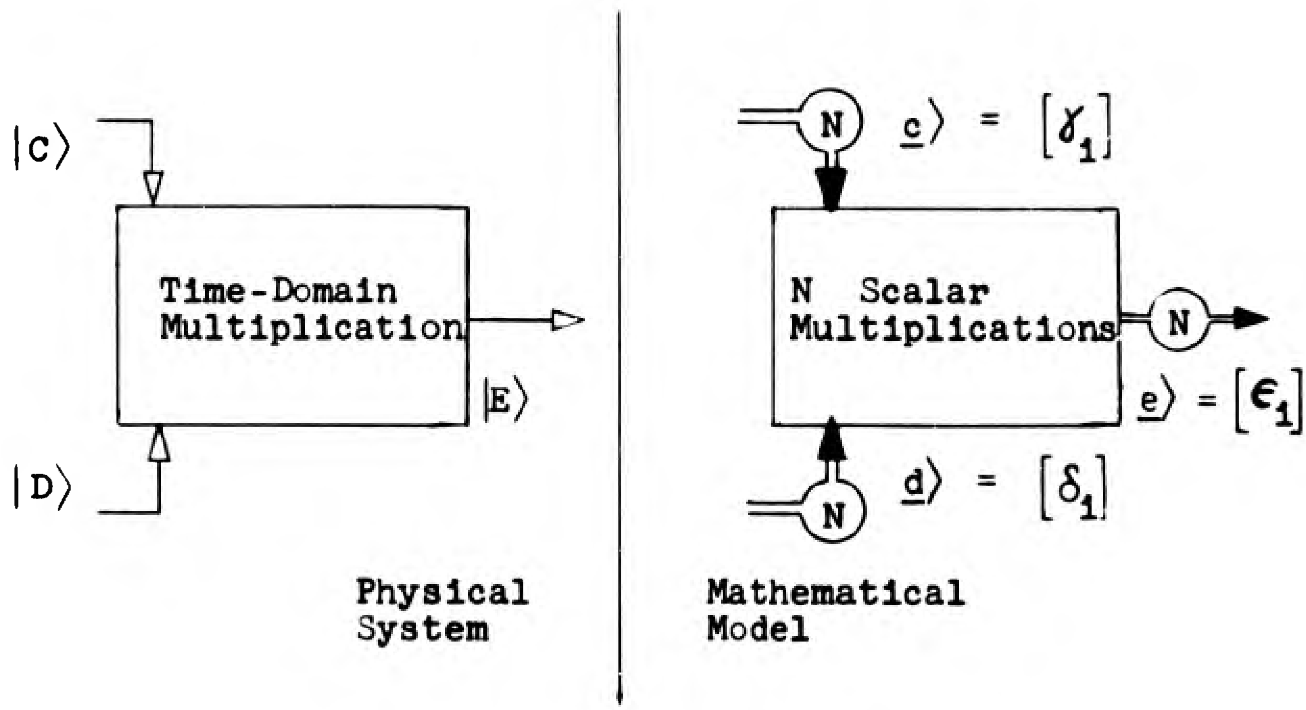


Figure 22 Time-Domain Multiplication

The operation of the instantaneous multiplier is defined by

$$\epsilon_i = \gamma_i \delta_i \qquad i = 1,2,\ldots,N \tag{92}$$

where N may be quite large, e.g., several hundred or more, in typical problems. We will suppose that $|C\rangle$ and $|D\rangle$ are known to lie in subspaces of relatively small dimensionality, e.g., ten or less. Thus,

$$|C\rangle \in \mathcal{A} \subset \mathfrak{H} \qquad m = \mathrm{Dim}\,\mathcal{A} < \mathrm{Dim}\,\mathfrak{H} = N$$

$$|D\rangle \in \mathcal{B} \subset \mathfrak{H} \qquad n = \mathrm{Dim}\,\mathcal{B} < \mathrm{Dim}\,\mathfrak{H} = N$$

Let $|\underline{A}$ and $|\underline{B}$ be orthonormal bases in $\mathcal{A}$ and $\mathcal{B}$ defined in terms of the basis-change matrices

$$\tilde{\underline{t}}|\underline{A} = \left[\alpha_{ij}\right] = \underline{A} \qquad j = 1,2,\ldots,m$$

$$\tilde{\underline{t}}|\underline{B} = \left[\beta_{ik}\right] = \underline{B} \qquad k = 1,2,\ldots,n$$

Then, if $\underline{F}\rangle$ and $\underline{G}\rangle$ represent the input signals on $|\underline{A}$ and $|\underline{B}$, we have

$$\underline{c}\rangle = \tilde{\underline{t}}|C\rangle = \tilde{\underline{t}}|\underline{A}\ \underline{F}\rangle$$

$$\underline{d}\rangle = \tilde{\underline{t}}|D\rangle = \tilde{\underline{t}}|\underline{B}\ \underline{G}\rangle$$

which may be rewritten in the conventional notation as follows:

$$\gamma_i = \sum_j \alpha_{ij}\, \phi_j \tag{93}$$

$$\delta_i = \sum_k \beta_{ik}\, \psi_k \tag{94}$$

Substituting (93) and (94) into (92) yields

$$\epsilon_i = \left[\sum_j \alpha_{ij}\phi_j\right]\left[\sum_k \beta_{ik}\psi_k\right]$$
$$= \sum_{(j,k)} (\alpha_{ij}\,\beta_{ik})(\phi_j\psi_k) \qquad (95)$$

which can also be written as the product of N x m n and m n x 1 matrices where the second factor is $\underline{F}\rangle \otimes \underline{G}\rangle$. This observation raises the question as to whether the first factor of the matrix product might also be put in Kronecker product form.

We note that

$$\underline{A} \otimes \underline{B} = \left[\alpha_{\mu j}\;\beta_{\nu k}\right] \qquad \mu, \nu = 1,2,\ldots,N \qquad (96)$$

which can be rewritten in the form

$$\underline{A} \otimes \underline{B} = \begin{bmatrix} \alpha_{ij}\;\beta_{ik} \\ \hline \alpha_{\mu j}\;\beta_{\nu k} \end{bmatrix} \begin{matrix} \uparrow N \\ \updownarrow N(N-1) \end{matrix} \qquad (97)$$

by rearranging rows. We observe that the upper submatrix is of the form desired in the matrix form of Equation (95).

Let $\mathcal{P}$ be the subspace of $\mathcal{S} \otimes \mathcal{S}$ spanned by the n elements of $|\underline{t} \otimes |\underline{t}$ of the form $|t_i\rangle \otimes |t_i\rangle$. Let $|\underline{P}$ be the basis in $\mathcal{P}$ formed by these elements. Then $|\underline{P}\,\underline{\tilde{P}}|$ is a perpendicular projector on $\mathcal{P}$. Also,

$$\underline{\tilde{P}}|\left[|\underline{t} \otimes |\underline{t}\right] = \underbrace{\left[\begin{array}{c|c} \underline{I} & \underline{0} \end{array}\right]}_{\leftarrow N \rightarrow \leftarrow N(N-1) \rightarrow} \updownarrow N \triangleq \underline{P} \qquad (98)$$

$$\underline{P}\left[\underline{A} \otimes \underline{B}\right] = \left[\alpha_{ij}\;\beta_{ik}\right] \qquad (99)$$

We will call $\mathcal{P}$ the <u>principal subspace</u> of $\mathcal{S} \otimes \mathcal{S}$. Note that $\mathcal{P}$ is <u>not</u> invariant under change of basis. If any basis in $\mathcal{S}$ other than $|\underline{t}$ is employed to define a principal subspace of $\mathcal{S} \otimes \mathcal{S}$, qualifying nomenclature must be included.

Equation (95) can now be rewritten as follows.

$$\underline{e}\rangle = \underline{P}\left[\underline{A} \otimes \underline{B}\right]\left[\underline{F}\rangle \otimes \underline{G}\rangle\right] \tag{100}$$

We conclude that the column $\underline{e}\rangle$ represents a vector $|E\rangle$, where

$$|E\rangle \in \mathcal{P} \subset \mathcal{S} \otimes \mathcal{S} \tag{101}$$

and not some vector in $\mathcal{S}$. Thus, the physical operation of time-domain multiplication can be modeled in vector algebra terms via the tensor product extension. We note that the tensor product interpretation satisfies the check on physical units, provided that we assign as the unit of $\mathcal{S} \otimes \mathcal{S}$ the square of the unit of $\mathcal{S}$. More generally, the physical unit of a tensor product space is the "product" of the physical units of its factor spaces.

The correspondence between time-domain multiplication apparatus and its tensor product model is shown in Figure 23. The following manipulations serve to illustrate further this important correspondence.

$$\begin{aligned}
\underline{e}\rangle &= \underline{P}\left[\underline{A} \otimes \underline{B}\right]\left[\underline{F}\rangle \otimes \underline{G}\rangle\right] \\
&= \tilde{\underline{P}}|\left[|\underline{t} \otimes |\underline{t}\right]\left[\tilde{\underline{t}}|\underline{A} \otimes \tilde{\underline{t}}|B\right]\left[\underline{F}\rangle \otimes \underline{G}\rangle\right] \\
&= \tilde{\underline{P}}|\left[|\underline{t}\,\tilde{\underline{t}}| \otimes |\underline{t}\,\tilde{\underline{t}}|\right]\left[|\underline{A}\,\underline{F}\rangle \otimes |\underline{B}\,\underline{G}\rangle\right]
\end{aligned}$$

$$= \underline{\tilde{P}}|\,I\,|\left[|C\rangle \otimes |D\rangle\right]$$

$$= \underline{\tilde{P}}|\left[|C\rangle \otimes |D\rangle\right]$$

Premultiplication by $|\underline{P}$ yields

$$|\underline{P}\ \underline{e}\rangle = |\underline{P}\ \underline{\tilde{P}}|\ \left[|C\rangle \otimes |D\rangle\right]$$

$$|E\rangle = |\underline{P}\ \underline{\tilde{P}}|\ \left[|C\rangle \otimes |D\rangle\right] \qquad (102)$$

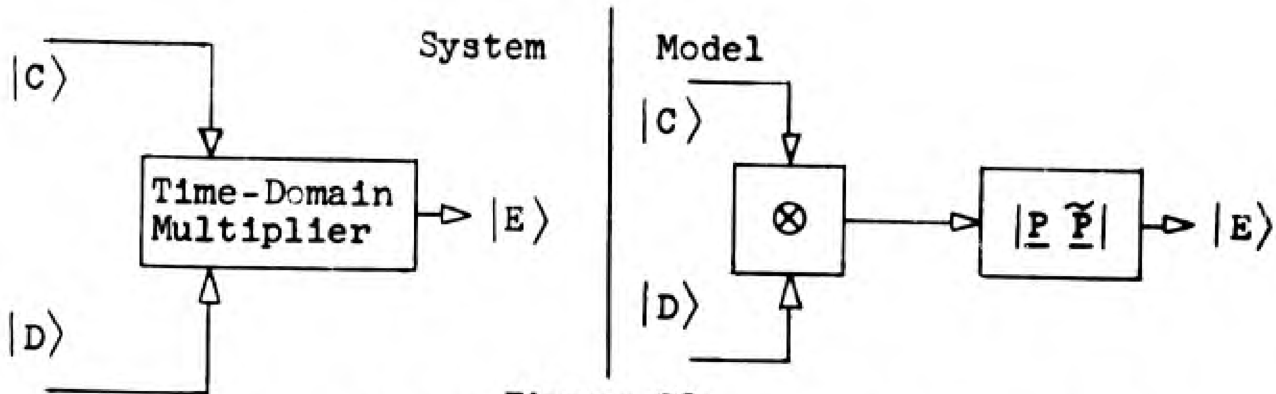


Figure 23
Tensor Product Model of Time-Domain Multiplication

It may seem rather curious that we have introduced a model involving N^2 dimensions to handle a problem which seems to require only N. However, this abundance need cause no concern, since there is a problem only when the situation is reversed and the model is deficient. A somewhat similar case is the familiar use of complex exponential functions of time to represent modulated sinusoidal carriers when the real part alone suffices. In both cases, the model includes "unnecessary" features which provide mathematical convenience and elegance. The tensor product model of time-domain multiplication is not only convenient, but is also a practical and useful tool for signal-processing system engineering.

The application problem discussed in Chapter 4 illustrates the finite-dimensional tensor product concepts which have been introduced. The existence of this detailed example, coupled with the simplicity associated with finite dimensionality, makes an elementary example unnecessary at this point.

However, the habit of viewing the world of signals through "continuum-colored glasses" is so prevalent that the finite-dimensional tensor product model of time-domain multiplication may seem strange. It may be desirable to connect our approach with the more familiar continuous time representation. The following example has been included to indicate such a connection. A reader convinced that Nature is inherently continuous may applaud this effort as a step toward reality, but that is not the view advocated here. Rather, the various models of any particular system stand on an equal footing until measurements on that system indicate that one of the models is preferred for the problem at hand.

<u>Example</u>

Let Dim $\mathcal{S} = 2$ and $\Gamma = \mathcal{S}$. Let $|B_1\rangle$, $|B_2\rangle$ be the elements of an orthogonal basis $|\underline{B}$ in $\mathcal{S}$ which are represented on the continuous time basis $|t$ by the graphs shown in Figure 24(a). Let the signals $|C\rangle$, $|D\rangle \in \mathcal{D}$ be represented on $|t$ by the graphs in Figure 24(c). Then the 4 time-domain products of all possible pairs of $\tilde{t}|B_1\rangle$ and $\tilde{t}|B_2\rangle$ are represented by the 3 graphs in Figure 24(b), and the time-domain product of $\tilde{t}|C\rangle$ and $\tilde{t}|D\rangle$ is represended by the graph in Figure 24(d).

The same problem may be treated efficiently in tensor product notation by representing $|C\rangle$ and $|D\rangle$ on $|\underline{B}$ by

$$\underline{C}\rangle = \begin{array}{c|c|} & \\ 1 & 2 \\ 2 & \gamma \end{array} \qquad \underline{D}\rangle = \begin{array}{c|c|} & \\ 1 & -1 \\ 2 & \alpha \end{array}$$

where $\alpha = \sqrt{3/2} = 1/\gamma$. Then, $|C\rangle \otimes |D\rangle$ is represented on $|\underline{B} \otimes |\underline{B}$ by

$$\underline{C}\rangle \otimes \underline{D}\rangle = \begin{array}{c|c|} (1,1) & -2 \\ (1,2) & 2\alpha \\ (2,1) & -\gamma \\ (2,2) & 1 \end{array}$$

Note that the graph in Figure 24(d) is equal to that given by summing the graphs in Figure 24(b) with weights: $-2,\ 2\alpha - \gamma,\ 1$. This observation completes the illustration of t-multiplication; however, additional insight regarding tensor product bases may be gained by further study of the example.

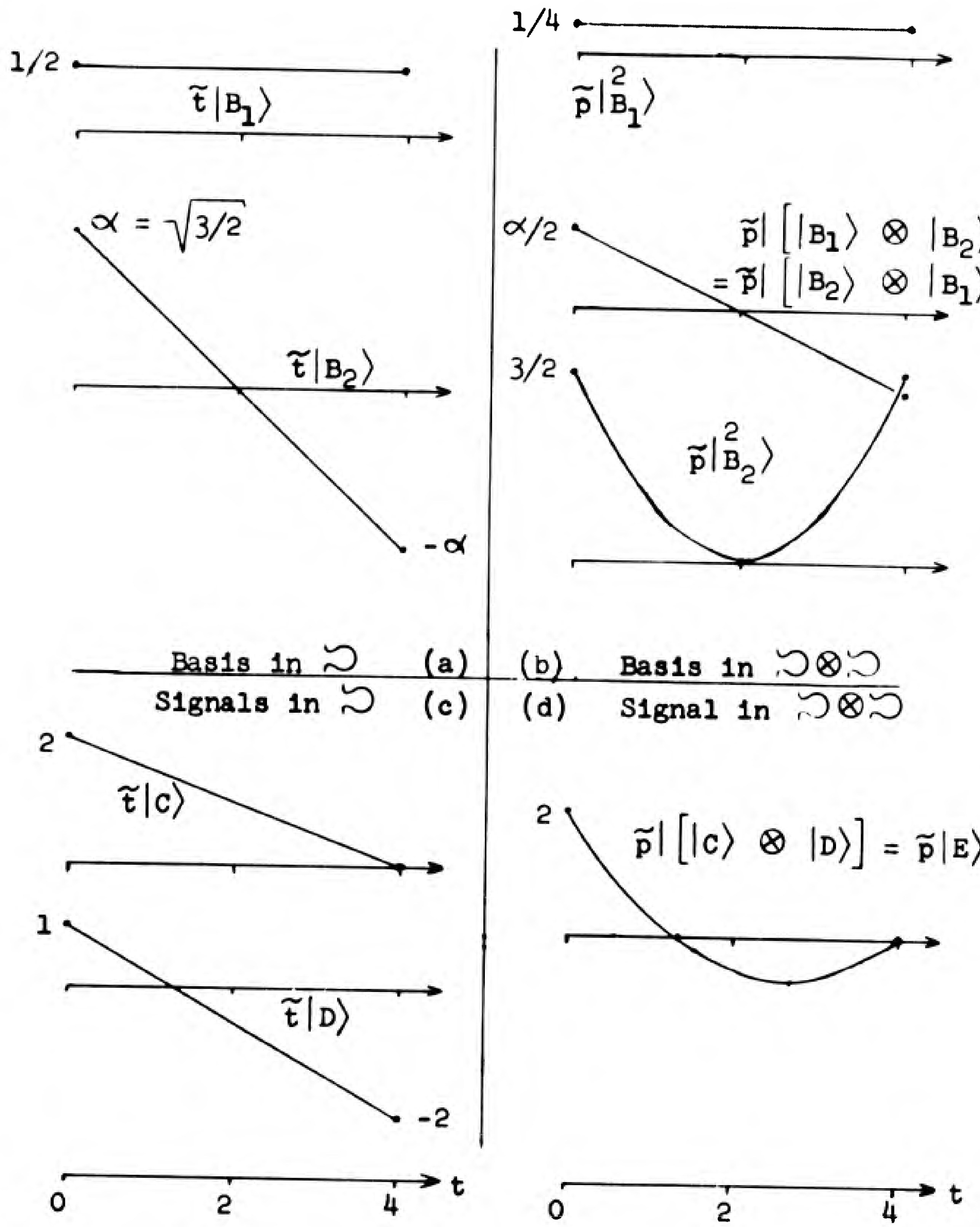


Figure 24 Classical View of Time-Domain Multiplication

Consider the tensor product of $|B_1\rangle$ and $|B_2\rangle$ represented on $|t \otimes |t$ by $\tilde{t}|B_1\rangle \otimes \tilde{t}|B_2\rangle$. The latter is the set of all possible products of pairs of coordinates, the first selected from the first factor and the second from the second factor. Let u, v represent the instants of time corresponding to the pair of coordinates selected. (Note that time is regarded as an index here, and not a coordinate.) It is convenient to graph the (u,v)th coordinate of $\tilde{t}|B_1\rangle \otimes \tilde{t}|B_2\rangle$ as a function of u and v in the usual way. The resulting graph is a plane parallel to the u-axis. Similarly, $\left[\tilde{t}| \otimes \tilde{t}|\right] \left[|B_2\rangle \otimes |B_1\rangle\right]$ corresponds to a plane parallel to the v-axis, $\left[\tilde{t}| \otimes \tilde{t}|\right] |\hat{B}_1\rangle$ to a horizontal plane of altitude 1/4, and $\left[\tilde{t}| \otimes \tilde{t}|\right] |\hat{B}_2\rangle$ to a saddle-shaped surface. An interesting exercise is to check the orthogonality of $|\underline{B} \otimes |\underline{B}$ by integration over u and v of each of the 16 possible products of pairs of representatives on $|t \otimes |t$ of the 4 elements of the tensor product basis.

Projection onto the principal subspace $\mathcal{P}$ corresponds to selecting only those coordinates for which u = v. Only one index, t, is needed from here on to index the principal sub-basis $|p$ of $|t \otimes |t$. The sets of coordinates resulting from the projection operation on the surfaces just described are the graphs shown in Figure 24(b) which are the intersections of the surfaces with the plane u = v.

The extension to the case of continuous representations for the tensor product model of time-domain multiplication has been illustrated. No use will be made here of the results of this brief departure from our objective of exploring finite-dimensional spaces.

CHAPTER FOUR

SATELLITE NAVIGATION SIGNAL PROCESSING

The Doppler shift of a sinusoidal carrier propagated between terminals in relative motion is useful in several applications. Doppler tracking of artificial satellites had an impromptu beginning at several laboratories within hours after the surprising appearance of 1957 α (Prenatt, Bentley and deBey 1958). A system of satellites as artificial radio "stars" has been proposed for use in celestial navigation (Kershner 1960) and in geodesy (Newton 1960). In this chapter, attention is confined to the navigation application, although the technique to be described appears to be sufficiently accurate to be applied to satellite tracking and perhaps even to geodesy.

At the beginning of the present study the primary goal was to simplify the signal-processing subsystem with the tacit assumption that reduced accuracy would necessarily accompany any major simplification. The results of the project show that a simple special-purpose processor, quite slow by modern standards, suffices for the navigation terminal with little or no reduction in accuracy as compared with the classical least-squares approach.

The satellite navigation system as described in the unclassified literature is presented in the first section of the chapter. The strategy of our design approach and the problems which it entails are discussed in the second section; the proposed processor configuration and its optimization are presented in the third. The fourth and last section is devoted to some of the numerical verification and error appraisal experiments which have been completed to date. Special terms and symbols used in the study of the satellite navigation problem are tabulated for convenient reference in Appendix 4.

4 - 1 SATELLITE NAVIGATION SYSTEM

The purpose of this section is to describe briefly the main features of a satellite navigation system. In the interest of brevity and simplicity, several restrictions are imposed. The purpose of the system will be restricted to the measurement of the latitude λ and longitude ϕ of a fixed point on the surface of a spherical planet of radius ρ_p rotating with angular velocity ω_p. We will consider a single satellite in circular polar orbit of radius ρ_s with angular velocity ω_s transmitting a sinusoidal carrier of frequency f_c. We will confine the study to the signal-processing portion of the navigation terminals, and all five of the parameters just introduced are assumed to be fixed and known. The system can be generalized by removing one or more of the restrictive terms underlined above, but we will generally ignore that fact in the present chapter.

Major Subsystems

The division of the navigation satellite system into major subsystems is indicated in Figure 25. The tracking computer maintains an up-to-date ephemeris (and rate of change of orbit) for the satellite by combining data from passages of the satellite near tracking receivers at several places and at several times. The ephemeris is then transmitted to navigation terminals along with accurate timing signals. Ephemeris communications may use the satellite itself so that the data is available when it is needed, or other means of communication may be used to provide the orbital data before the satellite passage.

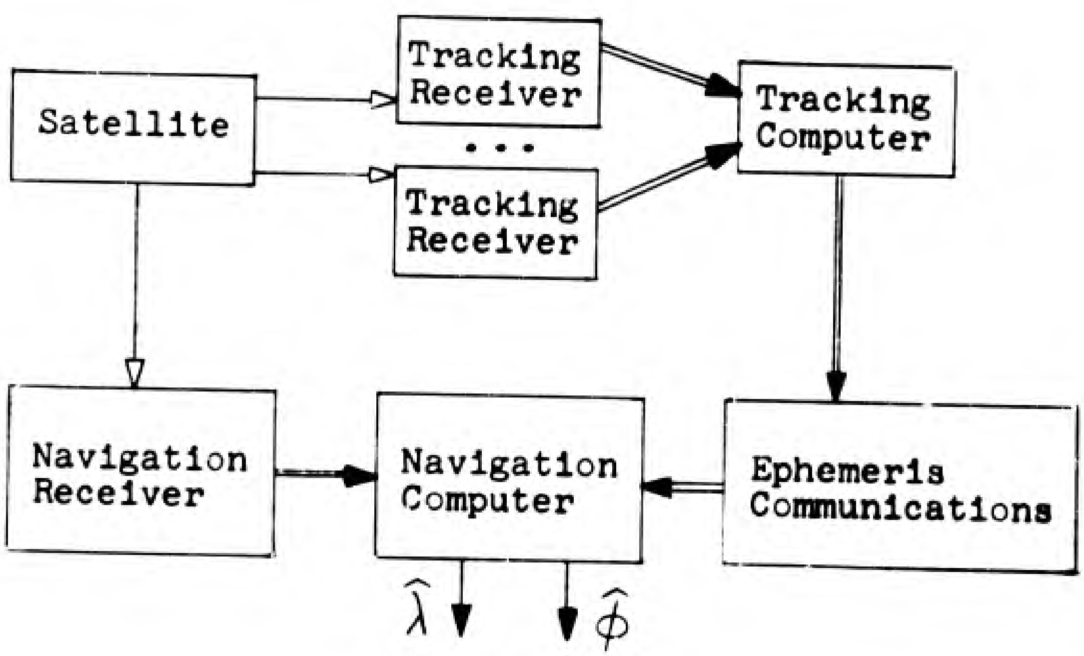


Figure 25 Satellite Navigation System

Instead of dealing with Doppler shift, represented on the continuous time basis by the familiar S -curve, we will deal with the range-rate $\dot{\rho}$. The graph of the latter vs. time looks much like the hyperbolic tangent function. The Doppler shift is, of course, a scalar multiple of $-\dot{\rho}$. The graph of the range ρ is almost symmetrical but not quite (unless $\omega_p = 0$). Large values of $\ddot{\rho}_o$ correspond to close approaches.

Limiting the Doppler data to some interval of time centered on the epoch $\mathcal{T}_o$ of closest approach is convenient. The duration of the pass, as far as data recording is concerned, will be limited here to an interval 2NT, N sampling intervals of duration T before and after the epoch, fixed for all passes such that data from low elevation angles, e.g., less than 10^o, is never used. Typical values of 2NT and T are 600 sec and 1 sec, for a total of $2N + 1 = 601$ data points (including the epoch) for a single pass. The magnitude of the Doppler shift at the ends of the pass will be slightly less than the shift corresponding to the orbital velocity of the satellite, e.g.,

$$(f_c) (\omega_s \rho_s/C) = (200 \text{ Mc}) \left[\frac{18{,}640 \text{ mile/hr}}{186{,}400 \text{ mile/sec}}\right] \left[\frac{1 \text{ hr}}{3600 \text{ sec}}\right]$$

$$= (1/180) \text{ Mc} = 5555 \text{ cps}$$

A block diagram of the navigation terminal is shown in Figure 26. Two radio receivers are shown with intermediate-frequency outputs mixed in order to produce a first-order compensation of the refraction effect of the ionosphere (G&W 1960, Weiffenbach 1960). This requires that the satellite transmit two sinusoidal carriers with frequencies related by a simple ratio. If the compensation is exact, then the signal at the tracking filter input is noise plus a sinusoid of frequency $f_d + f_o$ where f_d is the Doppler shift and f_o is a fixed offset frequency sufficient to insure that $f_d + f_o$ is positive.

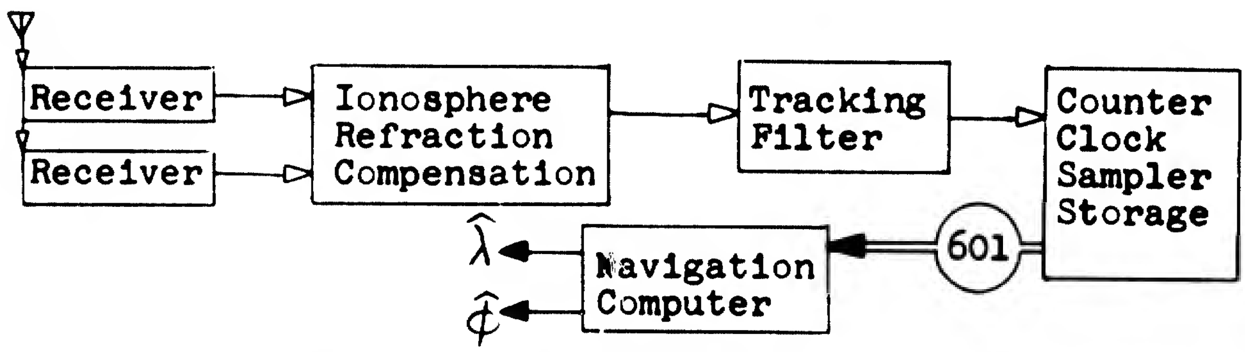


Figure 26 Navigation Terminal

The tracking filter is, in effect, a narrow band-pass filter with variable center frequency. Its function is to reduce noise by reducing band-width from that of the receiver, e.g., 15 kc, down to a few cps. In the absence of noise, the output of the tracking filter will differ by negligible dynamic lags from the input. Reference is made to the literature on tracking filters for the satellite navigation application (Richard 1958) and for theoretical and experimental studies of tracking filter behavior in general (Viterbi 1960).

The cycles of the tracking filter output are counted. Two variations on the system design are possible at this point. Either a clock can be read after some fixed number of cycles, or the counter can be read after some fixed interval of time. We select the latter scheme and will assume that the record of each pass consists of $2N + 1$ samples each consisting of the number of cycles counted over a sampling interval of duration T. In order to reduce quantization noise, it may be desirable to measure fractions of cycles, e.g., eighths. This might be done by using a frequency multiplier and operating the tracking filter at a harmonic, e.g., the eighth, of the compensator output.

The tracking receiver is similar to the navigation receiver, principal differences being environmental. Also, the tracking receiver must meet standards set by the entire system, while the navigation receiver must satisfy only one user.

The study on which this dissertation is based has resulted in a new design approach for the signal-processing portion of the navigation terminal. No new ideas were discovered of any consequence to the rest of the system up to and including the tracking filter output. Therefore, the discussion of the over-all system has been limited to that sufficient to provide a background for the signal-processing problem which will be treated in detail in other sections of this chapter. Articles are available on the development of the satellites (Schreiber and Wyatt 1960), on Doppler measurement techniques (Weiffenbach 1960), on computing techniques (Lorens 1959), and on ionospheric effects and other sources of error in the system (G&W 1960).

Analytic Model

Refraction effects will be neglected here on the assumption that the multiple-carrier compensation technique (G&W 1960) produces results sufficiently close to what would be obtained with a single carrier and a planet with no ionosphere. With this restriction and those mentioned before, the system can be accurately represented by a simple analytic model. It seems appropriate to note that the proper standard for testing the proposed processor, or any processor, is not how well it fits the analytic model but how accurately it operates in actual practice. The analytic model is a convenient basis for determining feasibility, estimated cost and estimated accuracy, but it cannot serve as the ultimate test of accuracy.

The geometry of the system is indicated in Figure 27 where N represents a navigation receiver at latitude λ

and longitude ϕ, $0 \le \lambda \le \pi/2, -\pi/2 \le \phi \le \pi/2$; S represents the satellite. The origin of time t is chosen at the north-bound equatorial crossing. The longitude scale is such that the prime meridian is directly under S at $t = 0$ and positive values of ϕ correspond to points east of the prime meridian. For convenience, we will limit attention to northbound passages of S.

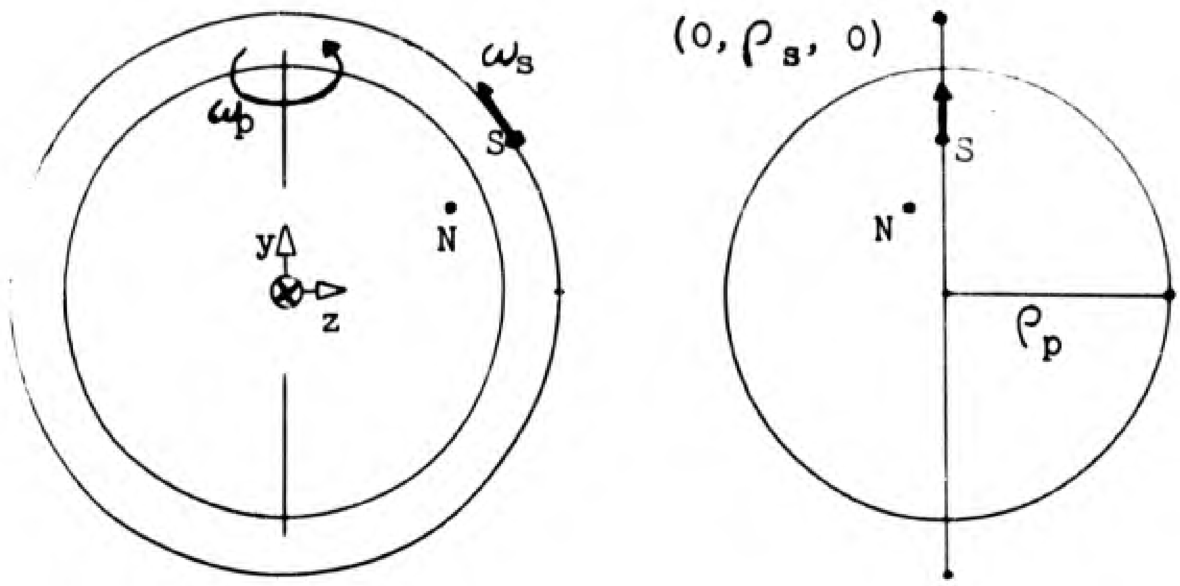


Figure 27 System Geometry

It turns out that the geodetic coordinates λ and ϕ are unwieldy choices of independent variables. The angles α and θ defining the navigator's location in inertial coordinates are convenient.

$$\alpha = \omega_s \tau_o \qquad \theta = \omega_p \tau_o + \phi \qquad (103)$$

Note that α is the latitude of S at the instant of closest approach where $t = \tau_o$, and θ is the longitude of N relative to the orbital plane of S at the same instant. The signal-processing scheme to be developed requires that signal measurements be referred to some <u>epoch</u>, and we choose the instant $t = \mathcal{T}_o$ of closest approach, i.e., when $\dot{\rho}_o = 0$. Time referred to the epoch will be symbolized by $\mathcal{T}$.

$$t = \mathcal{T}_o + \mathcal{T} \qquad -NT \leq \mathcal{T} \leq NT \tag{104}$$

Signal measurements will refer to the epoch and $\mathcal{T}$; the ephemeris will refer to the origin and t. The satellite latitude provides a convenient alternate to the time scale, so we will often deal with $\omega_s t$, $\omega_s \mathcal{T}$ and α in place of t, $\mathcal{T}$ and $\mathcal{T}_o$. We also introduce the angle γ corresponding to the duration of the recorded data.

$$\gamma = 2NT\,\omega_s \tag{105}$$

The basic equations of the model are derived by introducing a coordinate system with origin at the planet center, z-axis through S at t = 0, y-axis through the north pole, and x-axis so as to make a right-hand system. Then, at time t, the satellite and navigator are located at

$$\rho_s \begin{bmatrix} 0 \\ \sin \omega_s t \\ \cos \omega_s t \end{bmatrix} \quad \text{and} \quad \rho_p \begin{bmatrix} \cos\lambda \ \sin(\omega_p t + \phi) \\ \sin\lambda \\ \cos\lambda \ \cos(\omega_p t + \phi) \end{bmatrix}$$

and the square of the slant range is given by

$$\rho^2 = \rho_s^2 + \rho_p^2 - 2\rho_s\rho_p \left[\cos\lambda \cos\omega_s t \cos(\omega_p t + \phi) + \sin\lambda \sin\omega_s t\right] \tag{106}$$

Differentiating with respect to time yields

$$\rho\dot{\rho}/\rho_s\rho_p = \omega_p \cos\lambda \cos\omega_s t \sin(\omega_p t + \phi) - \omega_s \sin\lambda \cos\omega_s t + \omega_s \cos\lambda \sin\omega_s t \cos(\omega_p t + \phi) \qquad (107)$$

At the epoch, $t = \tau_0$, $\dot{\rho} = \dot{\rho}_0 = 0$ and

$$0 = \omega_p \cos\lambda \cos\alpha \sin\theta + \omega_s \cos\lambda \sin\alpha \cos\theta - \omega_s \sin\lambda \cos\alpha$$

$$\tan\lambda = (\omega_p/\omega_s) \sin\theta + \tan\alpha \cos\theta \qquad (108)$$

$$\phi = \theta - (\omega_p/\omega_s)\alpha \qquad (109)$$

Thus, λ and ϕ are easily found if α and θ are known.

We will usually deal with $\dot{\rho}$ rather than with Doppler shift. In these terms, the operation of the counter used as the measuring device in the navigation terminal is accurately represented by

$$\text{Counter output at } t_k = \int_{t_k - T}^{t_k} dt \left[\dot{\rho}\right] = \rho_k - \rho_{k-1} \qquad (110)$$

provided that the counter is reset at the beginning of each sampling interval of duration T or, equivalently, the first differences of the counter readings are recorded. Equation (110) shows that an accurate model of the measured data is obtained by taking first differences of the range ρ given by the analytic model. For the sake of simplicity, we will assume here that the epoch occurs at the midpoint of one of the sampling intervals so that the epochal measurement is zero and

$$t_k = \tau_o + \left[k + (1/2)\right] T$$

$$k = -N, \ldots, -1, 0, 1, \ldots, N$$

This restriction can be removed easily.

A program called EPOCH was written to provide tabulations of the epochal values of several of the important quantities in the satellite navigation system as functions of α and θ. A brief description of this program is given in Appendix 2. Expressions free of physical units for use in EPOCH and other programs were obtained by dividing distances by ρ_p and angular velocities by ω_p. This is equivalent to using the planet radius as the unit of distance and the planet angular velocity as the unit of angular velocity. The corresponding unit of time is the time required for the planet to turn through one radian. All of the final runs of EPOCH and other programs used $(\rho_s/\rho_p) = 1.1$, $\delta = 24^\circ$ and $(\omega_s/\omega_p) = 14.798$ corresponding roughly to the system parameters given in the literature (G&W 1960).

Classical Least-Squares Approach

The approach to the navigation signal-processor design described in the literature is based on the use of classical least-squares techniques to fit the range-rate measurements to the analytic model (G&W 1960). The latter is obtained by combining Equations (106) and (107) to obtain Equation (111) which equates the following expression to $\dot{\rho}/\omega_s \rho_s \cos\lambda$.

$$\frac{(\omega_p/\omega_s)\cos\omega_s t \sin(\omega_p t+\phi) + \sin\omega_s t \cos(\omega_p t+\phi) - \tan\lambda\cos\omega_s t}{\sqrt{(\rho_s/\rho_p)^2 + 1 - 2(\rho_s/\rho_p)\cos\lambda\left[\cos\omega_s t\cos(\omega_p t+\phi) + \tan\lambda\sin\omega_s t\right]}}$$

The least-squares procedure consists of the following steps.

(1) Form the $(2N+1) \times 2$ matrix $\underline{W}$ of partial derivatives of $\dot{\rho}$ with respect to λ and ϕ evaluated at each of the sampling instants t_k for some initial estimated values (λ_1, ϕ_1).

(2) Form the $(2N+1)$ -column $\underline{e}_1\rangle$ of elements $\dot{\rho}(t_k)$ for the initial estimates (λ_1, ϕ_1).

(3) Subtract $\underline{e}_1\rangle$ from the data column $\underline{d}\rangle$ to form the column of residuals.

(4) Solve for $\Delta\lambda, \Delta\phi$ in

$$\begin{bmatrix}\Delta\lambda \\ \Delta\phi\end{bmatrix} = (\tilde{\underline{W}}\,\underline{W})^{-1}\,\tilde{\underline{W}}\left[\underline{d}\rangle - \underline{e}_1\rangle\right] \tag{112}$$

(5) Select (λ_2, ϕ_2) $(\lambda_1 + \Delta\lambda, \phi_1 + \Delta\phi)$ as an improved estimate of (λ, ϕ). The procedure may be iterated as many times as desired. A typical stopping rule would be to continue until the magnitude of both $\Delta\lambda$ and $\Delta\phi$ were within preset bounds. The limit of (λ_n, ϕ_n) as n increases without limit is $(\hat{\lambda}, \hat{\phi})$, the least-squares estimate of (λ, ϕ) given $\underline{d}\rangle$. The norm of the

residuals for $(\hat{\lambda}, \hat{\phi})$ is a minimum over all of $\{(\lambda, \phi)\}$.

The least-squares procedure can be described in terms of linear algebra with the aid of Figure 28. The vector $|D\rangle$ is represented by the data and the $|E_n\rangle$ by the sequence of $(2N+1)$-columns corresponding to the sequence of estimates (λ_n, ϕ_n). The space $\mathcal{L}$ is the 2-parameter hypersurface containing all $(2N+1)$-dimensional vectors represented by columns with elements $\dot{\rho}(t_k)$. The space $\mathcal{L}_1$ is a 2-dimensional vector space with origin at $|E_1\rangle$ and with basis elements represented by the $(2N+1)$-columns of elements $\partial\dot{\rho}(t_k)/\partial\lambda$ and $\partial\dot{\rho}(t_k)/\partial\phi$. Solution for $\Delta\lambda$ and $\Delta\phi$ corresponds to perpendicularly projecting $|D\rangle - |E_1\rangle$ onto $\mathcal{L}_1$ to obtain $|P_1\rangle$. The limiting vector $|E\rangle$ is the perpendicular projection of $|D\rangle$ onto $\mathcal{L}$ and will lie near the vector $|T\rangle$ corresponding to the "true" values of λ and ϕ if the noise, $|D\rangle - |T\rangle$, is not too large compared to the size of any nearby folds of $\mathcal{L}$.

The navigation computer must employ digital techniques in order to obtain the required precision. A measure of complexity, suitable for preliminary system design purposes, is the total number of computer instructions required per pass. Unpublished results of a preliminary study of the navigation computer done in 1961 at the IBM Space Guidance Center indicated that the number of arithmetic instruction required for three iterations would be about 300 + (2N + 1)(500). A rough check on this estimate is available via reports on the complexity of the tracking computation using the classical least-squares approach (Lorens 1959). This more difficult problem requires several thousand instructions per data point.

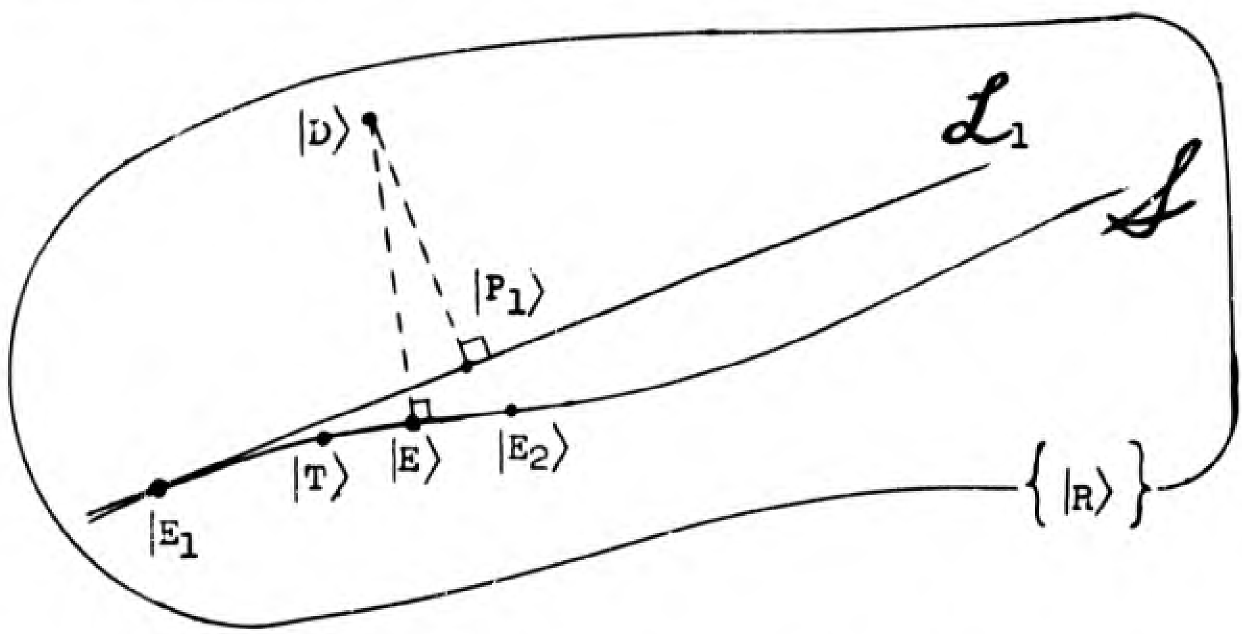


Figure 28 Least-Squares Estimation

4 - 2 THE SIGNAL-PROCESSING PROBLEM

One approach to the simplification of the navigation computation is to measure fewer points on the Doppler S-curve, with correspondingly larger effects of noise on the position estimates produced. In this section, we will present an approach to the design of the navigation terminal which uses all the data and yet provides a significant reduction in complexity.

Efficient Representation

The strategy employed here is based on the observation that the set of range-rate vs. time graphs appears to have a very simple structure. However, the standard approach uses several hundred dimensions to represent the set, and an extensive computation is required for each satellite passage. Perhaps the navigation computation ought to be factored into two steps as shown in Figure 29. The purpose of the first step is to represent the received signal efficiently, e.g., using 3 parameters rather than 601. The second step performs the same function as before, i.e., least-squares estimation of position given a received signal, and its complexity is greatly reduced since the least-squares fit is now to be carried out in 3-space rather than 601-space. Whether this strategy results in an actual simplification or a mere transfer of the problem depends on the complexity of the representation step.

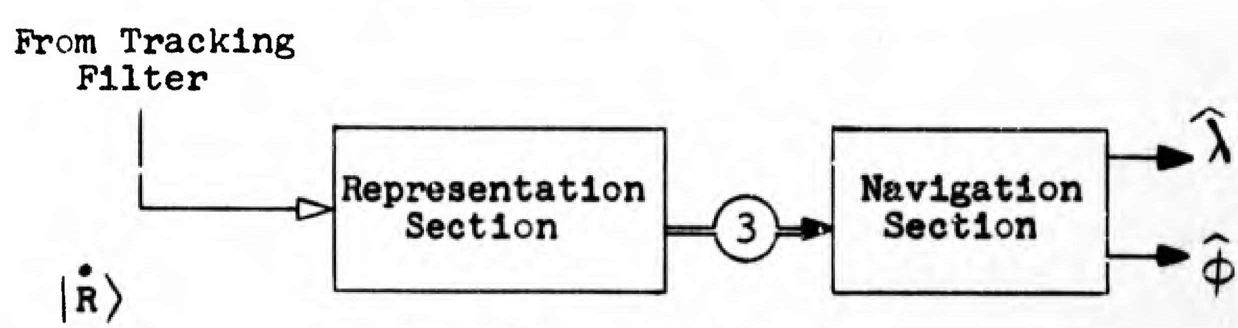


Figure 29 Strategy

Study of $\dot{\rho}$ vs. t graphs and of Equation (111) fails to show any simple choice of basis for a linear vector representation of the range-rate signal. We consider next the possibility of a linear basis of few dimensions for some simple function of $\dot{\rho}$, and we find that ρ^2 meets our requirements. This important result is emphasized by introducing the symbol σ for ρ^2, and is found from Equation (106) by letting $t = \mathcal{T}_o + \mathcal{T}$. We immediately find that σ can be represented as a linear combination of 7 terms.

$$\sigma(\mathcal{T}) = \sum_i \beta_i b_i(\mathcal{T}) \qquad i = 1,\ldots,7 \qquad (113)$$

Expressions for the 7 basis elements represented on $|\underline{t}$ and the corresponding 7 coordinates are given in Table 2. We may rewrite Equation (113) in our notation as

$$\underline{s}\rangle = \sum_i \underline{b_i}\rangle\, \beta_i \qquad (114)$$

Table 2 Representation of Range-Square

i	$b_i(\mathcal{T})$	$\beta_i/(2\,\rho_s/\rho_p)$
1	$\mathcal{T}^o$	$(1/2)\left[(\rho_s/\rho_p) + (\rho_p/\rho_s)\right]$
2	$\sin\,\omega_s\mathcal{T}$	$-\sin\lambda\,\cos\alpha$
3	$\sin\,\omega_s\mathcal{T}\,\sin\,\omega_p\mathcal{T}$	$-\cos\lambda\,\sin\alpha\,\sin\theta$
4	$\cos\,\omega_s\mathcal{T}\,\sin\,\omega_p\mathcal{T}$	$+\cos\lambda\,\cos\alpha\,\sin\theta$
5	$\cos\,\omega_s\mathcal{T}\,\cos\,\omega_p\mathcal{T}$	$-\cos\lambda\,\cos\alpha\,\cos\theta$
6	$\sin\,\omega_s\mathcal{T}\,\cos\,\omega_p\mathcal{T}$	$+\cos\lambda\,\sin\alpha\,\cos\theta$
7	$\cos\,\omega_s\mathcal{T}$	$-\sin\lambda\,\sin\alpha$

Typical variation of $\omega_s \tau$ is from -12° to 12°, and $\omega_p \tau$ from about $-3/4^\circ$ to $3/4^\circ$. Inspection of the $b_i(\tau)$ indicates that they are almost linearly dependent on some sub-basis of dimensionality considerably less than 7. A computer program called BASIS was written to investigate the $b_i(\tau)$ at 49 points, $-NT \leq \tau \leq NT$, $N = 24$, $NT = 12^\circ$. Results from this program showed clearly that elements 4 - 7 were almost exactly dependent on elements 1 - 3. The correlations between each of the first 3 elements and each of the 7 are given in Table 12 in Appendix 3. Thus, the range-square signal can be represented as a vector in 3 dimensions with small error and with no error at all in 7 dimensions. We will choose the 3-dimensional representation and, despite the fact that the error involved here is negligible, we will take the trouble later to compensate for it exactly. We will call the 7-dimensional basis $|\underline{A}$ and the 3-dimensional sub-basis $|\underline{B}$. Representation of the latter on $|\underline{P}$ is given in Table 14 and its utility in representing a typical ensemble $\{|S\rangle\}$ is shown in Table 15. Note that neither $|\underline{A}$ nor $|\underline{B}$ is orthonormal. The corresponding subspaces are $\mathcal{A}$ and $\mathcal{B}$ where $\mathcal{B} \subset \mathcal{A}$.

Presumably, we can also find some basis $|\underline{M}$ on which to represent the range-rate signal (and, thus, the range signal) such that Dim $\mathcal{M}$ is considerably less than 601.

We could proceed by selecting the elements of $|\underline{M}$ arbitrarily, e.g., by orthonormalizing the power functions of $\mathcal{T}$. Alternatively, we could construct an ensemble of typical range signals and find the largest eigencolumns of the covariance matrix of the ensemble. Both techniques were used in the computer programs to be described later. For the moment, we merely observe that these techniques exist and that we cannot expect them to lead to such an efficient representation as in the case of the range-square signals.

The signal-processing problem comes down to finding a way to reconcile two facts.

(1) Convenient and effective apparatus are available for measurement of range-rate.

(2) The most efficient linear representation known involves range-squared.

Tensor Product Approach

The two facts stated above were noted very early in the present study. A simple and effective way to take advantage of both facts did not appear until the problem had been considered as a possible application of finite-dimensional tensor product spaces. The direction in which to proceed is clear after the discussion on time-domain multiplication in Section 3 - 3.

Let $|\dot{R}\rangle$ and $|R\rangle$ be the range-rate and range signals in $\mathcal{S}$, where $\mathrm{Dim}\,\mathcal{S} = 2N + 1$. Let $|S\rangle$ be the range-square signal such that

$$|S\rangle \in \mathcal{A} \subset \mathcal{P} \subset \mathcal{H}\otimes\mathcal{H}$$

We will also be interested in the perpendicular projection of $|S\rangle$ onto $\mathcal{B}$. Therefore, if $\underset{2}{\underline{D}}|\underline{B} = \underline{I}$, we have

$$|\underline{B}\ \underline{D}|S\rangle \in \mathcal{B} \subset \mathcal{A} \subset \mathcal{P} \subset \overset{2}{\mathcal{H}} \subset \mathcal{H}\otimes\mathcal{H}$$

where $\overset{2}{\mathcal{H}}$ is the set of all elements of $\mathcal{H}\otimes\mathcal{H}$ of the form

$$R\rangle \otimes R\rangle = \overset{2}{R}\rangle$$

The dimensionality of $\overset{2}{\mathcal{H}}$ is given by

$$\text{Dim}\ \overset{2}{\mathcal{H}} = \frac{n(n+1)}{2} \tag{115}$$

if $\text{Dim}\,\mathcal{H} = n$.

Advantage of the restriction here to tensor squares of vectors in $\mathcal{H}$ can be taken by <u>symmetrizing</u> the basis in $\mathcal{H}\otimes\mathcal{H}$. Let $|\underline{t}$ be an orthonormal basis in $\mathcal{H}$ and let the corresponding tensor product basis in $\mathcal{H}\otimes\mathcal{H}$ be permuted to the sequence:

$$(1,\ 1),(2,\ 2),\ldots(n,\ n);\begin{Bmatrix}1, & 2\\ 2, & 1\end{Bmatrix},\begin{Bmatrix}1, & 3\\ 3, & 1\end{Bmatrix},\ldots,\begin{Bmatrix}n-1, & n\\ n, & n-1\end{Bmatrix}.$$

Note that the numbers of elements in the three sub-bases are: n; $n(n-1)/2$; $n(n-1)/2$. Let every element $|B_i\rangle \otimes |B_j\rangle$ in the second sub-basis be replaced with $(1/\sqrt{2})\left[|B_i\rangle \otimes |B_j\rangle + |B_j\rangle \otimes |B_i\rangle\right]$ and every element of the third with $(1/\sqrt{2})\left[|B_i\rangle \otimes |B_j\rangle - |B_j\rangle \otimes |B_i\rangle\right]$. The resulting basis is clearly orthonormal; it is called the <u>symmetrized basis</u> in $\mathcal{H}\otimes\mathcal{H}$. The first $n(n+1)/2$

elements of the symmetrized basis, i.e., the first two sub-bases given above, spans $\overset{2}{\mathcal{S}}$. All $n(n-1)/2$ coordinates on the third sub-basis of any vector of the form $|R\rangle \otimes |R\rangle$ are identically zero.

The symmetrizing step is accomplished by multiplying the permuted basis by the matrix

$$\underline{U} = \frac{1}{\sqrt{2}} \left[\begin{array}{c|c|c} \underline{I}\sqrt{2} & \underline{0} & \underline{0} \\ \hline \underline{0} & \underline{I} & \underline{I} \\ \hline \underline{0} & \underline{I} & -\underline{I} \end{array}\right]$$

where the partitioning is as given above. Note that $\underline{U}$ is both unitary and self-adjoint. A tensor product basis formed on any sub-basis in $\mathcal{S}$ can be symmetrized with the advantage of reducing the dimensionality required by almost a factor of two.

The basic scheme requires that we obtain a representative of $|R\rangle$ from the range-rate measurements representing $|\dot{R}\rangle$. We postpone discussing the "d-c restoration" problem of establishing the proper constant of integration. The subsequent steps in the representation process, described in abstract terms, are the following.

(1) Project $|R\rangle$ onto $\mathcal{M}$ to form $|\underline{M}\,\underline{\tilde{M}}|R\rangle$.

(2) Take the tensor product of the resulting vector with itself to form $|\overset{2}{\underline{M}}\,\overset{2}{\underline{\tilde{M}}}|\overset{2}{R}\rangle \in \overset{2}{\mathcal{M}} \subset \overset{2}{\mathcal{S}} \subset \mathcal{S}\otimes\mathcal{S}$.

(3) Project the resulting vector onto $\mathcal{B}$ (equiva-

lently: onto $\mathcal{P}$, then onto $\mathcal{A}$, then onto $\mathcal{B}$) to form $|\underline{B}\ \underline{D}|\overset{2}{\underline{M}}\ \overset{2}{\underline{M}}|\overset{2}{R}\rangle$.

The signal vector diagram of the processor is shown in Figure 30 where all of the operations on $|R\rangle$ are referred to the tensor product space. Note that all projections are perpendicular.

The navigation section can be designed to fit $|\underline{B}\ \underline{D}|S\rangle$ rather than $|S\rangle$ so that the small difference between these two vectors need have no effect on the final estimates of position. Thus, the representation error produced by the lack of completeness of $|\underline{M}$ is the difference between $|\underline{B}\ \underline{D}|\overset{2}{\underline{M}}\ \overset{2}{\underline{M}}|\overset{2}{R}\rangle$ and $|\underline{B}\ \underline{D}|S\rangle$. Since $\mathcal{B} \subset \mathcal{P}$, we have

$$|\underline{B}\ \underline{D}|S\rangle = |\underline{B}\ \underline{D}|\underline{P}\ \underline{P}|\overset{2}{R}\rangle = |\underline{B}\ \underline{D}|\overset{2}{R}\rangle \tag{116}$$

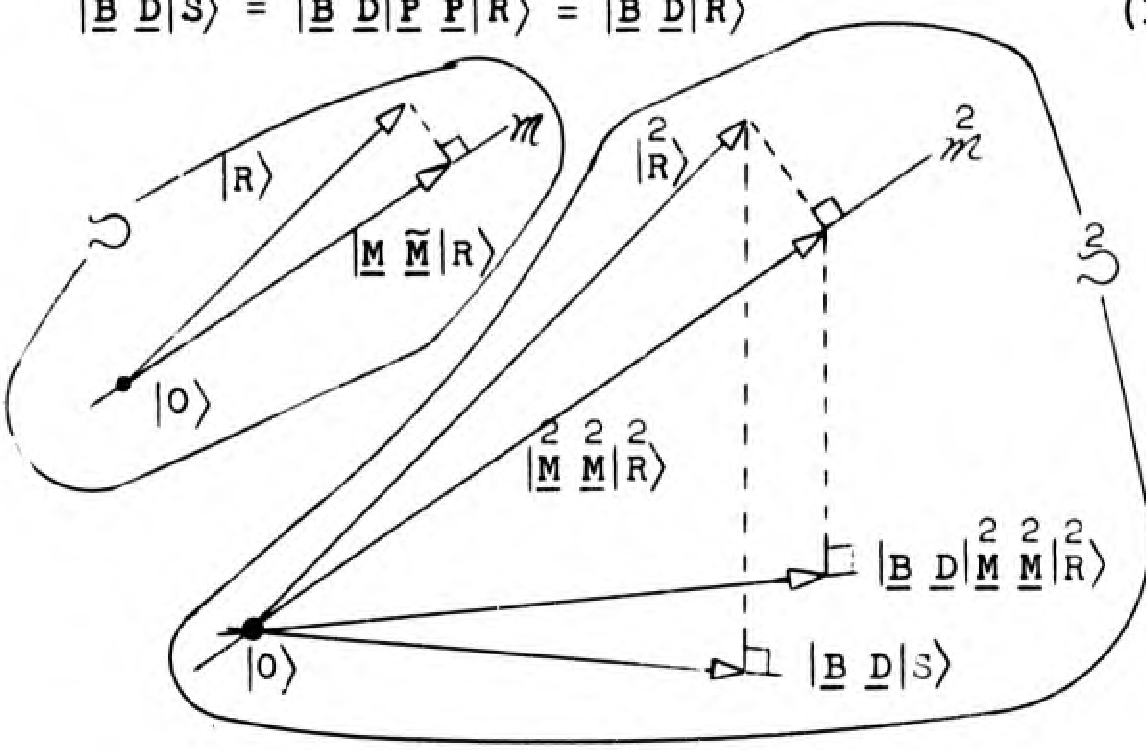


Figure 30 Vector Diagram of the Processor

The basic scheme of the representation section of the proposed processor is shown in Figure 31 where the d-c restoration problem has been ignored. Note that

$$|\underline{M}\ \tilde{\underline{M}}|R\rangle \otimes |\underline{M}\ \tilde{\underline{M}}|R\rangle = \left[|\underline{M} \otimes |\underline{M}\right]\left[\tilde{\underline{M}}| \otimes \tilde{\underline{M}}|\right]\left[|R\rangle \otimes |R\rangle\right] = |\underline{M}^{2}\ \tilde{\underline{M}}^{2}|\ R^{2}\rangle$$

This diagram is intended as an illustration of the abstract level of thinking about system concepts rather than actual hardware. If we were planning to instrument the various elements using analog techniques, we might call $|S|$ an "integrator", but a more appropriate term here is "accumulator".

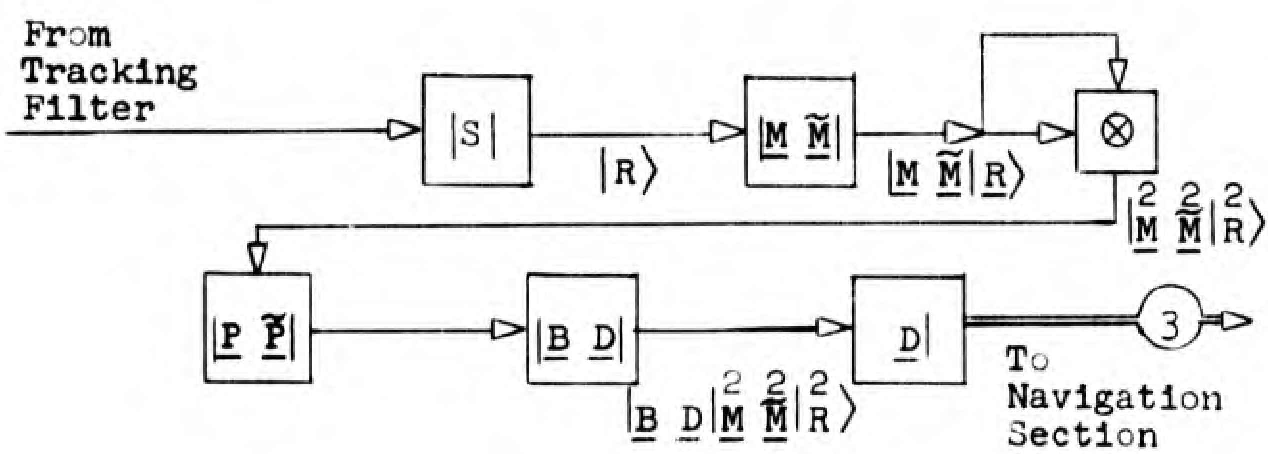


Figure 31 Processor Design - Step 1

D-C Restoration

The range signal $|R\rangle$ has a substantial d-c component (i.e., its representative on the time domain has a substantial constant term). We are given, not $|R\rangle$, but the column $\dot{\underline{r}}\rangle$ of data equal (neglecting noise) to the first differences of $\tilde{\underline{t}}|R\rangle = \underline{r}\rangle$. The vector represented by $\dot{\underline{r}}\rangle$ is $|\dot{R}\rangle$.

However, there are some constraints in the system under discussion which permit the restoration of the d-c term with accuracy sufficient to meet the system requirements. The proposed d-c restoration scheme is based on three points: (1) the epochal value of the second time derivative of range-square, i.e., $\ddot{\sigma}_0$, is practically a constant over all possible passes, (2) $\ddot{\rho}_0$ is easily measured, given $\tilde{\underline{M}}|S|\dot{R}\rangle$, (3) the following simple derivation.

$$\sigma = \rho^2 \qquad , \quad \dot{\sigma} = 2\rho\dot{\rho}$$
$$\ddot{\sigma} = 2\rho\ddot{\rho} + 2(\dot{\rho})^2 \quad , \quad \ddot{\sigma}_0 = 2\,\rho_0\ddot{\rho}_0 + 0$$
$$\rho_0 = \ddot{\sigma}_0/2\ddot{\rho}_0 \tag{117}$$

The near constancy of $\ddot{\sigma}_0$, as can be seen from Table 10 in Appendix 3, allows an accurate estimate of $\ddot{\sigma}_0$ to be obtained from the a priori estimate of position which any competent navigation system would always have available. If we attempted to measure $\ddot{\rho}_0$ directly from $\dot{\underline{r}}\rangle$, noise might cause difficulty. One of the reasons, besides simplicity, for introducing the measurement basis $|\underline{M}$ was to provide smoothing so that a reasonable estimate of $\ddot{\rho}_0$ could be found by operating on $\tilde{\underline{M}}|S|\dot{R}\rangle$, or $\tilde{\underline{M}}|R\rangle$, with the row $\langle\underline{F}$ defined by the elements

$$\frac{\langle \tilde{t}_{-1} | M_k \rangle - 2\langle \tilde{t}_0 | M_k \rangle + \langle \tilde{t}_1 | M_k \rangle}{T^2} \qquad k = 1, \ldots, m = \text{Dim} \, |\underline{M} \tag{118}$$

Thus, a reasonable approach to a solution of the d-c restoration problem is to use the procedure just described to find one point on the representative of range on the time basis, i.e., ρ_0. It is convenient to define $|S|$ so that it operates on $|\dot{R}\rangle$ to yield a vector which has zero for its epochal coordinate. Then, the d-c restoration step is effected by adding ρ_0 to all coordinates of $\underline{\tilde{t}}|S|\dot{R}\rangle$. Since we have assumed that projection onto $\mathcal{M}$ has already taken place, it is desirable that the d-c restoration step be accomplished by operations on the representative on $|\underline{M}$ rather than that on $|\underline{t}$. This is accomplished most conveniently by defining $|M_1\rangle$ to be d-c. In that case, the d-c restoration step consists merely in adding $\rho_0 \sqrt{2N+1}$ to the d-c coordinate of $\underline{\tilde{M}}|S|\dot{R}\rangle$, leaving the other m-1 coordinates unchanged. For this design, the first coordinate of $\langle\underline{F}$ is zero.

4 - 3 DEVELOPMENT OF THE PROCESSOR

In this section the basic design approach is developed into a proposed processor configuration of operations which are clearly realizable. This is accomplished by rewriting the abstract version of the system diagram in terms of representatives.

Bases and Subspaces

A brief review of the several bases and corresponding subspaces involved in the problem seems appropriate. In the tensor product space $\mathcal{S} \otimes \mathcal{S}$, we have the bases $|\underline{B}$, $|\underline{A}$, $|\underline{P}$, $|\overset{2}{\underline{M}}$ and $|\underline{t} \otimes |\underline{t}$ of dimensionality 3, 7, 601, $m(m+1)/2$ and $(601)^2$; in $\mathcal{S}$, the bases $|\underline{M}$ and $|\underline{t}$ of dimensionality m and 601. In general, Dim $|\underline{t} = 2N+1$. Initially, it seemed that $|\underline{M}$ would require not more than 10 dimensions; results of the computer experiments indicate that a design based on $m = 5$ yields accurate estimates. We have also introduced $\underline{D}|$, the dual of $|\underline{B}$.

All of these bases are referred to the epoch. The measured value of the epoch is also used in the navigation computation. The method proposed for this measurement is the determination of the instant when $\dot{\rho}(t) = 0$, i.e., the instant when the frequency of the tracking filter output equals the offset frequency.

Note that the epoch τ_o occurs almost exactly T/2 sec before the axis-crossing τ_1 defined by the data column $\dot{\underline{r}}\rangle$. This can be shown via Figure 32 where

$$\dot{\rho}(t) = t - \tau_o$$

for t near τ_o, and the time origin is shifted to the sampling instant prior to the epoch for convenience. The measured coordinates of $\dot{\underline{r}}\rangle$ at $t = T$ and $t = 2T$ are given by

$$(T - \tau_o)^2/2 - \tau_o^2/2 \quad \text{and} \quad (T/2)\left[(T - \tau_o) + (2T - \tau_o)\right]$$

Then, τ_o is found by

$$\frac{T - \tau_1}{(1/2)\left[T^2 - 2T\,\tau_o\right]} = \frac{2T - \tau_1}{(T/2)\left[3\,T - 2\,\tau_o\right]}$$

$$T^2 - 2T\,\tau_1 + 2T\,\tau_o = 0$$

$$\tau_o = \tau_1 - (T/2) \qquad (119)$$

In order to reduce the effect of noise on the epoch measurement, the axis-crossing defined by the best-fitting straight line defined by several data points in the vicinity of the epoch could be used. A convenient design results if several data points up to and including the first positive one are used to measure τ_o. In the computer experiments, τ_o was assumed to occur midway between sampling instants labeled -1 and 0.

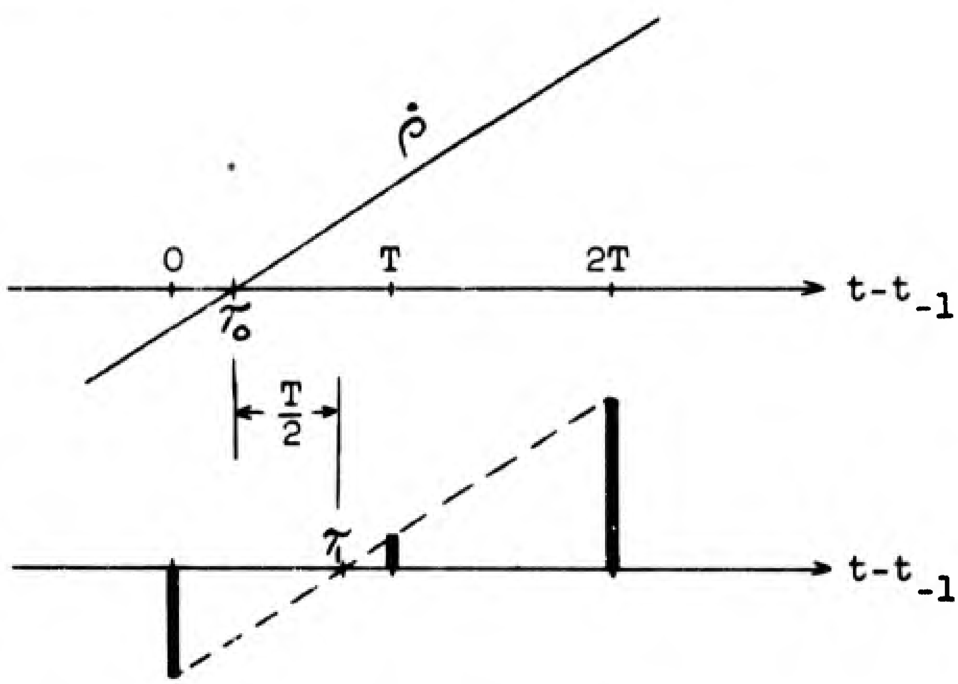


Figure 32 Epoch Measurement

The second step in evolving a processor design is obtained by inserting projection operators at appropriate points in the system diagram, moving $|\underline{M}$ through the tensor product step, using $\underline{D}|\underline{B}\,\underline{D}| = \underline{D}|$, regrouping factors, and inserting the d-c restoration step. The resulting configuration is shown in Figure 33. Note that $\tilde{\underline{t}}|$ symbolizes the combined operations of counting, sampling, epoch measuring and recording. The vectors and columns used as labels in system diagrams correspond to an idealized error-free situation.

The column $\overset{2}{\tilde{\underline{M}}}|\overset{2}{R}\rangle$ and the matrix $\tilde{\underline{P}}|\overset{2}{\underline{M}}$ are represented on the symmetrized basis defined by $|\underline{M}$. The symmetrized Kronecker product operation symbolized by $\otimes$ in the system diagram implies the following simple transformation.

$$\tilde{\underline{M}}|R\rangle = \begin{array}{c|c|} 1 & \mu_1 \\ 2 & \mu_2 \\ 3 & \mu_3 \\ 4 & \mu_4 \\ 5 & \mu_5 \end{array} \quad \Longrightarrow \quad \overset{2}{\tilde{\underline{M}}}|\overset{2}{R}\rangle = \begin{array}{c|c|l} (1,1) & \mu_1^2 & \uparrow \\ \vdots & \vdots & m \\ (5,5) & \mu_5^2 & \downarrow \\ (1,2) & \mu_1\mu_2\sqrt{2} & \uparrow \\ (1,3) & \mu_1\mu_3\sqrt{2} & \\ \vdots & \vdots & \frac{m(m-1)}{2} \\ (4,5) & \mu_4\mu_5\sqrt{2} & \downarrow \end{array}$$

This operation can be further simplified by absorbing the $\sqrt{2}$ factors in the subsequent matrix multiplication step.

The matrix $\tilde{\underline{t}}|S|\underline{t}$ accumulates $\dot{\underline{r}}\rangle$ outward from the

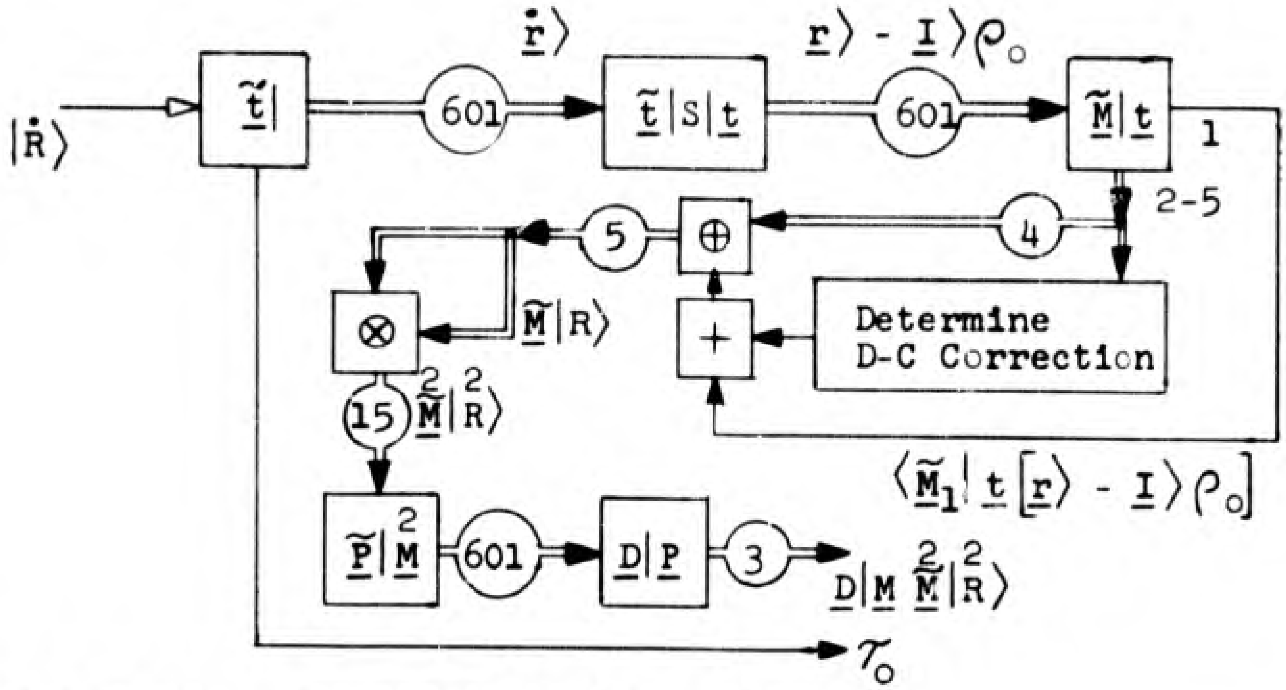


Figure 33 Processor Design - Step 2

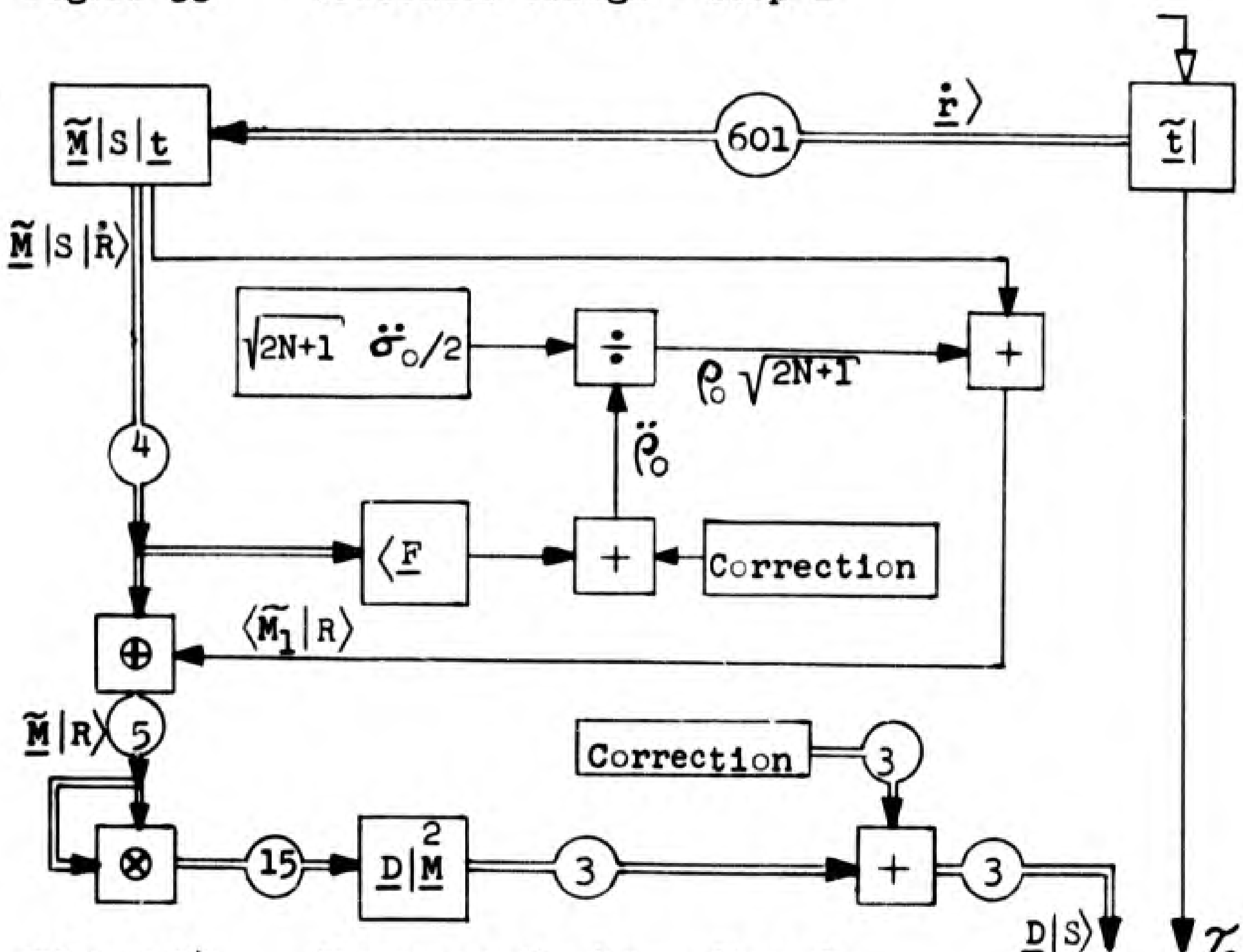


Figure 34 Processor Design - Step 3

epoch so that the epochal coordinate of $\underline{r}\rangle$ is zero.

$$\tilde{\underline{t}}|S|\underline{t}\ \ \dot{\underline{r}}\rangle = \underline{r}\rangle - \underline{I}\rangle\,\rho_o \qquad (120)$$

For N = 3, Equation (120) stands for

$$\begin{array}{c|ccccccc|}
 & -3 & -2 & -1 & 0 & 1 & 2 & 3 \\
\hline
-3 & 0 & -1 & -1 & -1 & & & \\
-2 & & 0 & -1 & -1 & & & \\
-1 & & & 0 & -1 & & & \\
0 & & & & 0 & & & \\
1 & & & & & 1 & & \\
2 & & & & & 1 & 1 & \\
3 & & & & & 1 & 1 & 1 \\
\hline
\end{array}
\quad
\begin{array}{c|c|}
\hline
-3 & \rho_{-3}-\rho_{-4} \\
\hline
-2 & \rho_{-2}-\rho_{-3} \\
\hline
-1 & \rho_{-1}-\rho_{-2} \\
\hline
0 & \rho_{0}-\rho_{-1} \\
\hline
1 & \rho_{1}-\rho_{0} \\
\hline
2 & \rho_{2}-\rho_{1} \\
\hline
3 & \rho_{3}-\rho_{2} \\
\hline
\end{array}
\quad = \quad
\begin{array}{c|c|}
\hline
-3 & \rho_{-3}-\rho_{0} \\
\hline
-2 & \rho_{-2}-\rho_{0} \\
\hline
-1 & \rho_{-1}-\rho_{0} \\
\hline
0 & \rho_{0}-\rho_{0} \\
\hline
1 & \rho_{1}-\rho_{0} \\
\hline
2 & \rho_{2}-\rho_{0} \\
\hline
3 & \rho_{3}-\rho_{0} \\
\hline
\end{array}$$

The d-c correction is obtained by

$$\langle \tilde{\underline{M}}_1|\underline{t}\ \underline{I}\rangle\,\rho_o = \rho_o \sum_{i=1}^{2N+1} \left[\frac{1}{\sqrt{2N+1}}\right] [\,1\,]$$

$$= \sqrt{2N+1}\ \rho_o = \sqrt{2N+1}\ \ddot{\sigma}_o/(2\,\ddot{\rho}_o) \qquad (121)$$

<u>The Proposed Processor</u>

The final configuration of the proposed processor is shown in Figure 34 and is obtained from the previous design by inserting two minor correction steps, showing the d-c restoration in more detail, and combining cascaded matrix multiplication steps. The latter is done in two places. The 5x601 matrix $\tilde{\underline{M}}|\underline{t}$ and 601x601 matrix $\tilde{\underline{t}}|S|\underline{t}$ are replaced by the 5x601 matrix $\tilde{\underline{M}}_2|S|\underline{t}$, and the 3x601 matrix $\underline{D}|\underline{P}$ and 601x15 matrix $\tilde{\underline{P}}|\underline{M}_2$ are replaced by the 3x15 matrix $\underline{D}|\underline{M}_2$.

The latter step is the culmination of a sequence of simplifications beginning with a square matrix of 361,201 rows! Fortunately, we have not had to calculate matrices of such proportions, but only to think about them.

The two correction steps take advantage of the a priori estimate of position assumed to be available. The magnitude of the corrections are a measure of the incompleteness of the basis $|\underline{M}$ and will be quite small for appropriate choices of $|\underline{M}$. The table of $\ddot{\rho}_o$ corrections is found by

$$\ddot{\rho}_o \text{ correction } = \ddot{\rho}_o - \langle \underline{F}\ \underline{\tilde{M}} | R \rangle \qquad (122)$$

via the analytic model evaluated at some convenient set of equispaced latitudes and longitudes. Similarly, the representation corrections are given by

$$\underline{D}|S\rangle \text{ correction } = \underline{D}|S\rangle - \underline{D}|\overset{2}{\underline{M}}\ \overset{2}{\underline{\tilde{M}}}|\overset{2}{R}\rangle \qquad (123)$$

Then, if the a priori position estimate happended to be exact, the error in the estimate produced by the processor would be due only to noise, which we will consider, plus ionospheric refraction and other sources of error neglected here.

The operation of the processor is described fairly completely by Figure 34 and previous discussion of various components with the exception of the data recording steps. The processor must include storage for the m coordinates of $\underline{\tilde{M}}|S|\dot{R}\rangle$ and for the 301 coordinates of $\dot{\underline{r}}\rangle$ through the epoch, since we assume that the epoch is not known (with sufficient accuracy) until it has occurred. At the beginning of the measurement cycle (prior to the instant $\mathcal{T} = -NT$), data points are measured at equal intervals of duration T and stored sequentially in the 301-register memory. The 302nd measurement is stored in register 1 in place of the first measurement, and the process continues to cycle until the epochal coordinate is recognized and stored. An index counter is used to record the number of the last register used.

During the post-epoch period, measurements continue to be made at equal intervals. After the +kth measurement, the m multiplications and additions connected with the +kth coordinate of $\underline{\dot{r}}\rangle$ in accumulating the m terms of

$$\underline{\widetilde{M}}|S|\dot{R}\rangle = \underline{\widetilde{M}}|S|\underline{t} \quad \underline{\dot{r}}\rangle$$

are executed along with the same steps associated with the -kth coordinate. All of this is to be done in the interval of duration T prior to the $(k+1)$th sampling instant. As each coordinate measured during the pre-epoch interval is used, the index count is reduced by one, and one register is freed for other purposes if needed.

Prior to the beginning of the pass, the a priori estimate of position is used by the processor to determine corresponding estimates of $\ddot{\sigma}_0 \sqrt{2N+1}\,/2$, the two corrections, and the initial values of the 1 x 3 and 3 x 1 matrices required for the navigation calculation. These steps are accomplished by table look-up and interpolation.

After coordinate +N has been measured and coordinates -N and +N used in completing the calculation of $\underline{\tilde{M}}|S|\dot{R}\rangle$, the subsequent operations shown in Figure 34 may be initiated. The remaining calculations in the representation section consist of 64 multiplications, 54 additions and 1 division. Each iteration of the least-squares procedure in the navigation section will require 6 multiplications and 11 additions for the actual calculation of the estimate. Each recalculation of the initial values of the required 1 x 3 and 3 x 1 matrices in the navigation section will require 6 table look-up and interpolation operations. Experience indicates that not more than 3 iterations will normally be required and the 1 x 3 matrix may not have to be recalculated for each iteration. Exactly what balance should be made between small tables and high-order interpolation vs. large tables and linear interpolation will not be considered here. We have probably already delved into more detail than necessary for the present purpose.

The point in presenting design details is to show that the proposed processor does actually simplify the navigation calculation. All but a small part of the representation calculation takes place during the measurement phase at the rather slow rate of 2m multiplications and 2m additions per sampling interval of duration T, e.g., 2m = 10 and T = 1 sec. This is slow even for a serial computer with a one-bit arithmetic element. Even when allowance is made for the calculations necessary to account for the lack of synchronism of the epoch with the sampling clock, the $\underline{\tilde{M}}|S|\dot{R}\rangle$ calculation does not tax any computer which might be considered for this application.

Consider the computing speed requirement dictated by the time which can be tolerated between the end of the pass and the availability of the newly computed position estimate. As we have seen, only a few hundred arithmetic operations need be performed in this interval, since the 3005 multiplications and 3005 additions involved in finding $\underline{\tilde{M}}|S|\dot{R}\rangle$ are completed during the pass. Since the simplest modern computers operate at several hundred multiplications per second, it is difficult to see how the delay required for post-pass computation could be a significant problem.

Thus, the proposed processor poses no problem as far as the cost of computation is concerned. Any computing capacity not otherwise needed can be devoted to more sophisticated interpolation in order to reduce storage needed for tables. Also, the system designer is free to increase the number of data points by a considerable factor if desired, without increasing the cost of the arithmetic element of the processor. On the other hand, the classical least-squares approach leads to a processor design in which the number of data points is limited somewhat by the cost of computation (G&W 1960 p 510).

Improvement of the Processor Design

Once the basic configuration of the processor is established, practically all design steps depend on the choice of measurement basis $|\underline{M}$. Some of the computer experiments were based on arbitrary choices of the elements of $|\underline{M}$. Results indicated that a satisfactory processor design could be developed in this manner.

However, it is fairly easy to obtain an improved processor design, for fixed Dim $|\underline{M}$, by judicious choice of the elements of $|\underline{M}$. If $|\underline{M}$ is to provide a best fit to the ensemble $\{|R\rangle\}$, then $|\underline{M}$ should involve the ensemble mean and the m-1 largest eigenvectors of the ensemble covariance. This approach was studied by means of a computer program called RANGE which generated a typical ensemble of signals, constructed the eigenvector basis, and calculated the representation errors. A modification of RANGE is included in all subsequent test programs, and results from this subprogram are given in

Table 20 in Appendix 3. These results indicate that Dim $|\underline{M}$ ought to be about 5 and that each of the representatives on $|\underline{t}$ of the mean vector and first 4 eigenvectors is almost symmetric about the epoch, i.e., almost an even function of τ.

Results of some of the early computer tests of various designs based on arbitrary choices for $\tilde{\underline{t}}|\underline{M}$ indicated that it ought to include at least one odd, or nearly odd, function of τ. Such a choice tends to concentrate the capability of measurement of the sign of θ, the relative longitude. As noted earlier, the d-c restoration portion of the processor is simplified if the first element of $|\underline{M}$ is d-c.

A compromise design used in the later phases of feasibility testing was developed using the $\tilde{\underline{t}}|\underline{M}$ defined by the following procedure.

(1) Let the first element be constant. Normalize τ^0 to obtain $\tilde{\underline{t}}|M_1\rangle$, i.e., $\tilde{\underline{t}}|M_1\rangle = \underline{I}\rangle\,(1/\sqrt{2N+1})$.

(2) Let the second element be linear in τ. Orthonormalize τ^1 to obtain $\tilde{\underline{t}}|M_2\rangle$.

(3) Project $\{|R\rangle\}$ on the partial basis, subtract the projections, find the mean of the residuals represented on $|\underline{t}$, and orthonormalize to find $\tilde{\underline{t}}|M_3\rangle$.

(4) Project residuals on the partial basis, subtract the projections, find the covariance matrix $\underline{C}$ of the

new residuals represented on $|\underline{t}$, find the largest eigencolumn of $\underline{C}$, and orthonormalize to find $\underline{\tilde{t}}|M_4\rangle$.

(5) Repeat the preceding step to find $\underline{\tilde{t}}|M_5\rangle$.
The fourth step can be repeated until the desired accuracy of representation is achieved.

The relative longitude is measured with some difficulty for near-overhead passes. This fact is pointed out in the literature (G&W 1960 p 511) and was confirmed by some of the early computer tests in the present project. In other words, the sign of θ is hard to measure when $\|\theta\|$ is small. Worst-case distance errors in the final estimates produced by the processor occur when $\|\theta\|$ is small, e.g., less than 1°. The difficulty in measuring small values of θ justifies giving it particular attention in designing the processor.

The compromise design of $|\underline{M}$ includes only one element, $|M_2\rangle$, which is designed to measure the sign of θ . The other four elements are either even or nearly so. Thus, improvement can be expected if any arbitrarily chosen $|M_2\rangle$ is replaced with one designed to measure the sign of θ when $\|\theta\|$ is small. Suppose we ignore, for the moment, the fact that one element has already been chosen and ask what is the best choice of a basis element $|B\rangle$, for our present requirement. The element we seek must be such that $\langle\tilde{B}|$ yields maximum difference in response to the two inputs shown in Figure 35 for fixed θ.

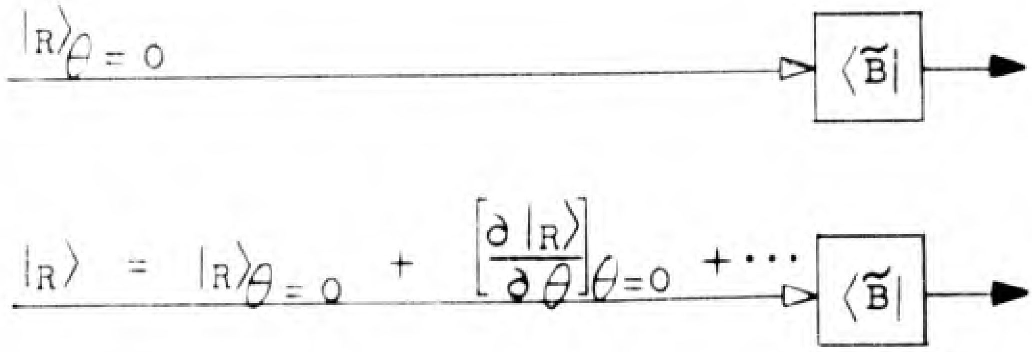


Figure 35 Measurement of Small θ

In other terms, we seek $|B\rangle$ satisfying

$$\max_{\langle\tilde{B}|} \left\{ \langle\tilde{B}| \left[\frac{\partial |R\rangle}{\partial\theta}\right]_{\theta=0} \right\} \quad \text{and} \quad \langle\tilde{B}|B\rangle = 1$$

From the matched-filter theorem (Mason and Zimmerman 1960) well-known to engineers, or from basic principles of linear algebra, it is clear that the best choice of $|B\rangle$ is obtained by letting

$$|B\rangle = \frac{\left[\frac{\partial |R\rangle}{\partial\theta}\right]_{\theta=0}}{\text{magnitude of numerator}} \tag{124}$$

The question of what α to use in Equation (124) is left open; $\langle\tilde{t}|M_2\rangle$ is nearly odd for small α and nearly even for large α. In the most recent runs of the feasibility test programs, a value of α corresponding to the center of the test ensemble was selected. The resulting basis represented on $|\underline{t}$ is given in Table 16, and its utility in representing a typical ensemble $\{|R\rangle\}$ is shown in Table 17.

Navigation Section

Little attention was given in the present project to the design of the navigation section of the proposed processor since little was needed. The novel features of the processor are all contained in the representation section. The classical least-squares procedure used in the navigation section is described in Section 4-1.

However, some work was done on the navigation section to permit transformation of errors found in various tests of the representation section into corresponding errors in final position estimates. For this reason, a program called L-S was written which generates, for several possible navigator positions, various partial derivatives useful in studying the navigation section alone or in combination with the representation section. This program is described in Appendix 2.

Two approaches to the design of the navigation section are possible. Elsewhere in this dissertation we limit attention to Method 1 where an estimate of epochal satellite latitude α is found by multiplying the measured epoch

τ_o by ω_s. The relative longitude θ is then estimated by a least-squares fit of $\underline{D}|S\rangle$ given by the analytic model to the 3 measured coordinates of $\underline{D}|S\rangle$ at the output of the representation section. Method 2 uses the least-squares procedure to fit both α and θ to all 4 measurements produced by the representation section. Both methods are considered in the program L-S. At this writing, it is not clear whether Method 2 provides sufficiently greater accuracy to justify the somewhat greater complexity required.

Instead of θ, it is convenient to deal with east-west distance ν_1 expressed as a fraction of ρ_p. The matrix used in the navigation section to transform the difference between measurement and initial estimate of $\underline{D}|S\rangle = [\sigma_1]$ into a correction in east-west distance ν_1 is the row $[\partial \nu_1 / \partial \sigma_1]$. It is found by taking the pseudo-inverse of the column $[\partial \sigma_1 / \partial \nu_1]$ which corresponds in principle to the matrix $\underline{W}$ of Equation (112). For example, the matrix which transforms small changes in $\underline{D}|S\rangle$ into corresponding changes in ν_1 for points near $(\alpha, \theta) = (30^\circ, 5^\circ)$ is

$$[\partial \nu_1 / \partial \sigma_1] = (\tilde{\underline{W}}\ \underline{W})^{-1}\tilde{\underline{W}}$$

$$= \begin{array}{ccc} 1 & 2 & 3 \\ \hline \multicolumn{1}{|c|}{602.} & \multicolumn{1}{c|}{-.423} & \multicolumn{1}{c|}{-8.46} \\ \hline \end{array}\ 10^{-3} \qquad (125)$$

Values of this matrix for other positions are given in Table 11 of Appendix 3.

The matrix $\tilde{\underline{W}}\,\underline{W}$ in the least-squares procedure is called the normal matrix and its determinant is an excellent measure of the sensitivity of the system to various sources of error. In regions where Det $\tilde{\underline{W}}\,\underline{W}$ is very small, large errors in the position estimates produced by the processor may be caused by small errors due to noise, etc. As shown in Table 13 in Appendix 3, Det $\tilde{\underline{W}}\,\underline{W}$ is small when $\|\theta\|$ is small, and this effect is most pronounced for $\alpha = 0$. Thus, if navigation error magnitudes due to any fixed cause are plotted as a function of α and θ, we can expect a steep ridge of error centered on the sub-track of the satellite with broad low areas on either side and a peak in the ridge at the equator. Thus, it may be necessary to reject longitude estimates and to use only the latitude estimates obtained from near-overhead passes, particularly those near the equator.

4 - 4 EVALUATION OF THE PROCESSOR

The various sources of error affecting the representation section of the processor and the tests which have been made on the magnitude of their effects are described in this section.

Feasibility Testing

In addition to the computer programs mentioned earlier in this chapter, a program called TEST was written. The purpose of this program was to design the representation section of the processor using an arbitrarily chosen measurement basis $|\underline{M}$, and to check the validity of the tensor product approach. The criterion employed in this early phase of testing was that

$$|\underline{B}\ \underline{D}|S\rangle - |\underline{B}\ \underline{D}|\overset{2}{\underline{M}}\ \overset{2}{\underline{M}}|\overset{2}{R}\rangle$$

be small for every $|R\rangle$ in a typical ensemble. As can be seen from Figure 30, this is equivalent to requiring that the lack of completeness of $|\underline{M}$ have tolerably small effects on the output of the representation section. TEST is included as a subprogram of subsequent versions of the feasibility test programs. Results of the validity check from the most recent runs are given in Table 22.

As soon as results of TEST showed that the tensor product approach was valid, various subprograms were added successively for the purpose of evaluating the effect of various sources of error on the output of the representation section. From this point on, the program has been developed in two versions: TRY and OPT; the former is based on an arbitrary $|\underline{M}$ and the latter on an improved choice of $|\underline{M}$. Results given in this dissertation are obtained from the two most recent runs of OPT. The processor design developed by OPT (see Equation (128) and Tables 18, 19, 21, 22) is optimal in the sense that no better one is known at this writing and that it may be difficult to find an appreciably better one. However, no claim is made here that the design cannot be improved by some completely different approach.

The sources of error considered in the feasibility tests to date are:

(1) Noise on the range-rate data.

(2) Epoch measurement error.

(3) Error in a priori estimate of position.

The d-c restoration step was given special attention in the tests, but it is not regarded as a source of error. Noise has a direct effect on the representation section output and also an indirect effect via the d-c restoration step. The other sources of error may be treated similarly. Error effects in addition to those considered in the present project are discussed in the literature (G&W 1960).

Noise

The largest error effect considered here is that due to range-rate noise. Other effects are smaller or can be made smaller by compensation schemes and/or iteration of the processor operation. The noise on each coordinate of $\underline{\dot{r}}\rangle$ is assumed to have mean zero and variance σ^2, and to be independent of the noise on other coordinates. The noise variance is actually somewhat larger for coordinates near the ends of the pass than for those near the epoch. This effect can easily be incorporated in the test programs, if desired. The distribution of the noise is approximately normal. Since the first step in the operation of the signal processor involves linearly combining several hundred data points, the noise distribution on each of the coordinates of $\underline{\tilde{M}}|S|\dot{R}\rangle$ would be approximately normal regardless of the distribution of the range-rate noise. The rms noise level of 1.91 x 10^{-6} used in the test program corresponds to the 1.0 cps value used by Guier and Weiffenbach in their simulation experiments (G&W 1960). The noise level is scaled so that the ratio of rms noise on each coordinate of $\underline{\dot{r}}\rangle$ to the satellite velocity $\omega_s \rho_s$ is preserved.

Let $|\dot{N}\rangle$ be the noise on $|\dot{R}\rangle$ and let $|N\rangle$ be the noise on $|R\rangle$. Note that $\underline{\tilde{t}}|\dot{N}\rangle = \underline{\dot{n}}\rangle$ has mean zero and covariance matrix $\sigma^2\underline{I}$. The first coordinate consists of $\langle\tilde{M}_1|S|\dot{N}\rangle$ and an additional term resulting from the d-c restoration step. The latter can be found almost exactly for small values of σ typical of the satellite navigation system. The error in $\ddot{\rho}_o$ due to noise (see Table 25) is

$$\langle\underline{F}\ \underline{\tilde{M}}|S|\underline{t}\ \underline{\dot{n}}\rangle \triangleq \nu$$

so that

$$\hat{\ddot{\rho}}_o = \ddot{\rho}_o + \nu = \ddot{\rho}_o\left[1 + (\nu/\ddot{\rho}_o)\right]$$
$$\hat{\rho}_o = \ddot{\sigma}_o/(2\hat{\ddot{\rho}}_o) \sim \rho_o\left[1 - (\nu/\ddot{\rho}_o)\right]$$

Thus, the noise on ρ_o is obtained by multiplying the noise on $\ddot{\rho}_o$ by $-\rho_o/\ddot{\rho}_o$. The total noise on $\langle \tilde{M}_1 | R \rangle$ is then found to be

$$\langle \tilde{M}_1 | N \rangle = \left[\langle \tilde{M}_1 | - \sqrt{2N+1} \, (\rho_o/\ddot{\rho}_o) \langle \underline{F} \, \tilde{\underline{M}} | \right] \, | S | \dot{N} \rangle \quad (126)$$

For any fixed position of the navigator, the bracketed expression in Equation (126) is a fixed functional. Thus, it is a straight-forward calculation to find a representation of $|N\rangle$ for any fixed signal.

Subsequent calculations on the effect of noise are complicated by the mixing of noise and signal in the tensor product step. Instead of calculating the covariance matrix of the representation error, Monte Carlo simulation over a typical ensemble of 24 noise vectors (see Table 23) was used for each signal in a test ensemble of 25. The test ensemble differs from the typical ensemble used to design the processor in that the test ensemble has a much larger proportion of near-worst cases than the typical ensemble.

The mean representation error due to noise is negligible, as is seen from the following calculation. The input to the Kronecker multiplier in Figure 34 is $\tilde{\underline{M}} | \left[|R\rangle + |N\rangle \right]$ and the output is

$$\overset{2}{\tilde{\underline{M}}} | \left[|\overset{2}{R}\rangle + |R\rangle \otimes |N\rangle + |N\rangle \otimes |R\rangle + |\overset{2}{N}\rangle \right]$$

assuming that errors are due to noise alone. The output of the multiplication by the matrix $\underline{D} | \overset{2}{\underline{M}}$ is

$$\underline{D} | \overset{2}{\underline{M}} \, \overset{2}{\tilde{\underline{M}}} | \left[|\overset{2}{R}\rangle + |R\rangle \otimes |N\rangle + |N\rangle \otimes |R\rangle + |\overset{2}{N}\rangle \right]$$

The representation output, assuming no error in the a priori estimate of position, is given by

$$\underline{D}|\hat{S}\rangle = \underline{D}|S\rangle \quad \underline{D}|\underline{M}^2\,\underline{M}^2|\left[|R\rangle \otimes |N\rangle + |N\rangle \otimes |R\rangle + |N^2\rangle\right] \qquad (127)$$

Thus, the representation error due to noise alone (see Table 24) is the second term in Equation (127) which consists of three terms where the first two have zero mean. The third noise term is of the order of 4×10^{-12} times the signal (i.e., -228 db) and may be ignored.

Inspection of Equation (127) shows that the magnitude of the representation error, for a fixed noise input $|N\rangle$, depends mainly on the magnitude of the relative longitude θ. In the first place, $|R\rangle$ increases in magnitude as θ increases, slowly for small θ and linearly for large θ. Secondly, a major fraction of the magnitude of $|N\rangle$ is proportional to $\rho_o/\ddot{\rho}_o$ and, thus, to ρ_o^2. The latter is nearly independent of θ for small θ, and varies as θ^2 for large θ. Thus, the magnitude of the representation error for fixed $|N\rangle$ is nearly independent of θ for small θ and varies as the magnitude of θ^3 for large θ. These observations are checked by the results of OPT.

Other Sources of Error

Epoch measurement errors can arise due to noise and/or unpredicted drift in the frequency f_c of the satellite transmitter. Suppose that the cycle counts over intervals of duration T in the vicinity of the epoch were all high by about 1.5 cps (1.5σ) and that the slope of the Doppler S-curve were 20 cps/sec. The measured epoch would then be high by 0.075 sec and $\omega_s \tau_o$ would be high by 0.005° if the satellite latitude increased at the rate of 1°/15 sec. If the navigation section employs Method 1, an epoch error of such a magnitude would result directly in a latitude error of about 0.35 mile (on Earth) and indirectly in a small longitude error. The latter is evaluated in OPT for errors in $\omega_s \tau_o$ of -0.005, 0.000, 0.005, 0.010 degree.

Epoch errors of the order of 0.005° are higher than typical. Even if only two coordinates of $\dot{\underline{r}}\rangle$ are used to measure τ_o, it is unlikely that the average error in the two points will be off in the same direction by as much as 1.5σ; and there is no reason to employ such a crude epoch measurement. Also, an unpredicted component of drift of 1.5 cps is about 10^{-8} times the typical value of f_c. Stable oscillators for use in satellites can be obtained which would have unpredicted drifts of the order of 10^{-9} f_c over a period of an hour or two. Thus, epoch errors are appreciable but not large enough to affect the feasibility of the system.

The third source of error considered in OPT is the error in a priori estimate of position which affects the $\ddot{\rho}_o$ correction, $\ddot{\sigma}_o$, and the column of representation corrections. The last of these effects is least and small enough to be neglected. In fact, the magnitudes of the representation corrections are small enough (for m = 5) to raise doubt concerning the need for including the representation correction step in the processor design. Thus, error in a priori estimate of position affects the d-c restoration step which, in turn affects the representation section output. All of this is illustrated by numerical examples based on the results of OPT.

Operation of the Processor

Some of the steps in the operation of the representation section may be clarified by means of the following sample calculations. By setting $\theta = 5.0^\circ$, these calculations are excerpted from Appendix 3. Most of the latter is excerpted from OPT, with $|M_2\rangle$ designed to facilitate measurement of small θ, by setting $\alpha = 30^\circ$. Parameter values used in the computer programs are:

$N = 24$ $\qquad \rho_s/\rho_p = 1.100$

$\gamma = 24^\circ$ $\qquad \omega_s/\omega_p = 14.798$

Range-rate noise (per coordinate) = 1.910×10^{-6} rms

The processor design is given by Tables 18, 19, 21, 22 and the following equation

$$\langle \underline{F} = (10^5) \begin{array}{cccc} 2 & 3 & 4 & 5 \\ \hline -.106271 & -.194373 & -.268333 & .889200 \end{array} \qquad (128)$$

For $(\alpha, \theta) = (30.0, 5.0)$, we find that $\rho_o = 0.127259$, $\ddot{\rho}_o = 1879.46$, $\ddot{\sigma}_o = 479.571$, $\lambda = 30.16$, $\phi = 2.97$. The corresponding column $\underline{r}\rangle$ is found in OPT by evaluating ρ at 49 equally-spaced values of $\omega_s \tau$ from -12° to +12°. The column $\underline{\dot{r}}\rangle$ could be found in OPT (but isn't) by taking first differences of $\underline{r}\rangle$ (plus one more point at -12.5°). The column corresponding to accumulation of $\underline{\dot{r}}\rangle$ outward from the epoch could be found in OPT (but isn't) by subtracting 0.127259 from every coordinate of $\underline{r}\rangle$. The column $\underline{\tilde{M}}|R\rangle$, obtained in the processor by $\left[\underline{\tilde{M}}|S|\underline{t}\right]\underline{\dot{r}}\rangle$ plus the d-c restoration step, is found in OPT by $\left[\underline{\tilde{M}}|\underline{t}\right]\underline{r}\rangle$. In the present example,

$$\underline{\tilde{M}}|R\rangle = \begin{array}{c|r|} 1 & 1.238808 \\ 2 & -.274608 \\ 3. & .053484 \\ 4 & -.000725 \\ 5 & -.000231 \end{array} \qquad (129)$$

The magnitude of the error in $\underline{\tilde{M}}|R\rangle$ is 0.000532 from Table 17.

Multiplication of $\langle\underline{F}$ and $\underline{\tilde{M}}|R\rangle$ in OPT, gives the same result as multiplication of $\langle\underline{F}$ and $\underline{\tilde{M}}|S|\dot{R}\rangle$ in the processor. Addition of the correction yields the measured value of $\ddot{\rho}_0$ which fits the theoretical model exactly in this error-free example.

$$\ddot{\rho}_0 = 1877.65 + 1.81 = 1879.46$$

Table 21 indicates that the correction will not be off by more than about 0.80 for any 1° error in a priori estimate of position, although more detailed tabulation of this and other tables is needed to make this and similar statements more precise. Table 26 indicates that the error in $\ddot{\rho}_o$ due to noise will rarely exceed 0.30 in magnitude. The mean error in $\ddot{\rho}_o$ due to noise is zero and the sample standard deviation is 0.178, for the sample of 24 noise vectors used in OPT.

The a priori estimate of $\ddot{\sigma}_o$ is 479.571 which yields 479.571/(2) (1879.46) = 0.127582 for ρ_o and $\sqrt{(2)\ (24) + 1}$ (0.127582) = 0.893074 to be restored to $\langle \tilde{M}_1 | R \rangle$. This has already been included in $\underline{\tilde{M}}|R\rangle$ as given in Equation (129). Inspection of Table 10 indicates that $\ddot{\sigma}_o$ will never be off by more than about 1.30 for any 1° error in a priori estimate of position. In this example, the effects of error in a priori estimate of position on $\ddot{\sigma}_o$ and $\ddot{\rho}_o$ tend to cancel in their combined effect on the d-c restoration step, and the total error fraction is about (1.3/480) - (0.8/1880) = (1/440) per degree error.

Corresponding outputs of the Kronecker multiplication step and multiplication by the matrix $\underline{D}|\overset{2}{\underline{M}}$ are shown here.

$\overset{2}{\underline{M}}|\overset{2}{R}\rangle =$

(1,1)	1.53464
(2,2)	.75409
(3,3)	.00286
(4,4)	.00000
(5,5)	.00000
(1,2)	-.48110
(1,3)	.09370
(1,4)	-.00127
(1,5)	-.00040
(2,3)	-.02077
(2,4)	.00028
(2,5)	.00009
(3,4)	-.00005
(3,5)	-.00002
(4,5)	.00000

$\underline{D}|\overset{2}{\underline{M}}\ \overset{2}{\underline{\tilde{M}}}|\overset{2}{R}\rangle =$

1	.162610
2	-.000148
3	.162567

Addition of the representation correction column

$$\begin{array}{c|c|} 1 & .00000004 \\ 2 & .00000030 \\ 3 & .00000006 \end{array} \quad \text{yields} \quad \underline{D}|S\rangle = \begin{array}{c|c|} 1 & .162610 \\ 2 & -.000148 \\ 3 & .162568 \end{array}$$

as the output of the representation section. This result is accurate to 6 decimals in this example in which no sources of error are present.

Sample Error Analysis Calculation

The error analysis performed in OPT is illustrated by the following sample calculations. The calculations are based on a navigator position typical of those situations in which the processor produces estimates of high accuracy, i.e., $(\alpha, \theta) = (30.0, 5.0)$. The magnitudes of the various perturbations employed in the error analysis are also typical.

If the d-c restoration step had resulted in a value of $\langle \widetilde{M}_1 | R \rangle$ that was high by 0.1% and the remaining 4 coordinates of $\underline{\widetilde{M}}|R\rangle$ were exact, the representation error would be

$$\begin{array}{c|c|} 1 & 468.7 \\ 2 & -0.1 \\ 3 & 154.2 \end{array} \; (10^{-6}) \qquad (130)$$

(see Table 24) and the corresponding east-west distance error in the output of the navigation section is approximately

$$\left[(602.)(468.7) + (.423)(.1) - (8.46)(154.2)\right] (10^{-9})$$
$$= (.281)(10^{-3}) \approx \hat{\nu}_1 - \nu_1 \qquad (131)$$

This result is obtained by premultiplying the representation error by $(\widetilde{\underline{W}}\ \underline{W})^{-1}\ \widetilde{\underline{W}}$ from Equation (125). The approximation is based on the assumption that the points $|E\rangle$ and $|T\rangle$ in Figure 28 are so close together that the rows $(\widetilde{\underline{W}}\ \underline{W})^{-1}\ \widetilde{\underline{W}}$ corresponding to the two points are practically the same.

For ρ_p of about 4000 miles, the error evaluated in Equation (131) corresponds to a distance error of about 1.1 mile. A likely source of the d-c restoration error assumed here would be an a priori position error of about 1/2° (or about 30 miles on Earth). Thus, one iteration of the processor operation might be expected to bring this error well within tolerance for navigation purposes (0.1 to 0.5 mile). Convergence rates of this sort are typical of all navigator positions except those very near the orbital plane.

The first noise sample (see Table 23) given by

$$\widetilde{\underline{M}}|S|\dot{N}\rangle = \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} \begin{bmatrix} -.019630 \\ -.294657 \\ .235789 \\ .235334 \\ -.120141 \end{bmatrix} (10^{-4}) \qquad (132)$$

reduces $\ddot{\rho}_o$ by 0.203 (see Table 25) so that the d-c restoration term is high by

$$(.203/1879.46)(.893074) = (.9646)(10^{-4})$$

and the total noise on $\widetilde{\underline{M}}|R\rangle$ is

$$\underset{\sim}{\tilde{M}}|N\rangle = \begin{array}{c|c|} 1 & .9450 \\ 2 & -.2947 \\ 3 & .2358 \\ 4 & .2353 \\ 5 & -.0201 \end{array} \quad (10^{-4}) \qquad (133)$$

The direct and indirect effects of noise are combined as illustrated above in the most recent runs of OPT. The representation error produced by the first noise sample

$$\begin{array}{c|c|} 1 & 14.1 \\ 2 & -1.8 \\ 3 & 36.5 \end{array} \quad (10^{-6})$$

(see Table 24) leads to an east-west distance error of

$$\left[(602.)(14.1) + (.423)(1.8) - (8.46)(36.5)\right] (10^{-9})$$
$$= (.0818)(10^{-3}) \approx \hat{\nu}_1 - \nu_1 \qquad (134)$$

(about 0.33 mile). This error, as in the case in all of the results given by OPT, corresponds to the <u>initial</u> operation of the processor.

One iteration of the processor operation appears to be necessary and probably sufficient, although further study of this question is needed. Each such iteration would accomplish the d-c restoration step using a value of $\langle \widetilde{M_1} | R\rangle$ obtained from the position estimate produced by the preceding operation of the processor. Representation error due to noise in each iteration would, therefore, include only the direct effect of noise and not the indirect effect of noise on the d-c restoration step. Thus,

$$\underset{\sim}{\tilde{M}}|N\rangle = \underset{\sim}{\tilde{M}}|S|\dot{N}\rangle \qquad (135)$$

for each <u>iteration</u> of the processor operation. From earlier runs of OPT in which the effect of noise on the d-c restoration step was omitted, the representation error due to the first noise sample is

$$\begin{matrix}1\\2\\3\end{matrix}\boxed{\begin{matrix}-3.3\\-12.3\\-13.4\end{matrix}}\ (10^{-6}) \qquad (136)$$

and the error in ν_1 is $(0.0019)(10^{-3})$.

An epoch measurement error, referred to the α scale, of 0.002° leads to a representation error of

$$\begin{matrix}1\\2\\3\end{matrix}\boxed{\begin{matrix}-2.4\\0.1\\0.4\end{matrix}}\ (10^{-6})(.002/.005) \approx \begin{matrix}1\\2\\3\end{matrix}\boxed{\begin{matrix}-1.0\\0.0\\0.1\end{matrix}}\ (10^{-6})$$

and an error in ν_1 of $(0.00060)(10^{-3})$.

Results

Extensive listings are available from OPT of the errors expected in the output of the representation section at the end of the initial operation of the processor. A significant fraction of these results is given in Appendix 3. The sample error analysis calculation described earlier was performed for several cases and the results are given in the following tables.

The calculated error fraction in $\langle \tilde{M}_1 | R \rangle$ due to a 5° error in the a priori estimate of longitude is given in Table 3 for various latitudes. In each case, the correct and estimated values of ϵ were taken as 5° and 10° (see Tables 10 and 21). Typical errors in initial position estimates would be much less than 5°.

Table 3 D-C Error Due to 5° Error in Initial Position

	Error		Error Fraction	
α	$\ddot{\sigma}_o$	$\ddot{\rho}_o$	ρ_o	$\langle \tilde{M}_1 \| R \rangle$
0.0	-5.54	-3.37	-.0096	-.0071
15.0	-6.54	-2.51	-.0122	-.0090
30.0	-6.56	-0.12	-.0136	-.0098
45.0	-5.59	2.17	-.0128	-.0089
60.0	-3.86	0.84	-.0084	-.0057

Representation errors due to an error fraction in $\langle \tilde{M}_1 | R \rangle$ of 0.001 for $\alpha = 30^{\circ}$ and for several values of θ are given in Table 24 along with the representation error due to the first noise sample. The corresponding east-west distance errors, expressed as a fraction of the planet radius, are found by premultiplying the representation errors by $\left[\partial \nu_1 / \partial \sigma_j\right]$ from Table 11 and the results are given in Table 4.

Table 4 Analysis of Navigation Error

$\alpha = 30.0^{\circ}$	Cross-Track Distance Error			
θ	Noise Sample 1		$\langle \tilde{M}_1 \| R \rangle$ 0.1% High	
-2.0	-.578	E-6	-.541	E-3
-0.5	1.127	E-6	-2.08	E-3
0.0	-2773.	E-6	13.82	E-3
5.0	8.18	E-6	.281	E-3
10.0	25.8	E-6	.257	E-3

Inspection of Tables 3 and 4 indicates that the processor operation converges (i.e., the effect of a priori position error is attenuated by the processor) for all positions except those within a few tenths of a degree of the orbital plane. For example, $\langle \widetilde{M}_1 | R \rangle$ is off by 0.1% for an initial position error of about 0.5° (roughly 30 miles) and this produces an error in the output estimate of about 1.0, 1.1, 2.2, 8.3, 55. miles for θ = 10.0, 5.0., 2.0, 0.5, 0.0. Table 13 indicates that the interval of divergence widens for navigator positions nearer the equator.

The indirect effect of an error in $\omega_s \tau_o$ of 0.005 on the output of the representation and navigation section is given in Table 5 for $\alpha = 30.0^{\circ}$ and several values of θ . In addition, such an error in epoch measurement would have a direct effect of about 0.005 distance units on the estimate of the latitude of the navigator. Clearly, the effect of epoch error on the longitude of the final estimate of position is negligible compared to the effect on latitude.

The most significant test of processor performance to date is the determination of typical longitude errors in the final estimates of position resulting from noise. This can be found easily for each possible signal by calculating the standard deviation of the east-west distance error over an ensemble of noise vectors. If $\underline{C}$ is the covariance matrix of the representation error, then

$$\text{Variance of } (\hat{\nu}_1 - \nu_1) = (\underline{\tilde{W}}\ \underline{W})^{-1}\underline{\tilde{W}}\ \underline{C}\ \underline{W}\ (\underline{\tilde{W}}\ \underline{W})^{-1} \quad (137)$$

Table 5 Effect of Error in Epoch Measurement

$\alpha = 30.0^\circ$ $\qquad \omega_s(\hat{T}_o - T_o) = 0.005^\circ$

θ	$10^6 [\hat{\sigma}_1 - \sigma_1]$	$10^6(\hat{\nu}_1 - \nu_1)$
-2.0	42.4 .1 .0	-63.6
-0.5	15.0 .0 .1	-89.6
0.0	.2 .0 .3	-26.9
5.0	-2.4 .1 .4	-1.44
10.0	8.0 .1 .5	24.2

The calculation of standard deviation of the error in estimates of ν_1 has been done for $\alpha = 30.0^\circ$ and several values of θ using the results of L-S and OPT given, in Tables 13 and 26 in Appendix 3. The results of this calculation correspond to errors in the estimates produced by the initial operation of the processor. Using output from earlier runs of OPT in which the indirect effect of noise via the d-c restoration step was omitted, the ultimate error due to noise was calculated. These results along with roughly comparable values read from Figure 1 of

Guier and Weiffenbach's paper are given in Table 6. The only significantly different conditions between the two situations is the satellite altitude; Guier and Weiffenbach used $\rho_s - \rho_p = 400$ where we have used 343.7 nautical mile.

Table 6 Cross-Track Error Due to Noise

$\alpha = 30.0^\circ$ $\sigma = (1.91)(10^{-6})$

ρ_p = 3437 nautical mile

$\|\theta\|$		0.0	0.5	2.0	5.0	10.0	degree
Cross-	Initial(x10 ^{6})	1575.	111.8	30.9	19.8	29.7	planet
Track	Ultimate(x10 ^{6})	889	38.1	9.8	4.5	9.4	radius
Error	Ultimate	3.1	0.1	<0.1	<0.1	<0.1	
(rms)	G&W 1960	>3.0	0.5	0.1	<0.1	<0.1	nautical
Distance to Subtrack		0.0	25.8	103.	258.	516.	mile

The preliminary feasibility tests indicate that the accuracy of the proposed processor based on the tensor product approach is comparable to that based on classical least-squares. Thus, the original objective of simplification of the signal processor for the navigation terminal can be satisfied without any accompanying relaxation of performance specifications.

Further optimization of the processor can be obtained by whitening the representation error. The whitening required is a function of navigator position. Since the matrices used in the navigation section must be found for each location, there is no increase in complexity imposed by modifying the stored tables of these matrices to include the whitening operation. Preliminary calculation on one case indicates that whitening might reduce cross-track error due to noise by about a factor of two.

OPT provides for a repetition of the noise tests with Dim $\mathbb{M}$ reduced from 5 to 4. This change produces no significant change in the representation error due to noise. In some cases, the errors are increased slightly, and in other cases, decreased slightly. The main effect of reducing Dim $\mathbb{M}$ is that representation errors due to error in the a priori estimate of position are markedly increased. Changing Dim Ↄ from 49, as in OPT, to 601 or more, as in an actual satellite navigation system using the proposed processor would result in a reduction in the error due to noise by a factor of 3.5 or more.

Much more work must be done on the proposed processor in accounting for numerous details which have been omitted from the preliminary tests. However, sufficient results have been obtained to establish the tensor product approach as being worthy of consideration by satellite navigation system engineers.

CHAPTER FIVE

AREAS OF FURTHER STUDY

Several areas of further study relating to the satellite navigation problem in particular and to ignal theory in general are suggested in this chapter. Some of these areas have been considered briefly in the course of the research project, but results were not sufficiently complete for inclusion in the dissertation.

Satellite Navigation

The existing versions of OPT and L-S ought to be combined into a single program including steps corresponding to the sample calculations described in Chapter 4. Concurrently, the processor designed by OPT ought to be improved by whitening the representation error. This can be done by replacing $\underline{D}|^{2}_{\underline{M}}$ with $\underline{C}^{-\frac{1}{2}}\ \underline{D}|^{2}_{\underline{M}}$ and $\underline{W}$ with $\underline{C}^{-\frac{1}{2}}\ \underline{W}$. Equation (137) then reduces to

$$\text{Variance of } (\hat{\nu}_1 - \nu_1) = (\underline{\tilde{W}}\ \underline{C}^{-1}\underline{W})^{-1}\underline{\tilde{W}}\ \underline{C}^{-\frac{1}{2}}\underline{I}\ \underline{C}^{-\frac{1}{2}}\underline{W}\ (\underline{\tilde{W}}\ \underline{C}^{-1}\underline{W})^{-1}$$
$$= (\underline{\tilde{W}}\ \underline{C}^{-1}\underline{W})^{-1} \qquad (138)$$

An equivalent approach which avoids additional complexity is to leave $\underline{D}|^{2}_{\underline{M}}$ alone and replace $(\underline{\tilde{W}}\ \underline{W})^{-1}\underline{W}$ with $(\underline{\tilde{W}}\ \underline{C}^{-1}\underline{W})^{-1}\underline{\tilde{W}}\ \underline{C}^{-1}$.

A complete Monte Carlo simulation of the processor should be programmed to test the effect of interactions of the various sources of error. Provision should be included for non-polar orbits of small ellipticity, descending passes, several hundred data points, epoch at an arbitrary instant on the sampling time scale, parallel testing of Methods 1 and 2, replacement of "bad" data points, and larger ensembles of signal and noise vectors than used in the preliminary feasibility tests described in this dissertation.

The change from polar to non-polar orbits has no effect on Dim $|\underline{A}$ and, thus, no effect on the complexity of the processor. Preliminary study of the effect of non-zero ellipticity ϵ indicates that Dim $|\underline{A}$ would have to be increased from 7 to 15 to account for the signal components proportional to ϵ. However, it seems reasonable to predict that a factor analysis of $\{|S\rangle\}$ would show that the effective dimensionality required for $|\underline{B}$ would be only 4 or 5. There is no reason to expect increases of more than 1 or 2 in Dim $|\underline{M}$ due to non-zero ϵ.

A study of the proper number of iterations of the processor operation ought to be made. This number will depend on the a priori estimate of θ. My prediction is that the proper number of iterations is one for all but very small values of θ.

The processor design can be further improved by using the measured $\underline{\tilde{M}}|S|\dot{R}\rangle$ to obtain an improved epoch measurement. It seems reasonable to expect that an appropriate addition to $|\underline{M}$ would be an element matched to signal differences corresponding to small changes in epoch, i.e., an element obtained from

$$\left[\frac{\partial |R\rangle}{\partial \alpha}\right]_{\alpha = 0}$$

by orthonormalization.

Techniques appropriate for compensation of navigator motion (with respect to the planet) should be developed and analyzed. Preliminary studies indicate that such compensation steps can be included in the proposed processor and that the bulk of the additional computation required can be done between sampling intervals. The only significant increase in complexity appears to be the necessity of recording relative motion of the navigator at each sampling instant.

In the proposed processor, the accumulation step precedes the projection from $\mathcal{D}$ onto $\mathcal{M}$ (e.g., Dim $\mathcal{D}$ = 601, Dim $\mathcal{M}$ = 5). Consideration should be given to a design based on the opposite sequence of these operations. Intuition suggests that the effect of noise would be reduced by projecting and then accumulating, but rough preliminary calculations indicate that this is not always true. The reversed-sequence design would require that $|\underline{M}$ be designed to fit $\{|\dot{R}\rangle\}$ rather than $\{|R\rangle\}$.

The tracking subsystem should be studied with a view to the use of the tensor product approach to signal processing. One benefit which would result from the use of a modified version of the representation section of the processor at each tracking station would be a reduction in the amount of information sent to the central tracking computer by a factor of more than 100. The main objective of a study of this possibility would be to determine whether the savings provided by the reduction in communications would justify the additional cost of signal processing at the tracking sites.

Signal Theory

Modulation systems involve time-domain multiplication and might be profitably treated in tensor product terms. An operation on a modulated signal would then be regarded as the tensor product of an operator on the modulating signal and an operator on the carrier. Radar ambiguity functions may be another fruitful application of tensor product concepts as is suggested by the product-basis representation of these functions (Sussman 1962).

Vector space concepts and ad hoc bases efficiently represent systems modeled classically by sets of ordinary differential equations, and tensor product concepts extend the approach to systems presently described by sets of partial differential equations. Processing of generalized space-time signals as in arrays of antennas, hydrophones or seismometers are natural areas for further application of tensor product spaces.

APPENDIX

The notation and nomenclature proposed for general use in signal theory is summarized in the first section of the Appendix. The second section contains some details of the various computer programs which were written to evaluate the proposed satellite navigation signal processor. Design values, time-domain representatives of typical signals, typical errors of various types and other results obtained from the computer experiments have been selected and presented in the third section of the Appendix. The particular symbols used in Chapter 4 are listed in Appendix 4.

Complete output listings of FORTRAN code and results of execution are available in The Johns Hopkins University Library for all of the programs used in the preparation of this dissertation (Ross 1964), however, reference to these listings is not necessary for an understanding or evaluation of the design approach. A copy of each of these programs in both FORTRAN and binary form is also available on punched cards at the Computation Center of The Johns Hopkins University.

A - 1 SIGNAL THEORY NOTATION

The three most important objectives of the notation and nomenclature developed for use in signal theory are:

(1) Patterns, signals and operators ought to be clearly and conveniently distinguished.

(2) A representative ought to be clearly and conveniently distinguished from the entity which it represents.

(3) The notation should admit the convenient use of several bases in the same problem without leading to confusion among different representatives of the same entity.

The first objective is satisfied by means of the Dirac bra, ket and vertical bar. Representatives are indicated by omitting the bar or bars from the corresponding symbol for the entity being represented. All remaining requirements are met through the appropriate use of common type fonts, as shown in Table 7. The similarity of the first two rows of Table 7 emphasizes the fact that the mathematical structure chosen here as corresponding to the physical system is an abstract linear vector space (and its dual). Table 7 shows that the notation extends conveniently to include the classical models of infinite dimensionality. The notation shown for infinite discrete time bases may also be used for finite time bases of large dimensionality.

Greek capitals which differ from Roman letters are used to denote sets, for example, the field Ω over which our vector spaces are constructed. The latter are denoted by turning the field symbol on its side, one way for the signal space and the opposite way for the pattern space. Subspaces of $\rotatebox{90}{$\Omega$}$ will be indicated by script capitals. For example, $|F\rangle \in \mathcal{B} \subset \rotatebox{90}{Ω}$, $\mathcal{B} \oplus \mathcal{B}^{\perp} = \rotatebox{90}{Ω}$ and $\mathcal{B} \cap \mathcal{B}^{\perp} = |0\rangle$.

Table 7 Basic Scheme of the Notation

Physical entity		$\|F\rangle$	$\|H\|$	$\langle G\|$
Abstract model		$\|F\rangle$	$\|H\|$	$\langle G\|$
Repr'tive	Continuous frequency basis	$F\rangle$	H	$\langle G$
	Continuous time basis	$f\rangle$	h	$\langle g$
	Discrete time basis	$\underline{f}\rangle$	$\underline{h}$	$\langle \underline{g}$
	Finite-dimensional basis	$\underline{F}\rangle$	$\underline{H}$	$\langle \underline{G}$
Scalars		$\alpha, \beta, \gamma, \ldots, \omega$		

Standard notation is employed in the dissertation for scalars, functions, sets and relations. For the sake of completeness, the basic mathematical notation is summarized in Table 8. Most of the basic formulas in Chapter 2 and 3 continue to apply if the field is complex, as might be convenient in problems involving amplitude-

Table 8 Basic Mathematical Notation

$=$	equals	$\in$	belongs to
$\approx$	approximates	$\subseteq$	is included in
$\triangleq$	$=$, by definition	$\subset$	is properly included in
$<$	is less than	$\bigcup_i$	union over i of
$\leq$	is not greater than	$\bigcap_i$	intersection over i of
$+$	addition	$\sqrt{\ }$	square root $(+)$ of
$-$	subtraction	Dim	dimensionality of
$\phi\theta$	product of ϕ and θ	$\sum_i$	sum over i of
$/$	division	$\prod_i$	product over i of
ϕ^*	conjugate of ϕ	$\Rightarrow$	implies
...	ellipsis	$\Leftrightarrow$	if and only if
Pr	probability of	E_i	expectation over i of
Tr	trace of	Det	determinant of
$\perp$	orthogonal to	δ_{ij}	1 if $i=j$, 0 if $i\neq j$
$\hat{\theta}$	estimate of θ	$\dot{\rho}$	time derivative of ρ
$\{\ \}$	set of elements listed or defined within the brackets		
$[\phi_{ij}]$	matrix of m rows and n columns of elements ϕ_{ij}		
Dg $\underline{H}$	matrix formed by zeroing non-diagonal elements of $\underline{H}$		

and/or phase-modulated sinusoidal carriers. However, in the application study of Chapter 4, Ω is the field of real numbers.

The letters i, j, k will normally be reserved for indices and n, m, N for maximum indices. Indices over the natural numbers will be indicated by the ellipsis and the absence of a maximum index.

A fairly complete set of examples of the use of the notation, including the extension to tensor product spaces, is presented in Table 9. Note that $\otimes$ may be included or omitted in the symbol for the matrix $\underline{F}\rangle\otimes\langle\underline{G}$ since it may be regarded either as the Kronecker product of $\underline{F}\rangle$ and $\langle\underline{G}$ or as the ordinary matrix product of $\underline{F}\rangle$ and $\langle\underline{G}$. Similar remarks apply to the operator $|F\rangle\langle G|$. In Table 9 and Figure 36, Ↄ and Γ are finite-dimensional vector spaces over a field Ω; Ⅽ and $\urcorner$ are the duals of Ↄ and Γ.

Matrices in which every element is equal to one are useful in representing d-c components (constant in time) of signals and in calculating ensemble means and covariances. Such matrices may be written conveniently in one of the forms: $\underline{I}\rangle$, $\langle\underline{I}$, $\underline{I}\rangle\langle\underline{I}$ or $\underline{I}_m\rangle\langle\underline{I}_n$.

The mixed symbols for bases are $|\underline{B}$ in Ↄ and $\underline{D}|$ in Ⅽ are important and useful. They are defined by

Table 9 Examples of the Use of the Notation

Abstract Entities		Representatives	
signal vector in Ɔ	$\|F\rangle$	$\underline{F}\rangle$	column, or nx1 matrix
pattern vector in C	$\langle G\|$	$\langle\underline{G}$	row, or 1xn matrix
linear operators on Ɔ, or on C	$\|L\|,\ \|H\|$	$\underline{L},\ \underline{H}$	nxn matrices
inverse of $\|H\|$	$\|H^{-1}\|$	$\underline{H}^{-1}$	inverse of "
operator multiplication	$\|L\|H\|$	$\underline{L}\ \underline{H}$	matrix multiplication
match of $\|F\rangle, \langle G\|, \|H\|$	$\langle\tilde{F}\|, \|\tilde{G}\rangle, \|\tilde{H}\|$	$\langle\tilde{\underline{F}},\ \tilde{\underline{G}}\rangle,\ \tilde{\underline{H}}$	adjoint of $\underline{F}\rangle, \langle\underline{G}, \underline{H}$
identity operator	$\|I\|$	$\underline{I}$	unit matrix $= [\delta_{ij}]$
null vector, pattern, operator	$\|0\rangle, \langle 0\|, \|0\|$	$\underline{0}\ ,\ \underline{0}, \underline{0}$	null column, row, matrix
direct sum of $\|F\rangle$ in Ɔ and $\|A\rangle$ in Γ	$\|F\rangle \oplus \|A\rangle$	$\underline{F}\rangle \oplus \underline{A}\rangle$	Kronecker sum of $\underline{F}\rangle$ and $\underline{A}\rangle$
direct sum of $\langle G\|$ in C and $\langle X\|$ in Γ	$\langle G\| \oplus \langle X\|$	$\langle\underline{G} \oplus \langle\underline{X}$	Kronecker sum of $\langle\underline{G}$ and $\langle\underline{X}$
tensor product of $\|F\rangle$ and $\|A\rangle$	$\|F\rangle \otimes \|A\rangle$	$\underline{F}\rangle \otimes \underline{A}\rangle$	Kronecker product of $\underline{F}\rangle$ and $\underline{A}\rangle$
tensor product of $\|F\rangle$ and $\|F\rangle$	$\|\overset{2}{F}\rangle$	$\overset{2}{\underline{F}}\rangle$	Kronecker product of $\underline{F}\rangle$ and $\underline{F}\rangle$
tensor product of $\langle G\|$ and $\langle X\|$	$\langle G\| \otimes \langle X\|$	$\langle\underline{G} \otimes \langle\underline{X}$	Kronecker product of $\langle\underline{G}$ and $\langle\underline{X}$
tensor product of $\|H\|$ and $\|K\|$	$\|H\| \otimes \|K\|$	$\underline{H} \otimes \underline{K}$	Kronecker product of $\underline{H}$ and $\underline{K}$
product of $\|F\rangle$ and $\langle G\|$	$\|F\rangle\langle G\|$	$\underline{F}\rangle\langle\underline{G}$	product of $\underline{F}\rangle$ and $\langle\underline{G}$
linear functional $\langle G\|$ evaluated at $\|F\rangle$	$\langle G\|F\rangle = \beta = \langle\underline{G}\ \underline{F}\rangle$		product of row $\langle\underline{G}$ and column $\underline{F}\rangle$
norm of $\|F\rangle, \langle F\|$ or $\|F\|$	$\\|F\\| = \phi = \\|\underline{F}\\|$		norm of $\underline{F}\rangle$, $\langle\underline{F}$ or $\underline{F}$

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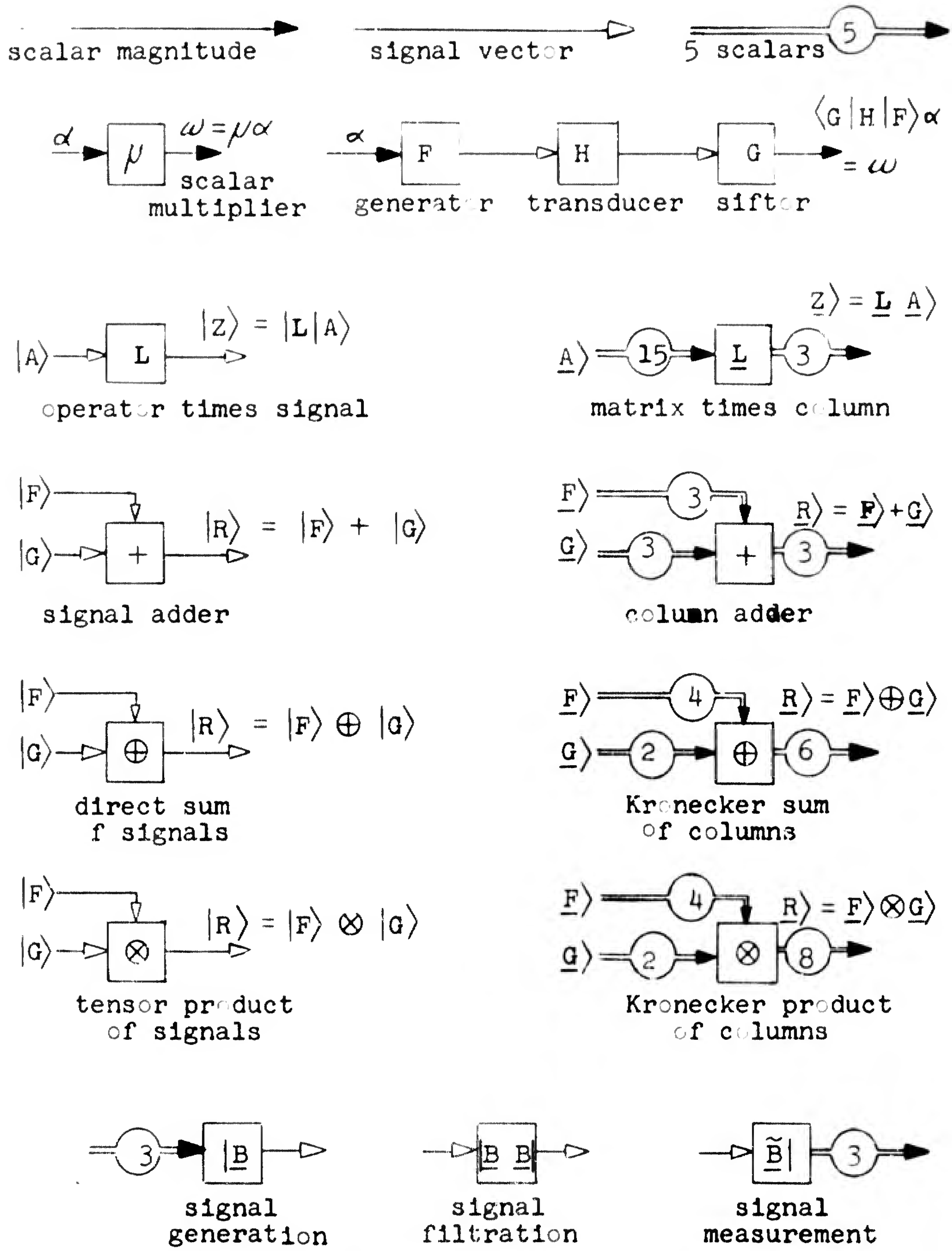

Figure 36 Notation for System Diagrams

$$|\underline{B} = \begin{array}{|c|c|c|}\hline |B_1\rangle & \cdots & |B_n\rangle \\ \hline\end{array} \; ; \; \underline{D}| = \begin{array}{|c|}\hline \langle D_1| \\ \hline \vdots \\ \hline \langle D_n| \\ \hline\end{array}$$

If $\underline{D}|$ is the dual of $|\underline{B}$, then $|\underline{BD}| = |I|$ and $\underline{D}|\underline{B} = \underline{I}$. If $\mathcal{C}$ and $\mathcal{D}$ are inner-product spaces, we frequently confine our attention to bases which are orthonormal with respect to the inner product and then $\underline{D}| = \underline{\tilde{B}}|$, $\underline{\tilde{B}}|\underline{B} = \underline{I}$.

The general section on signal theory notation and nomenclature is closed with the set of examples of system diagram notation shown in Figure 36. Extensions of our notation suitable for use with continuous time and frequency bases and tying in closely with the classical approach have been developed and are described in the literature (Lai 1960, Huggins 1963). For example, idealized impulse generators symbolized by the ket and samplers symbolized by the bra are useful for system analysis purposes. These idealized elements are represented in the time domain by the Dirac delta function.

A - 2 COMPUTER PROGRAMS

Brief descriptions of the programs developed in this project along with major subroutines are presented in Appendix 2. The names of these programs are: EPOCH, BASIS, RANGE, TRY, OPT and L-S. The dissertation is based on results of EPOCH, OPT and L-S. The features of BASIS and RANGE are effectively included in OPT, and TRY is based on a processor design inferior to that used in OPT.

All programs were written in FORTRAN and all runs used one of the IBM 7094 computers at the Applied Physics Laboratory of The Johns Hopkins University except one on the 709 at the IBM Washington Systems Center. OPT, the largest of the set, consists of 997 FORTRAN statements and comments and requires 0.04 hr to execute (for N = 24) and 0.08 hr to compile on the 7094.

EPOCH

The purpose of the EPOCH program is the preparation of tables of the several important quantities involved in the satellite navigation problem. Epochal values of range, range-square and their second derivatives are tabulated along with the altitude of the satellite above the horizontal plane at each end of the pass. These values are found for each of 208 navigator positions given in geodetic coordinates (λ, ϕ) and inertial coordinates (α, θ). The latter serve as independent variables in all programs. The 208 positions are defined by all combinations of $\alpha = 0,5,\ldots,75$ and $\theta = -30,\ldots, -5,0,5\ldots,30$ degrees. Input data to EPOCH consists of ρ_s/ρ_p, ω_s/ω_p and ; the output consists of tabulations of 8 quantities as functions of (α, θ) plus a listing of basic assumptions, input data, etc.

BASIS

The purpose of the BASIS program is the study of the complete 7-dimensional basis $|\underline{A}$ of range-square signals. The program represents $|\underline{A}$ on $|\underline{P}$, normalizes $|\underline{A}$, and finds the Gram matrix of the normalized basis. Inspection of this Gram matrix shows that a 3-dimensional sub-basis $|\underline{B}$ is effectively complete. The program continues by calculating $\underline{D}|\underline{P}$ and $\underline{D}|\underline{A}$ in both normalized and unmodified forms. Finally, the basis $|\underline{B}$ is tested by finding the errors involved in projecting a typical ensemble $\{|S\rangle\}$ of 25 range-square signals onto $\mathcal{B}$. The 25 signals correspond to 25 navigator positions: α = 0, 15, 30, 45, 60 and θ = 10, -5, 0, 5, 10 degrees.

In all of the program executions, N = 24 and Dim $|\underline{P}$ = 49 rather than 601 as in the proposed system. This reduction in dimensionality does not affect the validity of the feasibility tests of the proposed processor. The reduction was made to avoid unnecessary computer time. The particular choice of 49 was based on the number of lines which can be conveniently tabulated on each page of printout and on the fact that 49 is a perfect square. The choice here of 49 data points and $\rho_s/\rho_p = 1.1$ permits rough comparison with results given in the literature (G&W 1960) based on 47 data points and $(\rho_s - \rho_p) = 400$ nautical miles.

RANGE

The purpose of the RANGE program is to explore the effective dimensionality of a typical ensemble $\{|R\rangle\}$ of 25 range signals. The program represents the ensemble on $|\underline{t}$. Then $\underline{I}\rangle \rho_o$ is subtracted from $\underline{r}\rangle$. The resulting columns correspond to a typical $\{\underline{\tilde{t}}|S|\dot{R}\rangle\}$. The covariance matrix $\underline{C}$ of this ensemble and its eigenvalues are found. Results of this program indicated that a basis for the measurement of $\{|R\rangle\}$ would probably not require more than 5 dimensions.

The RANGE program is included in OPT in a modified form where considerations other than least-squares approximation are used in selecting the first two elements of $|\underline{M}$. The third element is designed to fit the mean of the residuals of $\{|R\rangle\}$ after the components on the first two elements of $|\underline{M}$ are removed. The eigenvector technique is then applied to the residual covariance after the components on the first three elements are removed. In both RANGE and the corresponding subprogram of OPT, the basis $|\underline{M}$ is tested by finding the errors in using it to represent $\{|R\rangle\}$.

TRY

The tensor product approach to the design of the satellite navigation signal processor ought to be valid for a wide variety of choices of measurement basis $|\underline{M}$.

The purpose of the program TRY was to test the feasibility of a processor design for an arbitrary choice of $|\underline{M}$, in particular, the basis obtained by orthonormalizing τ^k over 49 instants in the interval $-NT \leq \tau \leq NT$. Several runs were made with different choices for the exponents. The basis defined by k = 0, 1, 2, 4, 6 gave best results of the bases tested. Development of TRY was dropped as soon as OPT was debugged. To obtain a version of TRY comparable to the present form of OPT, the simplest approach would be to rewrite the section of OPT in which $|\underline{M}$ is designed.

<u>OPT</u>

The purpose of the OPT program is to design and test a processor with the measurement basis $|\underline{M}$ chosen so as to improve one or more aspects of performance. The outline of OPT is as follows.

1. Read 8 parameters from data card and initialize.
2. Execute subprogram BASIS.
3. Construct typical $\{|R\rangle\}$ of 25 cases.
4. Design 5-dimensional basis $|\underline{M}$, find $\{\underline{\tilde{M}}|R\rangle\}$ and errors.
5. Design $\underline{D}|\overset{2}{\underline{M}}$, $\langle\underline{F}$ and typical $\ddot{\rho}_o$ corrections.
6. Find typical $\{\overset{2}{\underline{\tilde{M}}}|\overset{2}{R}\rangle\}$ and outputs of $\underline{D}|\overset{2}{\underline{M}}$.

7. Design typical representation corrections.
8. Find typical representation section outputs and errors.
9. Print results of zero-error case for typical ensemble.
10. Design $\underset{\sim}{\tilde{M}}|S|\underline{t}$.
11. Construct test ensemble $\{|R\rangle\}$ by replacing 10 typical cases with near-worst cases. Modify corrections and standards accordingly.
12. Repeat steps 6, 8, 9 for test ensemble with $\langle\tilde{M}_1|R\rangle$ multiplied by 0.999, 1.000, 1.001.
13. Construct noise ensemble of (2N+1)-columns.
14. Premultiply each noise column by $\underset{\sim}{\tilde{M}}|S|\underline{t}$.
15. Find effect of noise on $\ddot{\rho}_o$.
16. Clear 25 arrays for covariance matrices of representation error due to noise.
17. Add each noise column to each $\underset{\sim}{\tilde{M}}|R\rangle$ in the test ensemble. Repeat steps 6, 8, 9 and accumulate covariances.
18. Complete the computation of the covariance matrices.
19. Reduce Dim $|\underline{M}$ from 5 to 4 by dropping $|M_5\rangle$ and repeat steps 17, 18.

20. Replace test ensemble with typical ensemble and restore $|M_5\rangle$ to $|\underline{M}$.

21. Repeat steps 3, 6, 8, 9 for epoch error on α scale of -.005, .000, .005, .010.

All results described above are printed with the exception that printing of the representation output and error due to noise is limited to the first 5 noise samples. The number of noise samples is controlled by the parameter NSAMP on the data card. This parameter also controls the design of $|M_2\rangle$. If NSAMP is odd, $\tilde{\underline{t}}|M_2\rangle$ is linear in τ; if even, $|M_2\rangle$ is optimized for the measurement of small ϵ.

The eight parameters in the data card are: N, ρ_s/ρ_p, ω_s/ω_p, γ, DLAT, DLON, SIGMA and NSAMP. The spacing of the navigator positions used in defining the typical and test ensembles is controlled by DLAT and DLON. The rms noise on each range-rate coordinate is controlled by SIGMA.

L-S

The purpose of the program L-S is to calculate matrices which can be used to transform representation error into corresponding navigation error for two possible designs of the navigation section. We restrict attention in this dissertation to the simpler design in which the measured α is taken as the estimated α and $\hat{\theta}$ is the least-squares estimate of θ based on the measurement of the column $\underline{D}|S\rangle$. Considering only the matrices associated with he simpler design, the scalars and matrices computed by L-S are: $\lambda, \phi, \rho_o, \ddot{\rho}_o, \ddot{\sigma}_o, \partial\nu_1/\partial\phi, \partial\phi/\partial\theta, \partial\rho_o/\partial\theta, \partial\nu_1/\partial\theta, \partial\theta/\partial\nu_1, [\partial\alpha_1/\partial\theta], [\partial\sigma_1/\partial\nu_1] = \underline{W}, \tilde{\underline{W}}\,\underline{W}, (\tilde{\underline{W}}\,\underline{W})^{-1}, [\partial\nu_1/\partial\sigma_j], [\partial\phi/\partial\sigma_j], [\partial\rho_o/\partial\sigma_j], [\partial\rho_o/\partial\mu_j]$. These are printed for each of the 25 positions corresponding to the test ensemble in OPT. Also, Det $\tilde{\underline{W}}\,\underline{W}$ is printed for α = 0, 15, 30, 45, 60 and θ = -0.20, ...,-0.02, 0.00, 0.02,..., 0.20 degrees.

Input to L-S consists of the first 5 parameters on the same data card used with OPT. In addition, the matrix $\underline{D}|\underline{A}$ and the 5 columns of $\underline{D}|\overset{2}{\underline{M}}$ associated with $|M_1\rangle$ are required from OPT. At present, the latter is accomplished by program statements copied from results of OPT. To avoid this nuisance and for other reasons, OPT and L-S ought to be combined into a single program. An even better arrangement would be to divide OPT into several subroutines which would then be controlled along with L-S by a simple executive program.

Subroutines

OPT is the only one of the programs which uses any subroutines not included in the FORTRAN System. Three computational subroutines obtained from the program library of The Johns Hopkins University for use with OPT are: EIGEN (7.02.01), RNV (8.03.04) and S (7.01.04).

EIGEN finds eigenvalues and eigencolumns of a real symmetric matrix using Givens' method. The subroutine is a modification of SHARE routine 664. EIGEN was checked by comparing the sum of the eigenvalues with the trace and by noting that the basis $|\underline{M}$ designed with the aid of EIGEN fits $\{|R\rangle\}$ quite well.

RNV is a pseudo-random number generator based on summations of 48 consecutive terms of a Fibonacci sequence. The distribution of the numbers generated by RNV is approximately normal with zero mean and unit variance. The mean and variance of the noise coordinates generated as well as the correlation between adjacent noise coordinates was checked in OPT. The sample mean, standard deviation and correlation of adjacent pseudorandom numbers were found to be 0.0127, 0.994 and 0.0076 for the sample of 1176 numbers generated for an ensemble of 24 noise columns of 49 elements each. RNV was written by K. R. Wander at the Applied Physics Laboratory.

The subroutine S provides for many of the common matrix operations but it was used in OPT for inversion only. The Crout method with modifications by Wilkinson is used in S to calculate the inverse $\underline{B}$ of a given matrix $\underline{A}$. The inverse given by S was improved in OPT by 3 iterations of the algorithm

$$\underline{B}_{k+1} = \underline{B}_k \left[2\,\underline{I} - \underline{A}\,\underline{B}_k \right]$$

A - 3 SAMPLES OF COMPUTED RESULTS

Presented here are selected results produced by some of the programs described in Appendix 2. Table 10 is an excerpt from EPOCH, Tables 11 and 13 from L-S, and the remainder of Appendix 3 is from OPT with $|M_2\rangle$ optimized for the measurement of small θ. Many of the tables are photographically copied from the computer printouts. Floating point results are given in the standard FORTRAN form, e.g., -0.139698E-08 means $(-0.139698)(10^{-8})$. Magnitudes of various quantities are generally accurate to at least 6 significant digits; values of various errors and corrections are of considerably less accuracy, of course.

Table 10 $\delta_o(\alpha, \theta)$

α \ θ	-5	0	5	10
0	0.482108E+03	0.483958E+03	0.482108E+03	0.476573E+03
5	0.482597E+03	0.483941E+03	0.481613E+03	0.475643E+03
10	0.483067E+03	0.483891E+03	0.481129E+03	0.474820E+03
15	0.483504E+03	0.483810E+03	0.480670E+03	0.474126E+03
20	0.483893E+03	0.483700E+03	0.480249E+03	0.473583E+03
25	0.484223E+03	0.483565E+03	0.479879E+03	0.473207E+03
30	0.484484E+03	0.483408E+03	0.479571E+03	0.473008E+03
35	0.484668E+03	0.483234E+03	0.479334E+03	0.472993E+03
40	0.484769E+03	0.483049E+03	0.479177E+03	0.473162E+03
45	0.484783E+03	0.482858E+03	0.479102E+03	0.473513E+03
50	0.484711E+03	0.482667E+03	0.479113E+03	0.474036E+03
55	0.484553E+03	0.482482E+03	0.479210E+03	0.474715E+03
60	0.484316E+03	0.482308E+03	0.479389E+03	0.475531E+03

All angles tabulated in Appendix 3 are given in degrees.

Table 11 $\partial \nu_1 / \partial \sigma_j$ for $\alpha = 30°$

θ	j=1	j=2	j=3
-2.0	-1.50	-.00264	-.0135
-0.5	-5.97	-.0421	-.350
0.0	-.000631	-9.59	-89.7
5.0	.602	-.000423	-.00846
10.0	.302	-.000105	-.00324

Table 12 Correlation Between $|A_i\rangle$ and $|A_j\rangle$

i \ j	1	2	3
1	1.000000	0.000000	0.000000
2	0.000000	1.000000	0.000000
3	0.000000	0.000000	1.000000
4	0.000000	0.999992	0.000000
5	1.007675	0.000000	-0.010317
6	0.000000	1.000000	0.000000
7	1.007640	0.000000	-0.010270

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Table 13 Det $\underline{\tilde{W}}\ \underline{W}$ for Near-Overhead Passes

θ \ α	0.0	15.0	30.0	45.0	60.0
-0.16	0.381100E-02	0.358172E-02	0.296537E-02	0.213008E-02	0.130134E-02
-0.14	0.291823E-02	0.274967E-02	0.229810E-02	0.168674E-02	0.108064E-02
-0.12	0.214450E-02	0.202867E-02	0.171997E-02	0.130270E-02	0.889529E-03
-0.10	0.148980E-02	0.141872E-02	0.123099E-02	0.977966E-03	0.727993E-03
-0.08	0.954141E-03	0.919796E-03	0.831139E-03	0.712519E-03	0.596033E-03
-0.06	0.537515E-03	0.531899E-03	0.520403E-03	0.506355E-03	0.493644E-03
-0.04	0.239926E-03	0.255021E-03	0.298775E-03	0.359464E-03	0.420823E-03
-0.02	0.613717E-04	0.891525E-04	0.166243E-03	0.271838E-03	0.377566E-03
0.	0.185371E-05	0.342865E-04	0.122796E-03	0.243467E-03	0.363868E-03
0.02	0.613717E-04	0.904150E-04	0.168422E-03	0.274344E-03	0.379727E-03
0.04	0.239926E-03	0.257530E-03	0.303113E-03	0.364401E-03	0.425139E-03
0.06	0.537515E-03	0.535624E-03	0.526855E-03	0.513808E-03	0.500100E-03
0.08	0.954141E-03	0.924688E-03	0.839638E-03	0.722378E-03	0.604607E-03
0.10	0.148980E-02	0.142472E-02	0.124145E-02	0.990163E-03	0.738657E-03
0.12	0.214450E-02	0.203570E-02	0.173229E-02	0.131715E-02	0.902245E-03
0.14	0.291823E-02	0.275763E-02	0.231213E-02	0.170335E-02	0.109537E-02
0.16	0.381100E-02	0.359050E-02	0.298098E-02	0.214873E-02	0.131803E-02

Table 14 $\tilde{\underline{P}}|\underline{B} = \left[\langle \overset{2}{\tilde{t}}_i | B_j \rangle\right]$

i \ j	1	2	3
-24	0.09999999E-00	-0.20791168E-00	0.29425036E-00
-23	0.09999999E-00	-0.19936793E-00	0.27040463E-00
-22	0.09999999E-00	-0.19080898E-00	0.24754470E-00
-21	0.09999999E-00	-0.18223552E-00	0.22567607E-00
-20	0.09999999E-00	-0.17364816E-00	0.20480212E-00
-19	0.09999999E-00	-0.16504759E-00	0.18492683E-00
-18	0.09999999E-00	-0.15643445E-00	0.16605094E-00
-17	0.09999999E-00	-0.14780940E-00	0.14817446E-00
-16	0.09999999E-00	-0.13917309E-00	0.13131466E-00
-15	0.09999999E-00	-0.13052619E-00	0.11545374E-00
-14	0.09999999E-00	-0.12186933E-00	0.10061400E-00
-13	0.09999999E-00	-0.11320321E-00	0.86754473E-01
-12	0.09999999E-00	-0.10452846E-00	0.73970145E-01
-11	0.09999999E-00	-0.95845748E-01	0.62173714E-01
-10	0.09999999E-00	-0.87155731E-01	0.51337801E-01
-9	0.09999999E-00	-0.78459091E-01	0.41641624E-01
-8	0.09999999E-00	-0.69756471E-01	0.32991905E-01
-7	0.09999999E-00	-0.61048536E-01	0.25260913E-01
-6	0.09999999E-00	-0.52335954E-01	0.18418076E-01
-5	0.09999999E-00	-0.43617395E-01	0.12815066E-01
-4	0.09999999E-00	-0.34899495E-01	0.82334... E-02
-3	0.09999999E-00	-0.26176947E-01	0.46311045E-02
-2	0.09999999E-00	-0.17452405E-01	0.20533402E-02
-1	0.09999999E-00	-0.87265351E-02	0.51461940E-03
0	0.09999999E-00	-0.	0.
1	0.09999999E-00	0.87265351E-02	0.51461940E-03
2	0.09999999E-00	0.17452405E-01	0.20533402E-02
3	0.09999999E-00	0.26176947E-01	0.46311045E-02
4	0.09999999E-00	0.34899495E-01	0.82334... E-02
5	0.09999999E-00	0.43617395E-01	0.12815066E-01
6	0.09999999E-00	0.52335954E-01	0.18418076E-01
7	0.09999999E-00	0.61048536E-01	0.25260913E-01
8	0.09999999E-00	0.69756471E-01	0.32991905E-01
9	0.09999999E-00	0.78459091E-01	0.41641624E-01
10	0.09999999E-00	0.87155731E-01	0.51337801E-01
11	0.09999999E-00	0.95845748E-01	0.62173714E-01
12	0.09999999E-00	0.10452846E-00	0.73970145E-01
13	0.09999999E-00	0.11320321E-00	0.86754473E-01
14	0.09999999E-00	0.12186933E-00	0.10061400E-00
15	0.09999999E-00	0.13052619E-00	0.11545374E-00
16	0.09999999E-00	0.13917309E-00	0.13131466E-00
17	0.09999999E-00	0.14780940E-00	0.14817446E-00
18	0.09999999E-00	0.15643445E-00	0.16605094E-00
19	0.09999999E-00	0.16504759E-00	0.18492683E-00
20	0.09999999E-00	0.17364816E-00	0.20480212E-00
21	0.09999999E-00	0.18223552E-00	0.22567607E-00
22	0.09999999E-00	0.19080898E-00	0.24754470E-00
23	0.09999999E-00	0.19936793E-00	0.27040463E-00
24	0.09999999E-00	0.20791168E-00	0.29425036E-00

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Table 15 Typical {|S⟩} Represented on |B

ALPHA	THETA	ERROR ENERGY	ERROR/SIGNAL	COORDINATES		
0.	-10.00	0.130385E-07	0.694597E-07	0.435562	0.000236	0.161545
0.	-5.00	0.931323E-08	0.130387E-06	0.183936	0.000119	0.163422
0.	0.	0.884756E-08	0.191559E-06	0.099838	0.000000	0.164049
0.	5.00	0.838190E-08	0.117349E-06	0.183936	-0.000119	0.163422
0.	10.00	0.931323E-08	0.496141E-07	0.435562	-0.000236	0.161545
15.0	-10.00	0.558794E-08	0.316171E-07	0.415295	0.000187	0.162609
15.0	-5.00	0.558794E-08	0.800436E-07	0.178540	0.000077	0.163894
15.0	0.	0.931323E-08	0.201727E-06	0.099839	-0.000034	0.164000
15.0	5.00	0.130385E-07	0.188433E-06	0.178079	-0.000145	0.162937
15.0	10.00	0.111759E-07	0.644834E-07	0.411637	-0.000254	0.160720
30.0	-10.00	0.	0.	0.355375	0.000120	0.163634
30.0	-5.00	0.651926E-08	0.100465E-06	0.163253	0.000030	0.164228
30.0	0.	0.884756E-08	0.191866E-06	0.099838	-0.000059	0.163865
30.0	5.00	0.130385E-07	0.204013E-06	0.162610	-0.000148	0.162568
30.0	10.00	0.111759E-07	0.796882E-07	0.350245	-0.000236	0.160345
45.0	-10.00	0.745058E-08	0.704436E-07	0.271266	0.000051	0.164345
45.0	-5.00	0.111759E-07	0.191687E-06	0.142160	-0.000009	0.164332
45.0	0.	0.111759E-07	0.242747E-06	0.099837	-0.000069	0.163682
45.0	5.00	0.130385E-07	0.227550E-06	0.141664	-0.000128	0.162412
45.0	10.00	0.130385E-07	0.127978E-06	0.267275	-0.000186	0.160521
60.0	-10.00	0.558794E-08	0.769412E-07	0.185819	0.000001	0.164542
60.0	-5.00	0.931323E-08	0.179054E-06	0.120988	-0.000030	0.164177
60.0	0.	0.107102E-07	0.233008E-06	0.099836	-0.000059	0.163499
60.0	5.00	0.107102E-07	0.208986E-06	0.120772	-0.000089	0.162512
60.0	10.00	0.931323E-08	0.132288E-06	0.184073	-0.000118	0.161207

Table 16 $\underline{\tilde{t}}|\underline{M} = \left[\langle \tilde{t}_i | M_j \rangle\right]$

i \ j	1	2	3	4	5
-24	.142857	-.191698	.446069	.091705	.287598
-23	.142857	-.175114	.383726	.031918	.142559
-22	.142857	-.158232	.324762	-.021887	.020313
-21	.142857	-.141048	.269364	-.069393	-.061622
-20	.142857	-.123565	.217726	-.110290	-.122170
-19	.142857	-.105785	.170058	-.144291	-.156738
-18	.142857	-.087711	.126585	-.171134	-.167316
-17	.142857	-.069352	.087536	-.190606	-.156257
-16	.142857	-.050720	.053151	-.202551	-.126916
-15	.142857	-.031835	.023670	-.206902	-.083150
-14	.142857	-.012726	-.000685	-.203688	-.029500
-13	.142857	.006565	-.019706	-.193076	.028883
-12	.142857	.025979	-.033237	-.175405	.086108
-11	.142857	.045434	-.041186	-.151190	.136447
-10	.142857	.064817	-.043577	-.121200	.174135
-9	.142857	.083975	-.040575	-.086437	.194144
-8	.142857	.102709	-.032555	-.048170	.192521
-7	.142857	.120758	-.020148	-.007939	.167813
-6	.142857	.137794	-.004316	.032529	.120953
-5	.142857	.153435	.013724	.071270	.056390
-4	.142857	.167223	.032346	.106354	-.018686
-3	.142857	.178674	.049736	.135887	-.094180
-2	.142857	.187305	.064011	.158221	-.158823
-1	.142857	.192693	.073456	.172114	-.202600
0	.142857	.194532	.076788	.176805	-.218029
1	.142857	.192675	.073361	.172165	-.202536
2	.142857	.187158	.063259	.158651	-.158894
3	.142857	.178188	.047248	.137305	-.094294
4	.142857	.166104	.026605	.109629	-.019034
5	.142857	.151318	.002866	.077460	.055745
6	.142857	.134269	-.022397	.042831	.120047
7	.142857	.115369	-.047788	.007810	.166301
8	.142857	.095002	-.072081	-.025657	.190261
9	.142857	.073481	-.094402	-.055781	.191091
10	.142857	.051061	-.114130	-.081024	.170115
11	.142857	.027947	-.130879	-.100129	.131396
12	.142857	.004295	-.144460	-.112098	.079811
13	.142857	-.019776	-.154821	-.116187	.021163
14	.142857	-.044179	-.162023	-.111892	-.038713
15	.142857	-.068848	-.166193	-.098899	-.094038
16	.142857	-.093736	-.167508	-.077051	-.139664
17	.142857	-.118809	-.166167	-.046331	-.171076
18	.142857	-.144042	-.162385	-.006829	-.184253
19	.142857	-.169419	-.156381	.041291	-.176083
20	.142857	-.194929	-.148365	.097806	-.144014
21	.142857	-.220563	-.138545	.162441	-.086153
22	.142857	-.246316	-.127117	.234911	-.001133
23	.142857	-.272187	-.114264	.314885	.112102
24	.142857	-.298172	-.100156	.402050	.253974

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Table 17 Typical $\{|R\rangle\}$ Represented on $|\underline{M}$

ALPHA	THETA	ERROR ENERGY	COORDINATES				
0.	-10.00	0.506639E-06	1.703796	-0.201131	0.042735	0.007036	0.000282
0.	-5.00	0.223517E-06	1.283651	-0.266856	0.052231	0.000830	-0.000269
0.	0.	0.208616E-06	1.101464	-0.310345	0.056435	-0.007119	0.000246
0.	5.00	0.312924E-06	1.283651	-0.266778	0.052635	0.000600	-0.000244
0.	10.00	0.506639E-06	1.703796	-0.201001	0.043401	0.006657	0.000322
15.0	-10.00	0.506639E-06	1.675414	-0.205715	0.043592	0.006809	0.000240
15.0	-5.00	0.312924E-06	1.273645	-0.269613	0.052644	0.000441	-0.000266
15.0	0.	0.238419E-06	1.101372	-0.310266	0.056489	-0.007150	0.000249
15.0	5.00	0.283122E-06	1.271092	-0.268512	0.052843	0.000261	-0.000242
15.0	10.00	0.447035E-06	1.667396	-0.204188	0.043935	0.006425	0.000280
30.0	-10.00	0.536442E-06	1.585261	-0.218267	0.045720	0.006019	0.000106
30.0	-5.00	0.193715E-06	1.243011	-0.276517	0.053491	-0.000626	-0.000249
30.0	0.	0.208616E-06	1.101116	-0.310076	0.056510	-0.007160	0.000251
30.0	5.00	0.283122E-06	1.238808	-0.274608	0.053484	-0.000725	-0.000231
30.0	10.00	0.447035E-06	1.572447	-0.215537	0.045723	0.005692	0.000143
45.0	-10.00	0.387430E-06	1.447286	-0.239177	0.048921	0.004225	-0.000104
45.0	-5.00	0.238419E-06	1.198535	-0.286529	0.054560	-0.002336	-0.000185
45.0	0.	0.238419E-06	1.100769	-0.309325	0.056491	-0.007148	0.000250
45.0	5.00	0.193715E-06	1.194048	-0.284228	0.054398	-0.002336	-0.000175
45.0	10.00	0.298023E-06	1.434428	-0.235659	0.048638	0.004030	-0.000073
60.0	-10.00	0.208616E-06	1.289279	-0.267484	0.052585	0.000755	-0.000257
60.0	-5.00	0.193715E-06	1.151257	-0.297569	0.055554	-0.004447	-0.000038
60.0	0.	0.193715E-06	1.100419	-0.309581	0.056438	-0.007117	0.000247
60.0	5.00	0.193715E-06	1.147718	-0.295459	0.055334	-0.004370	-0.000041
60.0	10.00	0.253320E-06	1.280187	-0.263916	0.052164	0.000751	-0.000240

Table 18 $\underline{\mathbf{T}}|S|\underline{M} = \left[\langle t_i|S|M_j\rangle\right]$

i \ j	1	2	3	4	5
-24	0.	0.	0.	0.	0.
-23	-.142857	.191698	-.446069	-.091705	-.287598
-22	-.285714	.366813	-.829795	-.123623	-.430157
-21	-.428571	.525044	-1.154557	-.101735	-.456470
-20	-.571429	.666093	-1.423921	-.032342	-.394848
-19	-.714286	.789658	-1.641646	.077948	-.272678
-18	-.857143	.895443	-1.811704	.222239	-.115940
-17	-1.000000	.983154	-1.938289	.393373	.051377
-16	-1.142857	1.052505	-2.025826	.583980	.207633
-15	-1.285714	1.103225	-2.078977	.786531	.334550
-14	-1.428571	1.135060	-2.102647	.993433	.417700
-13	-1.571429	1.147786	-2.101962	1.197121	.447200
-12	-1.714286	1.141221	-2.082256	1.390197	.418317
-11	-1.857143	1.115242	-2.049019	1.565601	.332209
-10	-2.000000	1.069808	-2.007833	1.716791	.195762
-9	-2.142857	1.004991	-1.964256	1.837991	.021627
-8	-2.285714	.921016	-1.923681	1.924428	-.172517
-7	-2.428571	.818307	-1.891126	1.972598	-.365038
-6	-2.571428	.697549	-1.870978	1.980537	-.532851
-5	-2.714286	.559755	-1.866662	1.948008	-.653804
-4	-2.857143	.406320	-1.880385	1.876739	-.710194
-3	-3.000000	.239097	-1.912732	1.770384	-.691508
-2	-3.142857	.060423	-1.962467	1.634498	-.597328
-1	-3.285714	-.126881	-2.026478	1.476276	-.438505
0	-3.428571	-.319575	-2.099934	1.304162	-.235905
1	3.428571	-.514108	-2.176722	1.127356	-.017876
2	3.285714	-.706783	-2.250083	.955191	.184660
3	3.142857	-.893941	-2.313342	.796540	.343554
4	3.000000	-1.072129	-2.360590	.659235	.437848
5	2.857143	-1.238233	-2.387195	.549607	.456882
6	2.714286	-1.389551	-2.390061	.472147	.401136
7	2.571428	-1.523820	-2.367665	.429316	.281090
8	2.428571	-1.639189	-2.319877	.421505	.114788
9	2.285714	-1.734191	-2.247795	.447162	-.075473
10	2.142857	-1.807671	-2.153393	.502943	-.266564
11	2.000000	-1.858733	-2.039262	.583967	-.436679
12	1.857143	-1.886680	-1.908382	.684096	-.568075
13	1.714286	-1.890975	-1.763924	.796194	-.647886
14	1.571429	-1.871199	-1.609102	.912381	-.669050
15	1.428571	-1.827020	-1.447080	1.024274	-.630336
16	1.285714	-1.758172	-1.280887	1.123172	-.536299
17	1.142857	-1.664437	-1.113379	1.200223	-.396635
18	1.000000	-1.545628	-.947213	1.246554	-.225559
19	.857143	-1.401586	-.784828	1.253384	-.041306
20	.714286	-1.232166	-.628447	1.212093	.134776
21	.571429	-1.037238	-.480082	1.114287	.278790
22	.428571	-.816675	-.341537	.951846	.364943
23	.285714	-.570358	-.214420	.716935	.366076
24	.142857	-.298172	-.100156	.402050	.253974

Table 19 Projection Matrix $\underline{D}|\overset{2}{\underline{M}}$ (Transposed)

	1	2	3
(1, 1)	.204082	.000000	.000000
(2, 2)	.122961	.057689	.079178
(3, 3)	-.061154	-.101395	.258883
(4, 4)	.093107	.044573	.108317
(5, 5)	.181090	-.001192	.022441
(1, 2)	.313861	-.037318	-.306344
(1, 3)	-.074591	-.191437	.072804
(1, 4)	-.023998	.108826	.023423
(1, 5)	-.003900	-.011436	.003807
(2, 3)	.086673	.195292	-.084597
(2, 4)	.265782	-.129750	-.259417
(2, 5)	.034457	.010498	-.033631
(3, 4)	.118534	-.014802	-.115695
(3, 5)	-.153735	-.015599	.150053
(4, 5)	-.213569	.002724	.208454

Table 20 Residual Covariances of $\{|R\rangle\}$

		After Step 4	After Step 5
Trace $\underset{\sim}{C}$		.2374E-4	.4990E-7
Spectrum $\underline{C}$	1	.2369E-4	.4975E-7
	2	.9252E-7	.4975E-7
	3-29	.4626E-7	.4975E-7
	30-35	.0000	.4975E-7
	36-49	.0000	.0000

Table 21 $\{ \ddot{\rho}_o - \langle \underline{F}\ \tilde{\underline{M}} | \dot{R} \rangle \} = \ddot{\rho}_o$ Corrections

α \ θ	-10.0	-5.0	-2.0	-0.5	0.0	5.0	10.0
0.0	-1.52	2.09	2.40	5.44	5.76	2.46	-0.91
15.0	-0.70	1.98	2.62	5.53	5.82	2.33	-0.18
30.0	1.31	1.52	3.11	5.60	5.82	1.81	1.69
45.0	3.16	1.06	3.85	5.64	5.78	1.24	3.41
60.0	2.31	1.67	4.68	5.65	5.72	1.67	2.51

Table 22 (Representation Corrections)(10^6)

α \ θ	-10.0	-5.0	-2.0	-0.5	0.0	5.0	10.0
	0.09	0.07	0.11	0.19	0.06	0.09	0.11
0.0	-2.99	-0.54	-0.07	-0.07	-0.05	0.28	2.25
	0.12	0.02	-0.08	-0.14	0.00	0.02	0.10
	0.10	0.07	0.10	0.19	0.06	0.07	0.13
15.0	-2.35	-0.36	-0.05	-0.07	-0.05	0.36	2.33
	0.09	0.04	-0.09	-0.15	0.01	0.02	0.06
	0.21	0.01	0.11	0.19	0.06	0.04	0.20
30.0	-1.40	-0.17	-0.04	-0.06	-0.06	0.30	1.80
	0.01	0.07	-0.10	-0.15	0.01	0.06	-0.01
	0.19	-0.02	0.15	0.19	0.06	-0.01	0.20
45.0	-0.60	-0.04	-0.04	-0.06	-0.06	0.16	0.97
	-0.04	0.10	-0.10	-0.16	0.01	0.10	-0.06
	0.07	-0.04	0.16	0.19	0.06	-0.04	0.07
60.0	-0.13	-0.01	-0.06	-0.06	-0.05	0.04	0.28
	0.02	0.11	-0.13	-0.14	0.01	0.11	0.01

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Table 23 Noise Ensemble Added to $\{|R\rangle\}$

$\sigma = (1.910)(10^{-6})$

Noise Sample Number	Coordinates of Representatives on $\mathbb{M}$				
1	-0.196298E-05	-0.294657E-04	0.235789E-04	0.235334E-04	-0.201412E-05
2	-0.108524E-05	0.652689E-05	-0.181119E-04	0.115753E-04	-0.108642E-04
3	-0.245607E-05	-0.287709E-05	0.952689E-05	0.309839E-04	-0.219368E-04
4	0.152509E-05	-0.426705E-04	0.226476E-04	-0.746967E-05	0.390579E-05
5	0.143267E-06	0.170266E-04	-0.887691E-05	-0.750996E-05	-0.724232E-05
6	-0.160617E-06	-0.469829E-04	0.213192E-04	0.890773E-05	0.963294E-05
7	-0.619749E-06	0.159202E-04	0.844869E-05	-0.774859E-05	-0.101050E-04
8	0.295572E-05	-0.354697E-04	-0.615436E-07	-0.461622E-04	0.312730E-04
9	0.669392E-06	-0.168376E-04	0.156758E-04	-0.218192E-04	0.307908E-05
10	0.360076E-05	0.352536E-04	-0.279584E-04	-0.531685E-04	0.169061E-04
11	-0.534086E-06	-0.329511E-04	0.333570E-04	0.116007E-04	0.216132E-05
12	-0.314572E-06	-0.120338E-04	-0.129923E-04	0.601890E-05	0.233323E-05
13	-0.181710E-05	0.209616E-05	0.302290E-05	0.336502E-04	-0.166540E-04
14	-0.201147E-05	-0.258962E-04	0.816322E-05	0.316291E-04	-0.248318E-05
15	-0.357292E-06	-0.678662E-05	0.100261E-05	0.618940E-05	-0.115186E-04
16	0.513854E-06	-0.156108E-04	-0.976265E-05	-0.543693E-05	0.110275E-04
17	0.189271E-05	-0.175368E-05	-0.116081E-04	-0.180641E-04	0.598219E-05
18	-0.966944E-06	0.899965E-05	-0.205822E-04	0.668657E-06	-0.721796E-05
19	-0.124901E-05	-0.131466E-06	0.603038E-06	0.135127E-04	-0.299493E-05
20	0.237136E-05	-0.336492E-04	-0.281831E-05	-0.131709E-04	0.242849E-04
21	0.543476E-06	-0.180185E-04	0.361832E-05	-0.233029E-05	-0.567813E-05
22	0.117553E-06	0.141541E-04	-0.217711E-04	-0.610698E-05	0.111009E-04
23	0.815399E-06	0.178473E-04	0.224500E-06	-0.225459E-04	-0.709941E-05
24	0.246115E-05	-0.612046E-05	0.262167E-05	-0.213881E-04	0.153198E-04

Table 24 Analysis of Representation Error ($\times 10^6$)

λ / ϕ	α / θ	Source of Error: Noise Sample 1	Source of Error: D-C Restoration $\langle \tilde{M}_1 \vert R \rangle$ 0.1% High
29.88	30.00	0.1	359.1
-4.03	-2.00	-1.7	0.0
		32.0	154.5
29.97	30.00	-2.0	338.7
-2.53	-0.50	-1.7	0.0
		31.1	154.2
30.00	30.00	-2.1	337.3
-2.03	0.00	-1.7	0.0
		31.1	154.1
30.16	30.00	14.1	468.7
2.97	5.00	-1.8	-0.1
		36.5	154.2
30.13	30.00	86.0	851.4
7.97	10.00	-2.2	-0.2
		49.6	154.5

Table 25 Effect of Noise on D-C Restoration

Noise Sample	Error in $\ddot{\rho}_o$
1	-.203
2	-.097
3	.141
4	-.017
5	.148
6	-.044
7	.080
8	.060
9	-.274
10	.079
11	.243
12	-.254
13	.268
14	-.094
15	.018
16	.324
17	.216
18	-.152
19	.237
20	-.228
21	-.023
22	.111
23	.014
24	-.273

In Tables 24 and 25, RMS noise on each range-rate coordinate = $(1.91)(10^{-6})$.

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Table 26 Covariance Matrices of Representation Error($\times 10^6$) over Ensemble of 24 Noise Vectors $\alpha = 30^\circ$

	θ	Covariance Matrix		
Initial Operation	-2.0	425.62	4.21	-25.47
		4.21	33.84	-24.71
		-25.47	-24.71	333.35
	-0.5	355.43	4.62	-48.52
		4.62	33.08	-23.71
		-48.52	-23.71	315.50
	0.0	351.29	4.65	-49.82
		4.65	33.02	-23.65
		-49.82	-23.65	314.24
	5.0	1088.91	-0.01	168.73
		-0.01	37.69	-30.11
		168.73	-30.11	435.18
	10.0	9733.61	-34.35	1739.15
		-34.35	51.79	-48.52
		1739.15	-48.52	835.95
Subsequent Iteration	-2.0	43.10	1.24	-2.28
		1.24	128.53	13.77
		-2.28	13.77	95.93
	-0.5	40.70	1.24	-2.29
		1.24	125.63	13.41
		-2.29	13.41	94.11
	0.0	40.53	1.23	-2.28
		1.23	125.40	13.38
		-2.28	13.38	93.98
	5.0	56.18	1.23	-2.33
		1.23	143.10	15.44
		-2.33	15.44	104.86
	10.0	103.40	1.45	-3.63
		1.45	196.54	21.62
		-3.63	21.62	138.79

A - 4 SYMBOLS USED IN CHAPTER FOUR

In Appendix 4, the various special assignments of symbols used in Chapter 4 and in the computer programs are presented. Time measured from the origin (north-bound equatorial crossing) is symbolized by t. Time measured from the epoch ($t = \tau_0$) is symbolized by τ. In the computer programs, $\omega_s t$ and $\omega_s \tau$ are used in place of t and τ. Note that the subscript zero refers to the epoch. Angles are tabulated in degrees throughout the dissertation. Range of satellite from navigator is symbolized by ρ. The symbol σ is used for ρ^2 and also for rms noise.

Table 27 Parameters of the Navigation Problem

Symbol	Meaning	Unit
ρ_p	Planet radius	m
ρ_s	Satellite radius	m
ω_p	Planet angular velocity of rotation	rad/sec
ω_s	Satellite angular velocity of revolution	rad/sec
λ	Navigator latitude	rad
ϕ	Original navigator longitude east of orbit	rad
τ_0	Interval from equatorial crossing to epoch	sec
α	Satellite latitude at epoch, $\omega_s \tau_0$	rad
θ	Epochal navigator longitude east of orbit	rad
ν_1	Distance eastward from orbital plane	m
ν_2	Distance northward from equator	m
T	Sampling interval	sec

Table 28 Bases of the Satellite Navigation Problem

Subspace	Dimensionality	Basis	Description
$\mathcal{T}$	2N+1=601	$\vert\underline{t}$	Finite time basis
$\mathcal{M}$	m=5	$\vert\underline{M}$	Measurement basis for $\{\vert R\rangle\}$
$\tilde{\mathcal{T}}$	2N+1	$\tilde{\underline{t}}\vert$	Dual of $\vert\underline{t}$
$\tilde{\mathcal{M}}$	m	$\tilde{\underline{M}}\vert$	Dual of $\vert\underline{M}$
$\mathcal{T}\otimes\mathcal{T}$	$(2N+1)^2$	$\vert\underline{t}\otimes\vert\underline{t}$	$\{\vert t_i\rangle\otimes\vert t_j\rangle\}$
$\mathcal{P}$	2N+1	$\vert\underline{P}$	$\{\vert t_i\rangle\otimes\vert t_i\rangle\}$
$\mathcal{M}\otimes\mathcal{M}$	m^2	$\vert\underline{M}\otimes\vert\underline{M}$	Tensor square of $\vert\underline{M}$
$\overset{2}{\mathcal{T}}$	(2N+1)(N+1)	$\vert\overset{2}{\underline{t}}$	Tensor square of $\vert\underline{t}$, symmetrized
$\overset{2}{\mathcal{M}}$	m(m+1)/2	$\vert\overset{2}{\underline{M}}$	Tensor square of $\vert\underline{M}$, symmetrized
$\mathcal{A}$	7	$\vert\underline{A}$	Complete basis (not orthonormal) for $\{\vert S\rangle\}$
$\mathcal{B}$	3	$\vert\underline{B}$	Effectively complete sub-basis of $\vert\underline{A}$
$\mathcal{D}$	3	$\underline{D}\vert$	Dual of $\vert\underline{B}$
$\tilde{\overset{2}{\mathcal{M}}}$	m(m+1)/2	$\tilde{\overset{2}{\underline{M}}}\vert$	Dual of $\vert\overset{2}{\underline{M}}$
$\tilde{\mathcal{P}}$	2N+1	$\tilde{\underline{P}}\vert$	Dual of $\vert\underline{P}$
$\mathcal{C}$	1	$\vert M_1\rangle$	Basis for d-c signals
$\tilde{\mathcal{C}}$	1	$\langle\tilde{M}_1\vert$	Dual of $\vert M_1\rangle$

All of the bases are orthonormal except for $|\underline{A}$ and those bases derived from $|\underline{A}$. The bases in $\mathcal{T}$ are defined on the interval of duration 2NT centered on the epoch.

Table 29 Signals, Operators and Matrices of the Satellite Navigation Problem

Symbol	Dimensionality	Description
$\|S\rangle$	7	Range-square signal
$\|R\rangle$	2N+1=601	Range signal
$\|\underline{B}\ \underline{D}\|S\rangle$	3	Projection of $\|S\rangle$ onto $\mathcal{B}$
$\|\underline{M}\ \underline{\tilde{M}}\|R\rangle$	m=5	Projection of $\|R\rangle$ onto $\mathcal{M}$
$\overset{2}{\|R\rangle}$	(2N+1)(N+1)	Tensor square of $\|R\rangle$
$\|\dot{R}\rangle$	2N+1	Range-rate signal
$\|S\|$	(2N+1)x(2N+1)	Accumulator (outward from epoch)
$[\alpha_i]$	7	Representative of $\|S\rangle$ on $\|\underline{A}$
$\underline{D}\|S\rangle = [\sigma_i]$	3	Representative of $\|\underline{B}\ \underline{D}\|S\rangle$ on $\|\underline{B}$
$[\hat{\sigma}_i]$	4	Output of representation section $\hat{\sigma}_4 = \omega_s \hat{\tau}_o = \hat{\alpha} = \hat{\alpha}_8$
$\underline{\tilde{P}}\|S\rangle = \underline{s}\rangle$	2N+1	Representative of $\|S\rangle$ on $\|\underline{P}$
$\underline{\tilde{t}}\|R\rangle = \underline{r}\rangle$	2N+1	Representative of $\|R\rangle$ on $\|\underline{t}$
$\underline{\tilde{t}}\|\dot{R}\rangle = \underline{\dot{r}}\rangle$	2N+1	Representative of $\|\dot{R}\rangle$ on $\|\underline{t}$
$\underline{\tilde{M}}\|R\rangle$	m	Representative of $\|\underline{M}\ \underline{\tilde{M}}\|R\rangle$ on $\|\underline{M}$
$\underline{\tilde{M}}\|S\|\dot{R}\rangle$	m	Representative of $\|S\|\dot{R}\rangle$ on $\|\underline{M}$
$\langle\underline{F}$	m-1	Row which operates on $\underline{\tilde{M}}\|R\rangle$ or $\underline{\tilde{M}}\|S\|\dot{R}\rangle$ to yield $\ddot{\rho}_o$ approximately
$\underline{\tilde{M}}\|S\|\underline{t}$	m x (2N+1)	Representative of $\|S\|$ on $\|\underline{M}$ and $\underline{\tilde{t}}\|$
$\|\underline{B}\ \underline{D}\|\overset{2}{\underline{M}}\ \overset{2}{\underline{\tilde{M}}}\|$	3 x m(m+1)/2	Projector cascade, $\overset{2}{\mathcal{D}} \to \overset{2}{\mathcal{M}} \to \mathcal{P} \to \mathcal{A} \to \mathcal{B}$
$\underline{D}\|\overset{2}{\underline{M}}$	3 x m(m+1)/2	Representative of projector cascade on $\|\underline{B}$ and $\overset{2}{\underline{\tilde{M}}}\|$

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BIOGRAPHICAL INFORMATION

Dan Connor Ross graduated from high school in 1940 at Whiteland, Indiana, attended Purdue University, served in the U.S. Army from 1943 to 1946, and received the BSEE degree in 1946 and the MSEE in 1949 from Purdue. He served on the full-time staff at Purdue until 1951. He then served as Instructor of Electrical Engineering, USMA, West Point, and studied at Columbia University on a part-time basis.

Mr. Ross joined IBM in 1953 as a systems planning engineer on the SAGE system. He directed the system design of the production models of the SAGE computers. Other assignments at the engineering laboratory at Kingston, New York included: Manager of Systems Engineering, Manager of Air Traffic Control Project and Manager of Technical Planning. From 1959 to 1962, Mr. Ross was assigned to full-time graduate study at The Johns Hopkins University. Since 1962, he has divided his time between his research project and management of the Communications Analysis Department of the IBM Federal Systems Division.

Several patents have been issued to Mr. Ross and he has published papers on position telemetry and signal theory. He belongs to Eta Kappa Nu, Tau Beta Pi, Sigma Xi, IEEE, AAAS and other professional societies. He has been a member of the Joint AIEE-IRE-ACM Computer Committee.

CORRECTIONS

Page	Place	
24	Last 3 lines	The paragraph beginning "Since we ..." ends with "... in Chapter 3.".
30	Lines 13, 25	Add small tilde to $\|G\rangle$.
35	Line 10 from end	... of n linearly independent signals...
39	Line 12	Enclose $\|B_1\rangle$ and $\langle\tilde{B}_1\|$ in braces.
58	Line 13	Insert 3 pairs of brackets.
81	Lines 22, 23	Insert - .
86	Line 3 from end	The conjugation applies to the entire expression in both cases.
90	Equation (57)	Replace μ with J and J with μ.
122	Line 7 from end	Insert = .
123	Lines 17, 18	... compared to the distance to any nearby fold of $\mathcal{L}$.
124	Figure 28	Insert $\rangle$ after $\|E_1$.
127	Lines 9-11	The sentence ought to read: Projections on the 3-dimensional sub-basis of each of the 7 basis elements (normalized) are given in Table 12 in Appendix 3.
129	Line 8	Insert 3 vertical bars.
131	Equation (116)	Add small tilde to $\underline{P}\|_2$.
131	Lines 2,12,20,21	Add small tilde to $\underline{M}\|$.
141	Lines 12, 13	... the kth post-epoch measurement, the m multiplications ...
147	Figure 35	The derivative term should be multiplied by θ .
155	Equation (127)	Insert + .

Page	Place	
159	Line 18	Add small tilde to $\tilde{M}\|$.
165	Lines 16, 17	... about 0.005° on the estimate ...
165	Lines 19, 20	... position is somewhat less than the effect on latitude.
166	Line 7 from end	... 11 and 26 ...
179	Line 2	Change F, H, G to $\|F\rangle$, $\|H\|$, $\langle G\|$.
179	Line 3	Change L to $\|L\|$.
190	Table 12	Change caption to: Relation of $\|\underline{A}$ to the Sub-basis $\|\underline{B}$
190	Table 12	Add note: Row 1 represents the projection on $\|\underline{B}$ of the normalized $\|A_1\rangle$.
205	Before last line	Insert the line: <u>IRE Transactions on Audio</u>
208	After line 3	Insert 3 lines: A. Lichnerowicz Éléments de Calcul Tensoriel Colin, Paris, 1962

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