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A Missile Targeting Problem

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Summary

The following missile assignment problem is considered. Missiles are to be assigned to targets in two distinct steps. First, each missile is programmed so that it can be fired at any one of a small number of targets, the number of targets being the missile capability. The programming of the missiles is represented by a qualification matrix Q. Second, if battle occurs, all missiles are to be assigned to targets and launched. Each missile must be assigned to a target for which it is programmed. It is assumed that only a random subset X of the missiles will actually be available for battle, and so the assignment must be made for a reduced qualification matrix $Q(X)$. The questions considered are "what is an optimal assignment given the reduced qualification matrix $Q(X)$?", and "what can be expected from this assignment?" Use of a damage function is proposed. An optimal assignment is one which maximizes the value of the damage function. The damage function may be chosen to represent a wide variety of optimization requirements. The main part of the paper describes Monte Carlo procedures for estimating the expected damage and the probability that the damage will be at least c for any number c. In an appendix several examples are given to illustrate the use of the damage function. Another paper [1] describes an algorithm for finding an assignment which maximizes the damage function.

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Introduction

This note deals with certain mathematical problems connected with the assignment of missiles to targets. The problems are those which arise because the missiles have a multiple target capability. The assignment of such missiles to targets occurs in two distinct steps. First, each missile is programmed so that it can be fired at any one of a small set of targets. The number of targets for which the missile may be programmed is its target capability. Second, when battle actually occurs and the missile is to be fired, it is assigned to one of the targets for which it is programmed. The second step, i.e. the assignment of the missile to a target, takes almost no time. However, the first step, the programming of the missile for a set of targets, is very time consuming. Therefore, we assume that if battle actually occurs, the missile cannot be reprogrammed but must be fired at one of the targets for which it is already programmed.

When battle occurs, it may be that not all of the missiles are available. Still because of the multiple target capability, there will be many different ways of assigning the available missiles to targets. One problem is to pick the best possible assignment. Another problem is to evaluate the effectiveness of the original missile programming. Because of the complexity of the final assignment procedure (the procedure which picks out the best assignment of available missiles), this problem must be done by the Monte Carlo method.

To describe the problems more precisely and introduce some of our terminology, let us look at a simple example.

Consider a group of nine missiles, each missile having a capability of two targets, and suppose that there are six targets. The group might be set up with the missiles programmed as follows:

Missile	Targets	Missile	Targets
1	1,5	6	3,6
2	1,6	7	3,4
3	1,2	8	4,5
4	2,5	9	4,6
5	2,3		

We will represent this programming of missiles by a <u>qualification</u> <u>matrix</u> Q:

$$Q = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

If $q_{ij} = 1$, this means that missile j is qualified, i.e. programmed, to go to target i. Thus, for Q, we have $q_{11} = q_{12} = q_{13} = 1$, so missiles 1, 2, and 3 can all be fired at target 1.

Now consider the situation if battle actually occurs. It may well be that not all missiles are available; some might be out of order, disable by enemy action, or already used. Suppose that missiles 3, 4, and 8 have been eliminated. Then the situation is represented by this qualification matrix:

$$Q_2 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

The first problem to consider is: Find an optimal assignment of missiles to targets given a qualification matrix such as Q_2. We will call this the basic assignment problem. In this basic assignment problem we assume that all missiles are to be launched; we do not consider the possibility of keeping some missiles in reserve.

Clearly, the first thing we must do in attacking this problem is to define the term optimal. We propose the use of a function that gives for each assignment a number called the damage for that assignment. An optimal assignment is then one for which the damage is a maximum. A natural idea is to make the damage equal to the expected value of the loss which the enemy will suffer if the missiles are launched with the given assignment of targets. However, by suitably choosing the damage function, many different optimization criteria may be represented. The appendix contains several examples to illustrate this. It is hoped that these examples will help the reader gain familiarity with the damage function concept. Example 4, which shows how to maximize the probability that all targets are destroyed, is especially interesting. Precise definitions of the terms "damage function" and "basic assignment problem" are given in Section 1. However, no discussion of the solution of the problem is given. For that, the reader must turn to Reference [1], which he should certainly read along with this document. There an efficient algorithm for solving the basic assignment problem is described.

The second problem we consider is: Evaluate the effectiveness of the original qualification matrix. If we knew that all missiles would be available for battle, there would be nothing to this problem. The measure of its effectiveness would simply be the damage for the best assignment, this being found by the algorithm mentioned above. However, we do not expect all the missiles to be available for battle, and here a random element enters the calculation. It seems reasonable to assume that each missile has probability p of being available. We then ask what is the expected value of the damage resulting from the best possible assignment of the available missiles. Except in some simple cases, it seems impossible to compute this expected damage by analytic methods. Instead, we turn to the Monte Carlo method. The problem could be attacked by straightforward simulation, but a better approach is possible. This is given in Section 2. In addition to asking for the expected damage, one might want to know the probability that the damage would be at least a certain value. For example, one might ask for the probability that all targets would be destroyed. This may also be calculated by the Monte Carlo method. The calculation is described in Section 3.

1. Damage functions and the basic assignment problem

In this section, we give a precise statement of the basic assignment problem. The terms defined here will be used freely in the following sections. No description of the algorithm for solving the problem is given here. The reader must obtain all information about that from

Reference [1], and it is strongly recommended that he do that before going on to Section 2. Examples illustrating the use of the damage function will be found in the appendix. The reader might also wish to look at some of these before continuing to the discussion of Monte Carlo methods in Sections 2 and 3.

We begin with some definitions. A <u>qualification matrix</u> Q is a zero-one matrix, i.e. a matrix in which every element is either zero or one, which has at least one non-zero element in each column. A qualification matrix is assumed to represent the programming of a set of missiles as described in the introduction. The restriction that every column has a non-zero element means simply that every missile is qualified for some target.

An <u>assignment matrix</u> A <u>belonging to</u> a qualification matrix Q is a zero-one matrix with the following properties.

(1.1) If $q_{ij} = 0$, then $a_{ij} = 0$.

(1.2) For each j there exists exactly one i such that $a_{ij} = 1$.

An assignment matrix is supposed to give the targets at which the available missiles will be launched. Condition (1.1) means that a missile can be launched only at a target for which it is qualified. Condition (1.2) says first that a missile can be launched at only one target, i.e. it cannot go in two directions at once, and second that it must be launched at some target. This last condition we mentioned in the introduction; to repeat: "<u>in the basic assignment problem all missiles are to be launched at once; none are to be kept in reserve</u>."

For an assignment matrix A, we let $r_i(A) = \sum_j a_{ij}$, i.e. $r_i(A)$ is the number of missiles assigned to target i.

Let Q be an $n \times m$ qualification matrix. A <u>damage function</u> for Q is a function $g(i,j)$ defined for all i with $1 \le i \le n$ and all j with $0 \le j \le m$ which has the following properties:

(1.3) $g(i,0) = 0$ for all i.

(1.4) $g(i,j)$ is for each i an increasing function of j.

(1.5) $g(i,j)$ is for each i a concave function of j.

If we let $\delta(i,j) = g(i,j) - g(i,j-1)$ for $j \ge 1$, Condition 1.5 is equivalent to

(1.6) If $j \le k$, then $\delta(i,k) \le \delta(i,j)$.

The quantity $g(i,j)$ may be interpreted as the expected loss which the enemy will suffer if j missiles are launched at target i. If we assume that the enemy's total loss is the sum of the losses for each target, then the expected loss to the enemy resulting from an assignment according to the assignment matrix A will be

$$(1.7) \qquad d(A) = \sum_{i=1}^{n} g(i, r_i(A)).$$

This leads us to the following formulation of the <u>basic assignment problem</u>: <u>Given a qualification matrix</u> Q <u>and damage function</u> g <u>for</u> Q, <u>find an assignment matrix</u> A <u>which maximizes the damage</u> $d(A)$ <u>given by (1.7)</u>.

An efficient algorithm for the solution of this problem is described in Reference [1]. In that paper, the problem is described in terms of men and tasks, but translation into the terms of the missile problem is easy. For the term "man" in [1], substitute "missile", for "job" substitute "target", for "output" substitute "damage". The output or damage in [1] is given by a set of functions $f_1(j),\dots,f_n(j)$. The relation to the damage function $g(i,j)$ is simply $f_i(j) = g(i,j)$.

Interpreting $g(i,j)$ as an expected loss, the Conditions (1.3) and (1.4) are obvious. The Condition (1.5) is, however, a significant restriction on the class of damage functions which we allow. It says, roughly, that for each target the expected additional damage done by sending one more missile to that target decreases as the number of missiles assigned to that target is increased. It is possible to consider damage functions without the restriction (1.5), but this condition is necessary for use of the algorithm of Reference [1].

2. Expected damage for a given qualification matrix

In both this section and the next, we will let X_p denote a random subset from the set of integers $1,2,\dots,m$. The number m will be assumed to be given and fixed throughout the argument. Each integer has independently probability p of being in X_p. Thus p is a parameter in the probability distribution of X_p. The number of elements in X_p will be denoted by $\alpha(X_p)$. The probability that $\alpha(X_p) = r$ will be denoted by q_r, which is just a binomial probability, i.e.

$$(2.1) \qquad \mathrm{Prob}(\alpha(X_p) = r) = q_r = \binom{m}{r} p^r (1-p)^{m-r}.$$

Note that the value of q_r is a function of p, although this is not made clear by the notation.

The purpose of this section and the next is to investigate the following situation. A damage function $g(i,j)$ and an $n \times m$ qualification matrix Q are given. However, for the conjectured attack each missile has only probability p of being available. Thus, the qualification matrix of the available missiles is a random variable, $Q(X_p)$, where $Q(X_p)$ is the matrix obtained from Q by dropping out those columns which are not in X_p. The damage for an optimal assignment belonging to $Q(X_p)$ will be denoted by $d(Q(X_p))$. We are interested in the properties of the random variable $d(Q(X_p))$.

In this section we study the problem of estimating $\mathbf{E}[d(Q(X_p))]$.* For convenience, we will write $E(Q,p) = \mathbf{E}[d(Q(X_p))]$. The expected damage $E(Q,p)$ is a natural measure of the effectiveness of a qualification matrix. If the value of p is chosen realistically, we might well choose between two qualification matrices Q_1 and Q_2 by comparing $E(Q_1,p)$ and $E(Q_2,p)$.

Here is a simple example of calculating the expected damage $E(Q,p)$. Take Q to be the matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Suppose the damage function is

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* For any random variable Y, $\mathbf{E}[Y]$ will denote the expected value of Y.

$$g(i,0) = 0 \qquad\qquad \text{all } i$$

$$g(i,j) = 1 \qquad 1 \leq j, \qquad \text{all } i.$$

That is, the damage is just the number of targets covered by the assignment. Now it is easy to see that $d(Q(X_p)) = \alpha(X_p)$. The probability that X_p contains exactly k missiles is

$$\binom{3}{k} p^k (1 - p)^{3-k}.$$

Therefore, $E(Q,p) = \sum_{k=0}^{3} k\binom{3}{k} p^k (1 - p)^{3-k}$. For $p = 1/3$ this is

$$E(Q,p) = 0 + 1 \times 3 \times (\tfrac{1}{3}) \times (\tfrac{2}{3})^2 + 2 \times 3 \times (\tfrac{1}{3})^2 \times (\tfrac{2}{3}) + 3 \times 1 \times (\tfrac{1}{3})^3 \times 1$$

$$= 1.0.$$

For $p = 2/3$, this is

$$E(Q,p) = 0 + 1 \times 3 \times (\tfrac{2}{3}) \times (\tfrac{1}{3})^2 + 2 \times 3 \times (\tfrac{2}{3})^2 \times (\tfrac{1}{3}) + 3 \times 1 \times (\tfrac{2}{3})^3 \times 1$$

$$= 2.0.$$

Unfortunately, an exact calculation of the expected damage can be made only in very simple cases such as that given in the example. For large qualification matrices and more complication damage functions, the calculation must be made by the Monte Carlo method. A simple approach would be to carry out a direct simulation of the problem. One could generate a sequence of independent random sets $X_p^1, \ldots, X_p^N$ and form the estimate $\bar{Y} = \frac{1}{N} \sum_{i=1}^{N} d(Q(X_p^i))$. The expected value of $\bar{Y}$ is clearly $E(Q,p)$. The variance of $\bar{Y}$ could be estimated by

$$\frac{1}{N - 1} \left(\frac{1}{N} \sum_{i=1}^{N} [d(Q(X_p^i))]^2 - \bar{Y}^2\right).$$

Two objections can be raised against this procedure. First of all, it yields an estimate of $E(Q,P)$ only for one value of p. To find $E(Q,P)$ for a different value of p, the entire calculation would have to be done again, using the new value of p in generating the random sets X_p^i. Secondly, this procedure takes no account of the operation of the algorithm which finds the optimal assignment for $Q(X_p^i)$ and computes $d(Q(X_p^i))$. As well as finding $d(Q(X_p^i))$, the algorithm obtains as intermediate results the values $d(Q^j(X_p^i))$ for $j = 1,\ldots,\alpha(X_p^i)$, where $Q^j(X_p^i)$ is the matrix containing only the first j columns of $Q(X_p^i)$. The direct simulation procedure makes no use of the values $d(Q^j(X_p^i))$ for $j < \alpha(X_p^i)$.

A procedure which avoids both of these objections has proved quite efficient in practice. It makes use of a random permutation σ of the integers $1,2,\ldots,m$, i.e. a permutation chosen at random from all the $m!$ permutations of the integers $1,2,\ldots,m$. For the permutation σ, we let $Q(\sigma)$ denote the qualification matrix obtain from Q by permuting the columns of Q according to σ. Thus, if the elements of $Q(\sigma)$ are q^*_{ij}, we have $q^*_{ij} = q_{i\sigma(j)}$, where $\sigma(j)$ is the integer into which the permutation σ takes j. For $j \leq m$, we let $Q^j(\sigma)$ denote the $n \times j$ qualification matrix consisting of the first j columns of $Q(\sigma)$.

<u>To estimate</u> $E(Q,p)$ <u>proceed as follows: Generate a sequence</u> $\sigma_1,\sigma_2,\ldots,\sigma_N$ <u>of independent random permutations of the integers</u> $1,2,\ldots,m$. <u>Let</u>

$$\bar{G}_j = \frac{1}{N}\sum_{i=1}^{N} d(Q^j(\sigma_i)). \tag{2.2}$$

Then for any probability p, an unbiased estimate of $E(Q,p)$ is

(2.3) $$\bar{W} = \sum_{j=0}^{m} q_j \bar{G}_j ,$$

where q_j is given by (2.1).

Note that the parameter p enters into the calculation only through the q_j in (2.3). Thus, the number of calculations involving the parameter p is independent of the number of permutations used. Moreover, one may save the values of the $\bar{G}_j$. Then, at some later date, the estimate $\bar{W}$ may be calculated for various values of p without generating any additional random permutations.

The natural measure of the error in a Monte Carlo estimate such as (2.3) is the standard deviation of the estimate. This is defined as follows. The variance of the random variable $\bar{W}$ is $\mathbf{V}[\bar{W}] = \mathbf{E}[(\bar{W} - \mathbf{E}[\bar{W}])^2]$. The standard deviation of $\bar{W}$ is $\sqrt{\mathbf{V}[\bar{W}]}$. The usual procedure in a Monte Carlo calculation is to make an unbaised estimate of $\mathbf{V}[\bar{W}]$ and then take the square root of this as an estimate of the standard deviation. Therefore, in the remainder of this paper we will only discuss the problem of estimating the variance.

Unfortunately, if we desire to estimate the variance of $\bar{W}$, it seems that intermediate computations involving p must be made. Let

(2.4) $$W_i = \sum_{r=0}^{m} q_r d(Q^r(\sigma_i)),$$

and let

(2.5) $$G_{rs} = \frac{1}{N} \sum_{i=1}^{N} d(Q^r(\sigma_i)) d(Q^s(\sigma_i)).$$

An unbiased estimate of $\mathbf{V}[\bar{W}]$ may be computed by using either of the following two expressions

$$(2.6)\qquad \bar{Z} = \frac{1}{N}\left(\frac{N}{N-1}\left[\frac{1}{N}\sum_{i=1}^{N}(W_i - \bar{W})^2\right]\right) = \frac{1}{N-1}\left[\frac{1}{N}\sum_{i=1}^{N} W_i^2 - \bar{W}^2\right] \quad \text{or}$$

$$(2.7)\qquad \bar{Z} = \frac{1}{(N-1)}\left[\sum_{r,s=0}^{m} q_r q_s G_{rs} - \bar{W}^2\right].$$

The two expressions on the right are, in fact, equal.

The expression (2.7) might be used to avoid intermediate computations with the parameter p. Unfortunately, many computations will be needed to find the G_{rs}. In fact, it will require $m(m-1)/2$ multiplications and additions for each random permutation.

The remainder of this section will be devoted to proving that the estimates $\bar{W}$ and $\bar{Z}$ do have the desired properties. Note that we can write

$$(2.8)\qquad \bar{W} = \frac{1}{N}\sum_{i=1}^{N}\left[\sum_{r=0}^{m} q_r d(Q^r(\sigma_i))\right] = \frac{1}{N}\sum_{i=1}^{N} W_i.$$

Thus, $\bar{W}$ is the average of the identically distributed random variables $W_1, W_2, \ldots, W_N$. Thus we have

$$(2.9)\qquad \mathbf{E}[\bar{W}] = \mathbf{E}[W_1] = \sum_{r=0}^{m} q_r \mathbf{E}[d(Q^r(\sigma))]$$

where σ is a random permutation. Moreover, by familiar argument from the theory of statistics, we know that an unbiased estimate of $\mathbf{V}[\bar{W}]$ is just $\bar{Z}$ as given by (2.6). To verify that $\mathbf{E}[\bar{W}]$ is equal to $\mathbf{E}[d(Q(X_p))]$, we need only note that

$$(2.10)\qquad \mathbf{E}[d(Q^r(\sigma))] = \mathbf{E}[d(Q(X_p))\,|\,\alpha(X_p) = r].$$

Here, the term on the right denotes the conditional expectation of the random variable $d(Q(X_p))$ given that $\alpha(X_p) = r$, i.e. that the number of elements in X_p is equal to r. The expression on the right of (2.10) is independent of p, because all sets with the same number of elements have equal probability. Now by definition q_r is equal to $\mathrm{Prob}[\alpha(X_p) = r]$. Thus (2.9) becomes

$$(2.11) \qquad \mathbf{E}[\bar{W}] = \sum_{r=0}^{m} \mathbf{E}[d(Q(X_p))|\alpha(X_p) = r]\mathrm{Prob}[\alpha(X_p) = r] = \mathbf{E}[d(Q(X_p))], \text{ q.e.d.}$$

The only remaining statement to prove is that (2.6) and (2.7) are equivalent, i.e. that

$$(2.12) \qquad \frac{1}{N}\sum_{i=1}^{N} W_i^2 = \sum_{r,s=0}^{m} q_r q_s G_{rs}.$$

This is obtained by squaring the expression (2.4). We get

$$\frac{1}{N}\sum_{i=1}^{N} W_i^2 = \frac{1}{N}\sum_{i=1}^{N}\Big[\sum_{r,s=0}^{m} q_r q_s d(Q^r(\sigma_i))d(Q^s(\sigma_i))\Big]$$

$$= \sum_{r,s=0}^{m} q_r q_s \Big[\frac{1}{N}\sum_{i=1}^{N} d(Q^r(\sigma_i))d(Q^s(\sigma_i))\Big]$$

$$= \sum_{r,s=0}^{m} q_r q_s G_{rs}, \text{ q.e.d.}$$

3. Probability that the damage exceeds a given value

The notation of the first two paragraphs of Section 2 applies also in this section. We wish to study further the random variable $d(Q(X_p))$. The expected value $E(Q,p)$, discussed in Section 2, is certainly the most useful single number associated with the random variable. It is possible, however, to get more detailed information about the distribution of $d(Q(X_p))$. For any number c, we can estimate $\mathrm{Prob}[d(Q(X_p)) \geq c]$, i.e.

the probability that $d(Q(X_p))$ will be at least c. Knowing $\mathrm{Prob}[d(Q(X_p)) \geq c]$ for several values of c would give us a good idea of the distribution of the random variable $d(Q(X_p))$. In some cases, knowing $\mathrm{Prob}[d(Q(X_p)) \geq c]$ for a single value of c would be very useful. For example, if the damage is just the number of targets covered by the assignment and c is the total number of targets, then $\mathrm{Prob}[d(Q(X_p)) \geq c]$ is the probability of covering all targets.

With notation the same as in Section 2, we can describe the Monte Carlo procedure to estimate $\mathrm{Prob}[d(Q(X_p)) \geq c]$ as follows:

Let the damage function $g(i,j)$, the qualification matrix Q, and the number c be given. Generate a sequence of independent random permutations $\sigma_1, \sigma_2, \ldots, \sigma_N$. For $1 \leq i \leq N$ and $0 \leq r \leq m$, set

(3.1)
$$F_r(\sigma_i) = 1 \text{ if } d(Q^r(\sigma_i)) \geq c,$$
$$F_r(\sigma_i) = 0 \text{ if } d(Q^r(\sigma_i)) < c.$$

Let

(3.2)
$$\bar{F}_r = \frac{1}{N} \sum_{i=1}^{N} F_r(\sigma_i).$$

Given any probability p, let $q_r = \binom{m}{r} p^r (1-p)^{m-r}$. An unbiased estimate of $\mathrm{Prob}[d(Q(X_p)) \geq c]$ is

(3.3)
$$\bar{Y} = \sum_{r=0}^{m} q_r \bar{F}_r.$$

An unbiased estimate of $\mathbf{V}(\bar{Y})$ is

(3.4)
$$\bar{Z} = \frac{1}{(N-1)} \left[\sum_{r=0}^{m} t_r \bar{F}_r - \bar{Y}^2 \right], \text{ where}$$

(3.5)
$$t_r = q_r^2 + 2\left[\sum_{s=r+1}^{m} q_r q_s \right].$$

Note that for fixed i, $d(Q^0(\sigma_i)) \leq \cdots \leq d(Q^m(\sigma_i))$. Therefore, for each permutation σ_i, there is a critical number $n(\sigma_i)$ such that

(3.6)
$$F_r(\sigma_i) = 0 \qquad r < n(\sigma_i)$$
$$F_r(\sigma_i) = 1 \qquad r \geq n(\sigma_i).$$

To determine $F_r(\sigma_i)$ for all r, it is only necessary to find $n(\sigma_i)$.

In order to verify that (3.3) is indeed an unbiased estimate of $\text{Prob}[d(Q(X_p)) \geq c]$, note that

(3.7) $$\mathbf{E}[Y] = \sum_{r=0}^{m} q_r \mathbf{E}[F_r(\sigma)], \text{ where}$$

σ is a random permutation. Further, we have

(3.8) $$\mathbf{E}[F_r(\sigma)] = \text{Prob}[d(Q(X_p)) \geq c | \alpha(X_p) = r]$$

and

(3.9) $$\text{Prob}(\alpha(X_p) = r) = q_r.$$

(Recall that $\alpha(X_p)$ is the number of missiles in X_p; and that given that $\alpha(X_p)$ is equal to r, X_p is simply a random subset of r elements, all such subsets being equally probable.) Combining (3.7) - (3.9), we have

(3.10) $$\mathbf{E}[Y] = \sum_{r=0}^{m} \text{Prob}(\alpha(X_p) = r)\text{Prob}[d(Q(X_p)) \geq c | \alpha(X_p) = r]$$
$$= \text{Prob}[d(Q(X_p)) \geq c], \text{ q.e.d.}$$

To see that (3.4) is an unbiased estimate of $\mathbf{V}[\bar{Y}]$, let

$$Y_i = \sum_{r=0}^{m} q_r F_r(\sigma_i).$$

Then the Y_i are independent identically distributed random variables, and $\bar{Y} = \frac{1}{N} \Sigma Y_i$. Therefore, just as in expression (2.6), we have that an unbiased estimate of $\mathbf{V}[\bar{Y}]$ is

$$(3.11) \qquad \bar{Z} = \frac{1}{(N-1)} \left[\frac{1}{N} \sum_{i=1}^{N} Y_i^2 - \bar{Y}^2\right].$$

To prove that this is equivalent to (3.4), we expand Y_i^2

$$(3.12) \qquad Y_i^2 = \left(\sum_{r=0}^{m} q_r F_r(\sigma_i)\right)^2 = \sum_{r=0}^{m} \left[q_r^2 F_r^2(\sigma_i) + 2 \sum_{s=r+1}^{m} q_r q_s F_r(\sigma_i) F_s(\sigma_i)\right].$$

Now $F_r(\sigma_i)$ is either zero or one, and for $r < s$, $F_r(\sigma_i) \leq F_s(\sigma_i)$. Therefore, (3.10) becomes

$$(3.13) \qquad Y_i^2 = \sum_{r=0}^{m} \left[\left(q_r^2 + 2 \sum_{s=r+1}^{m} q_r q_s\right) F_r(\sigma_i)\right] = \sum_{r=0}^{m} t_r F_r(\sigma_i).$$

Thus, we have

$$\frac{1}{N} \sum_{i=1}^{N} Y_i^2 = \sum_{r=0}^{m} t_r \bar{F}_r,$$

and this proves that (3.11) is equivalent to (3.4), q.e.d.

Appendix

In this appendix we present four examples to illustrate the use of the damage function idea. The first example gives the natural construction of a damage function based on the enemy's expected loss. The next two examples show how the damage function may be used to define an assignment problem based on the idea of target priorities. This includes the simple special case of assigning one missile to each target. The last example deals with the problem of maximizing the probability that all targets be destroyed. This is easily done using a suitable damage function and the algorithm of Reference [1], yet it does not appear on the surface to be a problem accessible to the damage function approach.

Example 1. Damage function by analysis of target values and kill probabilities

There are six targets, which have the following values to the enemy:

Targets	Value each target
1, 2, 3	20
4, 5, 6	10.

The value is not assumed to be in any concrete measure such as men or dollars, although it may be; it is simply a measure of the relative importance of the target. Thus target 2, with a value of 20, is twice as important as target 5, which has a value of only 10. From this, we conclude that an action which would destroy target 2 with probability 1/2 is equivalent to one which would destroy target 5 with certainty. In each case, the expected loss to the enemy is 10.

To construct the damage function, it is necessary to have the kill probability for each target, i.e. the probability that a single missile will destroy the target. The next table gives these kill probabilities.

Targets	Kill probability
1, 6	.2
2, 5	.5
3, 4	.8.

We assume that if several missiles are fired at a target their effects are independent. Each missile either destroys the target or leaves it undamaged. Then, if p_i is the probability that a single missile will destroy target i, the probability that a salvo of j mis·iles will destroy target i is

$$1 - (1 - p_i)^j .$$

Thus, if the value of target i is V_i, the expected damage to the enemy from a salvo of j missiles at target i is

$$g(i,j) = V_i[1 - (1 - p_i)^j].$$

Note that

$$\delta(i,j) = g(i,j) - g(i,j - 1) = (1 - p_i)^{j-1}(V_i p_i).$$

Thus $\delta(i,j)$ decreases with j; $g(i,j)$ is concave in j.

The next table gives the value of $g(i,j)$ for $j \leq 6$. Values for j greater than 6 could be easily computed from the information given above.

Table 1

i/j	0	1	2	3	4	5	6
1	0	4	7.2	9.76	11.81	13.46	14.76
2	0	10	15	17.5	18.75	19.38	19.69
3	0	16	19.2	19.84	19.99	20.00	20.00
4	0	2	3.6	4.88	5.90	6.72	7.38
5	0	5	7.5	8.75	9.38	9.69	9.84
6	0	8	9.6	9.92	9.99	10.00	10.00

The value of the damage for any assignment matrix may be easily computed from this table. Suppose, for example, we have nine missiles with this qualification matrix Q

$$Q = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

The following assignment matrix A belonging to Q assigns two missiles to each of the targets with value 20 (numbers 1, 2, and 3) and one to each of the other targets.

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

For $i = 1, 2$, and 3, $r_i(A) = 2$; and for $i = 4, 5$, and 6, $r_i(A) = 1$. Thus the damage $d(A)$ is given by

$$d(A) = \sum_{i=1}^{6} g(i,r_1(A)) = \sum_{i=1}^{3} g(i,2) + \sum_{i=4}^{6} g(i,1)$$

$$= 7.2 + 15.0 + 19.2 + 2.0 + 5.0 + 8.0 = 56.4.$$

Is this an optimal assignment for Q? This is not easy to determine using Table 1. Much more convenient for this purpose is a table of the difference $\delta(i,j)$. The difference table corresponding to Table 1 is:

Table 2

i/j	1	2	3	4	5	6
1	4.00	3.20	2.56	2.05	1.65	1.30
2	10.00	5.00	2.50	1.25	.63	.31
3	16.00	3.20	.64	.15	.04	.00
4	2.00	1.60	1.28	1.02	.82	.66
5	5.00	2.50	1.25	.63	.31	.15
6	8.00	1.60	.32	.07	.01	.00

The next table reproduces the first 3 columns of Table 2. In addition, certain entries are starred. The number of starred entries in a row is the number of missiles assigned to that target by the matrix A.

Table 3

i/j	1	2	3
1	4.00*	3.20*	2.56
2	10.00*	5.00*	2.50
3	16.00*	3.20*	.64
4	2.00*	1.60	1.28
5	5.00*	2.50	1.25
6	8.00*	1.60	.32

The damage for the assignment A is just the sum of the starred entries in Table 3. Now look at the two underlined entries $\delta(4,1) = 2.00$ and $\delta(1,3) = 2.56$. The value of the assignment matrix would clearly be higher if $\delta(1,3)$ was starred instead of $\delta(4,1)$. An assignment which effects this improvement is:

$$B = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The damage for B is 56.96, which is .56 greater than that for A, i.e. just the difference between $\delta(1,3)$ and $\delta(3,1)$. The assignment matrix B is in fact an optimal assignment matrix belonging to Q. Moreover, no assignment matrix for nine missiles can improve on B whatever the qualification matrix may be. To see this, simply note that in Table 3 with $\delta(1,3)$ starred instead of $\delta(4,1)$ the nine largest values are starred. Now the damage for any assignment of nine missiles will be a sum of nine terms taken from Table 2. Since the damage for B is the sum of the nine largest terms, it is the maximum possible.

An interesting point to note about Table 2 is that $\delta(3,3)$ is less than $\delta(4,3)$ even though the value of target 4 is only half that of target 3. This occurs because target 3 is relatively soft and will almost certainly be destroyed by a salvo of two missiles. In fact, the probability of the missiles destroying target 3 is .96, so there is very little to gain by adding a third missile to this target.

Example 7. <u>One missile to each target with a set of target priorities</u>

The problem of assigning one missile to each target is a special case of the basic assignment problem. Suppose there are six missiles and six targets with the following qualification matrix

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} .$$

There is an extra row in Q. This has been introduced as a dummy target, for which all missiles are qualified, to take care of missiles not otherwise assigned. It is necessary, because at most one missile is to be assigned to each real target.

Suppose that it is desired to find an assignment which assigns at most one missile to each of the targets 1-6 and covers as many of these targets as possible. If the following damage function is used, any optimal assignment for the basic assignment problem will have the desired properties.

$$g(i,0) = 0 \qquad \text{all } i$$

(1) $$g(i,j) = 1 \qquad 1 \le i \le 6, \qquad 1 \le j$$

$$g(7,j) = j/2 \qquad 1 \le j .$$

Note that $\delta(i,j) = 0$ for $i \le 6$ and $j \ge 2$, and $\delta(7,j) = 1/2$ for all j. Therefore, because all missiles are qualified for target 7, an optimal assignment will assign at most one missile to each of the targets 1-6. If n missiles are assigned to n of the targets 1-6 and the remainder to target 7, the dummy target, the damage will be $n + 1/2(6 - n)$. Clearly this will be a maximum when n is a maximum. Thus any optimal assignment has the desired properties.

Here are two optimal assignment matrices for this problem

$$(2) \qquad A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

The assignment A fails to cover target 2 while B fails to cover target 1. It is not possible for an assignment belonging to Q to cover both of these targets.

It is possible to construct a damage function for which any optimal matrix will have the properties given above, and in addition, the assignment matrix A will be preferred over B. That is to say, a set of priorities can be introduced for the targets so that target 1 is more important than target 2. To give an exact statement of a priority scheme, it is necessary to give an ordering for the assignment matrices. An optimal matrix belonging to Q is then one which is greater than all other matrices belonging to Q.

Suppose that target 1 is most important, target 2 is next most important, and so on to target 6, the least important target. An assignment problem for this set of targets and priorities might be stated in this way:

Priority Assignment Problem

Consider only matrices belonging to Q for which $r_i(A) \leq 1$ for $1 \leq i \leq 6$. For two such matrices A and B, we say that A is greater than or equal to B and write $B \lesssim A$ if one of the following conditions holds:

(i) $\sum_{i=1}^{6} r_i(B) < \sum_{i=1}^{6} r_i(A)$,

(ii) $\sum_{i=1}^{6} r_i(B) = \sum_{i=1}^{6} r_i(A)$ and

for some $k \leq 6$, $r_k(B) < r_k(A)$ while $r_i(A) = r_i(B)$ for all $i < k$,

(iii) $r_i(B) = r_i(A)$ for all $i \leq 6$.

The problem is to find an assignment matrix A belonging to Q and with $r_i(A) \leq 1$ for $i \leq 6$ such that any other matrix B with these properties satisfies $B \lesssim A$.

Consider the assignment matrices A and B of (2). The assignment matrix A is optimal. We have, in fact, $\sum_{i=1}^{6} r_i(A) = 5$, $r_1(A) = 1$, $r_2(A) = 0$, $r_i(A) = 1$ for all i with $3 \leq i \leq 6$. It was pointed out above that no matrix belonging to Q could assign to both targets 1 and 2. Hence, the maximum possible value of $\sum_{i=1}^{6} r_i(A)$ is 5; and if $r_1(A) = 1$, $r_2(A)$ must be zero. The matrix A is greater than B, because $r_1(A) = 1$ while $r_1(B) = 0$.

A damage function for which any optimal matrix will be a solution to the target priority problem is easily constructed by modifying the damage function (1). For the function of (1), the quantities $\delta(i,1)$, $1 \leq i \leq 6$ are all equal to one. It is only necessary to make these increases in order of target priority, provided that we keep $\delta(i,1) > \delta(7,1)$ for $1 \leq i \leq 6$. For example, we can use

$$g(i,0) = 0 \qquad \text{all } i$$

(4) $$g(i,j) = 7 - i \qquad 1 \leq i \leq 6, \qquad 1 \leq j$$

$$g(7,j) = j/2 \qquad 1 \leq j.$$

The fact that a damage function of this type solves the priority assignment problem is proved in [1].

Example 3. A more complicated set of priorities

Consider the damage function $g(i,j)$ whose differences $\delta(i,j)$ are given by this matrix

(5) $$\begin{pmatrix} 6.0 & 6.0 & 0.0 \\ 5.0 & 5.0 & 0.0 \\ 4.0 & 2.0 & 0.0 \\ 3.0 & 0.0 & 0.0 \\ 1.5 & 0.0 & 0.0 \\ 1.5 & 0.0 & 0.0 \end{pmatrix}$$

Write the targets in a list entering target i in the list one time for each non-zero $\delta(i,j)$. Start with the largest $\delta(i,j)$ and work downward. The result is this list

(6) 1, 1, 2, 2, 3, 4, 3, 5, 6.

Suppose that n missiles are available, each qualified for all targets. An optimal assignment for the damage function (5) will assign missiles to the first n targets in the list (6), each target getting one missile for each of its occurences in the first n elements of the list. For example, if five missiles are available, they will be assigned to targets 1, 1, 2, 2, and 3, i.e. two missiles to target 1, two missiles to target 2, and one missile to target 3. With seven missiles available, targets 1, 2, and 3 would each receive two missiles, and one missile would be assigned to target 4.

When the missiles are not qualified for every target, the assignment will in a certain sense come as close as possible to assigning missiles to the first n targets in the list. To be more precise, an optimal assignment A will have the following properties:

(i) If possible A will assign two missiles to target 1. If that is not possible and one missile can be assigned to target 1, then it will be.

(ii) In addition to satisfying (i), A will assign, if possible, two missiles to target 2. If that is not possible but one missile can be assigned to target 2, then it will be.

(iii) In addition to satisfying (i) and (ii), A will if possible assign a missile to target 3.

(iv) In addition to satisfying (i) - (iii), A will if possible assign a missile to target 4.

(v) In addition to satisfying (i) - (iv), A will if possible assign a second missile to target 3.

(vi) and so on.

The relationship between this set of conditions and the list of targets (6) should be clear to the reader. We call the list (6) a <u>list of priorities</u> and say that an assignment matrix which is optimal for the damage function (5) is optimal with respect to this list of priorities. The list of priorities is not merely an ordering of the targets. It specifies how many missiles should be assigned to each target and in what order the possibly conflicting requirements should be met. For example, with the list (6), one missile will be assigned to target 3 rather than target 4. However, assigning a missile to target 4 is preferred over assigning the second missile to target 3. Moreover, no preference is made between target 5 and 6. Let us see how all this will work out with a qualification matrix Q.

$$Q = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

An optimal assignment matrix belonging to Q with the damage function (5) is

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

This achieves the maximum possible damage of 34.

If missiles 1, 2, and 7 of Q are not available, the qualification matrix is:

$$Q_2 = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

An optimal assignment matrix for this is

$$A_2 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The damage for this assignment is 26.5.

If missiles 3, 4, 8, and 9 are not available from Q, the qualification matrix is

$$Q_3 = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

An optimal assignment matrix belonging to Q_3 is

$$A_3 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The damage for A_3 is 26.

Now as is indicated in [1], the magnitudes of the $\delta(i,j)$ are not important in determining the list of priorities. Only the ordering of the $\delta(i,j)$ is important. Thus, the damage function $g_*(i,j)$ with different $\delta_*(i,j)$ given below, results in the same optimal assignments as $g(i,j)$ above.

$$\delta_*(i,j) = \begin{pmatrix} 20 & 20 & 0 \\ 18 & 17 & 0 \\ 15 & 2 & 0 \\ 3 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

However, for the assignment matrices A_2 and A_3, the damages using $g_*(i,j)$ are 79 and 90 respectively. Thus, using $g_*(i,j)$ it seems better to have missiles 1, 2, 5, 6, and 7 available (qualification matrix Q_3) than to have missiles 3, 4, 5, 6, 8, and 9 available (matrix Q_2). On the other hand, the reverse is true if we use the damage function $g(i,j)$.

The point of all this is that it is not really sufficient to specify a list of target priorities. If we are to be able to compare the effectiveness of different qualification matrices, we must have more indication of the relative values of the various targets. Constructing the damage function by a procedure like that of Example 1 will give such information.

Example 4. Maximizing the probability of destroying all targets

Assume that for each target i the kill probability p_i is known. Then the probability that target i will be destroyed if missiles are launched according to an assignment matrix A is $1 - (1 - p_i)^{r_i(A)}$. Therefore, the probability that all targets will be destroyed is

$$(7) \qquad P_A = \prod_{i=1}^{n} (1 - (1 - p_i)^{r_i(A)}).$$

Given a qualification matrix Q, we ask for an assignment matrix A belonging to Q for which $P(A)$ is maximum.

At first it might appear that this problem cannot be formulated using a damage function as described in Section 1. However, note that $P(A)$ is a maximum if and only if $\log(P(A))$ is a maximum, and

$$(8) \qquad \log P(A) = \sum_{i=1}^{n} \log(1 - (1 - p_i)^{r_i(A)}).$$

Now this looks very much like a damage function expression. Let us define $g(i,j)$ as follows:

$$g(i,0) = 0 \qquad \text{all } i$$

$$g(i,j) = c + \log(1 - (1 - p_i)^j) \qquad \text{for } j \geq 0,$$

where c is chosen so that for all i and k

(9) $g(i,1) \geq g(k,2) - g(k,1)$.

We claim that $g(i,j)$ is a damage function, i.e. it satisfies (1.3) - (1.5). Clearly $g(i,0) = 0$. To verify that $g(i,j)$ is concave in j, let $f(t) = (1 - (1 - p_i)^t)$ for $t > 0$ and $h(t) = \log f(t)$. In Example 1 it was proved that f is concave, i.e. that $f''(t) \leq 0$. Clearly $f'(t) \geq 0$ and $f(t) \geq 0$. Therefore, for $h(t)$, we have

(10) $h'(t) = f'(t)/f(t) \geq 0$

and

(11) $h''(t) = \frac{f''(t)}{f(t)} - [\frac{f'(t)}{f(t)}]^2 \leq 0$.

From (10), we conclude that for all i and $j \geq 1$, $g(i,j)$ is increasing in j, and from (11), we conclude that for all i and $j \geq 1$, $g(i,j)$ is concave in j. The choice of c so that (9) holds insures that $g(i,j)$ is concave and increasing in j for all $j \geq 0$.

Now suppose that it is possible to find an assignment matrix A^* belonging to Q such that for all i, $r_i(A^*) \neq 0$. Then we claim that $\log P(A)$ is a maximum if and only if

$$d(A) = \sum_{i=1}^{n} g(i, r(A))$$

is a maximum. To see this, simply note that because of Condition (9) $d(A)$ cannot be a maximum if $r_i(A) = 0$ for some i. But with $r_i(A) \neq 0$ for all i, we have

$$d(A) = \log (P(A)) + nc.$$

Now if no matrix A belonging to Q exists for which $r_i(A) \neq 0$ for all i, then $P(A) = 0$ for all matrices A belonging to Q. Hence

we can conclude that for an assignment matrix A belonging to Q,
P(A) is a maximum if the damage d(A) is a maximum.

Reference

[1] D. W. Walkup and M. D. MacLaren, "A Multiple-Assignment Problem," Boeing Document D1-82-0346, April 1964.