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MEMORANDUM

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**NUMERICAL SOLUTIONS OF
FUNCTIONAL EQUATIONS BY MEANS
OF LAPLACE TRANSFORM-VII:
THE MELLIN TRANSFORM**

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PREFACE

This Memorandum, with Parts I-VI, comprises a study of the use of computing machines to resolve the complex equations which arise in biology and medicine. Part VII discusses the question of the numerical inversion of the Mellin transform.

SUMMARY

In this series of papers, we have so far considered various analytic and computational aspects of the numerical inversion of the Laplace transform and, in addition, some applications to scientific problems. In this paper we wish to consider the question of the numerical inversion of the Mellin transform,

$$F(s) = \int_0^{\infty} u(t)t^{s-1}dt.$$

This transform also plays an important role in the study of functional equations of the type that arise in mathematical physics.

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NUMERICAL SOLUTIONS OF FUNCTIONAL EQUATIONS BY MEANS OF LAPLACE TRANSFORM—VII: THE MELLIN TRANSFORM

1. INTRODUCTION

In this series of papers, we have so far considered various analytic and computational aspects of the numerical inversion of the Laplace transform and, in addition, some applications to scientific problems. In this paper we wish to consider the question of the numerical inversion of the Mellin transform,

$$(1.1) \quad F(s) = \int_0^{\infty} u(t)t^{s-1}dt.$$

This transform also plays an important role in the study of functional equations of the type that arise in mathematical physics.

2. A PRELIMINARY TRANSFORMATION

Let us make the change of variable

$$(2.1) \quad t = \frac{r}{1-r}, \quad r = \frac{t}{1+t}.$$

Then (1.1) assumes the form

$$(2.2) \quad \begin{aligned} F(s) &= \int_0^1 \frac{u(r/(1-r))}{(1-r)^2} \left(\frac{r}{1-r}\right)^{s-1} dr \\ &= \int_0^1 g(r) \left(\frac{r}{1-r}\right)^{s-1} dr, \end{aligned}$$

upon simplifying the notation by setting

$$(2.3) \quad g(r) = u(r/(1-r))/(1-r)^2.$$

We suppose that $u(t) \rightarrow 0$ as $t \rightarrow \infty$ rapidly enough so that $F(s)$ is defined for $\text{Re}(s) > 0$, and thus that $g(1) = 0$.

Applying an N -th order Gaussian quadrature as in the earlier papers, we have

$$(2.4) \quad F(s) = \sum_{i=1}^N w_i g(r_i) \left(\frac{r_i}{1-r_i} \right)^{s-1},$$

where the r_i are the roots of the shifted Legendre polynomial, $P_N\left(\frac{1+x}{2}\right) = \psi_N(x)$.

Setting $s = 1, 2, \dots, N$, we obtain N linear algebraic equations for the N quantities $w_i g(r_i)$, $i = 1, 2, \dots, N$. Since the matrix $([r_i(1-r_i)]^j)$ is ill-conditioned, this is not necessarily equivalent to obtaining meaningful values for the $w_i g(r_i)$.

3. EXPLICIT MATRIX INVERSION

Let us then employ a device used in [1] to obtain an explicit analytic solution of the corresponding system obtained in connection with the Laplace transform. From (2.4), we have

$$(3.1) \quad \sum_{k=1}^N a_k F_k = \sum_{i=1}^N w_i g(r_i) \left[\sum_{k=0}^{N-1} a_{k+1} \left(\frac{r_i}{1-r_i} \right)^k \right]$$

for any N constants a_1, a_2, \dots, a_N . We can write (3.1) in the form

$$(3.2) \quad \sum_{k=1}^N a_k F_k = \sum_{i=1}^N w_i g(r_i) \varphi \left(\frac{r_i}{1-r_i} \right),$$

where

$$(3.3) \quad \varphi(x) = \sum_{k=0}^{N-1} a_{k+1} x^k,$$

a polynomial of degree $N - 1$.

Let us choose this polynomial in a convenient fashion. Suppose that

$$(3.4) \quad \varphi \left(\frac{r_i}{1-r_i} \right) = \delta_{ij}.$$

Then, if $a_{k+1,j}$, $k = 1, 2, \dots, N$, are the corresponding coefficients in (3.3), we have

$$(3.5) \quad \sum_{k=1}^N a_{kj} F_k = w_j g(r_j).$$

Thus, $(a_{k+1,j})$ is the required inverse matrix.

To obtain the required polynomial $\varphi_j(x)$, consider the function

$$(3.6) \quad \pi_N(x) = \psi\left(\frac{x}{\Gamma + x}\right),$$

where $\psi_N(x)$ is the function introduced in Sec. 2. We see that

$$(3.7) \quad \pi_N\left(\frac{r_i}{\Gamma - r_i}\right) = \psi_N(r_i) = 0, \quad i = 1, 2, \dots, N.$$

However, $\pi_N(x)$ is a rational function, not a polynomial. To remedy this, consider the new function

$$(3.8) \quad M_N(x) = (1 + x)^N \pi_N(x) = (1 + x)^N \psi_N\left(\frac{x}{\Gamma + x}\right),$$

a polynomial of degree N . Then

$$(3.9) \quad \varphi_j(x) = \frac{M_N(x)}{\left(x - \frac{r_j}{\Gamma - r_j}\right) M'_N\left(\frac{r_j}{\Gamma - r_j}\right)}$$

is the required polynomial of degree $N - 1$. Writing

$$(3.10) \quad \varphi_j(x) = \sum_{k=0}^{N-1} a_{k+1,j} x^k,$$

we obtain the desired weighting factors.

4. COMPUTATIONAL ASPECTS

A computer program will be required, as in the case of the Laplace transform (see [1]), to determine the

$a_{k+1,j}$. They can, in this way, be obtained to arbitrary precision. We shall report on this, together with applications, in a subsequent paper.

REFERENCE

1. Bellman, R., H. Kagiwada, R. Kalaba, and M. Prestrud, Invariant Imbedding and Time-dependent Transport Processes, The RAND Corporation, R-423-ARPA, February 1964; American Elsevier Publishing Company, New York, 1964.