Department of Physics
Western Reserve University
Cleveland, Ohio 44106

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THEORY OF THE MANY-BODY BOSE SYSTEM

P. R. Zilsel

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The purpose of our work has been to study the collective elementary excitations, as well as the ground state properties, of the non-relativistic many-body system of bosons interacting with short range forces. The ultimate aim is the theory of superfluid helium, and we have therefore been concerned with avoiding the limitations to low density and/or weak forces which characterize the Bogoliubov theory of the non-ideal Bose gas and those treatments which derive from it. In contrast to the fermion case, where the statistics tend to suppress the many-particle correlation effects of short range forces, there is in the boson case no reason to suppose that a free particle or pair approximation provides an adequate starting point for a perturbation procedure, except in the low density limit. Our basic approach, therefore, has been to explore the use of field theoretic representations which include from the start at least some collective many-body effects of the strong short-range interaction.

Quasi-Hydrodynamic Representation

The many-body Hamiltonian for interacting bosons (in the second-quantization formalism),

$$H = \frac{\hbar^2}{2m} \int (\vec{\nabla} \Psi^\dagger) \cdot (\vec{\nabla} \Psi) \, d\tau \; +$$

$$+ \frac{1}{2} \iint V(\vec{r} - \vec{r}') \Psi^\dagger(\vec{r}) \Psi^\dagger(\vec{r}') \Psi(\vec{r}') \Psi(\vec{r}) \, d\tau \, d\tau' \tag{1}$$

can be put into a form paralleling the Hamiltonian of a classical irrotational fluid by going over to the action and angle variable operators ρ, S, given by

$$\Psi = e^{iS/\hbar} \rho^{1/2} \; ; \quad \Psi^\dagger = \rho^{1/2} e^{-iS/\hbar} \quad . \tag{2}$$

In the c-number limit the transformation (2) is canonical; however, as an operator transformation it is only approximately unitary: in treating $\rho(\vec{r})$, $S(\vec{r})$, or their Fourier transforms $\rho_{\vec{k}}$, $S_{\vec{k}}$, as operators satisfying canonical commutation relations

$$[\rho(\vec{r}) , \ S(\vec{r}')] = \frac{\hbar}{i} \delta(\vec{r} - \vec{r}')$$

$$[\rho_{\vec{k}}, \ S_{\vec{k}'}] = \frac{\hbar}{i} \delta_{\vec{k}, -\vec{k}'} \tag{3}$$

one is, in fact, ignoring the positive-semidefinite character of ρ . Thus the representation is valid only for states in which the density fluctuations are small compared to the mean density, i.e.

$$\langle \rho_{\vec{k}} \ \rho_{\vec{k}}^{\dagger} \rangle \ll \rho_0 \tag{4}$$

In the case of __weak__ interaction, for states exhibiting Bose condensation, the quasi-hydrodynamic representation can be shown to yield an approximation equivalent to the Bogoliubov treatment[1] of the weakly non-ideal Bose gas, the Bogoliubov quasi-particles corresponding to density fluctuations.

For the case of __strong__ __interaction__ including a repulsive core of radius _a_ , the condition (4) is clearly violated for $k \gtrsim 2\pi/a$, since the repulsion imposes strong anti-correlations in the density over distances of the order of _a_ . However, at high densities one can obtain an approximate representation by working with smoothed density operators $\bar{\rho}$, averaged over volumes of the order of r_0^3 ($r_0 \approx a$) , and expanding the Hamiltonian in powers of the $\bar{\rho}_k$, $\bar{S}_k$. This amounts to an expansion in __inverse__ powers of $(\rho_0 \, r_0^3)^{1/2}$, in contrast to the low density expansion of Lee-Huang-Yang[2] for the hard-sphere Bose gas, which is in positive powers of $(\rho_0 \, a^3)^{1/2}$. In lowest (quadratic) order in the $\bar{\rho}_k$, $\bar{S}_k$ one obtains a sum of harmonic oscillator terms describing uncoupled density fluctuations (phonons). The excitation spectrum is of

the general form obtained by Feynman[3] and by Bogoliubov and Zubarev,[4] but differs in detail because of the smoothing procedure. It reproduces semi-quantitatively the experimentally observed excitation spectrum for liquid He II, including the "roton" minimum at the observed wavenumber $k_0 = 1.95 \times 10^8$ cm^{-1}.

This work was reported at the VIIth International Conference on Low-Temperature Physics,[5] and the basic ideas of the quasi-hydrodynamic representation are also discussed elsewhere.[6]

Even though in lowest order the quasi-hydrodynamic representation gives an adequate semi-quantitative description of the collective elementary excitations in a Bose fluid at high densities, it does not turn out to provide a usable starting point for a perturbation treatment applicable to the case of liquid helium. The reason lies in the largeness of the density fluctuations near the roton minimum k_0 which, in turn, is an effect of the hard core repulsion. Only for wavenumbers $k \ll k_0$, that is for distances much larger than a, is a description in terms of uncoupled density oscillators an adequate starting point for a perturbation treatment. It thus appears that an adequate treatment of liquid helium must include essential density correlation effects of the hard core repulsion from the start.

Pseudospin Model

Considerations of the kind outlined above led us to investigate the many-body ground state and elementary excitations of a simple "pseudospin" cell model, which had been used previously by Matsubara and Matsuda[7] for treating lambda transition in liquid helium. Even though the model has some unphysical features--primarily an artificial "band" structure due to the use of lattice quantization, resulting in anisotropy of and a quasi-momentum cutoff in the excitation spectrum--it has the advantage of allowing us to treat exactly, in

zero order, important many-body effects of the hard core repulsion, thus overcoming some of the limitations of the earlier treatments.

The essential features of the pseudospin model and of our treatment are these:

(1) The hard core repulsion is treated kinematically, that is by algebraic relations between the field operators, instead of as the limit of a repulsive potential in the Hamiltonian. The algebra of the model represents an approximation to Siegert's[8] exact but highly untractable algebra for hard core boson fields. In the cell approximation this algebra turns out to be equivalent to that of a set of Pauli spins, and the dynamics of the model is equivalent to that of a Heisenberg ferromagnet with an interaction which is anisotropic in pseudospin space. In this model the transformation from field amplitudes to densities, corresponding to the action-angle variable transformation (2), is indeed unitary: it is simply a rotation in pseudospin space. Different densities ρ_0 correspond to different orientations (in pseudospin space) of the total "magnetization".

(2) Our treatment splits the Hamiltonian into two parts: an isotropic "unperturbed" part H_o , describing a system with hard core repulsion plus a square-well attraction of depth h^2/md^2 (d = size of the cubical cell) between nearest-neighbor cell pseudospins, and a perturbation consisting of the attraction's deviation from this value. The unperturbed interaction corresponds to zero two-particle scattering length f_o , and the basic idea of our perturbation treatment is thus similar to that employed by Moszkowski and Scott[9] for the nucleon system.

(3) The pseudospin algebra of the field operators permits construction, by rotation of the total pseudospin from the vacuum, of a "unperturbed quasi-particle vacuum" (a many-body ground state of H_o) and corresponding quasi-particle

operators for any mean density of the system. These serve as starting point for treating the full Hamiltonian by the equations-of-motion method in the random phase approximation. In this approximation the ground state of the full Hamiltonian is obtained from the quasi-particle vacuum (not the fully Bose-condensed free-particle ground state) by a transformation of the Bogoliubov type. We obtain expressions for the excitation spectrum and the ground state energy:

$$E_o/N = (2_\pi \hbar^2 \rho f_o/m) \left\{ 1 + (128/15\pi^{1/2})(\rho f_o^3)^{1/2} (1 - \zeta_o)^{5/2} \right.$$
$$\left. + (4\pi C_2/3) (\rho f_o^3)^{1/3} \zeta^{2/3} (1 - \zeta_o/2) \right\} \quad , \tag{5}$$

where $\zeta_o = \rho d^3$ is the fractional depletion of the Bose condensate in the unperturbed ground state (an "excluded volume" effect of the hard core repulsion), and $C_2 = 0.516$ is a numerical constant. In the low density limits, where $\zeta_o \to 0$, our results agree exactly with the well known hard sphere results of Lee, Huang, and Yang,[2] but the higher density corrections reflect the many-body correlations due to the hard core which are included in our unperturbed ground state.

This work has also been reported.[10]

Two-Fluid Hydrodynamics

We have investigated the nature of the velocity constraint introduced by C. C. Lin[11] into the Eckart variational-principle derivation of two-fluid hydrodynamics for liquid He II,[12] and its effects on the two-fluid equations of motion and the condition

$$\text{curl } v_s = 0 \quad .$$

It appears that, while a Lin type velocity constraint must be introduced if

the Eckart principle is to be used in connection with ordinary (one-fluid) hydrodynamics, the form this constraint should take in the two-fluid model is somewhat arbitrary and depends on the basic assumptions underlying the model. In Lin's treatment the center of mass velocity but not the relative velocity of the two fluids is constrained. As a result the equations of motion differ from those obtained earlier (without the Lin constraint) and the superfluid motion is no longer necessarily irrotational. On the other hand the basic ideas underlying the two-fluid model suggest that a Lin type constraint should be imposed on the normal fluid velocity v_r , but not on v_s . In this case the equations of motion are identical with those obtained earlier, and curl v_s remains zero. This work has been reported in an interim technical report.[13]

As a by-product of this investigation Mr Whitlock developed a variational principle formulation of the laws of motion for a classical perfect compressible plasma, which has also been reported.[14]

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SCIENTIFIC PERSONNEL

P. R. Zilsel, Professor of Physics, principal investigato. (June 1959-June 1963)
   B.Sc., College of Charleston, 1943
   M.A., University of Wisconsin, 1945
   Ph.D., Yale University, 1948

J. M. Pearson, Instructor in Physics, associate investigator (December 195'-March 1960)
   B.Sc., University of London, 1954
   Ph.D., McMaster University, 1959

B. A. Green, Jr, Assistant Professor of Physics, associate investigator (September 1961-June 1962)
   B.S., University of Alabama, 1949
   M.S., University of Alabama, 1950
   Ph.D., Johns Hopkins University, 1956

R. T. Whitlock, graduate assistant in physics (June 1959-June 1963) and National Science Foundation cooperative graduate fellow in physics (June 1960-June 1962); now Assistant Professor of Physics, Thiel College
   B.S., Capital University, 1958
   M.S., Western Reserve University, 1961
   Ph.D., Western Reserve University, 1963

E. H. Synn, graduate assistant in physics (July 1959-June 1962)
   B.S., Waynesburg College, 1958
   M.S., Western Reserve University, 1962

T. L. Chow, graduate assistant in physics (July 1961-June 1962)
   B.S., National Taiwan University, 1958
   M.S., Western Reserve University, 1963

H. A. Ali, graduate assistant in physics (September 1960-June 1964)
   B.S., Antioch College, 1958
   M.S., Western Reserve University, 1963

V. F. Walters, graduate assistant in physics (September 1953-June 1964) and National Science Foundation cooperative graduate fellow in physics (June 1959-June 1961)
   B.A., Smith College, 1947
   M.A., Western Reserve University, 1958