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THE ATTENUATION OF ACOUSTIC WAVES IN A TWO-PHASE MEDIUM

BY

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Hercules Powder Company
Allegany Ballistics Laboratory
Cumberland, Maryland
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Condensed Report on

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SUMMARY

A study is made of the attenuation of acoustic waves by a suspension of fluid droplets in a fluid medium. Special attention is given to the case of small droplets for which the effect of the surface tension is not negligible. Both the droplets and the surrounding fluid medium are considered to be viscous and thermal conducting. The droplets are allowed to execute large translational motion and to undergo a small deformation from a spherical shape. It is shown that the result of Epstein and Carhart on the attenuation of sound waves in a gas with the suspension of liquid droplets is applicable even when the displacement of the droplet is large compared to its radius. The effect of surface tension is to increase sound attenuation in two-phase medium by increasing the thermal dissipation. This effect is important in the suspension of gaseous bubbles in liquid for small droplets and is negligible in the case of a gaseous medium containing liquid droplets. The explicit forms for attenuation, the drag force on droplets, and the heat transfer rate between phases are given for the case which is applicable to a gas containing liquid and solid droplets. The expression for the attenuation which is applicable to the suspension of gaseous bubbles in a liquid is also given and is found to be completely dominated by the thermal dissipation.

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SYMBOLS

a_0	sound speed of undisturbed medium
$\vec{A}$	viscous wave potential
C	surface tension per unit length
C_v, C_p	specific heat at constant volume and constant pressure
e	internal energy per unit mass
f	frequency of the incident wave
H_n, J_n	Hankel and Bessel functions
k_1, k_2, k	wave numbers of acoustic, thermal and viscous waves
n	number of droplets per unit volume
$N = \frac{4}{3} + \frac{\eta}{\mu}$	
p	pressure
P_n, P_n^m	Legendre and associated Legendre functions
r_0	radius of the droplet
r	radial coordinate
R_1, R_2	principal radii of curvature at a given point of the droplet surface
T	temperature
u	fluid velocity
u_p	velocity of the mass center of the droplet
x_i	rectangular cartesian coordinate
α	attenuation coefficient
α_v	coefficient of volume expansion

$\gamma = \frac{c_p}{c_v}$	specific heat ratio
ξ_p , ξ_g	displacements of the droplet and gaseous medium
η	coefficient of dilatational viscosity
θ	polar angle
κ	thermal conductivity
λ_1	acoustic wavelength
μ , ν	dynamic and kinematic viscosities
ξ	radial displacement of a point on the surface of the droplet
$\Pi_{ij} = -p\delta_{ij} + \tau_{ij}$	stress tensor
ρ_0	density
$\Phi_{(i)}$	incident wave potential
Φ_1 , Φ_2	acoustic and thermal potentials
φ	azimuthal angle
Ψ	dissipation function
$\omega = 2\pi f$	angular frequency
$\Omega = \frac{\kappa}{\rho_0 c_p}$	thermal diffusivity
$P_r = \frac{\nu}{\Omega}$	Prandtl number
$(\quad)_0$	equilibrium quantities
$\mathcal{R}(\quad)$	real part
$(\quad)_{av.}$	time average
$(\quad)'$	variables inside the droplet
KC	kilocycle per second

I. INTRODUCTION

The attenuation of acoustic waves in two-phase medium has attracted attention repeatly since the publication of Sewell's paper[(1)*]. Until present time, the analytical treatment[(1), (2), (3)] is limited to the cases where the displacement of the droplet is small compared with the size of the droplet and the effect of the surface tension can be neglected. These assumptions were consistently mentioned as the possible causes of the discrepancies between the existing theories and the experimental results[(4), (5), (6)]. In view of this, a generalized theory will be presented by introducing the surface tension effect and allowing the free movement of the droplets by formulating the basic equations with respect to a moving coordinate system fixed in the droplet. Both media are considered to be viscous and heat conducting and a small deformation of the droplet is allowed. Attention will be confined to the cases where the incident wavelength is much greater than the size of the droplet and there is no interaction between the droplets (low volume concentration).

The attenuation coefficient, drag force on the droplet, heat transfer rate between phases, and the ratio of the droplet displacement to the particle displacement of the surrounding fluid medium are obtained. The explicit expressions of these quantities are given for the case applicable to the suspension of liquid droplets in a gas and the comparison is made with Epstein and Carhart's theory. Also, the expression for the attenuation which is applicable to a liquid containing gas bubbles is obtained.

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* Numbers in parentheses refer to References at the end of the paper.

II. FORMULATION OF THE PROBLEM

Consider a train of acoustic waves described by a potential $\Phi_{(i)} = \exp[i(\kappa x_3 - \omega t)]$ impinging upon a spherical droplet. The linearized equations governing the motion of a viscous and heat conducting fluid in a fixed cartesian coordinate system assume the form:

$$\frac{\partial \rho}{\partial t} + \rho_0 \frac{\partial u_i}{\partial x_i} = 0 \qquad i = 1, 2, 3 \tag{1}$$

$$\frac{\partial u_i}{\partial t} = \frac{1}{\rho_0} \frac{\partial \Pi_{ij}}{\partial x_j} \qquad j = 1, 2, 3 \tag{2}$$

$$\rho_0 \frac{\partial e}{\partial t} = -p \frac{\partial u_i}{\partial x_i} + \kappa \frac{\partial^2 T}{\partial x_i \partial x_i} \tag{3}$$

Instead of referring these equations to axes fixed in space, they shall be referred to axes originating at the center of the droplet and moving with the velocity $u_p(t)$ in the x_3-direction. This transformation enables one to write the boundary conditions in a simpler form in the case of large oscillatory motion of the droplet when a spherical coordinate system is used. Retaining the same set of dependent variables, one find that the form of the governing equations and the incident potential $\Phi_{(i)}$ are <u>unchanged</u> by referring to the moving coordinate system. In this transformation the terms containing space and time derivatives remain the same because of the linearization. However, it should be noted that u_i still denotes the fluid velocity as observed in the coordinate system <u>fixed in space</u>.

The boundary conditions to be satisfied on the surface of the droplet are the equality of the normal and tangential velocity components, the temperature, the heat flux, and the normal and tangential stresses with due account of the effect of surface tension. For a small droplet, the effect of surface tension is to limit the deformation of the droplet to small derivation from spherical shape. One may thus evaluate the boundary conditions at $r = r_0$. Let the superscript prime refer to the variables inside the droplet and $\xi(\theta, t)$ be the radial displacement of a point on the surface of the droplet. The boundary conditions are:

$$u_r = u_r' \tag{4}$$

$$u_\theta = u_\theta' \tag{5}$$

$$T = T' \tag{6}$$

$$\kappa \frac{\partial T}{\partial r} = \kappa' \frac{\partial T'}{\partial r} \tag{7}$$

$$\Pi_{r\theta} = \Pi_{r\theta}' \tag{8}$$

$$\Pi_{rr} = \Pi_{rr} + \frac{c}{r_0^2}\left[2 - 2\xi - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial \xi}{\partial\theta}\right)\right] \tag{9}$$

the second term in the right hand side of Eq. 9 is the surface force, $c\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$. It is obtained by assuming the shape of the droplet is slightly different from the spherical one $\left(\frac{\xi}{r_0} << 1\right)$. A workable form of Eq. 9 can be obtained by taking the derivative with respect to time and setting $\frac{\partial \xi}{\partial t} = u_r\big|_{r=r_0} - u_p \cos\theta$

The mathematical forms of the basic equations and initial conditions are same as Epstein and Carhart[3] although the independent variables have a different physical meaning. Furthermore, except for $\frac{c}{r_0^2}$ term in Eq. 9, the boundary conditions are also identical to Epstein and Carhart[3]. Indeed, the problem may be solved in a similar way. The calculation is, of course, long and tedious. For clarity sake, the detailed calculation will not be reproduced here. However, the method of solution is outlined below and the principal results are given.

The solution of our boundary value problem can be expressed in terms of three potentials, namely acoustic ϕ_1, thermal ϕ_2, and viscous $\vec{A}$. Once these three potentials are known, the various physical quantities can be calculated.

$$\vec{u} = -\nabla\phi + \nabla\times\vec{A} \qquad A_r = A_\theta = 0 \tag{10}$$

$$u_r = -\frac{\partial\phi}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A) \tag{11}$$

$$u_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} - \frac{1}{r}\frac{\partial}{\partial r}(rA) \tag{12}$$

$$T = \alpha_1(\phi_{(i)} + \phi_1) + \alpha_2\phi_2 \tag{13}$$

$$p = -i\omega\rho_0\left[\gamma_1(\phi_{(i)} + \phi_1) + \gamma_2\phi_2\right] \tag{14}$$

$$\Pi_{r\theta} = \mu\left\{-\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial\phi}{\partial r} - \frac{\phi}{r^2}\right) - \left(\frac{\partial^2 A}{\partial r^2} - \frac{2A}{r^2}\right) + \frac{1}{r^2}\frac{\partial}{\partial\theta}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(A\sin\theta)\right]\right\} \tag{15}$$

$$\Pi_{rr} = \mu k^2\left[\beta_1(\phi_{(i)} + \phi_1) + \beta_2\phi_2\right] + 2\mu\left\{-\frac{\partial^2\phi}{\partial r^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\left(-\frac{A}{r^2} + \frac{1}{r}\frac{\partial A}{\partial r}\right)\right]\right\} \tag{16}$$

$$u_\varphi = \Pi_{r\varphi} = 0 \tag{17}$$

where $\phi = \phi_{(i)} + \phi_1 + \phi_2$ and α_1, α_2, β_1, β_2, γ_1, γ_2 are functions of the properties of the two-phase medium and the frequency of the incident wave (see Eq. 18). Since the main interest is the attenuation due to the presence of small droplets, it is assumed that the attenuation of the incident wave by viscosity and heat conduction is small in the fluid medium in the absence of droplets. In other words, the study is limited to the case where the amplitude of the incident wave decreases relatively little over the region occupied by the droplets ($\ell_1 = \frac{2a_0}{\omega}\left[\frac{N\nu\omega}{a_0^2} + \frac{\Omega\omega}{a_0^2}(\gamma-1)\right]^{-1} \gg 2r_0$).

This means that parameters $\frac{N\nu\omega}{a_0^2} \approx \frac{\Omega\omega}{a_0^2} \ll 1$. In such cases, $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1$, and γ_2 are given approximately as follows.

$$\alpha_1 = -\frac{i\omega T_0 \alpha_v}{C_p} = -\frac{i\omega(\gamma-1)}{\alpha_v a_0^2}, \qquad \alpha_2 = -(\alpha_v \Omega)^{-1} \tag{18}$$

$$\beta_1 = 1 + i\frac{3}{2}\frac{N\nu\omega}{a_0^2}, \quad \beta_2 = 1 - \frac{3}{2}NP_r, \qquad \gamma_1 = 1 + \frac{iN\nu\omega}{a_0^2},$$

$$\gamma_2 = 1 - NP_r .$$

The equations governing ϕ_1, ϕ_2, and A are,

$$(\nabla^2 + K_1^2)\,\phi_1 = 0 \tag{19}$$

$$(\nabla^2 + K_2^2)\,\phi_2 = 0 \tag{20}$$

$$(\nabla^2 + K^2)\,A = 0 \tag{21}$$

where

$$K_1^2 = \frac{\omega^2}{a_0^2}\left\{1 + i\left[\frac{N\nu\omega}{a_0^2} + \frac{\Omega\omega}{a_0^2}(\gamma-1)\right]\right\}, \quad K_1 = \frac{\omega}{a_0}\left\{1 + i\frac{1}{2}\left[\frac{N\nu\omega}{a_0^2} + \frac{\Omega\omega}{a_0^2}(\gamma-1)\right]\right\}$$

$$K_2^2 = \frac{i\omega}{\Omega}, \qquad K_2 = (1+i)\left(\frac{\omega}{2\Omega}\right)^{\frac{1}{2}},$$

$$K^2 = \frac{i\omega}{\nu}, \qquad K = (1+i)\left(\frac{\omega}{2\nu}\right)^{\frac{1}{2}}.$$

and the solution of the system satisfying the initial and boundary conditions (including the surface tension) can be written in series form:

$$\phi_{i\omega} = \sum_{n=0}^{\infty} i^n (2n+1)\, J_n(K_1 r)\, P_n(\cos\theta)\, \exp(-i\omega t) \tag{22}$$

$$\phi_1 = \sum_{n=0}^{\infty} i^n (2n+1)\, H_n(K_1 r)\, P_n(\cos\theta)\, B_n \exp(-i\omega t) \tag{23}$$

$$\phi_2 = \sum_{n=0}^{\infty} i^n (2n+1) H_n(k_2 r) P_n(\cos\theta) C_n \exp.(-i\omega t) \tag{24}$$

$$A = \sum_{n=0}^{\infty} i^n (2n+1) H_n(kr) P_n^{I}(\cos\theta) D_n \exp.(-i\omega t) \tag{25}$$

$$\phi_1' = \sum_{n=0}^{\infty} i^n (2n+1) J_n(k_1' r) P_n(\cos\theta) B_n' \exp.(-i\omega t) \tag{26}$$

$$\phi_2' = \sum_{n=0}^{n=\infty} i^n (2n+1) J_n(k_2' r) P_n(\cos\theta) C_n' \exp.(-i\omega t) \tag{27}$$

$$A' = \sum_{n=0}^{n=\infty} i^n (2n+1) J_n(k' r) P_n^{I}(\cos\theta) D_n' \exp.(-i\omega t) \tag{28}$$

where J_n, H_n, P_n, and P_n^{I} are Bessel, Hankel, Legendre and associated Legendre (order 1) functions respectively. The coefficients B_n, C_n, D_n, B_n', C_n' and D_n' are determined from the six boundary conditions given by Eqs. 4 to 9. The values of these coefficients for $n = 0$ and $n = 1$ are given in Appendix. It is sufficient to consider first two terms of the expansion because of the rapid convergence of the series for $\frac{r_0}{\lambda_1} << 1$. If a_1 denotes $\frac{2\pi f r_0}{\lambda_1} (= r_0 k_1 << 1)$ each term in the series after $n = 1$ is an order a_1^2 times smaller than the preceeding. In terms of the small parameter a_1, the relative order of magnitude of the coefficients are

$$\begin{aligned} &B_0 \sim a_1^3, \quad && B_0' \sim \delta, \quad && C_0 \sim a_1^2 \\ &C_0' \sim a_1'^2, && B_1 \sim a_1^3, && B_1' \sim \delta \\ &C_1 \sim a_1^3 && C_1' \sim a_1'^3 && D_1 \sim a_1 \\ &D_1' \sim a_1' \end{aligned} \tag{29}$$

where $\delta = \frac{\rho_0}{\rho_0'}$. The physical interpretation of these coefficients is that for a given incident wave of unit amplitude these coefficients give the magnitude of amplitudes for reflected waves outside and transmitted waves inside the droplet.

The deformation of the droplet ξ is

$$\xi = \sum_{n=0}^{\infty} A_n P_n(\cos\theta) \exp. -i\omega t \tag{30}$$

where A_n can be determined from the relation $\frac{\partial \xi}{\partial t} = u_r|_{r=r_0} - u_p \cos\theta$

Hence $n = 0$ corresponds the pulsation of the droplet and $n = 1$ gives the translational motion of the droplet as a whole. Therefore, the term due to surface tension is not present for $n = 1$. Terms with $n > 1$ correspond to the change of the shape of the droplet and describe the higher modes of deformation.

III. ATTENUATION

When a sound wave propagates through a medium, its intensity decreases in proportion to $\exp(-\alpha x)$ where α is the attenuation coefficient and x is the distance traversed by the wave. When there are n droplets per unit volume, the total energy loss per unit volume per unit time will be $n\frac{dE}{dt}$, where $\frac{dE}{dt}$ is the average time rate of the energy dissipation per single droplet. The expression of α is

$$\alpha = \frac{n}{E_0}\frac{dE}{dt} \tag{31}$$

where $E_0 = \frac{1}{2}\rho_0 a_0 \overline{U_{(i)}^2}$, the intensity of the incident wave.

Let Ψ_μ and Ψ_κ be the viscous and thermal dissipation functions per unit volume respectively. Then the energy dissipation is

$$\frac{dE}{dt} = \int (\Psi_\mu + \Psi_\kappa)_{av.} dV \tag{32}$$

where $\Psi_\mu = \frac{\partial u_i}{\partial x_i}\tau_{ij}$ and $\Psi_\kappa = \frac{\kappa}{T_0}(\nabla T)^2$. Integrating Eq. 32 over a volume surrounding the droplets, we obtain

$$\frac{dE}{dt} = -2\pi\rho_0 a_0 \sum_{n=0}^{\infty} (2n+1) \mathcal{R}(B_n + B_n B_n^*) \tag{33}$$

$$\alpha = -\frac{4\pi n}{k_1^2}\sum_{n=0}^{\infty}(2n+1)\,R\,(B_n + B_n B_n^*) \tag{34}$$

where B_n^* is complex conjugate of B_n.

Let α_μ, α_κ, and $\Delta\alpha_\kappa$ be the attenuation coefficients due to the presence of viscosity, heat conduction, and surface tension, respectively. Neglecting terms of $n > 1$, (they are small if $a_1 << 1$), the expression for α can be written as the combination of these three terms.

$$\alpha = \alpha_\mu + \alpha_\kappa + \Delta\alpha_\kappa \tag{35}$$

A. Attenuation Applicable to the Suspension of Liquid Droplets in a Gas, i.e. $\delta << 1$, $\sigma << 1$, $\chi << 1$.

The explicit attenuation coefficient α_μ, α_κ and $\Delta\alpha_\kappa$ are

$$\alpha_\mu = \frac{9}{2}\epsilon\left(\frac{\nu}{a_0 r_0^2}\right)\frac{16(1+y)\,y^4}{16y^4 + 72\delta y^3 + 81\delta(1+2y+2y^2)} \tag{36}$$

$$\alpha_\kappa = 3\epsilon\left(\frac{\nu}{a_0 r_0^2}\right)\frac{4(\gamma-1)(1+P_r^{\frac{1}{2}}y)\,P_r\,y^4}{4P_r^2 y^4 + 12\left(\delta\frac{C_p}{C_p'}\right)P_r^{\frac{3}{2}}y^3 + 9\left(\delta\frac{C_p}{C_p'}\right)} \tag{37}$$

$$\Delta\alpha_\kappa = \alpha_\kappa\sum_{n=1}^{\infty}\left(\frac{2\varsigma}{3r_0\rho_0' a_0'^2}\right)^n \tag{38}$$

where $P_r = \frac{\nu}{\kappa}$, $\delta = \frac{\rho_0}{\rho_0'}$, $\sigma = \frac{\mu}{\mu'}$, $\chi = \frac{\kappa}{\kappa'}$, $\epsilon =$ volume fraction of the droplets.

B. Attenuation Applicable to the Suspension of Gaseous Bubbles in a Liquid, i.e. $\delta >> 1$, $\sigma >> 1$, $\chi >> 1$.

The explicit attenuation coefficient α_μ, α_κ, and $\Delta\alpha_\kappa$ are

$$\alpha_\mu = 6\epsilon\left(\frac{\nu}{a_0 r_0^2}\right)\frac{2y^7+12y^6+6y^5}{2y^6+6y^5+2y^4+36y^3+81(1+2y+2y^2)} \tag{39}$$

$$\alpha_\kappa = 3\epsilon\left(\frac{\nu'}{a_0 r_0^2}\right)(\gamma'-1)\delta\left(\frac{a_0}{a_0'}\right)^2 P_r^{-1}\left(P_r'^{\frac{1}{2}} y' \frac{\sinh 2P_r'^{\frac{1}{2}}y' + \sin 2P_r'^{\frac{1}{2}}y'}{\cosh 2P_r'^{\frac{1}{2}}y' - \cos 2P_r'^{\frac{1}{2}}y'} - 1\right) \tag{40}$$

$$\Delta\alpha_\kappa = \alpha_\kappa \sum_{n=1}^{\infty}\left(\frac{2c}{3r_0\rho_0' a_0^2}\right)^n \tag{41}$$

IV. GENERAL DISCUSSION OF THE RESULTS

A. Attenuation

The effect of surface tension is to alter the thermal dissipation resulting from heat conduction, while no effect on viscous dissipation is produced. This effect is found to be negligible in the case of the suspension of waterdroplets in air, i.e. $\frac{c}{r_0\rho_0' a_0'^2} \ll 1$. Hence, if the droplets differ only slightly from the spherical shape in the course of motion, Epstein and Carhart's results for the sound attenuation in a gas with the suspension of liquid droplets apply even if the displacement of the droplet is large compared to its radius. On the other hand, for water containing air bubbles, the presence of the surface tension is to increase the thermal attenuation by a factor of 1.5 for $r_0 = 10^{-4}$ cm. (see Fig. 1). This effect becomes less pronounced as the size of the droplets increase.

For air containing waterdroplets, the ratio of thermal attenuation to the viscous attenuation is

$$\frac{\alpha_\kappa}{\alpha_\mu} \xrightarrow[y \to 0]{} (\gamma-1)\frac{3}{2} P_r \left(\frac{c_p'}{c_p}\right)^2 \sim 7.2 \tag{42}$$

$$\frac{\alpha_\kappa}{\alpha_\mu} \xrightarrow[y \to \infty]{} (\gamma-1)\frac{2}{3} P_r^{-\frac{1}{2}} \sim 0.3 \tag{43}$$

It is seen that the thermal attenuation is predominant at low frequency range while the viscous attenuation is more important at higher frequency range (see Fig. 2).

For the suspension of air bubbles in water, it is seen from Eqs. 39 and 40 that the thermal attenuation completely dominates the viscous attenuation for all values of y and y'. $\left(\frac{\alpha_\kappa}{\alpha_\mu} \sim 10^5\right)$. This attenuation is caused by thermal dissipation inside the bubbles and is approximately given by

$$\alpha_\kappa = 3\epsilon\left(\frac{\mu'}{a_0 r_0^2}\right)(\gamma'-1)\,\delta\left(\frac{a_0}{a_0'}\right)^2 P_r' \frac{4}{45} y'^4 \left[1 + 11520\, P_r'^2 y'^4 \left(\frac{1}{10!} - \frac{1}{(6!)^2}\right) + O(y'^8) + \cdots \right] \tag{44}$$

$$\Delta\alpha_\kappa = \alpha_\kappa \sum_{n=1}^{\infty} \left(\frac{2C}{3 r_0 \rho_0' a_0'^2}\right)^n \tag{45}$$

for $y' \ll 1$.

B. Droplet Displacement

The equation of motion of the droplet is

$$m_p \frac{du_p}{dt} = \int \left(\Pi_{rr}\cos\theta - \Pi_{r\theta}\sin\theta\right) dS \tag{46}$$

where m_p and S are the mass and surface area of the droplet respectively. Π_{rr} and $\Pi_{r\theta}$ are given by Eqs. 15 and 16.

Evaluating the integral for $n = 1$, and dropping the terms of $O(a_1^2)$ and higher, one obtains expressions for the drag force F_D and the amplitude of the droplet velocity $\overline{U}_p$:

$$F_D = \frac{4}{3}\pi \rho_0 r_0^2 \omega \left[-a_1 + 3 i a_1^{-2} B_1 + 6 H_1(b) D_1 \right] \tag{47}$$

$$\overline{U}_p = \delta r_0^{-1} i \left[-a_1 + 3 i a_1^{-2} B_1 + 6 H_1(b) D_1 \right] \tag{48}$$

where $b = k r_0$. Eq. 48 can be reduced to the following form which is applicable to air containing water droplets.

$$\overline{U}_p = \frac{-3 i K_1 \delta H_2(b)}{3\delta H_2(b) + 2(\delta - 1) H_0(b)} \tag{49}$$

The above expression can also be obtained by calculating the surface velocity of the droplet.

$$\overline{U}_p = (\overline{U}_r \cos\theta - \overline{U}_\theta \sin\theta)_{r=r_0} = \frac{3i}{r_0}\Big[-\frac{a_1}{3} + B_1 H_1(a_1)(2\cos^2\theta - \sin^2\theta) + D_1 H_1(b)(2\cos^2\theta - \sin^2\theta) + D_1 b H_0(b) \sin^2\theta \Big] \tag{50}$$

This expression is identical to Eq. 49 upon substitution for B_1 and D_1.

The ratio of the droplet displacement ζ_p to the radius of the droplet is

$$\frac{\zeta_p}{r_0} = \frac{\overline{U}_\omega}{\omega r_0} \frac{3\delta H_2(b)}{3\delta H_2(b) + 2(\delta - 1) H_0(b)} \tag{51}$$

In the limit of audibility, the range of the sound intensity level[7] is from 0 to 135 db(decibele) at 1 KC based on threshold sound pressure of 2×10^{-4} dyne per sq. cm. This corresponds to sound intensity E_0 of 10^{-9} to 10^{4} erg. per sec. per sq. cm.

and a velocity amplitude of 7×10^{-5} to 70 cm. per sec. in air. For $\omega = 1$ KC and droplet size of 2 microns ($y = 6 \times 10^{-3}$), it is found that in the case of the suspension of the water droplets in air, the droplet remains relatively motionless ($\frac{\xi_p}{r_0} \leq \frac{1}{5}$) at an intensity level of 52 db and $\frac{\xi_p}{r_0} \sim 2$ at 72 db.

The ratio of the droplet displacement to that of the surrounding medium is

$$\frac{\xi_p}{\xi_g} = \frac{3\delta H_2(b)}{3\delta H_2(b) + 2(\delta - 1) H_0(b)} \tag{52}$$

$$\frac{\xi_p}{\xi_g} = 1 - \frac{2(\delta-1) H_0(b)}{3\delta H_2(b)} + O(y^3) + \cdots \qquad y << 1 \tag{53}$$

This ratio is plotted as a function y (see Fig. 2) for air containing water droplets. $\frac{\xi_p}{\xi_g}$ approaches to unity as y approaches to zero which agrees with our intuition that at low frequencies there is little relative motion between the droplet and the surrounding gas. It approaches $\frac{3\rho_0}{\rho_0 + 2\rho_0'}$ as $y \longrightarrow \infty$. This agrees with the classical result for a sphere set in motion by an oscillating non-viscous fluid[8].

C. Limiting Cases for Attenuation, Drag Force, and Heat Transfer Rate

1. Attenuation: In case of low frequencies ($y << 1$), the attenuation coefficients α_μ and α_k for $\delta << 1$

are approximately given by

$$\alpha_{\mu} = N\frac{9y^4}{8\delta}\left[1 - y + O(y^2) + \cdots\right] \tag{54}$$

$$\alpha_{k} = N(\gamma-1)\frac{P_r y^4}{L^2}\left[1 + P_r^{\frac{1}{2}} y + O(y^3) + \cdots\right] \tag{55}$$

where $N = \frac{\epsilon\nu}{a_0 r_0^2}$, and $L = \delta\frac{c_p}{c_p'}$.

2. Drag Force: The force acting on the droplet is given by Eq. 47 and can be reduced to the following form which is applicable to the suspension of water droplets in air.

$$F_D = -4\pi\rho_0 r_0^3 \omega k_1 \frac{H_2(b)}{3\delta H_2(b) + 2(\delta-1)H_0(b)} \tag{56}$$

For low frequencies ($y \ll 1$), F_D is approximately given by

$$F_D = 6\pi\mu r_0\left(\frac{-\bar{U}_{(i)}2b^2}{9\delta}\right) + O(y^5) + \cdots \tag{57}$$

On the other hand, for low frequencies

$$\bar{U}_{(i)} - \bar{U}_p = \frac{-\bar{U}_{(i)}2b^2}{9\delta} + O(y^5) + \cdots \tag{58}$$

Hence

$$F_D = 6\pi\mu r_0(\bar{U}_{(i)} - \bar{U}_p) + O(y^5) + \cdots \tag{59}$$

It is recognized immediately that the first term of the expression for F_D is identical to Stoke's formula of a sphere moving with a velocity of $\bar{U}_{(i)} - \bar{U}_p$ in a viscous medium. Furthermore, the first term of the coefficient of viscous attenuation given by Eq. 54 agrees with the dissipation function calculated on the assumption of validity of Stoke's formula.

3. Heat Transfer Rate: The heat transfer rate between the droplet and the surroudding gaseous medium per unit time and per unit area is

$$
\begin{aligned}
q &= \frac{1}{S}\int \kappa \left.\frac{\partial T}{\partial r}\right|_{r=r_0} dS \\
&= \kappa r_0^{-1}\left\{\alpha_1\left[a_1 \dot{J}_0(a_1) + a_1 \dot{H}_0(a_1) B_0\right] + \alpha_2 a_2 \dot{H}_0(a_2) C_0\right\}
\end{aligned}
\tag{60}
$$

where S is the surface area of the droplet and dot on Bessel and Hankel functions denote the differentiation with respect to their respective arguments. In case of low frequency range, q is approximately given by

$$
\begin{aligned}
q = \kappa r_0^{-1}\Big[-\alpha_1 \frac{a_1^2}{3} &+ i\alpha_1 a_1 H_0(a_1) B_0 + i\alpha_2 a_2 H_0(a_2) C_0 \\
&- \alpha_1 H_0(a_1) B_0 - \alpha_2 H_0(a_2) C_0 \Big]
\end{aligned}
\tag{61}
$$

Let $T_{(i)}$ and T_s denote the temperature of the gaseous medium at infinity and the surface temperature of the droplet, respectively. Then the temperature difference $T_{(i)} - T_s$ takes the following form

$$
T_{(i)} - T_s = -\alpha_1 H_0(a_1) B_0 - \alpha_2 H_0(a_2) C_0 \tag{62}
$$

Then in low frequency range

$$q = \kappa r_0^{-1} (T_{(1)} - T_s) + O(y^5) + \cdots \tag{63}$$

The heat transfer coefficient $h \left(= \frac{q}{T_{(1)} - T_s}\right)$ corresponding to the first term of the expansion is κr_0^{-1}, which is equal to Nusset No. $\left(= \frac{2 h r_0}{\kappa}\right)$ of 2 a agrees with the result for the heat transfer to a sphere when Reynolds No. tends to zero and heat exchange is by conduction only [9]. At low frequencies, it is noted from Eq. 64 that the temperature difference $T_{(1)} - T_s$ is proportional to $O(\omega^2)$.

APPENDIX

The assumptions made for obtaining the coefficients are (1) the acoustic damping length ℓ_1 is large compared with the region occupied by the droplets, i.e. $k_1 = \frac{\omega}{a_0}$, $k_1' = \frac{\omega}{a_0'}$. (2) The droplet is small compared with the acoustic wavelength $r_0 << \lambda_1$.

$n = 0$

$$B_0 = \frac{i\frac{a_1^3}{3}\left[\delta\left(\frac{a_1'}{a_1}\right)^2 - 1\right] + i a_1 \frac{\alpha_1}{\alpha_2}\left(\delta\frac{\alpha_1'}{\alpha_1} - 1\right)^2 \frac{a_2 H_1(a_2)}{H_0(a_2)(1 - \chi Z)}}{1 - \frac{2C}{3 r_0 \rho_0' a_0'^2}}$$

$$B_0' = \delta\left(1 - \frac{2C}{3 r_0 \rho_0' a_0'^2}\right)^{-1}$$

$$C_0 = \frac{\frac{\alpha_1}{\alpha_2}\left(\delta\frac{\alpha_1'}{\alpha_1} - 1\right)}{H_0(a_2)(1 - \chi Z)}\left(1 - \frac{2C}{3 r_0 \rho_0' a_0'^2}\right)^{-1}$$

$$C_0' = \chi\left(\frac{\alpha_2}{\alpha_2'}\right)\frac{a_2 H_1(a_2)}{a_2' J_1(a_2')} C_0$$

$n = 1$

$$B_1 = i\frac{a_1^3}{3}(1 - \delta) G$$

$$B_1' = \frac{a_1}{a_1'} G'$$

$$C_1 = \frac{a_1}{3}\frac{\alpha_1}{\alpha_2}\frac{\left[2 + \chi - 3\frac{\alpha_1'}{\alpha_1}G' + 2(1-\chi)(1-\delta)G\right] J_1(a_2') + \left[\frac{\alpha_1'}{\alpha_1}G' - 1 - (1-\delta)G\right] a_2' J_0(a_2')}{2(\chi - 1) J_1(a_2') H_1(a_2) + a_2' J_0(a_2) H_1(a_2) - \chi J_1(a_2') a_2 H_0(a_2)}$$

$$C_1' = -\frac{a_1}{3}\frac{\alpha_1}{\alpha_2}\frac{\left[(1 + 2\chi)\frac{\alpha_1'}{\alpha_1}G' - 3\chi\right] H_1(a_2) + \chi\left[1 - (1-\delta)G - \frac{\alpha_1'}{\alpha_1}G\right] a_2 H_0(a_2)}{2(\chi - 1) J_1(a_2') H_1(a_2) + a_2' J_0(a_2) H_1(a_2) - \chi J_1(a_2') a_2 H_0(a_2)}$$

$$D_1 = \frac{a_1}{b}(\delta-1)M\left\{[3\delta H_2(b)+2(\delta-1)H_0(b)]M-\sigma(\delta+2)bH_1(b)J_2(b)\right\}^{-1}$$

$$D_1' = \frac{-a_1\left(1+\delta+\frac{3i\delta B_1}{a_1^2}\right)+D_1[2bH_0(b)-6\delta H_1(b)]}{2b'J_0(b')-6J_1(b')}$$

where $\delta = \frac{\rho_0}{\rho_0'}$, $\sigma = \frac{\mu}{\mu'}$, $\chi = \frac{\kappa}{\kappa'}$, $a_1 = k_1 r_0$, $a_2 = k_2 r_0$,

$$b = k r_0, \qquad Z = \frac{J_0(a_2')\,a_2 H_1(a_2)}{a_2' J_1(a_2') H_0(a_2')}$$

$$G = \frac{H_2(b)[b'J_1(b')-2(1-\sigma)J_2(b')]-\sigma H_1(b)J_2(b')}{[3\delta H_2(b)+2(\delta-1)H_0(b)]M-\sigma(\delta+2)bH_1(b)J_2(b')}$$

$$G' = 3\delta G, \qquad M = b'J_1(b')-2(1-\sigma)J_2(b)$$

In obtaining D_1 and D_1', we neglect the terms containing C_1 and C_1', since these terms are small compared with the others.

For $\delta \le 1$, $\sigma \ll 1$, and $|b| \le 2$, the expression of B_1 and D_1 can be reduced to the following forms:

$$B_1 = i a_1^3(1-\delta)\frac{H_2(b)}{3\delta H_2(b)+2(\delta-1)H_0(b)}, \qquad D_1 = \frac{3iB_1}{a_1^2\, b H_2(b)}.$$

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TABLE I

Physical Constants of Water and Air at 20° C and Atmospheric Pressure Used in Preparation of Figures 1 and 2 and in Equations 36, 37, 40, 41, and 62.

	a_0 $\frac{cm.}{sec}$	ρ_0 $\frac{g}{cm^3}$	ν $\frac{cm^2}{sec}$	Ω $\frac{cm^2}{sec}$	C_p $\frac{cal.}{g\,°C}$	γ
Water	1.45×10^5	1.00	0.011	1.43×10^{-3}	1.0	1.00336
Air	3.30×10^4	1.29×10^{-3}	0.141	0.187	0.24	1.4

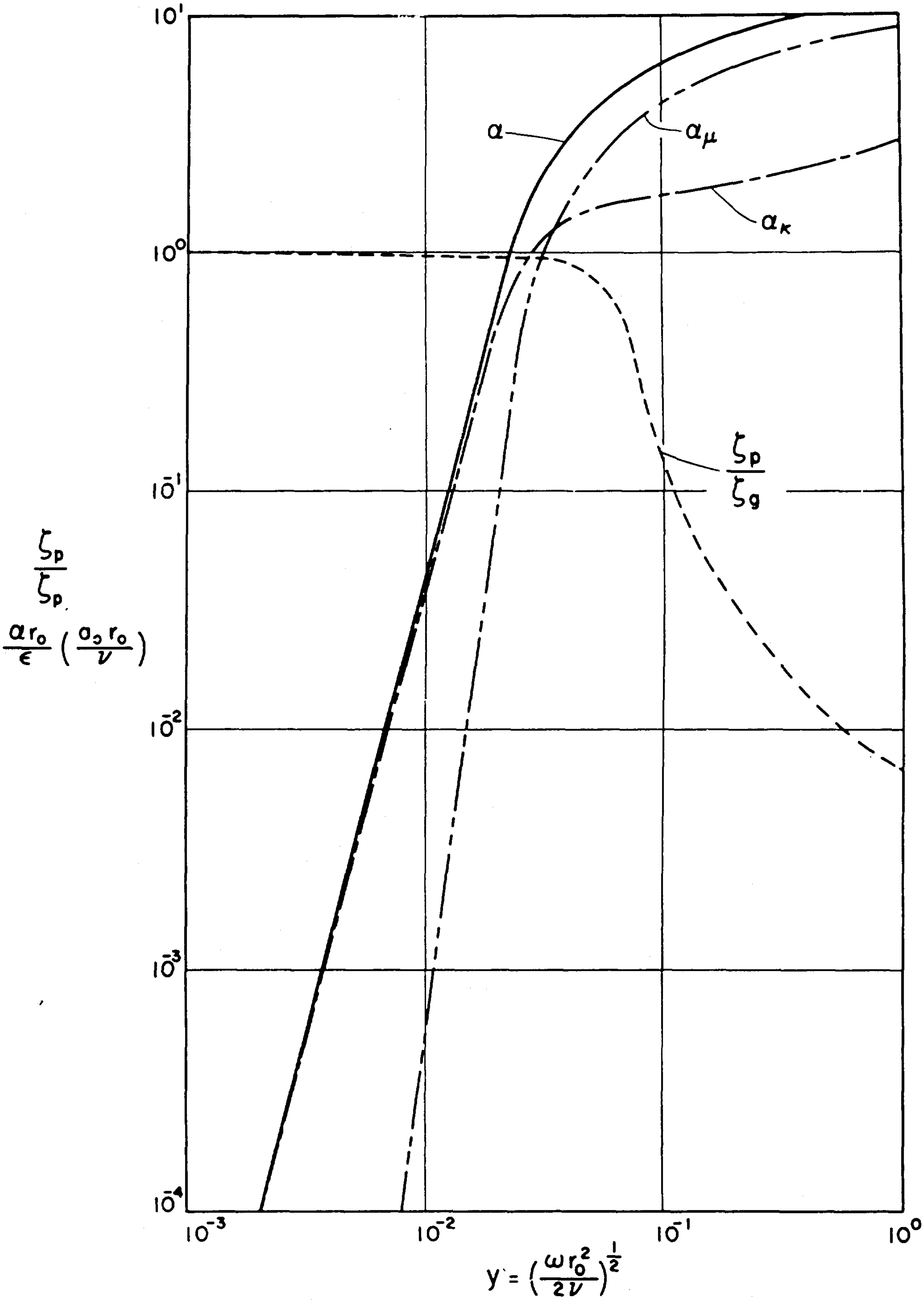

FIG. 1 THEORETICAL ATTENUATION PARAMETER AND DISPLACEMENT RATIO FOR WATER DROPLETS IN AIR

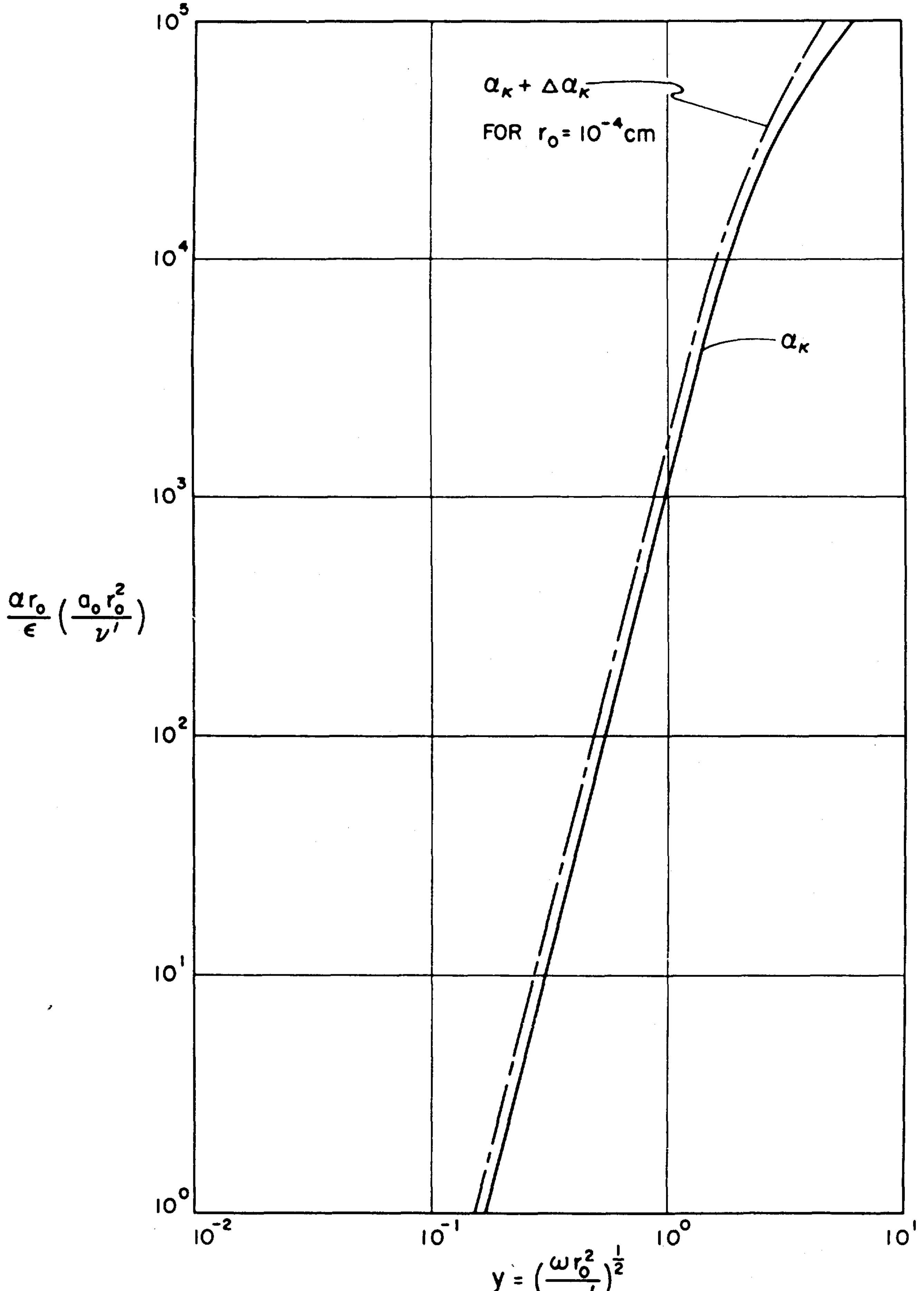

FIG. 2 THEORETICAL ATTENUATION PARAMETER FOR AIR BUBBLES IN WATER