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Orbit Determination by Angular Measurements

29 JANUARY 1964

Prepared by

P. O. BELL

Prepared for COMMANDER SPACE SYSTEMS DIVISION

UNITED STATES AIR FORCE

Inglewood, California

AEROSPACE CORPORATION

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Systems Research and Planning Division
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Aerospace TDR-269(4550-10)-3, entitled "Orbit Determination by Angular Measurements," dated January 1964, by P.O. Bell, carries an incorrect Air Force report number. Where it appears, change SSD-TDR-63-259 to SSD-TDR-63-359.

ABSTRACT

From a tracking station of known geocentric location, the measurements of the azimuth and elevation of a satellite at three instants of time t_1, t_2, t_3 provide tracking data sufficient for a preliminary determination of the orbit of the satellite about the earth. From such data, the components (in a geo-centered inertial coordinate system) of the position vectors $\bar{r}_1$, $\bar{r}_2$, $\bar{r}_3$ (corresponding to t_1, t_2, t_3) of the satellite may be calculated and in terms of these components, all of the orbital elements may be expressed. In view of the given positional data of the tracking station and the directional data of the satellite's position, only the slant ranges ρ_1, ρ_2, ρ_3 are needed to complete the determination of the vectors $\bar{r}_1$, $\bar{r}_2$, $\bar{r}_3$. To solve for the slant ranges, an iterative procedure is introduced whereby successive systems of linear equations are solved whose solutions rapidly converge to ρ_1, ρ_2, ρ_3. Subsequently, all of the orbital elements are easily computed. Fundamental use is made of the formulas of Gibbs (see Reference) which provide approximate values of the ratios B_{23}/B_{13}, B_{12}/B_{13}, in which B_{ij} denotes the area of the triangle bounded by the vectors $\bar{r}_i$, $\bar{r}_j$, $\bar{r}_j - \bar{r}_i$. The method has been tested for accuracy by applying it to compute the orbital parameters of many parameter families of hypothetical orbits. On comparing the computed values of the orbital elements with the corresponding hypothetical values, the differences were found to be exceedingly small. The average machine time for computing all elements of an orbit was approximately 0.05 minute.

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I. DYNAMICAL BASIS OF THE METHOD

A new method is introduced in the present paper for the determination of satellite orbits (or missile trajectories) about a spherical earth by means of angular measurements only.

Let $\bar{s}$, $\bar{\rho}$ denote the vectors from a tracking station at time t to the geocenter and the orbiting satellite, respectively. The satellite position vector $\bar{r}$ relative to a geocentered inertial frame is therefore defined by the vector equation

$$\bar{r} = \bar{\rho} - \bar{s} \tag{1}$$

Let ρ, $\bar{p}$ denote the slant range of the vehicle from the tracking station and the unit vector in the direction of $\bar{\rho}$, respectively, so that Equation (1) may be written in the form

$$\bar{r} = \rho\,\bar{p} - \bar{s} \tag{2}$$

Let vector and scalar functions of t_i, $i = 1, 2, 3$ be denoted simply by the use of the subscript i. Examples are $\bar{s}\,(t_i) \overset{\Delta}{=} \bar{s}_i$, $\rho\,(t_i) \overset{\Delta}{=} \rho_i$. Exceptions to the use of this notation will be the time intervals $t_2 - t_1$, $t_3 - t_2$ which will be denoted by Δ_1, Δ_2, respectively.

Expanding $\bar{r}_2$ by means of a Taylor's series, we have in terms of $\bar{r}_1$ and its derivatives

$$\bar{r}_2 = \bar{r}\,(t_1 + \Delta_1) = \bar{r}_1 + \dot{\bar{r}}_1 \Delta_1 + \frac{\ddot{\bar{r}}_1 \Delta_1^{\,2}}{2} + \frac{\dddot{\bar{r}}_1 \Delta_1^{\,3}}{6} + \frac{\ddddot{\bar{r}}_1 \Delta_1^{\,4}}{24} + \ldots \tag{3}$$

On substituting the dynamical relation

$$\ddot{\bar{r}}_1 = -\frac{\mu\,\bar{r}_1}{r_1^{\,3}} \tag{4}$$

into Equation (3), the equation becomes

$$\bar{r}_2 = \bar{r}_1 \left(1 - \frac{\mu \Delta_1^2}{2 r_1^3}\right) + \dot{\bar{r}}_1 \Delta_1 + \frac{\dddot{\bar{r}}_1 \Delta_1^3}{6} + \ldots \tag{5}$$

Vector multiplying the members of (5) by $\bar{r}_1$ yields the equation

$$\bar{r}_1 \times \bar{r}_2 = (\bar{r}_1 \times \dot{\bar{r}}_1) \Delta_1 + (\bar{r}_1 \times \dddot{\bar{r}}_1) \frac{\Delta_1^3}{6} + \ldots \tag{6}$$

Similarly,

$$\bar{r}_2 \times \bar{r}_3 = (\bar{r}_2 \times \dot{\bar{r}}_2) \Delta_2 + (\bar{r}_2 \times \dddot{\bar{r}}_2) \frac{\Delta_2^3}{6} + \ldots \tag{7}$$

Because of constancy of angular momentum ($\bar{r}_1 \times \dot{\bar{r}}_1 = \bar{r}_2 \times \dot{\bar{r}}_2$), the following equation results from a combination of Equations (6) and (7),

$$\frac{\bar{r}_1 \times \bar{r}_2}{\Delta_1} - \frac{\bar{r}_2 \times \bar{r}_3}{\Delta_2}$$

$$= \frac{\mu (\bar{r}_1 \times \dot{\bar{r}}_1)}{24} \left(4 \left[\frac{\Delta_2^2}{r_2^3} - \frac{\Delta_1^2}{r_1^3} \right] + 6 \left[\frac{\dot{r}_1 \Delta_1^3}{r_1^4} - \frac{\dot{r}_2 \Delta_2^3}{r_2^4} \right] + \ldots \right) \tag{8}$$

In view of Equation (5), the magnitudes r_1 and r_2 (of $\bar{r}_1$ and $\bar{r}_2$) differ by terms of order at least Δ_1. It follows that the right member of (8) is of order $\Delta_2^2 - \Delta_1^2$ if $\Delta_1 \neq \Delta_2$, but it is of order Δ_1^3 if $\Delta_1 = \Delta_2$. Consequently, in either case, if Δ_1 and Δ_2 are both small, the approximation results

$$\frac{\bar{r}_1 \times \bar{r}_2}{\Delta_1} \doteq \frac{\bar{r}_2 \times \bar{r}_3}{\Delta_2} \tag{9}$$

Let A_{ij} denote the area bounded by two position vectors $\bar{r}_i$, $\bar{r}_j$ and that portion of the orbit between $\bar{r}_i$, $\bar{r}_j$, i, j = 1, 2, 3, i < j, and let B_{ij} denote the area of the triangle bounded by the vectors $\bar{r}_i$, $\bar{r}_j$, $\bar{r}_j - \bar{r}_i$. The constancy of the rate of description of area assures that $A_{12}/\Delta_1 = A_{23}/\Delta_2$. From (9), $B_{12}/\Delta_1 \doteq B_{23}/\Delta_2$. The accuracy of (9), therefore depends upon how closely the ratio A_{12}/A_{23} is approximated by the ratio B_{12}/B_{23}. Closed form expressions for the ratios A_{12}/A_{23} and B_{12}/B_{23} can be derived in terms of orbit eccentricity and the eccentric anomaly angles associated with the positions of the vectors $\bar{r}_1$, $\bar{r}_2$, $\bar{r}_3$. They are

$$\frac{A_{12}}{A_{23}} = \frac{E_2 - E_1 - e(\sin E_2 - \sin E_1)}{E_3 - E_2 - e(\sin E_3 - \sin E_2)} \tag{10}$$

$$\frac{B_{12}}{B_{23}} = \frac{\sin(E_2 - E_1) - e(\sin E_2 - \sin E_1)}{\sin(E_3 - E_2) - e(\sin E_3 - \sin E_2)} \tag{11}$$

where

$$A_{ij} = \frac{a^2(1 - e^2)^{1/2}}{2} \left[E_j - E_i - e(\sin E_j - \sin E_i)\right],$$

$$B_{ij} = \frac{a^2(1 - e^2)^{1/2}}{2} \left[\sin(E_j - E_i) - e(\sin E_j - \sin E_i)\right],$$

i, j = 1, 2, 3 ; i ≤ j.

Moreover

$$A_{ij} - B_{ij} = \frac{a^2(1-e^2)^{1/2}}{2}\left[E_j - E_i - \sin(E_j - E_i)\right] \qquad (12)$$

$$< \frac{a^2(1-e^2)^{1/2}}{12}(E_j - E_i)^3$$

Suppose that, from a tracking station, angular measurements are made of the line of sight to an orbiting vehicle at times t_1, t_2, t_3. From these measurements, the inertial components of the corresponding unit vectors $\bar{p}_1$, $\bar{p}_2$, $\bar{p}_3$ may be determined. The corresponding system of equations

$$\bar{r}_i = \rho_i \bar{p}_i - \bar{s}_i \qquad (i = 1, 2, 3) \qquad (13)$$

may be written, in which the six vectors $\bar{p}_i$, $\bar{s}_i$ $(i = 1, 2, 3)$ are known. To solve for the unknowns ρ_1, ρ_2, ρ_3, use is made of Equation (9) together with the equation obtained from (9) by making the substitution

$$\begin{pmatrix} \bar{r}_1 & \bar{r}_2 & \Delta_1 & \Delta_2 \\ \bar{r}_1 & \bar{r}_3 & \Delta_3 & -\Delta_2 \end{pmatrix} \qquad (14)$$

where $\Delta_3 = \Delta_1 + \Delta_2$, e.g.,

$$\frac{\bar{r}_1 \times \bar{r}_3}{\Delta_3} \doteq \frac{\bar{r}_3 \times \bar{r}_2}{-\Delta_2} \doteq \frac{\bar{r}_2 \times \bar{r}_3}{\Delta_2} \qquad (15)$$

Since the vectors $\bar{r}_1$, $\bar{r}_2$, $\bar{r}_3$ are coplanar, constants c_1, c_2, c_3 exist, such that

$$c_1 \bar{r}_1 + c_2 \bar{r}_2 + c_3 \bar{r}_3 = 0 \tag{16}$$

Vector multiplications of Equation (16), first by $\bar{r}_2$ and then by $\bar{r}_3$, yield in virtue of (9) and (15) (for the equality sign) $c_1 \Delta_1 = c_3 \Delta_2$, $c_1 \Delta_3 = - c_2 \Delta_2$, so that (16) assumes the form

$$\Delta_2 \bar{r}_1 - (\Delta_1 + \Delta_2) \bar{r}_2 + \Delta_1 \bar{r}_3 = 0 \tag{17}$$

Substituting from Equation (13) into Equation (17) yields the vector equation

$$\Delta_2 \rho_1 \bar{p}_1 - (\Delta_1 + \Delta_2) \rho_2 \bar{p}_2 + \Delta_1 \rho_3 \bar{p}_3 = \Delta_2 \bar{s}_1 - (\Delta_1 + \Delta_2) \bar{s}_2 + \Delta_1 \bar{s}_3 \tag{18}$$

The system of equations obtained from the components of vector Equation (18) possesses a unique solution ρ_1, ρ_2, ρ_3 if and only if the determinant $(\bar{p}_1, \bar{p}_2, \bar{p}_3) \neq 0$, and not all of the components of the vector $\Delta_2 \bar{s}_1 - (\Delta_1 + \Delta_2) \bar{s}_2 + \Delta_1 \bar{s}_3$ vanish. The solutions assume the forms

$$\rho_1 = \frac{(\bar{\sigma}, \bar{p}_2, \bar{p}_3)}{D}, \quad \rho_2 = \frac{-\Delta_2 (\bar{p}_1, \bar{\sigma}, \bar{p}_3)}{\Delta_3 D}, \quad \rho_3 = \frac{\Delta_2 (\bar{p}_1, \bar{p}_2, \bar{\sigma})}{\Delta_1 D} \tag{19}$$

in which $\Delta_2 \bar{\sigma} = \Delta_2 \bar{s}_1 - \Delta_3 \bar{s}_2 + \Delta_1 \bar{s}_3$, $D = (\bar{p}_1, \bar{p}_2, \bar{p}_3)$. The condition of the vanishing of $\bar{\sigma}$ for non-vanishing values of Δ_1, Δ_2 implies that the three positions of the tracking station corresponding to the times t_1, t_2, t_3 are collinear. This can happen on the surface of the earth only if the tracking station is fixed in inertial space $\left(\text{i.e., } \bar{s}(t_1) = \bar{s}(t_2) = \bar{s}(t_3)\right)$. The only points where this could occur are at the poles.

Substituting the calculated values for ρ_1, ρ_2, ρ_3 (from (19)) into Equations (13) and performing the indicated arithmetic operations results in the numerical values of the components of the vectors $\bar{r}_1$, $\bar{r}_2$, $\bar{r}_3$. The orbital elements in turn can be easily calculated in terms of these. In fact, if θ_i denotes the angle from the line of intersection of the orbit plane and the $z = 0$ plane to the vector $\bar{r}_i$ (oriented so that $\theta_1 < \theta_2$) it may be easily shown that

$$\theta_i = \cos^{-1}\left\{\frac{-(\bar{r}_i)_1 (\bar{r}_1 \times \bar{r}_2)_2 + (\bar{r}_i)_2 (\bar{r}_1 \times \bar{r}_2)_1}{r_i\left[(\bar{r}_1 \times \bar{r}_2)_1^2 + (\bar{r}_1 \times \bar{r}_2)_2^2\right]^{1/2}}\right\}, \quad (i = 1,\ 2,\ 3) \tag{20}$$

in which r_i denotes the magnitude of $\bar{r}_i$ and a subscript j ($j = 1, 2$) following a vector enclosed in parentheses signifies the jth component of the vector. Moreover, the equation of the orbit may be written in the form

$$\frac{1}{r} = A + B \cos \theta + C \sin \theta \tag{21}$$

in which $A = 1/a\,(1 - e^2)$; $B = e \cos \theta_p / a\,(1 - e^2)$; $C = e \sin \theta_p / a\,(1 - e^2)$. Substituting into Equation (21) the values for θ_1, θ_2, θ_3 given by (20) and the values for r_1, r_2, r_3, three linear equations in A, B, C result which can be readily solved. Moreover, if Ω denotes the angle from the x-axis to the line of nodes, the elements a, e, θ_p, Ω are expressible by means of the formulas

$$a = \frac{A}{A^2 - (B^2 + C^2)}, \qquad e = \frac{(B^2 + C^2)^{1/2}}{A},$$

$$\theta_p = \tan^{-1}\frac{C}{B}, \qquad \Omega = \tan^{-1}\frac{(\bar{r}_2 \times \bar{r}_1)_1}{(\bar{r}_1 \times \bar{r}_2)_2} \tag{22}$$

II. HIGHER ORDER APPROXIMATIONS

The accuracies of Equations (9) and (15) are generally improved by reducing the time intervals between successive observations. Equation (9) becomes exact for circular orbits if $\Delta_1 = \Delta_2$. For near circular orbits it is therefore advantageous to choose $\Delta_1 = \Delta_2$.

The method of this paper can be modified so that it can be satisfactorily applied to cases in which the time intervals Δ_1, Δ_2 are much larger than formerly admissable, by making use of equations derivable from the formulas of Gibbs (see Reference) for the ratios of the triangular areas. In the terminology of this paper these equations are to third order terms

$$\frac{B_{23}}{B_{13}} = \frac{\Delta_2}{\Delta_3}\left[1 + \frac{\mu}{6r_2^3}\Delta_1(\Delta_2 + \Delta_3) + \frac{\mu}{12}\left(\frac{1}{r_1^3} - \frac{1}{r_2^3}\right)(\Delta_1\Delta_3 - \Delta_2^2) + \ldots\right]$$

$$\frac{B_{12}}{B_{13}} = \frac{\Delta_1}{\Delta_3}\left[1 + \frac{\mu}{6r_2^3}\Delta_2(\Delta_1 + \Delta_3)\right. \tag{24}$$

$$\left. - \frac{\mu}{12}\left(\frac{1}{r_1^3} - \frac{1}{r_2^3}\right)\left(\frac{\Delta_2}{\Delta_1}\right)(\Delta_2\Delta_3 - \Delta_1^2) + \ldots\right]$$

Equations (9) and (15) are first order approximations of the exact equations

$$\frac{\bar{r}_1 \times \bar{r}_2}{B_{12}} = \frac{\bar{r}_2 \times \bar{r}_3}{B_{23}} = \frac{\bar{r}_1 \times \bar{r}_3}{B_{13}} \tag{25}$$

in which the principal parts of the ratios B_{23}/B_{13}, B_{12}/B_{13} are employed. It follows that if good estimates of r_1, r_2 can be found, the substitutions of the ratios B_{12}/B_{23}, B_{23}/B_{13} (defined by (24)) for Δ_1/Δ_2, Δ_2/Δ_3 in the right members of Equations (19) will lead to much more accurate positional determinations. The following iterative process serves to determine good "initial estimates" of r_1, r_2. Since $r_2 \doteq r_1 + \dot{r}_1 \Delta_1$ it follows that $1/r_1^3 - 1/r_2^3 \doteq 3\dot{r}_1 \Delta_1 / r_1^4$. Consequently, the third terms of the right members of Equation (24) are of third order in the Δ's. Let B^*_{13}, B^*_{12}, B^*_{23} be defined such that

$$\frac{B^*_{23}}{B^*_{13}} = \frac{\Delta_2}{\Delta_3}\left[1 + \frac{\mu}{6 r_2^3} \Delta_1 (\Delta_2 + \Delta_3)\right]$$

$$\frac{B^*_{12}}{B^*_{13}} = \frac{\Delta_1}{\Delta_3}\left[1 + \frac{\mu}{6 r_2^3} \Delta_2 (\Delta_1 + \Delta_3)\right]. \qquad (26)$$

Select an estimate $\rho_2^{(1)}$ of the slant range ρ_2 and calculate the corresponding value $r_2^{(1)}$ by means of the equation

$$r_2^{(1)} = \left|\rho_2^{(1)} \bar{p}_2 - \bar{s}_2\right| \qquad (27)$$

Substitute $r_2 = r_2^{(1)}$ in Equation (26) and calculate the ratios $(B^*_{23}/B^*_{13})^{(1)}$, $(B^*_{12}/B^*_{13})^{(1)}$. The first and second equations of (19) with Δ_1/Δ_2 and Δ_2/Δ_3 replaced by $(B^*_{12}/B^*_{23})^{(1)}$ and $(B^*_{23}/B^*_{13})^{(1)}$, respectively, yield computed values for ρ_1 and ρ_2, say $\rho_1^{(2)}$ and $\rho_2^{(2)}$. Thus

$$\rho_1^{(2)} = \frac{(\bar{\sigma}^{(1)}, \ \bar{p}_2, \ \bar{p}_3)}{D}, \quad \rho_2^{(2)} = -\left(\frac{B_{23}^*}{B_{13}^*}\right)^{(1)} \frac{(\bar{p}_1, \ \bar{\sigma}^{(1)}, \bar{p}_3)}{D} \tag{28}$$

in which $D = (\bar{p}_1, \ \bar{p}_2, \ \bar{p}_3)$ and

$$\bar{\sigma}^{(1)} = \bar{s}_1 - \left(\frac{B_{13}^*}{B_{23}^*}\right)^{(1)} \bar{s}_2 + \left(\frac{B_{12}^*}{B_{23}^*}\right)^{(1)} \bar{s}_3$$

These slant range values are then used to calculate the corresponding values $r_1^{(2)}$, $r_2^{(2)}$ by means of the equations

$$r_1^{(2)} = \left| \rho_1^{(2)} \bar{p}_1 - \bar{s}_1 \right|, \quad r_2^{(2)} = \left| \rho_2^{(2)} \bar{p}_2 - \bar{s}_2 \right| \tag{29}$$

The values $r_1^{(2)}$, $r_2^{(2)}$ are then substituted in Equation (24) to determine ratios which are denoted by $(B_{12}/B_{23})^{(2)}$ and $(B_{23}/B_{13})^{(2)}$. This iteration process is continued making use of Equations (28), (29), (24) until the differences $(B_{12}/B_{23})^{(j)} - (B_{12}/B_{23})^{(j+1)}$ and $(B_{23}/B_{13})^{(j)} - (B_{23}/B_{13})^{(j+1)}$ become negligible. The values of the ratios B_{12}/B_{23}, B_{23}/B_{13}, thus determined are used for the final determinations of the slant ranges ρ_1, ρ_2, ρ_3 and position vectors $\bar{r}_1$, $\bar{r}_2$, $\bar{r}_3$ by means of the equations

$$\rho_1 = \frac{(\bar{\sigma}, \ \bar{p}_2, \ \bar{p}_3)}{D}, \quad \rho_2 = \frac{-B_{23}}{B_{13}} \ \frac{(\bar{p}_1, \ \bar{\sigma}, \ \bar{p}_3)}{D}$$

$$\rho_3 = \frac{B_{23}}{B_{12}} \frac{(\bar{p}_1, \ \bar{p}_2, \ \bar{\sigma})}{D}$$

in which

$$\bar{\sigma} = \bar{s}_1 - \frac{B_{13}}{B_{23}} \bar{s}_2 + \frac{B_{12}}{B_{23}} \bar{s}_3$$

and

$$\bar{r}_i = \rho_i \bar{p}_i - \bar{s}_i , \quad i = 1, 2, 3 \tag{31}$$

III. RESULTS OF DOUBLE PRECISION COMPUTATIONS OF ORBITS; SCOPE AND ACCURACIES OF THE DETERMINATIONS

The method of orbit determination was programmed for double precision computation on a digital computer. The means of testing the accuracies of the method consisted of computing the orbits (all of the orbit parameters) of various hypothetical orbits by using directional data, times of observation, and known positions of tracking stations, and comparing the computed results with the corresponding hypothetical values. A great multiplicity of hypothetical cases were considered in which the elements β, ω, a, e, θ_1, θ_2, θ_3, θ_p, α were selected from the following systems $\beta = 5^o, 15^o, 25^o, 35^o, \ldots, 90^o$; $\omega = 4^o, 12^o, 20^o, 28^o, \ldots, 68^o$; a = 3922.33, 4443.94, ..., 16,000 nautical miles; e = 0, 0.0125, 0.0150, ..., 0.85; $(\theta_1, \theta_2, \theta_3) = (6^o, 6.53^o, 7.07^o)$, $(18^o, 18.53^o, 19.07^o)$, ..., $(102^o, 102.53^o, 103.07^o)$, ..., $(0^o, 2^o, 4^o)$, ..., $(0^o, 5^o, 6^o)$; $\theta_p = -24^o, -12^o, 0^o, 12^o, 24^o, \ldots, 60^o$; $\alpha = 10(j-1)$, $j = 1, 2, 3, \ldots, 13$. The symbols, a, e have their usual connotation, β, ω denote latitude angle and the angle between the line of ascending node and the meridian plane of the tracking station at the time of the initial observation; θ_p is the true anomaly angle of perigee, θ_1, θ_2, θ_3, are the true anomaly angles of the satellite positions which correspond to times t_1, t_2, t_3 and α is the inclination angle of the orbit plane. In addition to the multiplicity of cases enumerated, similar systems of cases were considered which are near to a critical case. The critical case, in which the method is not applicable, occurs when the vectors from the tracking station to the three positions observed at t_1, t_2, t_3 are coplanar. In each of the cases which will be termed a near-critical case, the tracking station lies in the orbit plane at the time of one of the three observations. The accuracies of the results pertaining to the near-critical cases are significant because many of these cases were found to require double precision computations. The accuracies of the cases which are not near-critical are generally better than those of the near critical cases. But even for the near-critical cases, the differences between the hypothetical values of the elements and the values computed (from the observations) are very small. It will suffice, therefore, for the purpo~es of this report to describe the findings which pertain to the near-critical cases. Multiple parameter families of such cases were considered in which the ranges for the elements were as follows: $\beta = 10^o, 20^o, \ldots, 90^o$; $\alpha = 10^o, 20^o, \ldots, 120^o$; $\omega = \sin^{-1}$ (tan β cot α); $\theta_1 = \sin^{-1}$ (sin β csc α); $\theta_p = -24^o, -12^o, 0^o, \ldots, 60^o$; $\theta_2 = \theta_1 + 0.53^o$; $\theta_3 = \theta_2 + 0.54^o$; a = 3922.33, ..., 16,000 nautical miles; e = 0 , 0.0125, 0.0150, ..., 20.

The accuracies of the orbit determinations are, in general, better than those of the following typical near-critical cases:

	Actual Orbit Data	Computed Orbit Data
Case (1):	α = 30^o	α = 29.999952^o
	a = 3922.32999 n. mi.	a = 3922.2483 n. mi.
	e = 0.03	e = 0.029984383
	θ_p = -24^o	θ_p = -24.01641^o
	ρ_1 = 403.359695 n. mi.	ρ_1 = 403.58686 n. mi.
	ρ_2 = 405.10132 n. mi.	ρ_2 = 405.09120 n. mi.
	ρ_3 = 409.15499 n. mi.	ρ_3 = 409.14477 n. mi.
	θ_1 = 6^o	θ_1 = 6.000095^o
	θ_2 = 6.53^o	θ_2 = 6.530084^o
	θ_3 = 7.07^o	θ_3 = 7.070072^o

Case (2):	α	$= 10^\circ$	α	$= 10.000069^\circ$
	a	$= 3922.32999$ n. mi.	a	$= 3922.2591$ n. mi.
	e	$= 0.03$	e	$= 0.029986499$
	ρ_1	$= 914.40702$ n. mi.	ρ_1	$= 914.385$ n. mi.
	ρ_2	$= 922.4266$ n. mi.	ρ_2	$= 922.4048$ n. mi.
	ρ_3	$= 931.58933$ n. mi.	ρ_3	$= 931.5673$ n. mi.
	θ_p	$= -12^\circ$	θ_p	$= -12.01017^\circ$
	θ_1	$= 18^\circ$	θ_1	$= 18.001388^\circ$
	θ_2	$= 18.53^\circ$	θ_2	$= 18.531377^\circ$
	θ_3	$= 19.07^\circ$	θ_3	$= 19.071367^\circ$
Case (3):	α	$= 20^\circ$	α	$= 20.000279^\circ$
	a	$= 3922.32999$ n. mi.	a	$= 3922.2658$ n. mi.
	e	$= 0.03$	e	$= 0.029988690$
	ρ_1	$= 157.92217$ n. mi.	ρ_1	$= 157.91886$ n. mi.
	ρ_2	$= 158.42123$ n. mi.	ρ_2	$= 158.4179$ n. mi.
	ρ_3	$= 158.98764$ n. mi.	ρ_3	$= 158.9843$ n. mi.
	θ_p	$= 12^\circ$	θ_p	$= 11.991499^\circ$
	θ_1	$= 42^\circ$	θ_1	$= 42.000836^\circ$
	θ_2	$= 42.53^\circ$	θ_2	$= 42.530828^\circ$
	θ_3	$= 43.07^\circ$	θ_3	$= 43.070818^\circ$
Case (4):	α	$= 120^\circ$	α	$= 120.000165^\circ$
	a	$= 3922.32999$ n. mi.	a	$= 3922.3517$ n. mi.
	e	$= 0.03$	e	$= 0.030003371$
	ρ_1	$= 2636.2154$ n. mi.	ρ_1	$= 2636.2390$ n. mi.
	ρ_2	$= 2650.1334$ n. mi.	ρ_2	$= 2650.1571$ n. mi.
	ρ_3	$= 2664.5699$ n. mi.	ρ_3	$= 2664.5938$ n. mi.
	θ_p	$= 12^\circ$	θ_p	$= 11.998896^\circ$
	θ_1	$= 42^\circ$	θ_1	$= 41.999988^\circ$
	θ_2	$= 42.53^\circ$	θ_2	$= 42.529991^\circ$
	θ_3	$= 43.07^\circ$	θ_3	$= 43.069993^\circ$

Case (5):	α	$= 40^\circ$	α	$= 39.999986^\circ$
	a	$= 3922.32999$ n. mi.	a	$= 3922.4432$ n. mi.
	e	$= 0.01$	e	$= 0.010020000$
	ρ_1	$= 779.80816$ n. mi.	ρ_1	$= 779.80865$ n. mi.
	ρ_2	$= 806.28246$ n. mi.	ρ_2	$= 806.28279$ n. mi.
	ρ_3	$= 833.64203$ n. mi.	ρ_3	$= 833.64220$ n. mi.
	θ_p	$= 60^\circ$	θ_p	$= 60.118205^\circ$
	θ_1	$= 100^\circ$	θ_1	$= 99.999972^\circ$
	θ_2	$= 100.53^\circ$	θ_2	$= 100.529970^\circ$
	θ_3	$= 101.07000^\circ$	θ_3	$= 101.069968^\circ$
Case(6):	α	$= 30^\circ$	α	$= 29.999986^\circ$
	a	$= 16000$ n. mi.	a	$= 15999.892$ n. mi.
	e	$= 0.05$	e	$= 0.049994933$
	ρ_1	$= 11853.595$ n. mi.	ρ_1	$= 11853.585$ n. mi.
	ρ_2	$= 11857.068$ n. mi.	ρ_2	$= 11857.057$ n. mi.
	ρ_3	$= 11860.802$ n. mi.	ρ_3	$= 11860.792$ n. mi.
	θ_p	$= -20^\circ$	θ_p	$= -20.00322^\circ$
	θ_1	$= 10^\circ$	θ_1	$= 9.999997^\circ$
	θ_2	$= 10.53^\circ$	θ_2	$= 10.59997^\circ$
	θ_3	$= 11.07^\circ$	θ_3	$= 11.069997^\circ$
Case (7):	α	$= 30^\circ$	α	$= 30.000009^\circ$
	a	$= 16000$ n. mi.	a	$= 16001.358$ n. mi.
	e	$= 0.30$	e	$= 0.30005244$
	ρ_1	$= 8113.3207$ n. mi.	ρ_1	$= 8113.3793$ n. mi.
	ρ_2	$= 8126.2608$ n. mi.	ρ_2	$= 8126.3195$ n. mi.
	ρ_3	$= 8139.9188$ n. mi.	ρ_3	$= 8139.977$ n. mi.
	θ_p	$= -20^\circ$	θ_p	$= -20.003833^\circ$
	θ_1	$= 10^\circ$	θ_1	$= 9.99999^\circ$
	θ_2	$= 10.53^\circ$	θ_2	$= 10.529991^\circ$
	θ_3	$= 11.07^\circ$	θ_3	$= 11.069992^\circ$

Case (8):	$\alpha = 30^{\circ}$	$\alpha = 30.000001^{\circ}$
	$a = 16000$ n. mi.	$a = 16002.093$ n. mi.
	$e = 0.45$	$e = 0.45006593$
	$\rho_1 = 5737.7629$ n. mi.	$\rho_1 = 5737.8192$ n. mi.
	$\rho_2 = 5751.7785$ n. mi.	$\rho_2 = 5751.8349$ n. mi.
	$\rho_3 = 5766.6526$ n. mi.	$\rho_3 = 5766.7091$ n. mi.
	$\theta_p = 20^{\circ}$	$\theta_p = -19.99675^{\circ}$
	$\theta_1 = 10^{\circ}$	$\theta_1 = 9.999990^{\circ}$
	$\theta_2 = 10.53^{\circ}$	$\theta_2 = 10.529991^{\circ}$
	$\theta_3 = 11.07^{\circ}$	$\theta_3 = 11.069992^{\circ}$
Case (9):	$\alpha = 30^{\circ}$	$\alpha = 30.000021^{\circ}$
	$a = 16000$ n. mi.	$a = 16003.720$ n. mi.
	$e = 0.60$	$e = 0.60008761$
	$\rho_1 = 3294.5488$ n. mi.	$\rho_1 = 3294.6052$ n. mi.
	$\rho_2 = 3307.1798$ n. mi.	$\rho_2 = 3307.2364$ n. mi.
	$\rho_3 = 3320.7864$ n. mi.	$\rho_3 = 3320.8432$ n. mi.
	$\theta_p = -20^{\circ}$	$\theta_p = -19.99711^{\circ}$
	$\theta_1 = 10^{\circ}$	$\theta_1 = 9.999987^{\circ}$
	$\theta_2 = 10.53^{\circ}$	$\theta_2 = 10.529990^{\circ}$
	$\theta_3 = 11.07^{\circ}$	$\theta_3 = 11.069994^{\circ}$
Case (10):	$\alpha = 30^{\circ}$	$\alpha = 29.999965^{\circ}$
	$a = 16000$ n. mi.	$a = 15998.423$ n. mi.
	$e = 0.80$	$e = 0.79998041$
	$\rho_1 = 41.443169$ n. mi.	$\rho_1 = 41.442664$ n. mi.
	$\rho_2 = 44.263139$ n. mi.	$\rho_2 = 44.262486$ n. mi.
	$\rho_3 = 65.061448$ n. mi.	$\rho_3 = 65.060481$ n. mi.
	$\theta_p = -20^{\circ}$	$\theta_p = -20.00039^{\circ}$
	$\theta_1 = 10^{\circ}$	$\theta_1 = 10.000003^{\circ}$
	$\theta_2 = 10.53^{\circ}$	$\theta_2 = 10.529995^{\circ}$
	$\theta_3 = 11.07^{\circ}$	$\theta_3 = 11.069987^{\circ}$

Case (11):	α	=	30°	α	=	29.999979°
	a	=	-16000 n. mi.	a	=	-15998.314 n. mi.
	e	=	1.50	e	=	1.5000575
	ρ_1	=	12143.289 n. mi.	ρ_1	=	12143.362 n. mi.
	ρ_2	=	12169.763 n. mi.	ρ_2	=	12169.836 n. mi.
	ρ_3	=	12197.176 n. mi.	ρ_3	=	12197.248 n. mi.
	θ_1	=	10.0000°	θ_1	=	9.999996°
	θ_2	=	10.5300°	θ_2	=	10.529995°
	θ_3	=	11.0700°	θ_3	=	11.069995°
Case (12):	α	=	30°	α	=	29.99990°
	a	=	-16000 n. mi.	a	=	-15997.779 n. mi.
	e	=	20.0000	e	=	20.002653
	ρ_1	=	351905.91 n. mi.	ρ_1	=	351905.91 n. mi.
	ρ_2	=	353663.25 n. mi.	ρ_2	=	353663.25 n. mi.
	ρ_3	=	355449.83 n. mi.	ρ_3	=	355449.83 n. mi.
	θ_1	=	10.000°	θ_1	=	9.999992°
	θ_2	=	10.5300°	θ_2	=	10.529993°
	θ_3	=	11.0700°	θ_3	=	11.069992°

APPENDIX

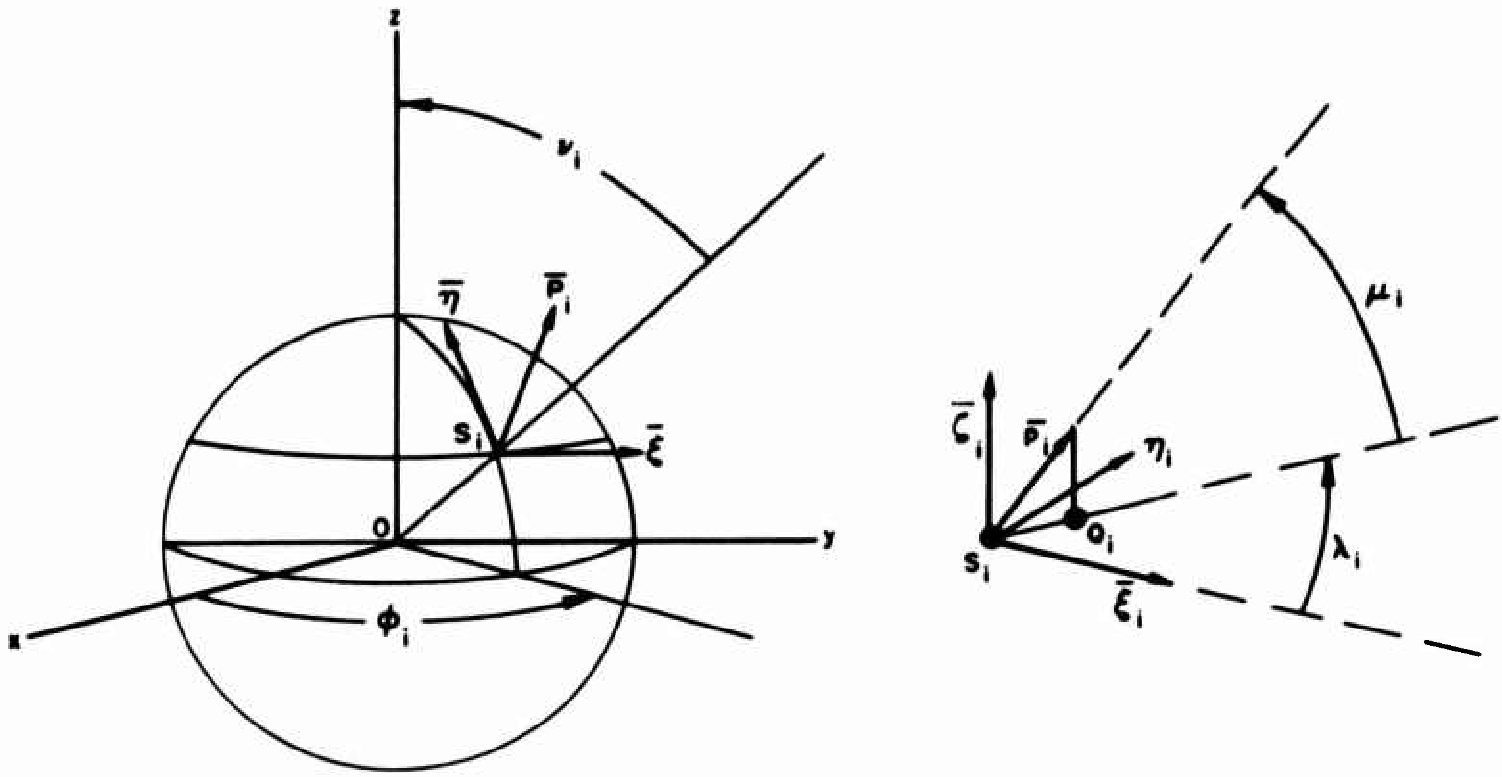

Figure 1. Geometry Relating Local Direction Angles of a Sattelite Position Vector $\overrightarrow{S_i P_i}$ to the Direction Cosines, in Inertial Coordinates, of the Unit Vector $\bar{p}_i = \overrightarrow{S_i P_i} / |\overrightarrow{S_i P_i}|$.

The vectors $\bar{p}_i$ ($i = 1, 2, 3$) employed in Equation (19) have been referred to a geocentered inertial coordinate frame. Direction angles of the vectors $\bar{p}_i$ will, however, be observed with reference to the horizontal plane through S_i and a vertical vector $\bar{\zeta}_i$ (see Figure 1). Let λ_i denote the angle formed by the orthogonal projection of the unit vector $\bar{p}_i$ on the horizontal plane through S_i and the east direction through S_i (if $0 < \lambda_i < \pi/2$ the bearing of $S_i Q_i$ is northeasterly). Let μ_i denote the elevation angle of the line of sight $S_i P_i$ above the horizontal plane at S_i. The direction angles λ_i, μ_i will be observed (measured) with respect to the unit vectors $\bar{\xi}_i$, $\bar{\eta}_i$, $\bar{\zeta}_i$ originating at S_i directed in the east, north, and vertical directions respectively. Let ν_i, ϕ_i denote the colatitude and longitude of the location S_i with respect to the inertial frame. As defined in Section I, $\bar{s}_i = \overrightarrow{S_i 0}$. Let the inertial components of $-\bar{s}_i$ be denoted by x_i, y_i, z_i. It follows from the geometry of Figure 1 that

$$x_i = R \sin \nu_i \cos \phi_i, \quad y_i = R \sin \nu_i \sin \phi_i, \quad z_i = R \cos \nu_i \tag{34}$$

The unit vectors $\bar{\xi}_i$, $\bar{\eta}_i$, $\bar{\zeta}_i$ can be expressed in terms of inertial coordinates by means of the equations

$$\bar{\xi}_i = \frac{-\dot{\bar{s}}_i}{|\dot{\bar{s}}_i|}, \qquad \bar{\eta}_i = \frac{\bar{s}_i \times \dot{\bar{s}}_i}{|\bar{s}_i||\dot{\bar{s}}_i|}, \qquad \bar{\zeta}_i = -\frac{\bar{s}_i}{|\bar{s}|} \tag{35}$$

in which the dot above the symbol denotes differentiation with respect to time t. As t varies, S_i moves (due to the earth's angular rotation rate $\dot{\phi}$) in a constant latitude circle such that the components of $-\dot{\bar{s}}_i$ are given by

$$\dot{x}_i = -R\dot{\phi} \sin \nu_i \sin \phi_i, \quad \dot{y}_i = R\dot{\phi} \sin \nu_i \cos \phi_i, \quad \dot{z}_i = 0. \tag{36}$$

The inertial components of the vectors $\bar{\xi}_i$, $\bar{\eta}_i$, $\bar{\zeta}_i$ can now be readily calculated by means of Equations (34) through (36). The results are found to be

$$\begin{aligned} &\xi_{i1} = -\sin \phi_i, \qquad \xi_{i2} = \cos \phi_i, \qquad \xi_{i3} = 0 \\ &\eta_{i1} = -\cos \nu_1 \cos \phi_i, \qquad \eta_{i2} = -\sin\phi_i \cos \nu_i, \qquad \eta_{i3} = \sin \nu_i \\ &\zeta_{i1} = \sin \nu_i \cos \phi_i, \qquad \zeta_{i2} = \sin \nu_i \sin \phi_i, \qquad \zeta_{i3} = \cos \nu_i \end{aligned} \tag{37}$$

From the definitions of the direction angles λ_i, μ_i it follows that the vector $\bar{p}_i$ is given by

$$\bar{p}_i = (\cos \mu_i \cos \lambda_i)\, \bar{\xi}_i + (\cos \mu_i \sin \lambda_i)\, \bar{\eta}_i + (\sin \mu_i)\, \bar{\zeta}_i \tag{38}$$

The inertial components of the vector $\bar{p}_i$ are found by substituting from Equations (37) into Equation (38) to be

$$\begin{aligned}
p_{i1} &= -\cos \mu_i \cos \lambda_i \sin \phi_i - \cos \mu_i \sin \lambda_i \cos \nu_i \cos \phi_i \\
&\quad + \sin \mu_i \sin \nu_i \cos \phi_i \\
p_{i2} &= \cos \mu_i \cos \lambda_i \cos \phi_i - \cos \mu_i \sin \lambda_i \sin \phi_i \cos \nu_i \\
&\quad + \sin \mu_i \sin \nu_i \sin \phi_i, \\
p_{i3} &= \cos \mu_i \sin \lambda_i \sin \nu_i + \sin \mu_i \cos \nu_i
\end{aligned} \tag{39}$$

Equations (39) provides the means of calculating the input data p_{ij} required by the method. Knowledge of the tracking station locations (in inertial space) at the instants of time t_1, t_2, t_3 furnishes the values of ϕ_i, ν_i, $i = 1, 2, 3$. The measurements of the observed local angles μ_i, λ_i furnish the remaining required data. The angle from the x-axis (of the inertial coordinate system) to the line of nodes is given by $\Omega = \phi_1 - \omega$.

REFERENCE

1. Danby, J. M. A., "Fundamentals of Celestial Mechanics," The Macmillan Company, New York (1962). For the formulas of Gibbs, see page 176, Equations (7.3.6) and (7.3.7).