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ON A STUDY OF SPECIFIC METHODS OF MEASURING LUNAR GRAVITY

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Harold Levenstein
Dr. Paul H. Savet
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ARMA DIVISION
American Bosch Arma Corporation
Garden City, New York

Contract No. AF19(628)-3855

FINAL REPORT
April 1964

Project 7600
Task 7600.07

Prepared for

AIR FORCE CAMBRIDGE RESEARCH LABORATORIES
Office of Aerospace Research
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ABSTRACT

A STUDY OF SPECIFIC METHODS OF MEASURING LUNAR GRAVITY

The measurement of the gravity field of the moon from an orbiting satellite is examined from the points of view of requirements for geodetic and geophysical application, and physical phenomena and instrumentation to meet these requirements.

It is shown that it is feasible to determine the gravity field by measuring the gravity gradient. Mathematical relations useful for data processing are stated. Examination of several principles for sensing gravity gradient leads to the recommendation of the vibrating string gradiometer as the best implementation with size, accuracy, range, and versatility being in its favor. In addition, the engineering status allows prototype design to begin immediately.

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SECTION 1

INTRODUCTION

The purpose of this program is to study the feasibility of measuring the gravity field of the moon from an orbiting satellite. The utility of such data lies in their application to geodesy (determining the shape of the moon) and to geophysics for a better understanding of the structure, density and internal processes of the moon. In immediate application it offers additional data for trajectory calculations for soft landing and manned exploration of the moon. The principles are also applicable to better definition of the earth gravity field for precision navigation of missiles, and for scientific planetary probes.

Direct methods for obtaining these data require landing a gravity meter on the surface of the moon and a geodetic reference to relate points of gravity observation. Without mobility the utility of even multiple meter landings is severely limited, but the difficulties of putting a vehicle on the moon's surface are too well known for further discussion of this as a technique. At some future date this will be both desirable and necessary.

The indirect methods involve analysis of the trajectory of an orbiting vehicle or the measurement of gravity gradient on board the vehicle. Investigation of this latter forms the basis for this report. Theoretically the measurement is sufficient to allow description of the gravity field, and the environment of the measurement is a low-noise one suitable for the measurement.

Instrumental design analysis further indicates it is feasible to make the necessary measurement. A vibrating-string gravity gradiometer is presented, which conservative estimates indicate will allow anomaly determination to the 10 milligal level; if the instrument preserves the dynamic range of its predecessor, the Arma vibrating string accelerometer, it will allow anomaly mapping at the 0.1 to 1 milligal level. Because this mapping is global and fast it appears to offer an ideal solution to lunar gravity determination.

SECTION 2

CONCLUSIONS AND RECOMMENDATIONS

2.1 Conclusions

Using gravity gradient measurements it is feasible to determine the gravity field of the moon from an orbiting satellite. The level of performance with which this can be accomplished provides the following information:

a) sufficient data to permit definition of the shape of the planet

b) at least two orders of magnitude improvement in the accuracy of current knowledge of the strength of the field

c) mapping of anomalous structure on the selenoid at the one to ten milligal level

It has been assumed that the probable orbit will be about 100 miles above the lunar surface. In recent discussions there have been indications that a forty mile orbit may be employed, in which case the anomaly discrimination level will be further improved by more than tenfold. Apart from generally improving with reduced orbital altitude, the system does not constrain the orbit in any way. Indeed we note that the natural eccentricity and precession of the orbit provides the global covering sample most useful for evaluation.

The data processing problem, the computation of gravity field from the gradient, is amenable to ordinary computer technology. Some computational techniques are suggested to illustrate feasibility in the report.

As primary instrumentation the vibrating string gradiometer is a more than satisfactory instrument for this purpose. A preliminary design analysis and performance summary is presented; the performance is predicated upon the Arma vibrating string accelerometer and the design is an engineering extrapolation of the knowledge and experience gained from the low range accelerometer now under test for RTD. A minimum sensitivity (dynamic range) of one part in 10^5 or better is indicated, compatible with the anomaly mapping goals.

As designed, the instrument also incorporates a low range accelerometer capability which may be employed in calibration and for ion-thrust engine, drag and solar radiation monitoring.

Note that the instrument is readily calibrated in orbit with high precision. Biases in the instrument are automatically isolated in the data reduction process, including time dependent drifts of the bias. The scale factor is readily determined by several methods, the simplest being the use of the gravity field of a nearby positionally controlled mass.

2.2 Recommendations

In accord with the above conclusions the following is recommended:

1 - Detailed design and fabrication of a vibrating string gradiometer should be carried out. The gradiometer should be fabricated for satellite application. The test program initially should include zero-g aircraft flight and recoverable flight testing in sounding rockets. Particular attention should be given to the possibility of designing a proof mass suspension system which would allow fully sensitive operation of the gradiometer in a one-g cross axis field. If this is achieved the gradiometer can be employed on a horizontal platform to measure gravity anomalies of the earth from an aircraft in level flight.

2 - A complete computational system should be worked out. In particular numerical methods for calculating the anomaly field more directly from gradient measurements than those indicated in this report should be developed. These will be of use in satellite, airborne and ground based gravity surveys.

3 - Further detailed study of the operational usage of the gradiometer and the system should be made. This can be done partly by simulation to examine alternative methods.

4 - The gradiometer provides information on the gravity field in different forms. For example, the curvature of the field is directly available. It appears desirable to explore direct methods of interpreting these findings as an alternative to the conventional anomaly maps. This entails deeper investigation into the geophysical interpretation of the data.

5 - A study should be made of the gravity gradient technique as a possible experiment, and as a working tool on a manned orbital research laboratory. The presence of the human monitor allows a number of operations to be carried out semi-automatically, and data can be evaluated on the spot. If the satellite is properly equipped the gradiometer could even be assembled in space and tension and proof mass variations could be tried.

The use of the gradiometer to indicate the local vertical can facilitate active attitude stabilization of the satellite laboratory. At the sensitivity level of the proposed instrument, and with the aid of the accelerometer function, short range non-radiating docking control functions can be carried out. Drag measurements can also be made, and, of course, the fundamental geophysical data that gravity offers can be obtained and interpreted.

SECTION 3

TECHNICAL DISCUSSION

Introduction

This section contains the important analytical and design findings of the study. They are presented in the following sequence:

a) The use of gravity data is first discussed. The magnitudes of gravity anomalies on the earth and typical geophysical phenomena are cited. The current status of knowledge of the moon and other planetary bodies is then summarized. Sensitivity requirements for gravity data to be useful are determined.

b) Equations of the gravity field are derived; relations to measurements by gradiometer are derived and methods are presented for deriving anomaly maps from the observations.

c) Instrumentation techniques for measuring the gravity gradient are reviewed, and a detailed analysis of a vibrating string gradiometer with preliminary design data is shown. Operational configurations for calibration and orientation are discussed. These are related to the requirements.

3.1 The Dependence of Geodesy, and Geophysics on Gravity Data

The gravity field of a planet is a unique function of the distribution of mass in the planet; that is it depends upon the shape, density and position of matter in the body. Through the process of hypothesizing density distributions and mass transfer mechanisms and then testing the predicted gravity field against the observed, the geophysicist is able to determine the most probable internal processes of the planet.

The uniqueness of the gravity field has been fundamental in the development of navigation techniques. The position of the local vertical projected on to the celestial sphere, and the time of day, suffice to determine location on the earth. To effect this, of course, the navigator must know the gravity field everywhere; the standard representation of this is a map - a correlation of geographic detail with coordinates on the celestial sphere. However, in order to determine lineal distance between points on the map, one must put the map not on the celestial sphere but on the real solid figure of the earth; the geodesy of the planet must be established. In determining the shape of the planet gravity again plays an important role. Strong correspondence between the geoid and the gravity equipotentials leads to a relatively simple description. For the earth, mean sea level is the equipotential and corresponds reasonably well to the ellipsoid now generally employed for the shape of the earth.

Nevertheless, current maps are imprecise. As Heiskanen points out in the Handbook of Geophysics the incompleteness of existing triangulation surveys leads to many geodetic systems, depending upon where the geodesist begins his survey. Since the continental masses are not now well tied together, the survey can not be closed globally. The data are good enough for ships but not as good as desired for highly precise navigation for, say, missiles.

Accordingly, wide-ranging gravity data, extended over the oceans, and into otherwise inaccessible regions has significant military and commerical value to the United States. Similar considerations apply to the moon.

In addition to its value for navigation, gravity data is of fundamental interest to the geophysicist. Basic theories of planetary origin and structure can be tested by observing the gravity fields. Keys to local subterranean geology of commercial importance are provided by gravity surveys.

In this program we are interested in both lunar geodesy and lunar geophysics. The following section discusses two of the geophysical aspects of the earth in more detail, as possible cues to the geophysics of the moon. These are the concept of isostasy and the nature of anomalies.

3.1.1 The Concept of Isostasy

During the course of making precise measurements it was noticed that the attraction of large masses such as mountains was less than predicted by the law of gravitation. It was concluded that the density of the mountains is less than that of the surrounding rock (this is still the present day conviction).

This, in turn, led to the concept of isostasy in which the earth's crust is considered to "float" on a liquid interior. The mountains, lowlands, and oceans are thought to exert the same pressure at depths not far (about 114 kilometers *) below sea level. These masses are said to be in "isostatic equilibrium" and the depth at which the pressure is considered to be uniform is called the "depth of isostatic compensation."

* There are other estimates of this level, The figure cited above is due to Hayford. Heiskanen has proposed a level ranging from 30 to 40 km, varying over the earth.

There are several isostatic systems, differing in the way in which it is assumed that mass compensation is achieved.

In one system it is assumed that mountains have low density earth crust directly beneath them and that oceans have high density earth crust directly beneath them in such a way that the higher the mountain the lower the density of the earth's crust and the deeper the ocean the higher the density of the earth's crust. If the earth's crust is assumed to be subdivided into columns floating upon the molten inner mass, then, according to this assumption, all columns at the same latitude have the same total mass. Consequently, the bases of all columns are at the same radial distance from the center of the earth, resulting in a uniform depth of isostatic compensation.

In another system it is assumed that the earth's crust is subdivided into columns as before; the columns now have unequal masses and therefore float upon the molten inner mass at different depths of displacement. Thus in this system, the depth of isostatic compensation is greatest under mountains and least under oceans.

In still another system it is assumed that the strength of the earth's crust is such as to prevent a floating system from being in complete equilibrium. This will result in the large mass of a mountain being compensated for in a region greater than its direct vertical projection. The crust below sea level is assumed to behave somewhat like an elastic plate floating upon a liquid and loaded locally by mountains, valleys, and oceans a negative load.

3.1.2 The Concept of Anomaly

If the earth were considered to be a perfect oblate spheroid without nonhomogeneities, all equigraviational surfaces would be concentric with the sea level datum. It would then be possible to calculate gravity very simply everywhere. However, as has been noted, the measurements of "g" have differed by varying amounts from the values calculated for an idealized earth as described above. These differences are attributed to the nonhomogeneous density distribution of the earth's crust and are called anomalies. The concept of isostasy was created in order to explain these anomalies, and the various isostatic systems described have been used one at a time or in combination in order to minimize them.

When elevation, topographic and isostatic corrections have been included in a particular theoretical determination of "g" on the earth's surface, the residual anomalies, determined by comparison with actual measurements, are relatively small. In the United States they are less than ±50 milligals (1 gal = 1 cm/sec^2) with few exceptions, the average determination being much lower. This corresponds, roughly, to about ±50 parts per million of "g" demonstrating that the earth exhibits a high degree of isostatic adjustment. In the few exceptions mentioned, the magnitudes of the anomalies may be much higher, running as high as hundreds of milligals. It has been demonstrated that some of these occur in regions of abnormal tectonic activity. Presumably, this would indicate a temporary deviation from isostatic equilibrium.

Typical residual gravitational anomalies on earth are:

Complete Tidal Cycle	- 0.2 to 0.3 milligal
Small Faults or Ore Bodies	- 0.1 to 0.5 milligal
Salt Domes or Buried Ridges	- 1 to 2 milligals
Small Craters	- Up to 15 milligals
Mountain Ranges, Ocean Deeps, or Rift Valleys	- Several Hundred milligals

Gravitational anomalies as determined from measurements aboard an orbiting vehicle do not uniquely fix the shape of the terrain below, since a mountain, for instance, and a subsurface high density mass can both produce positive anomalies of the same magnitude. On earth, topological, acoustical, magnetic, tectonic, and radiological data are used to minimize the residuals. These have led to conclusions regarding crustal and subcrustal phenomena such as stress releases in the crust, tilted fault planes (steps in the crust caused by shifting earth masses), and broken formations (separations in the crust caused by converging fault planes). If it is true, as has been conjectured, that the moon is a cold body, that is with a solid interior, any evidence of crustal and subcrustal phenomena as we know them on earth, will be useful in drawing conclusions regarding the origin of the moon. Under these circumstances there would be no need for concern regarding the constancy of geodetic determinations over a period of years, since the concept of isostasy would be inapplicable.

3.1.3 Gravity Measurements on the Moon

This study is concerned with the feasibility of determining the gravity field of the moon from an orbiting satellite. Before going onto the satellite a few words about other methods is in order.

The simplest concept is to land a gravity meter on the surface of the moon. This is technically feasible. However, the data would have limited utility since it provides an isolated value of the field. If properly placed and if sufficiently sensitive it might yield some information on tidal effects, and, from this, inferences can be made about rigidity of the crust. An improvement would be afforded if several meters were landed on the moon. But to use the data for geodesy a triangulation net would have to be set up.

A more meaningful experiment employs a probe on a lunar surface vehicle. This would permit data-taking over a wide area and through an on-board navigator, a suitable triangulation or geodetic net can be established. Currently, considering the rugged nature of the moon's topography and our uncertainty with respect to its surface composition (dust?), such a program appears out of the question. Someday it must be attempted. A surface program will be necessary to provide the geological detail required for fuller understanding of the moon.

However, it appears quite feasible to place a satellite in orbit around the moon and telemeter its data back to earth. Accordingly, the central question is the feasibility of instrumenting it for the collection of gravity data. Two general approaches may be followed. The first is observation of the trajectory and the determination of the planetary constants by calculation of the accelerations that appear to act on the satellite.

Although feasible in principle, the data processing involves the differentiation of data that are intrinsically noisy. Good observations of the principal term of the gravity field, that is the major attraction term, can be obtained in this way. However this is already fairly well known. It appears highly dubious that anomalies can be effectively detected. The other approach is that taken in the remainder of this report; the use of an on-board gravity gradiometer to provide the gravity data. The motion of the satellite provides an essentially gravity-free environment, and one free of environmental noise to a very high degree. Additionally, because of its nature a gradiometer is not affected by external forces on the satellite (such as radiation pressure).

Now it is evident that a very sensitive accelerometer mounted in the satellite would indicate the difference in gravity due to its separation from the center of mass of the satellite. However, this point is not easily found and is subject to change. Accordingly, the gradiometer technology considered inthis report leans toward measurements that do not depend upon knowledge of the c.g. As will be seen , this is quite feasible.

3.2 Gravity Gradient Theory

We define gravity gradient as follows. We first find the vector difference of the gravity forces on two nearby points in the gravity field. Then we take the component of this vector along the line joining the two points. Finally we consider the limiting process which results when the separation between the points passes to zero. Mathematically we may formalize the process by writing

$$(gg) = \frac{d\,\underline{g}}{|d\,\underline{r}|} \cdot \frac{d\,\underline{r}}{|d\,\underline{r}|} \qquad 3.2\text{-}1$$

The result is a scalar function of five variables, the three variables of position and two variables defining the direction along which the gradient is taken.

In this section we shall start with the gravity potential function V and set down the important relations between V, the vector gravity ($\underline{g}$), and the gravity gradient (gg).

The results will be presented in rectangular cartesian coordinates and in spherical coordinates. Some simple potential functions will be analyzed.

We then consider methods for recovering the potential function and the gravity vector from the gradient, and conclude with an estimate of the necessary sensitivity.

3.2.1 Gravity Gradient as a Function of the Potential

It is shown in classical mechanics that the forces produced by a gravitating body can be represented conveniently by means of operations on a scalar point function of position called the potential of the field. If the mass distribution is given, the potential of the field can be obtained by adding the separate potentials of each elementary piece of mass algebraically, or equivalently, by integrating over the volume. Potentials superimpose. Further these potentials are conservative. By this is meant that the work required to move a mass from one point to another in the field depends only upon the initial and final positions and not upon the path taken between the two points. For a unit mass this work is numerically equal to the change in potential between the two points. If two points have the same potential no net work is required to move from one point to the other. It is customary to take the reference level of potential as zero at infinity.

The potential of an elementary particle of mass m concentrated at a point is

$$V = -\frac{Gm}{r} \qquad 3.2\text{-}2$$

r is the distance from the mass
G is the universal gravitational constant, in units appropriate for V, m, and r.

When m is in grams, r is in centimeters, V is in ergs, and G then becomes 6.67×10^{-8} $cm^3/gm/sec^2$. This number is not known very exactly, although the associated gravitational acceleration on the earth is well defined.

We calculate the gravity force, $\underline{g}$, by taking the negative gradient of V. In traditional vector symbology

$$\begin{aligned} \underline{g} &= -\,\text{grad}\ V \\ &= -\nabla V. \end{aligned} \qquad 3.2\text{-}3$$

In this notation $\underline{g}$ is the force on a unit mass placed in the gravity field. For the particle field of equation (3.2-2)

$$\underline{g} = -1_r \frac{\partial}{\partial r}\left(-\frac{Gm}{r}\right) \qquad 3.2\text{-}4$$

$$= -1_r \left(\frac{Gm}{r^2}\right) \qquad 3.2\text{-}5$$

We shall designate unit vectors by the notation 1_q, meaning the unit vector associated with the q coordinate. Equation 3.2-5 is just Newton's equation for the forces of attraction between a unit mass and a mass m, inversely proportional to the square of their separation and directed toward each other.

For more general potential functions we may explicitly express $\underline{g}$ as a function of the partial derivatives of V with respect to the coordinates of the system of representation.

In rectangular cartesian coordinates we have:

$$\underline{g} = -\left[1_x \frac{\partial V}{\partial x} + 1_y \frac{\partial V}{\partial y} + 1_z \frac{\partial V}{\partial z}\right] \qquad 3.2\text{-}6$$

$$= 1_x g_x + 1_y g_y + 1_z g_z \qquad 3.2\text{-}7$$

In polar spherical coordinates we have:

$$\underline{g} = -\left[1_r \frac{\partial V}{\partial r} + 1_\theta \frac{1}{r} \frac{\partial V}{\partial \theta} + 1_\phi \frac{1}{r \sin\theta} \frac{\partial V}{\partial \phi}\right. \qquad 3.2\text{-}8$$

$$= 1_r g_r + 1_\theta g_\theta + 1_\phi g_\phi \qquad 3.2\text{-}9$$

Note that the point of observation is not moving in space. Thus the vector shown is not the vector which would be indicated by a plumb bob on the earth. It is the gravity vector which would be experienced by either a satellite, or a body attached to a non-rotating planet.

Following our definition of gravity gradient in the opening sentences of this section, we can evaluate it in terms of partial derivatives of the potential function and the direction cosines of the sensitive axis.

In rectangular coordinates we write (gg) in matrix form:

$$(gg) = C^T \left| V_{x_i x_j} \right| C \qquad 3.2\text{-}10$$

$$= (C_1 C_2 C_3) \begin{vmatrix} \frac{\partial^2 V}{\partial x^2} & \frac{\partial^2 V}{\partial x \partial y} & \frac{\partial^2 V}{\partial x \partial z} \\ \frac{\partial^2 V}{\partial x \partial y} & \frac{\partial^2 V}{\partial y^2} & \frac{\partial^2 V}{\partial y \partial z} \\ \frac{\partial^2 V}{\partial x \partial z} & \frac{\partial^2 V}{\partial y \partial z} & \frac{\partial^2 V}{\partial z^2} \end{vmatrix} \begin{vmatrix} C_1 \\ C_2 \\ C_3 \end{vmatrix} \qquad 3.2\text{-}11$$

Note that $\left| V_{x_i x_j} \right|$ is a symmetric, square matrix. The quantities C_1, C_2 and C_3 are the direction cosines of the sensitive axis on the X, Y, Z axes respectively. Their squares add to unity.

$$C_1^2 + C_2^2 + C_3^2 = 1 \qquad 3.2\text{-}12$$

Performing the indicated operations in rectangular coordinates

$$(gg) = C_1^2 \frac{\partial^2 V}{\partial x^2} + C_2^2 \frac{\partial^2 V}{\partial y^2} + C_3^2 \frac{\partial^2 V}{\partial z^2} + 2C_1C_2 \frac{\partial^2 V}{\partial x \partial y} + 2C_1C_3 \frac{\partial^2 V}{\partial x \partial z} + 2C_2C_3 \frac{\partial^2 V}{\partial y \partial z} \qquad 3.2\text{-}13$$

In spherical coordinates we go directly to the extended statement:

$$\begin{aligned}(gg) = &\left(\frac{C_\theta^2}{r} + \frac{C_\phi^2}{r}\right)\frac{\partial V}{\partial r} + \left(\frac{C_\phi^2 \cos\theta}{r \sin\theta} - \frac{2C_\theta C_r}{r}\right)\frac{1}{r}\frac{\partial V}{\partial \theta} \\ &- 2\left(\frac{C_\phi C_r}{r} + \frac{C_\theta C_\phi}{r}\cot\theta\right)\frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi} \\ &+ C_r^2 \frac{\partial^2 V}{\partial r^2} + \frac{2C_rC_\theta}{r}\frac{\partial^2 V}{\partial r \partial \theta} + \frac{2C_rC_\phi}{r\sin\theta}\frac{\partial^2 V}{\partial \phi \partial r} \\ &+ \frac{C_\theta^2}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{2C_\theta C_\phi}{r^2 \sin\theta}\frac{\partial^2 V}{\partial \phi \partial \theta} + \frac{C_\phi^2}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}\end{aligned} \qquad 3.2\text{-}14$$

Before continuing we want to note some other relations. An invariant of the potential field is the divergence of the gradient, which is identically zero wherever there is no mass. This is Laplace's equation.

$$\nabla^2 V = 0 \qquad 3.2\text{-}15$$

In rectangular coordinates

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \qquad 3.2\text{-}16$$

In spherical coordinates 3.2.-17

$$\frac{1}{r^2 \sin\theta}\left\{\sin\theta \frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2 V}{\partial \phi^2}\right\} = 0$$

If the potential is given everywhere on a closed surface the potential function is uniquely determined. This follows from the divergence theorem of vector calculus. (Dirichlet's problem).

If the normal derivative of the potential is given everywhere on a closed surface the potential function is uniquely determined. (Neumann's problem).

Note that equations (3.2-16, 17) holding in the mass-free space, can be solved by classical methods to provide literal solutions for V, and correspondingly for its gradient, the gravity field. Accordingly, if such a solution is set down, and if the literal coefficients can be evaluated from measurements of (gg) we have found a way to evaluate $\underline{g}$ outside of the body. This method will be examined below.

Consider a gradiometer at sequential points on a trajectory, the gradiometer holding a fixed attitude in X Y Z, that is, inertial space. If the gradiometer is oriented along the X axis, C_1 is unity and C_2 and C_3 are zero. Then

$$(gg) = \frac{\partial^2 V}{\partial x^2} \qquad 3.2\text{-}18$$

If we should employ a fixed orthogonal triad then the gradiometers would measure $\frac{\partial^2 V}{\partial x^2}$, $\frac{\partial^2 V}{\partial y^2}$, $\frac{\partial^2 V}{\partial z^2}$ and a check for behavior would yield a sum of zero. Note that the reference orientation is arbitrary. Therefore any inertially fixed triad would satisfy this condition, regardless of the trajectory, or the coordinate system.

3.2.2 Analytic Determination of the Gravity Field

We have earlier introduced spherical coordinates because of the approximate spherical shape of planetary objects, and the equipotential surfaces are approximately spherical. Accordingly we expect the principal terms in an analytical description of the field to be particularly simple if we use this coordinate system.

On this basis a literal general representation of the potential field in the space outside the gravitating body can be obtained as the sum of terms of the form

$$V_n = r^{-n-1} \, Y_n \qquad 3.2\text{-}19$$

The quantities Y_n are functions of θ and ϕ and n is an integer.

The spherical harmonics Y_n can be written as

$$Y_n = \sum_{m=0}^{m=n} a_{nm} P_n^m \cos m\phi + \sum_{m=0}^{m=n} b_{nm} P_n^m \sin m\phi \qquad 3.2\text{-}20$$

or, using complex variables

$$Y_n = \sum_{m=-n}^{m=+n} C_{nm} P_n^m e^{im\phi} \qquad 3.2\text{-}21$$

Thus the potential field is given by

$$V = \sum_{n=1}^{\infty} \left\{ r^{-n-1} \sum_{m=-n}^{n} C_{nm} P_n^m e^{im\phi} \right\} \qquad 3.2\text{-}22$$

The quantities P_n^m are trigonometric functions of θ alone, and are called associated Legendre functions. When m is zero they are called Legendre, simple, or zonal harmonics. They are most conveniently defined by introducing a new variable t.

$$t = \cos\theta \qquad 3.2\text{-}23$$

$$P_n^0 = \frac{1}{2^n n!} \frac{d^n (t^2-1)^n}{dt^n} \qquad 3.2\text{-}24$$

$$P_n^m = (t^2-1)^{m/2} \frac{d^m P_n^0}{dt^m} \qquad 3.2\text{-}25$$

$$\text{or } P_n^m = \frac{1}{2^n n!} (t^2-1)^{m/2} \left[\frac{d^{n+m} (t^2-1)^n}{dt^{n+m}} \right] \qquad 3.2\text{-}26$$

In the variable t these functions are polynomials, and t ranges over the interval $-1 \leq t \leq 1$.

Table I, taken from Heiskanen, lists some of the lower order terms. The special term P_n^n is called the sectorial harmonic, and the remaining terms for which m is neither zero nor n are called tesseral harmonics.

We shall use this description to find the gravity field. Suppose we have determined $\frac{\partial^2 V}{\partial r^2}$, the radial gravity gradient, at several positions on an orbit.

Differentiating our equation for V we find:

$$gg_r = \frac{\partial^2 V}{\partial r^2} = \sum (n+2)(n+1) r^{-n-3} \sum C_{nm} P_n^m e^{im\phi} \qquad 3.2\text{-}27$$

Thus for several values of r, θ, ϕ we have values for gg_r. We number each data point to distinguish it. Then we compute the value of P_n^m, r^{-n-3} and $e^{im\phi}$ at each point for the values of n and m we consider significant. We get a set of linear equations.

$$\begin{aligned} gg_{r_1} &= \sum C_{nm} k_{nm1} \\ gg_{r_2} &= \sum C_{nm} k_{nm2} \\ &\vdots \\ gg_{r_i} &= \sum C_{nm} k_{nmi} \end{aligned} \qquad 3.2\text{-}28$$

In these equations the k_{nmi} are known, and the gg_r terms are known. The C_{nm} terms are not known. Accordingly, we can solve the equations (invert the matrix) if we have a sufficient number of data points.

Table I. Spherical Harmonic Functions P_n, P_n^m, and P_n^n

Order n	Zonal P_n	m	Tesseral P_n^m	Sectorial P_n^n
0	1			
1	$\cos\theta$	..	....	$\sin\theta$
2	$\frac{3}{2}(\cos^2\theta - \frac{1}{3})$	1	$3\sin\theta\cos\theta$	$3\sin^2\theta$
3	$\frac{5}{2}(\cos^3\theta - \frac{3}{5}\cos\theta)$	1	$\frac{15}{2}\sin\theta(\cos^2\theta - \frac{1}{5})$	$15\sin^3\theta$
3	....	2	$15\sin^2\theta\cos\theta$	
4	$\frac{35}{8}(\cos^4\theta - \frac{6}{7}\cos^2\theta + \frac{3}{35})$	1	$\frac{35}{2}\sin\theta(\cos^3\theta - \frac{3}{7}\cos\theta)$	$105\sin^4\theta$
4	....	2	$\frac{105}{2}\sin^2\theta(\cos^2\theta - \frac{1}{7})$	
4	....	3	$105\sin^3\theta\cos\theta$	
5	$\frac{63}{8}(\cos^5\theta - \frac{10}{9}\cos^3\theta + \frac{5}{21}\cos\theta)$	1	$\frac{315}{8}\sin\theta(\cos^4\theta - \frac{2}{3}\cos^2\theta + \frac{1}{21})$	$945\sin^5\theta$
5	....	2	$\frac{315}{2}\sin^2\theta(\cos^3\theta - \frac{1}{3}\cos\theta)$	
5	....	3	$\frac{945}{2}\sin^3\theta(\cos^2\theta - \frac{1}{9})$	
5	....	4	$945\sin^4\theta\cos\theta$	

Further, once we have found the C_{nm}, we can find the radial component of the gravity vector by the relation

$$g_r = -\frac{\partial V}{\partial r} = \sum (n+1)\, r^{-n-2} \sum C_{nm}\, P_n^m\, e^{\,i m \phi} \qquad 3.2\text{-}29$$

The other two components are similarly available.

If we have more data than necessary, for the number of terms considered significant, we can make use of the redundancy to provide a least squares fit to the data.

By this means we can obtain a global description of the potential function and the gravity vector everywhere outside the planet from which we can get a reference surface or selenoid or geoid.

3.2.3 A Method for Mapping Anomalies

Potential functions, at least of gravity fields, are linear field properties. This is a characteristic of space. Accordingly, if we have a potential field we may partition it into several potential fields, since the sum of potential fields is itself a potential.

This property is useful, for example, in handling the reduction of the gradient data into gravity anomalies on a reference geoid. It has appeared implicity in the data reduction process described above, where we have taken a sum of functions (the spherical harmonics multiplied by appropriate powers of the radius) to represent the total potential.

Each of the terms $r^{-n-1} P_n^m e^{im\phi}$ is a potential function in its own right.

Assume that a satisfactory geoid has been found, for example, by following the method of the previous section. For this discussion we assume further that it accounts for the first few terms of the total potential.

If we compute the gravity gradient associated with this reference potential and subtract it from the data, the residuals constitute the anomalous gravity gradient data.

A characteristic of the series representation we have used is that the fine (anomalous) structure of the field requires many terms in the expansion to provide a satisfactory approximation. However, we do not really want the analytic description for most of the interpretive work of anomaly identification. What is required is a map of the anomalous behavior on the reference surface, in a local manner.

In the following paragraphs we will set down a theory.

We begin by returning to fixed rectangular coordinates and the properties of the potential function in this system.

Let V be a scalar potential function. Then its partial derivatives $\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z}$ which are the components of the gravity field vector are themselves potential functions. They satisfy Laplace's equation. This is demonstrated readily by direct computation.

$$\begin{aligned}\nabla^2 \frac{\partial V}{\partial x} &= \frac{\partial^2}{\partial x^2}\left(\frac{\partial V}{\partial x}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial V}{\partial x}\right) + \frac{\partial^2}{\partial z^2}\left(\frac{\partial V}{\partial x}\right) \\ &= \frac{\partial}{\partial x}\left[\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}\right] \\ &= \frac{\partial}{\partial x}\left[\nabla^2 V\right] = \frac{\partial}{\partial x}[0] \\ &= 0\end{aligned} \qquad 3.2\text{-}30$$

$$\therefore \nabla^2 \frac{\partial V}{\partial x} = \nabla^2 g_x = 0 \qquad 3.2\text{-}31$$

We have used the commutative property of partial differentiation to arrive at the result.

Now, suppose that we had measured the derivative of g_x in the x direction (the gravity gradient) on some plane on which x is constant. That is, we know $\frac{\partial g_x}{\partial x}$ as a function of y and z everywhere on the plane.

We can now use another relation of potential theory to find g_x itself on some other plane. The relation assumes we are concerned with two potential functions U and W, in mass-free space. Then, on a closed surface S,

$$\int_S \left(U\frac{\partial W}{\partial n} - W\frac{\partial U}{\partial n}\right) ds = 0 \qquad 3.2\text{-}32$$

Think of W as the g_x field. Then $\partial W/\partial X$ is the measured gravity gradient if part of the closed surface is the x plane. On this plane we would like $\partial U/\partial n$ (the normal derivative of U) to be zero. We accomplish this by an educated guess. We take U as the function

$$U = \left[(\alpha-x)^2+(\beta-y)^2+(\gamma-z)^2\right]^{-1/2} + \left[(\alpha+x)^2+(\beta-y)^2+(\gamma-z)^2\right]^{-1/2} \qquad 3.2\text{-}33$$

U is now a function of 6 quantities α, β, γ and x, y, z.

At the point $x = \alpha$ $y = \beta$ $z = \gamma$ in the volume above the x = 0 plane, U is singular, and we must somehow exclude it from the space surrounded by S. We do this by imagining a small sphere S' drawn around this point and attach this surface to S; the whole surface over which we integrate becomes S + S'. The normal to S is in the negative x direction, the normal to the sphere is into the sphere.

Since on the x plane $\partial U/\partial x$ is zero, we have

$$\int_S \frac{\partial W}{\partial x}(U)\,dz\,dy + \int_{S'}\left(U\frac{\partial W}{\partial n} - W\frac{\partial U}{\partial n}\right)ds' = 0 \qquad 3.2\text{-}34$$

Over the small sphere, U is constant in value and $\int \frac{\partial W}{\partial n}\,ds'$ is zero. On the other hand, $\int_{S'} W\frac{\partial U}{\partial n}\,ds'$ over the sphere S' is essentially equal to the constant value of W at the point (α, β, γ) and $\int \frac{\partial U}{\partial n}\,ds'$ is 4π .

Therefore, the final equation is

$$4\pi W(\alpha,\beta,\gamma) = -\int \frac{\partial W}{\partial x}\,U\,dz\,dy \qquad 3.2\text{-}35$$

Now taking g_x for W we have:

$$4\pi g_x = -\int \left\{ \begin{matrix} \left[(\alpha-x)^2+(\beta-y)^2+(\gamma-z)^2\right]^{-1/2} \\ +\left[(\alpha+x)^2+(\beta-y)^2+(\gamma-z)^2\right]^{-1/2} \end{matrix} \right\} (g g_x\,dz\,dy) \qquad 3.2\text{-}36$$

Note that if, by definition, we take the reference plane as x = 0 the result becomes more compact.

$$g_x = -\frac{1}{2\pi}\int \frac{g g_x(0,y,z)\,dz\,dy}{\left[\alpha^2+(\beta-y)^2+(\gamma-z)^2\right]^{1/2}} \qquad 3.2\text{-}37$$

Practically, we do not know $\mathscr{G}\mathscr{G}x$ continuously over the plane. At best we know it in terms of samples; and a discrete numerical integration process is taken to approximate the above operations.

This process is exactly that evaluated by Paterson in Geophysics, August '61. In the article Paterson deals with the problem of converting an airborne survey of gravity gradient onto the nearby earth as gravity force, and proposes a numerical technique for making the computations.

This appears to be a reasonable approach to mapping the anomaly field. However further work is needed, since the expected satellite observations will not be on a plane surface. It is suggested that equivalent theory for a spherical coordinate system be developed along with appropriate numerical methods. As an alternative the spherical harmonic representation can always be used.

3.2.4 An Estimate of Required Instrument Sensitivity

We now have a data reduction procedure from which we can develop a reference potential surface (geoid) and an approach to locally mapping the residual gravity field, the anomaly field, onto the geoid.

In order further to evaluate the instrumental problem, we must develop some estimates of the necessary sensitivity of the gradiometer. We would like to relate this to the more familiar gravity anomaly observed on the planetary surface. To do this we have employed a simple anomaly model, consisting of a large central planetary mass M, and a small mass m located near the surface of the planet. For an orbit at 100 miles above the planet we have calculated the radial gravity gradient above the anomaly and plotted this against the surface gravity anomaly, for various sites of the anomaly below the surface.

For this model we have

$$V = -\left(\frac{GM}{R} + \frac{Gm}{R - r_o}\right) \qquad 3.2\text{-}38$$

M = planetary mass; for the earth around 10^{21} tons; see table II for other planets.

	Mercury	Venus	Earth	Moon	Mars	Jupiter	Saturn	Uranus	Neptune	Pluto
Distance (minimum	28.6	66.8	91.4	91.2	128.5	460.7	837.4	1700.9	2772.65	2760.15
from Sun (mean	36.0	67.3	93.0	93.0	141.7	483.9	887.1	1784.8	2796.7	3675.3
miles x 10^6 (maximum	43.4	67.8	94.6	94.8	154.9	507.1	936.8	1868.7	2820.75	4590.45
Length of time to complete one revolution about Sun	87.97 days	224.7 days	365.26 days		687.0 days	11.86 years	29.46 years	84.02 years	164.8 years	247.7 years
Length of day	88 days	30 days	1 day	27.3217 days	1.026 days	9 hrs. 55 min.	10 hrs. 38 min.	10 hrs. 42 min.	15 hrs. 48 min.	Unknown
Orbital (speed, mi/sec)	29.76	21.75	18.517	0.6	14.975	8.45	5.965	4.225	3.355	2.98
Orbital (eccentricity)	0.206	0.007	0.017	0.055	0.093	0.048	0.056	0.047	0.0086	0.249
Orbital (inclination)	7°0'	3°24'	-	5°8'	1°51'	1°18'	2°29'	0°46'	1°47'	17°19'
Volume*	0.06	0.92	1	0.02	0.15	1318	736	64	39	0.094(?)
Mass*	0.04	0.82	1	0.0123	0.11	318.3	95.3	14.7	17.3	Unknown
Density*	0.69	0.89	1	0.606	0.70	0.24	0.13	0.23	0.29	Unknown
Surface Gravity*	0.27	0.86	1	0.16	0.37	2.64	1.17	0.92	1.44	Unknown
Escape velocity mi/sec	2.237	6.338	6.860	1.50	3.107	37.28	22.37	13.05	14.29	Unknown
Mean diameter, miles	3107	7705	7917.5	2160	4269	86,840	71,520	31,690	31,070	3600(?)
Oblateness	0.000	0.000	0.0034	0.000	0.0052	0.065	0.105	0.071	0.022	Unknown
Inclination of axis	Unknown	0°(?)	23°27'	6°30'	25°12'	3°7'	26°45'	98°	29°	Unknown
Albedo	0.07	0.59	0.29	0.07	0.15	0.44	0.42	0.45	0.52	0.04
Moons (known)	0	0	1	-	2	12	9	5	2	0
Distance from (max	136	160	-	0.252,948	236	600	1025	1950	2900	4650
Earth x 10^6 (min	50	26	-	0.221,593	21	367	745	1615	2700	2700

*Ratio to Earth

Table II. Physical and Positional Properties of the Planets /10/

m = excess mass associated with the anomaly. For visualization note that one cubic mile at mean earth density weighs 10^{11} tons.

R = altitude of orbit above planet center.

r_o = anomaly site above planetary center.

V = potential along line joining the two masses, beyond the surface.

h = $R - r_o$; altitude of orbit above the anomaly.

h_o = altitude of surface above the anomaly.

$$\text{Radial gravity} = g_r = -\frac{GM}{R^2} - \frac{Gm}{(R-r_o)^2} \qquad 3.2\text{-}39$$

$$= -\frac{GM}{R^2}\left[1 + \frac{m}{M}\left(\frac{R}{h}\right)^2\right] \qquad 3.2\text{-}40$$

$$\text{Radial gravity gradient} = gg_r = 2\frac{GM}{R^3} + \frac{2Gm}{(R-r_o)^3} \qquad 3.2\text{-}41$$

$$= 2\frac{GM}{R^3}\left[1 + \frac{m}{M}\left(\frac{R}{h}\right)^3\right] \qquad 3.2\text{-}42$$

In these equations if R is very closely the planetary radius then GM/R^2 is the normal surface gravity and $2GM/R^3$ is the normal surface gravity gradient. Further $\frac{m}{M}\left(\frac{R}{h}\right)^2$ is then the fraction of normal gravity due to the anomaly and $\frac{m}{M}\left(\frac{R}{h}\right)^3$ is the fraction of normal gravity gradient due to the anomaly. This latter defines the necessary effective sensitivity of the instrument. In the sense used here sensitivity is more often called "dynamic range", the ratio of the least detectible measurement to the measurement range. We define as the discrimination level the value of gravity gradient corresponding to S. Thus if the range of measurement is 10^{-7} g/foot and S is 10^{-5} the discrimination level is 10^{-12}g/foot.

In accord with this philosophy we now can derive a relation between the surface gravity anomaly and the corresponding change in gravity gradient at altitude. At the surface, the ratio anomalous to normal gravity is:

$$\frac{\delta g_{ro}}{g_{ro}} = \frac{m}{M}\left(\frac{R_o}{h_o}\right)^2 \qquad 3.2\text{-}43$$

At altitude the ratio of anomalous to normal gradient is:

$$S = \frac{\delta(gg)_r}{(gg)_r} = \frac{m}{M}\left(\frac{R}{h}\right)^3 = \frac{m}{M}\left(\frac{R_o}{h_o}\right)^2\left(\frac{h_o}{R_o}\right)^2\left(\frac{R}{h}\right)^2 \qquad 3.2\text{-}44$$

$$\approx \frac{\delta g_{ro}}{g_{ro}}\,\frac{h_o^2}{h^3}\,R_o \qquad 3.2\text{-}45$$

On earth g_{r_o} is 1 gravity $\cong$ 10^6 milligals. For an anomalous body or mass one mile below surface, producing a one milligal gravity deviation at the surface and for a 100 mile altitude orbit S is 4×10^{-9}.

S has been calculated for earth and lunar orbits, taking lunar gravity as 160, 000 milligals, and lunar radius as 1000 miles. The important results are tabulated below.

	Earth		Moon	
δg - milligals	h_o = 1 mile	5 miles	1 mile	5 miles
1	4×10^{-9}	10^{-7}	6×10^{-8}	1.5×10^{-6}
10	4×10^{-8}	10^{-6}	6×10^{-7}	1.5×10^{-5}
100	4×10^{-7}	10^{-5}	6×10^{-6}	1.5×10^{-4}
1000	4×10^{-6}	10^{-4}	6×10^{-5}	1.5×10^{-3}

Values of S

Table III

It appears that for a system with a sensitivity S of 10^{-6} the system would indicate lunar anomalies of one to 10 milligals at the surface, arising from anomalous structure one to five miles below the nominal planetary surface. At a level of 10^{-5} the surface anomalies would correspond to 10 to 100 milligals. This latter is considered to be a reasonable level to start the investigation of the lunar gravity field from an orbiting satellite. The discrimination level for an S of 10^{-5}, in a 10^{-7} g/foot field is simply 10^{-12}g/foot, or 0.03 E̊ (E ötvos units). There is no question but that this is an extremely delicate measurement, and potentially vulnerable to a vast variety of normally hidden noises. For example the potential field of a one kilogram mass of mean earth density material, at a distance of one foot produces a gravity gradient 500 times as great as that we seek to measure. A shifting of this mass by 0.1 mm can result in a signal change approximating the discrimination level. But this can be interpreted as a constraint on laboratory design. For example the gradiometer could be put on a boom so it is remote from the major mass. Then, mass motions within the satellite will not affect the gradiometer output.

The instrument must also be protected from developing thermal gradients along its length since this will result in a non-symmetrical configuration. If the end string lengths differ due to unequal expansion a biased output will result. Such a disturbance must be kept below the discrimination level.

Periodic, automatic calibrations of the instrument will allow for the compensation of temporal shifts in the instrument scale and bias. These shifts are unavoidable in any delicate instrument over a long period of time. It is reasonable to expect, however, that such shifts can be detected to the discrimination level and compensated systematically.

It is also important to note that the discrimination level is reduced by virtue of the repetitious data collection inherent in the orbiting measuring system. Random noise will be averaged to a low level by statistical techniques included in the data processing.

These problems of resolution, noise, thermal effects and statistical data processing have already been solved for the Arma Vibrating String Accelerometer.

Considering the foregoing, the dynamic range of the vibrating string gradiometer can be set so that the discrimination level, corresponding to lunar surface anomalies of 10 - 100 milligals will be achieved while the maximum gradiometer output will be at the level of the normal lunar gradient.

3.2.5 Effects of the Sun and the Earth

Two possibly important planetary ambient sources should be noted; those due to the earth and the sun. In computation of the lunar field these can and should be subtracted from the gradient data before processing.

The gradient field of the earth in the vicinity of the moon is approximately 4×10^{-13} g/foot and varies about 50%, from about 8×10^{-13} to 4×10^{-13}.

The gradient field of the sun is approximately 2×10^{-15} g/foot in the vicinity of the moon. This appears ignorable because it is two magnitudes less than the discrimination level we have been discussing.

Note that the attitude of the gradiometer relative to the radius to the earth must be entered into the corrective computation. This can be determined readily from the attitude of the sensor with respect to the moon.

3.2.6 Summary

In this section we have presented the basic gravity gradient theory and a method for converting gradient observations into gravity field. An analysis of the anomaly field of a point anomaly indicates that a system sensitivity of one part in 10^5 is desirable for anomaly mapping. It is further noted that at this level account must be taken of the gradient field of the earth. This is a simple computation.

3.3 Indications of a Gradiometer on a Rotating Platform

A useful physical picture of the gradient phenomenon is obtained if one imagines two masses fastened together by an ideal, short, infinitely slender rod and placed in the field of a gravitating body. If the rod is oriented radially, the gravitational force on the inner mass is greater than that on the outer and the rod is in tension by an amount equal to the radial gradient of the gravity force field multiplied by the length of the rod. If the rod is now held horizontally, the convergence of the acceleration forces toward the center of the gravitating body results in a force along the rod tending to bring the masses together, a compression. It is clear that this is a measure of the curvature of the gravity field at the point. Numerically, for the simple force field assumed, the compression is one half the tension. As the rod is turned, it passes from tension to compression and back again in one half turn.

Now visualize the same experiment performed on an orbiting satellite. To be specific, assume a circular orbit. If the rod - the gradiometer - is held fixed in inertial space, its indications will be independent of the orientation of the satellite, but will vary with time since the orientation of the rod with respect to the radius to the planet varies cyclically in orbit. Thus, the rod will sample the gravity gradient in various directions, as it did in the non-orbiting experiement described above. If the gradiometer should be continuously controlled to lie along the radius, then it will measure the radial gradient plus the square of the orbital angular velocity in inertial space. Because of the simple orbital geometry this angular term is exactly one half the gradient term and has the sense of tension. A gradiometer in the orbital plane lying normal to the radius would read zero (its compression is reduced by the equal tension due to the angular velocity). A gradiometer perpendicular to the orbital plane would read the usual compressive force due to gravity alone. Accordingly with the addition of rate gyroscopes, the three components of gravity gradient can be obtained separately. It can be shown that the sum of the three quantities measured by a triad of gradiometers is twice the square of the angular velocity of the gradiometers in inertial space.

If the gradiometer is on a rotating platform its indications are composed of two parts; the normal gravity gradient existing at the point in space, in the direction of the sensing axis plus a component due to angular velocity of the platform. If we follow the definition of gradient employed earlier we find that the term due to angular velocity is $W_s{}^2$ where $|W_s|$ is the magnitude of the component of angular velocity normal to the sensing axis. We compute W_s from the vector relation:

$$\underline{W_s} = 1_s \times \underline{W} \qquad 3.3\text{-}1$$

The sum of the squares of the three W_s terms is equal to $2W^2$. Accordingly, the sum of the 3 gradiometer outputs on a rotating platform corresponds to:

$$gg_1 + gg_2 + gg_3 = 2W^2 - \nabla^2 V.$$ In mass-free space $\nabla^2 V$ is zero.

If a gradiometer orthogonal triad is constrained so that one gradiometer tracks the geocentric radius and one tracks the local horizontal along the trajectory then the angular velocity of the platform is determined by the orbital parameters. For a simple central force field and a circular orbit the results are especially simple and useful for making system estimates. Denoting indications by I we have:

$$I_r = \frac{3}{2} gg_r \qquad \text{(radial)} \qquad 3.3\text{-}2$$

$$I_\theta = 0 \qquad \text{(along trajectory)} \qquad 3.3\text{-}3$$

$$I_\phi = -\frac{1}{2} gg_r \qquad \text{(normal to orbit plane)} \qquad 3.3\text{-}4$$

and

$$\sum I = gg_r = 2W^2 \qquad 3.3\text{-}5$$

The square of the orbital angular velocity is one half of the radial gravity gradient.

3.4 Instrumentation for Gradiometry

Several instruments which may serve as gradiometers are described below. Each instrument measures the difference between the forces affecting two (or more) proof masses when these masses are separated along the sensitive axis. The different proof masses are self evident in the discussion of matched accelerometers, the vibrating string gradiometer and the torsional gradiometer. For the accelerometer-on-a-boom and the San Marco experiment the orbiting satellite itself is one of the proof masses. The gas diffusion equipment is made up of innumerable proof masses whose relative motion may be analyzed statistically.

The gradiometer which appears most suitable for the above described measurements, is an extension of equipment already developed at the Arma Division. This is the vibrating string accelerometer which has been used so successfully in inertial guidance systems and which is also the basis for ruggedized thrust instrumentation recently delivered to NASA-Lewis Research Center. A low range accelerometer for ion propulsion measurements is now under evaluation for RTD.

3.4.1 Basic Theory of the Vibrating String Instrumentation

Single String, Single Mass System

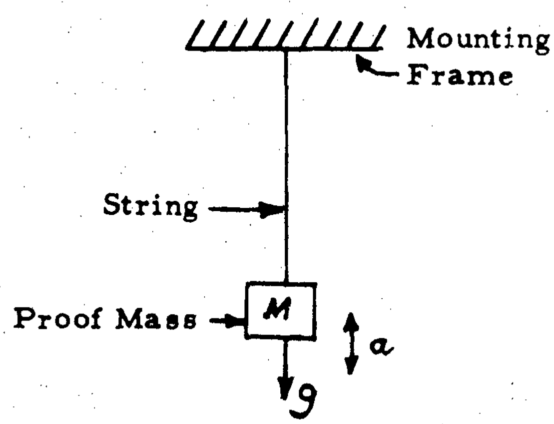
A string under tension will vibrate when excited. The frequency of vibration is a function of the dimensions of the string, the string density and the tension of the string. The frequency of vibration (f) is directly proportional to the square root of the tension of the string (T) and inversely proportional to the square root of the mass of the string (m) and its length (L).

$$f = 1/2 \sqrt{\frac{T}{mL}} \qquad 3.4\text{-}1$$

For a string supporting a mass M the tension on the string is equal to the initial tension (T_o), due to the mass attraction of the gravitational field, plus any other force acting on the sensitive mass.

$$T = T_o + Ma \qquad 3.4\text{-}2$$

In this case T_o is Mg and the inertial reaction force is Ma



From this it can be seen that for any particular string and mass, the frequency of vibration can be varied by changing the acceleration applied to the sensitive mass. If we consider the acceleration as variable, the frequency-tension relationship can be expressed as a Taylor Series expansion about zero acceleration.

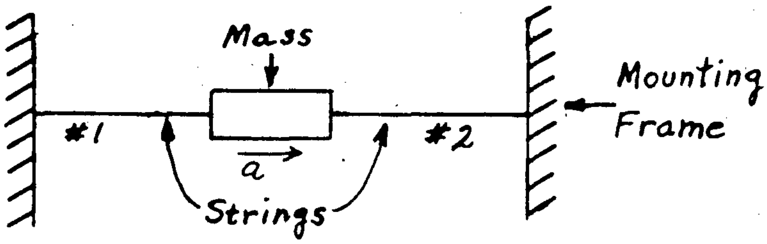
$$f = K(T_o + M_a)^{1/2} = F(o) + \frac{F'(o)a}{1!} + \frac{F''(o)a^2}{2!} + \frac{F'''(o)a^3}{3!} + \cdots \qquad 3.4\text{-}3$$

$$\text{or } f = K_o + K_1 a + K_2 a^2 + K_3 a^3 + \cdots$$

When **a**, the applied acceleration, is zero, $f = K_0$ which is called the bias frequency. With applied acceleration, there is a linear change in frequency plus other non-linear terms.

Double String, Single Mass Systems

The even order non-linear terms in the Taylor Series expansion can be reduced in magnitude by using two strings along one axis with the mass between them.



In this representation, the coefficients approach zero as the order of the term approaches infinity and the series converges. If we express the frequency of each string as a series, we have:

$$f_1 = K_{01} + K_{11}a + K_{21}a^2 + K_{31}a^3 + \cdots \quad \text{For } T = T_0 + Ma/2$$

$$f_2 = K_{02} - K_{12}a + K_{22}a^2 - K_{32}a^3 + \cdots \quad \text{For } T = T_0 - Ma/2$$

3.4-4

Since the K_1 terms are dominant, the frequency of one string will increase while the other will decrease. Taking the difference of these two frequencies we have:

$$f_1 - f_2 = (K_{01} - K_{02}) + (K_{11} + K_{12})a + (K_{21} - K_{22})a^2 + (K_{31} + K_{32})a^3 + \cdots$$

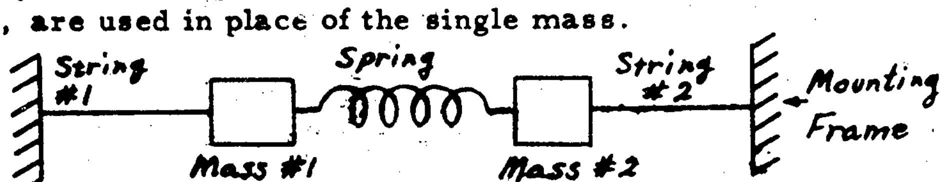
3.4-5

If the two strings are closely matched, all the even terms in the difference frequency equation become negligibly small and only the odd terms remain. Since the series converges rapidly, the only non-linear term of any significance is the K_3 term but this can be compensated if a is known approximately. This method is useful in achieving a substantially linear device. However, because the two strings are attached to a single mass, there may be coupling between the strings when they are near the same frequency.

The ameliorative is shown below.

Double String, Double Mass System

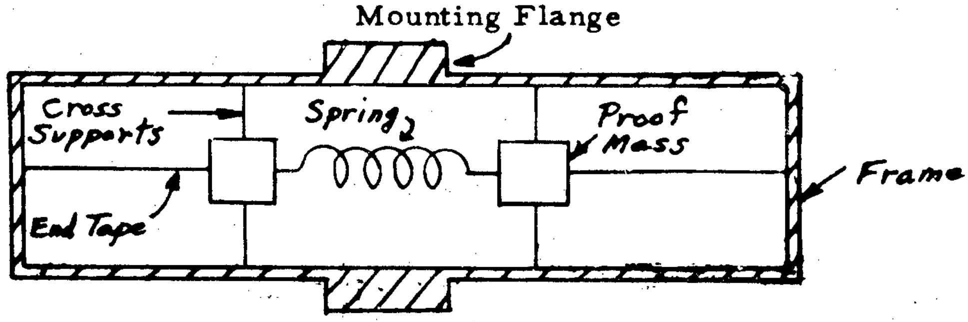
Two modifications are made on the double string, single mass system to minimize the effect of coupling between the strings. Two separate but equal sensing masses, joined with a relatively soft spring, are used in place of the single mass.



This reduces the transmission of energy from one string to the other while not causing any deterioration of the system. The second modification is the substitution of flat tapes for the strings. These tapes are oriented to vibrate in mutaully orthogonal planes and thus reduce the tendency to vibrate sympathetically.

Cross Centering System

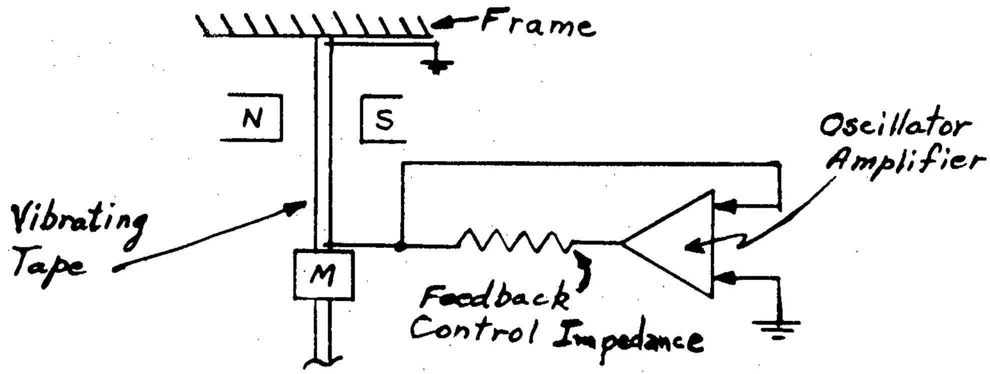
In order to reduce the effect of accelerations not directed along the axis of the tapes, it becomes necessary to restrain the masses. This is accomplished by cross supports placed perpendicular to the sensitive axis. The following diagram shows the basic Arma Vibrating String Accelerometer.



Electrical Considerations

Permanent magnets provide a means for laterally vibrating the tapes at their natural frequencies. If a current is passed through the tape in a magnetic field, a force is produced. This force tends to move the tape. If the current is reversed, the force is in the opposite direction and the tape is moved in that direction.

In this manner, the tapes can be vibrated. AC voltages are induced in the tapes by their motion in the magnetic fields. These voltages are regenerated through stable high gain amplifiers and returned to the tapes to provide the energy necessary to sustain oscillation. The tape is therefore part of a tank circuit which oscillates at its natural frequency.



The dynamics of the vibrating tapes in the magnetic fields provided by the permanent magnets have properties which are completely analogous to those of a parallel resonant electrical circuit. This feature permits the design of an appropriately controlled feedback amplifier to maintain a constant amplitude of oscillation.

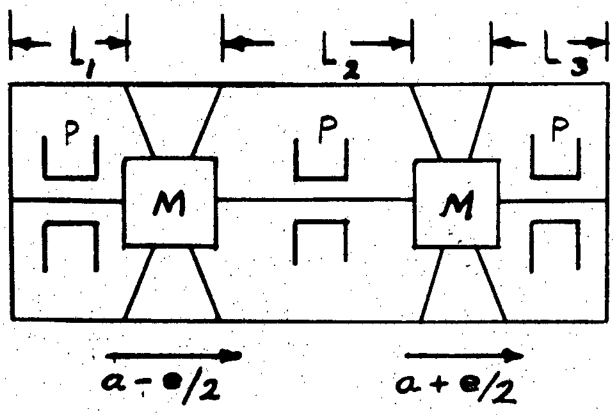
Accelerometer Operation

The two proof masses are free to move along the sensitive axis and are restrained in motion along the cross sensitive axes by the suspension system. An acceleration along the axis of the end tapes causes the masses to increase the tension of one tape and decrease the tension of the other tape by approximately an equal amount. Since the frequency of tape oscillation is proportional to the square root of tension, the instantaneous frequency difference of the tape oscillations is a measure of the acceleration.

Because of the square root relationship, the frequency difference is only approximately proportional to acceleration. The small effects introduced by higher order terms in an actual accelerometer can be compensated.

Vibrating String Gradiometer

It is a simple step from the Vibrating String Accelerometer to the Vibrating String Gradiometer. A third vibrating tape is inserted between the two proof masses and its tension variation will be a function of the difference in the forces affecting the two masses, in particular the gravity gradient. A simple sketch of the gradiometer is shown below.



L = string length

M = proof mass

p = pole magnets

a = applied acceleration

e = incremental acceleration
= L_2 x gravity gradient

Simplified sketch of Proposed Gradiometer

The gradiometer consists of two proof masses M held together along the sensitive axis by a long metal tape (string) L_2 and this combination held to the frame by two short end strings L_1 and L_3. Each string is caused to vibrate by passing an alternating current through it as it lies in a permanent magnetic field. The frequency of that current is equal to the natural mechanical frequency of the string and is established and maintained electrically through a feedback loop.

The natural frequency of the string is a function of its length, mass and the tension to which it is subjected. When the unit is held parallel to the local vertical, the lower proof mass is subjected to a greater gravitational force than the upper proof mass. Hence, the lower end string tension is reduced, the upper end string is increased, while the center string is increased by the difference, e, in the gravitational forces on the two proof masses. This difference (gradient times length) is obtained by processing the frequency data from the three strings. A brief mathematical description of the relationship between the instrument parameters, the differential (gradient) acceleration, e, and the ambient acceleration, a, follows.

$$f = \frac{1}{2L}\sqrt{\frac{T}{\rho}} \qquad 3.4\text{-}6$$

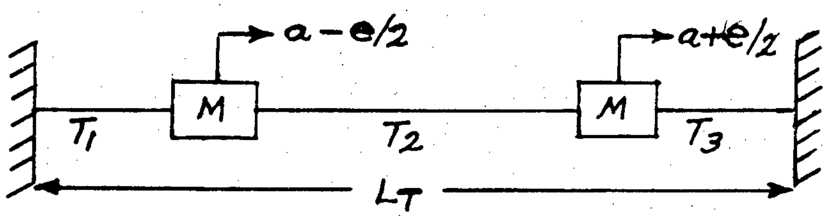
where f = natural mechanical frequency

L = length between clamps

T = tension (force)

ρ = linear mass density (mass per unit length)

We will apply this to the two mass, three string system shown in the figure below. Slightly different accelerations are applied to each proof mass. The tensions in each string, which were each T_o initially, change in such a manner as to balance the externally applied forces without changing the overall length of the system.



Therefore

$$T_1 = M(a-e/2)+T_2 \qquad 3.4\text{-}7$$

$$T_2 = M(a+e/2)+T_3 \qquad 3.4\text{-}8$$

$$L_1 + L_2 + L_3 = L_T \ ; \quad \Sigma \Delta L_i = 0 \qquad 3.4\text{-}9$$

These changes in tension are the new values of tension less T_o. The changes in length are directly related to the compliance, S, of the strings, the tension change and the original length.

$$\Delta L_i = S \, \Delta T_i \, L_i \qquad 3.4\text{-}10$$

Then

$$\Delta T_1 = 2Ma + \Delta T_3 \qquad 3.4\text{-}11$$

$$\Delta T_2 = Ma + Me/2 + \Delta T_3 \qquad 3.4\text{-}12$$

from equations (3.4-7, 8) and

$$L_1 \Delta T_1 + L_2 \Delta T_2 + L_3 \Delta T_3 = 0 \qquad 3.4\text{-}13$$

from equation (3.4-10)

Solving these last three equations for the string tension changes we find:

$$\Delta T_1 = -\frac{L_2}{L_T}\frac{Me}{2} - Ma \qquad 3.4\text{-}14$$

$$\Delta T_2 = \frac{L_1}{L_T} Me \qquad 3.4\text{-}15$$

$$\Delta T_3 = -\frac{L_2}{L_T}\frac{Me}{2} + Ma \qquad 3.4\text{-}16$$

The primary quantity of interest is e, the gravity gradient multiplied by the length of the second string, L_2. Note that a is the measure of the acceleration at the center of the instrument. If the instrument were on a satellite in orbit, a would be the gravity gradient at orbit altitude multiplied by the distance between the satellite center of mass and the instrument. The difference in end string tensions give the ambient acceleration while the tension of the second string gives the gravity gradient.

The string frequencies are related to the instrument parameters and acceleration environment through the following equations:

$$f_1^2 = \frac{1}{4L_1^2\rho}\left(T_0 - \frac{L_2}{L_T}\frac{Me}{2} - Ma\right) \qquad 3.4\text{-}17$$

$$f_2^2 = \frac{1}{4L_2^2\rho}\left(T_0 + \frac{L_1}{L_T} Me\right) \qquad 3.4\text{-}18$$

$$f_3^2 = \frac{1}{4L_1^2\rho}\left(T_0 - \frac{L_2}{L_T}\frac{Me}{2} + Ma\right) \qquad 3.4\text{-}19$$

These assume that the end strings are of equal length and that ρ, the linear mass density is the same for each string. Further, through the choice of string material and configuration the percentage change in length is made negligible compared to the percentage change in tension from their respective nominal values. Then the frequency depends only on the change in tension and not on the associated change in length.

If the frequency of the middle string is defined as f_0, when e is zero, the gravity gradient is given by:

$$(gg) = \frac{e}{L_2} = \left\{ f_2^{\,2} - f_0^{\,2} \right\} \frac{4\rho}{M} \quad \frac{L_2 L_T}{L_1} \qquad 3.4\text{-}20$$

The ambient acceleration is given by:

$$a = \left\{ f_3^{\,2} - f_1^{\,2} \right\} \frac{2 L_1^{\,2} \rho}{M} \qquad 3.4\text{-}21$$

In order to extract the gravity gradient from equation (3.4-20) it is necessary to know the nominal (bias) frequency f_o corresponding to the initial tension T_o. This frequency will be set when the instrument is manufactured and will be monitored as part of the periodic calibration procedure. It is possible also to monitor this value as an indication of instrument stability during its normal operation.

For this data processing function consider the relation:

$$f_1^{\,2} + f_3^{\,2} + \left(\frac{L_2}{L_1}\right)^3 f_2^{\,2} = T_o \ \frac{L_T}{4 L_1^{\,3} \rho} \qquad 3.4\text{-}21$$

This equation provides a measure of T_o, and can be used to test for off-nominal behavior.

In summary, it has been shown that by appropriate combinations of the three output frequencies, the gravity gradient and ambient acceleration can be obtained simultaneously. In addition, a monitoring technique is available to indicate whether or not the instrument is in need of re-calibration.

If the environment requires it, a thermal control may be added to the gradiometer. Also, with regard to operating conditions, if it is necessary to store the data for future processing, it should be noted that the frequency modulated nature of the output signals lend themselves readily to tape recording without intermediate coding steps.

3.4.2 Accelerometers for Gradiometry

A pair of high quality accelerometers, separated by an accurately known distance and having their outputs in opposition, give the measure of the gravity gradient over that distance (taking orientation relative to the radius vector into account). It may be possible, from an orbiting satellite, to obtain separations between the accelerometers of the order of hundreds of feet.

Suppose a 100 foot separation. Then a 10^{-5}g accelerometer could be used to measure 10^{-7}g/foot of gravity gradient. The accelerometer range is obviously shifted into a more desirable region. However, this does not improve the dynamic range, S, of the instrument, since the threshold is generally proportional to the maximum reading for a given instrument design and environment.

In addition the designer is now faced with the problem of matching performance and stability of two instruments over their entire operating range, in the face of different thermal environments. The separation must be continuously measured to high accuracy, and the operational problems of erecting and holding a long boom in different attitudes is unwarrantedly difficult.

3.4.3 Gradiometer on a Boom

One can obtain the geometric measure required for the field strength and gravitational acceleration determination without referring to the gravitating body itself. Consider a gradiometer which can be mounted on an extendable boom so that measurements can be taken both inside that satellite and possibly 100-300 feet away in the direction of the local vertical. This yields the gradient of the gradient, a very small difference in output. Nevertheless, it is theoretically possible to determine the gravity field strength from these two measurements. The outputs are given by

$$\Delta g_1 = K \frac{\mu}{r_1^3} \; ; \; \mu = GM \qquad 3.4\text{-}22$$

$$\Delta g_2 = K \frac{\mu}{r_2^3} \qquad 3.4\text{-}23$$

The difference between these two outputs is

$$\delta \Delta g = K\mu \left(\frac{1}{r_1^3} - \frac{1}{r_2^3} \right) \qquad 3.4\text{-}24$$

This may be approximated closely by

$$\delta \Delta g \approx 3 K \mu \frac{\delta r}{r_1^4} \qquad 3.4\text{-}25$$

where K = Gradiometer scale factor

r_1 = Nominal distance to the center of the gravitational field

δ r = Boom length

Taking the sum of the two gradiometer outputs yields

$$\Sigma \Delta g \approx \frac{K\mu}{r_1^3}\left(1 + \frac{1}{1 - \frac{3\delta r}{r_1}}\right) \approx 2\frac{K\mu}{r_1^3} \qquad 3.4\text{-}26$$

Then eliminating r_1 from equations (3.4-25 and 3.4-26) and solving for μ we get:

$$\mu = \frac{27(\Sigma \Delta g)^4}{16 K}\left(\frac{\delta r}{\delta \Delta g}\right)^3 \qquad 3.4\text{-}27$$

All parameters are theoretically available. Note that it is now possible to obtain r_1 explicitly from either of equations (3.4-25 or 3.4-26) which means that the satellite orbit parameters can be obtained. The trouble with this method is the fact that $\delta\Delta$ g is so small that small errors in Δ g give large errors in $\delta\Delta$ g and any errors in obtaining that latter value are further magnified by the sensitivity factor 3 which is related to the exponent of $\delta\Delta$ g.

3.4.4 Torsional Gradiometer

An instrument which relies on the winding or unwinding of a spring as compression or tension is applied along its length is presently being developed by the Texas Instrument Corporation. In the vertical orientation tension arises as the difference in gravitational attraction of two similar proof masses mounted at either end of the spring. The consequent angular rotation of one mass (a flat plate) relative to the other is the measure of the gradient.

As in the case of any gradiometer, the general advantages of this instrument is its insensitivity to externally applied acceleration, since the acceleration acts equally on each proof mass. The uncertain future of this instrument as evaluated by Texas Instrument arises from its early state of development. Their problems include size, weight, fragility and stability among others.

3.4.5 San Marco System

The San Marco experiment concerns the observation of the motion of an unsupported object within the confines of an orbiting body. The motion of this object relative to the body is due to the fact that it is in a different orbit from that of the body. The orbit of the latter can be defined in terms of the motion of its center of gravity relative to the gravity field. Since the test object is not located at the center of gravity its orbit differs from that of the satellite. Thus, the object will move within the satellite, as a free object in space until it strikes an obstruction within the satellite. By observing the motion of the object or by observing the magnitude and direction of restraints required to hold it fixed relative to the satellite, the gravity gradient can be obtained.

Consider a test mass attached to a satellite in orbit in a central force field. Let the test object be at some radial distance from the satellite c.g. Since it is attached, its initial velocity components are identical with those of the satellite (assume no rotation of the satellite about its own c.g.). It should be noted that if the mass were sited in the satellite on the trajectory of the c.g. no restraint would be necessary since both would then be on the same orbital path.

When the test object is released, and there is a radial displacement, the test object is not in the same orbit as the satellite and will begin to move with respect to the satellite. It is this motion which is the measure for the San Marco experiment and its interpretation is shown in the following analysis. The analysis takes into account the effect of releasing the test body with a non-zero velocity relative to the satellite.

The equations of motion of the test object in a coordinate system whose origin is at the satellite c. g., rotating at the angular velocity w of the satellite are:

$$\ddot{x} - 2w\dot{y} - y\dot{w} - xw^2 = gx/r \qquad 3.4\text{-}28a$$

$$\ddot{y} + 2w\dot{x} + x\dot{w} - yw^2 = 2gy/r \qquad 3.4\text{-}28b$$

Here, the forces producing relative motion are those due to the displacement from the c. g. multiplied by the gravity gradient at that point. The horizontal gradient is one of compression due to the curvature of the equipotential and the radial gradient is one of tension due to the di stance squared effect of gravitational force.

Assuming a circular orbit, we have two additional relations:

$$\dot{w} = 0 \qquad 3.4\text{-}29a$$

$$g = -w^2r \qquad 3.4\text{-}29b$$

so that the equations of motion become

$$\ddot{x} - 2w\dot{y} = 0 \qquad 3.4\text{-}30a$$

$$\ddot{y} + 2w\dot{x} = -3yw^2 \qquad 3.4\text{-}30b$$

The first of these equations can be integrated using, as initial conditions, $x = x_o$, $y = y_o$, $\dot{x} = u_o$, $\dot{y} = v_o$ at time zero.

$$\dot{x} - 2wy = u_o - 2wy_o \qquad 3.4\text{-}31$$

Solving for $\dot{x}$ and substituting in (3. 4-30b)

$$\ddot{y} + w^2y = 2w(2wy_o - u_o) \qquad 3.4\text{-}32$$

This undamped second order system can be solved by assuming the answer is in the form

$$y = A\cos wt + B\sin wt - 2u_o/w + 4y_o \qquad 3.4\text{-}33a$$

and solving for the coefficients in terms of the initial conditions.

Substituting this result in the equation (3.4-31) and integrating gives the horizontal displacement versus time.

$$x = x_0 + (4\,u_0/w - 6y_0)\sin wt + 2(v_0/w)(1-\cos wt) - 3t(u_0 - 2wy_0) \qquad 3.4\text{-}33b$$

It should be noted that this last result is divergent when, initially, there is either a vertical displacement or horizontal velocity.

For the test where only a vertical displacement exists at time zero the resulting motion, relative to the satellite cg as the origin of coordinates, is given by

$$x = 6\,y_0 \left\{ -\sin wt + wt \right\} \qquad 3.4\text{-}34a$$

$$y = y_0 \left\{ 1 + 3\,(1 - \cos wt) \right\} \qquad 3.4\text{-}34b$$

which is a spiraling motion in the x direction and eventually going beyond the bounds of the physical area of the test.

A special experiment could be made such that

$$u_0 = 2\,wy_0 \qquad 3.4\text{-}35$$

In this case, the divergent term in x would be zero and the motion of the test object would be bounded. The resulting elliptic motion is given by

$$x = -2\,y_0 \sin wt \qquad 3.4\text{-}36a$$

$$y = -y_0 \cos wt \qquad 3.4\text{-}36b$$

In either case, gravity gradient is determined from the angular velocity of the test object about the satellite c.g.

Problems associated with the San Marco experiment include:

a) the problem of determining the center of gravity of the satellite

b) the effects of the change in satellite cg due to movement of men and/or expendables in the satellite.

c) the effects of solar pressure, micrometeorites, etc. on the satellite which is one of the proof masses of the experiment.

3.4.6 Gas Diffusion Technique for the Determination of Gravity Gradient

A column of gas, carried on board the orbiting satellite will take on a density gradient related to the gravity gradient. Theoretically, a measure of that density variation, probably by optical means, will yield the desired data on the gravity gradient.

The force exerted by the gas is

$$\bar{F} = \frac{1}{\rho} \nabla p \qquad 3.4\text{-}37$$

where ρ (density) and p (pressure) are related by the equation of state for an ideal gas.

$$p = \rho \frac{R}{M} T \qquad 3.4\text{-}38$$

where R is the universal gas constant, M, the average molecular mass and T is the absolute temperature of the gas. Then

$$\bar{F} = \frac{R}{M} T \nabla (\log p) \qquad 3.4\text{-}39$$

For the vertical direction, the force component is

$$F_z = \frac{RT}{M} \frac{d}{dz} (\log p) \qquad 3.4\text{-}40$$

This is to be balanced against the force of gravity in the same direction.

$$F_z = \frac{\partial g}{\partial z} z = \frac{2 g_0 r_0^2}{(r_0 + z)^3} z = \frac{RT}{M} \frac{d}{dz} (\log p) \qquad 3.4\text{-}41$$

This can be integrated (Peirce #31) to obtain the variation of pressure as a function of altitude. The result is

$$-\frac{g z^2}{r_0} = \frac{RT}{M} \log (p/p_0) \qquad 3.4\text{-}42a$$

In exponential form we have

$$p/p_0 = \exp\left[-(M/RT)(g z^2/r_0)\right] \qquad 3.4\text{-}42b$$

Replacing the pressures by densities from the equation of state one obtains

$$\rho = \rho_0 \frac{T_0 M}{T M_0} \exp\left[-(M/RT)(g z^2/r_0)\right] \qquad 3.4\text{-}43$$

Thus, by a careful determination of the variation in density of a suitable gas constrained in a vertical column one can find the variation of gravity along that column.

There exists a variety of techniques to measure the density of the gas. In any case, the detection equipment to perform such measurements are either relatively large or extremely delicate. Extreme care must be taken to maintain temperature control everywhere in the gas column. Temperature variation here constitutes the major source of "noise" which may mask the gradient.

3.4.7 Summary of Gravity Gradient Instrumentation

The preceding discussion concerns several instruments and systems from which the gravity gradient can be obtained in an orbiting vehicle. Instruments include the Arma Vibrating String Gradiometer, a matched pair of accelerometers, the gradiometer on a boom and the Texas Instruments' Torsional Gradiometer. Systems yielding the gravity gradient include the San Marco system and the Gas diffusion System.

The instrumentation recommended for gradiometry measurements is the vibrating string gradiometer discussed in section 3.4.1 above. This recommendation is based on extensive development, manufacturing and testing experience with the closely related vibrating string accelerometer. Our experience leads to the conclusion that the three string gradiometer can be produced as a rugged, compact, highly sensitive device for the measurement of gravity gradient in an orbiting satellite.

An error analysis of the gravity gradient system shows that the vibrating string gradiometer is suitable for the measurement of gravitational anomalies in the order of ten milligals. A growth potential exists which will substantially improve the data to be measured from an orbiting vehicle.

3.5 Systems Analysis

This section deals with the manner in which the gradiometer is introduced into the lunar gravity environment and the interrelationships among the instrument, its carrier and the environments which affects the determination of the lunar gravity field.

Specifically, it is shown that relative and non-dimensional data will be collected and that orbit parameters may be obtained from the instrument in this environment. An example of a gravity measuring system is given and an error analysis is described wherein the influences of the system elements on the system accuracy are derived. An error budget gives the results in quantitative form for central force systems.

3.5.1 Relative Versus Absolute Gravitational Data

An important aspect of gravitational phenomena is the point that mass is relative, and its magnitude depends on the value chosen for the universal gravitation constant G. The second point is that gravity gradient technology yields data which have the dimension of time and we combine them with an independent geometric measurement to determine the gravity field strength and related functions.

Consider the value of the universal gravitational constant G which is the proportionality constant of Newton's Law of Mass Attraction

$$F = G \frac{M_1 M_2}{r^2} \qquad 3.5\text{-}1$$

It may be worth pointing out that centuries of investigation have failed to yield a precise value of G (i.e. precision to within a few parts per million). Instead, the current value of G is "adopted". Assuming distance, r, can be measured to the precision required, the adoption of a value of G forces the magnitude of mass to fit equation (3.5-1). In other words, the absolute magnitude of mass is not known any better than the absolute value of G. The characteristics which can be found to a high degree of precision are the ratios of masses and the functions of gravitational field strength GM namely potential, attractive force and gradient. The ability to determine data on the product GM results from the fact that tests are made in the precisely determined dimensions of time and length. Thus, two masses can be compared by "weighing" them in the same (Earth's) gravitational field without knowing the magnitude of that field. Nothing will be determined in absolute terms. Gravitational force is determined by noting the acceleration to which a falling test body is subjected. This is independent of the mass of the test body as Galileo showed long ago.

One method for determining the ratio of the masses of the Moon and the Earth has been the determination of the location of the center of gravity of the Earth-Moon system which is in orbit around the Sun. It is not the center of the Earth itself which follows an elliptic path about the Sun but rather the center of gravity (barycenter) of the Earth-Moon system. Both the Earth's center of gravity and the Moon's center of gravity revolve about the barycenter once per month. From astronomical observations of displacements of celestial bodies periodically, relative to the orbital motion of the Earth, the barycenter is found to be at an average distance of 2903 miles from the center of the Earth. The cg of the Moon is 81.3 times further from the barycenter. Hence the ratio of the masses is also 1:81.3.

The ratio of the masses can also be found from the constants of precession and nutation of the Moon in the Earth's gravitational field. H. S. Jones, in 1941, found the value of 1:81.27 ± 0.02. Then, the accuracy to which the Moon's mass is known is equal to that to which the Earth's mass is known. Similarly, one can determine the magnitude of the "average" lunar gravitational acceleration.

Obviously, the best way to measure the Moon's gravitational field is by use of gravimeters located at various points of the Moon's surface, just as it is done on Earth. However, this is not feasible at the present time. The next best method would involve tests from a lunar orbiting satellite.

The present state-of-the-art of satellite orbiting is just about at the stage where a vehicle could be put into a reasonably precise orbit around the Moon. Such an experimental station would greatly enhance man's knowledge of the Moon's characteristics and it is a mission that should be performed before landing a man on the Moon. Many experiments would be conducted from an orbiting satellite including the determination of atmosphere content (to the extent to which it exists) radiation levels, magnetic and gravitation fields. Complete three dimensional mapping would be undertaken as well as absorption and reflection experiments to determine the nature of the moon's surface.

It is possible to sense the strength of the gravitational field from within an orbiting satellite with accelerometers sensing the gradient of the field. These measurements correlated with other appropriate measures will yield the gravitational field strength of the Moon without the necessity for landing on it.

A problem with regard to the Moon's gravitational field is its non-spherical shape. This causes the gravitational field to be a function of angle as well as radius. The fact that the Moon shows approximately the same face to the Earth throughout the month indicates that it is subject to gravity gradient attitude stabilization. That is, where a satellite has unequal moments of inertia, the gravity gradient effect will cause the axis of least moment of inertia to be aligned toward the center of the gravitating body, i.e. along the local vertical. It is possible to improve our knowledge of the shape of the Moon by gravity field measurements. Jeffreys, 1937, has shown that the longest axis of the Moon is 3478.43 Km, the polar diameter is 3476.25 Km and the equatorial diameter normal to the longest axis is 3475.55 Km. Equipotential surfaces will follow this contour. Note that the Earth's corresponding dimensions are: long axis, in equatorial plane 12,756.94 Km; shorter axis in equatorial plane 12,756.62 Km; and the polar diameter is 12,713.82 Km. Thus the departure from sphericity of the Earth's gravitational field is normal to its equatorial plane. It does not have a dumbell analog like the Moon. The Earth's analog is a toroid.

3.5.2 Effect of Orbit Eccentricity on Gradiometer Output

The eccentric, elliptic, satellite orbit is characterized by a variable distance between the satellite and the center of the gravity field. The locus of the satellite is

$$r = A/(1 + e \cos \theta) \qquad 3.5\text{-}2$$

where A = semi-latus rectum

θ = true anomaly

e = eccentricity

The maximum and minimum radial distances are apogee and perigee (designation with respect to earth)

$$r_a = A/(1 - e) \qquad (\theta = 180^\circ) \qquad 3.5\text{-}3a$$

$$r_p = A/(1 + e) \qquad (\theta = 0^\circ) \qquad 3.5\text{-}3b$$

The radial gradiometer outputs vary, then, during the orbit between the values

$$\Delta g_a = K \frac{\mu}{A^3} (1 - e)^3 \qquad 3.5\text{-}4a$$

$$\Delta g_p = K \frac{\mu}{A^3} (1 + e)^3 \qquad 3.5\text{-}4b$$

where K is the scale factor.

Therefore, one can determine the eccentricity of the orbit from the cube root of the ratio of maximum to minimum outputs.

$$(\Delta g_a/\Delta g_p)^{1/3} = (1-e)/(1+e) \approx (1-e)^2 \approx 1-2e \qquad 3.5\text{-}5$$

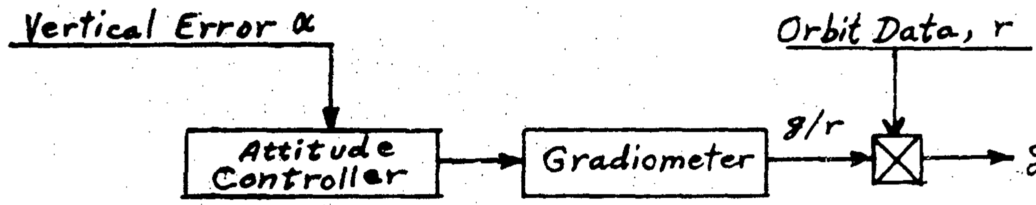
for small eccentricities

$$e = \frac{1}{2}\left[1 - (\Delta g_a/\Delta g_p)^{1/3}\right] \qquad 3.5\text{-}6$$

This is a non-dimensional characteristic and can be determined from gradiometer measurements alone. Where compensation for eccentricity is required, this parameter can be readily determined.

3.5.3 Example of System for Determining Gravitational Acceleration

The output of the gradiometer is a function of gravitational field strength and distance from the center of the gravity field in a form whose dimension is the reciprocal of time squared. In order to obtain gravitational acceleration, the distance factor is introduced explicitly. The system also includes a gimbal support system which aligns the sensitive axis of the gradiometer to the local vertical.



For an approximately circular orbit the gradiometer output is of the form

$$\Delta g = K_o + K_1 \frac{g}{r} \cos^2 \alpha \qquad 3.5\text{-}7$$

Where K_o = bias of instrument

K_1 = scale factor of instrument

α = attitude of sensitive axis relative to local vertical

Assuming that the orbiting satellite is controlled in attitude to the local vertical, so that $\alpha = 0$ and assuming adequate calibration methods, the gravity field can be measured continually in orbit as a function of position obtained from the navigation equipment.

Thus, a map of the gravitational field will be obtained, including measures of anomalies in that field and the perturbations due to other bodies in space.

As a first order model, one may consider that the satellite continually reproduces its path in inertial space. It is then reasonable and easy to correlate data from successive circuits. This correlation will improve the theoretical accuracy of the system in proportion to the square root of the number of correlated independent data points.

A more elaborate model such as suggested in section (3.2.2) might involve a three dimensional description of the gravity field in terms appropriate to the least squares estimation of coefficients in the model. This would require more tracking data and would eliminate time as an explicit parameter. This technique is similar to current techniques for evaluating the earth's gravity field by observation of satellites.

In addition to the vertically oriented gradiometer, the system may also include gradiometers aligned normal to the local vertical in the orbit plane. The output then is

$$\Delta g_H = K_0 + K_1 \frac{g}{r} (\cos^2 \alpha - \frac{1}{3} \sin^2 \alpha \sin^2 \phi) \qquad 3.5\text{-}8$$

This output indicates the orientation of the gravity equipotential, relative to the local horizontal and the angle ϕ from the orbit plane. For a homogeneous spherical body, the equipotentials would be concentric spheres and everywhere coincident with the geocentric horizontal. For a homogeneous oblate or prolate spheroid the equipotentials would be symmetric to the gravitating body near its surface and gradually become spherical as distance from the body increased. Where inhomogeneities in the figure or the mass distribution occur, the equipotential will be most distorted near those departures from symmetry, gradually disappearing as distance increases. Thus, for a gradiometer maintained aligned to the geocentric horizontal, in a satellite in orbit close to the surface of the body being investigated, anomalies will be detected and measured for relative magnitude and extent, in an otherwise noiseless configuration. By holding the gradiometer in the nominally null position, its sensitivity to the change in curvature of the equipotentials is very high. Note that the third gradiometer aligned normal to the two mentioned above, will give the third dimension to the magnitude and orientation measurement of the equipotential surfaces through which the orbiting satellite passes. This system will yield sufficient data to test the gravity models of the geophysicist.

3.5.4 Error Analysis

The measurement of the gravity gradient by an instrument whose attitude is maintained along the local vertical, while in a near circular orbit, is combined with position information to yield gravity acceleration. The errors in such a system are determined from the derivation of the system equation 3.5-7.

$$g = r \frac{(\Delta g - K_o)}{K_1 \cos^2 \alpha} \quad 3.5\text{-}9$$

$$dg = \frac{1}{K_1 \cos^2 \alpha} \left[(\Delta g - K_o) dr + r (d\Delta g - dK_o) \right] \quad 3.5\text{-}10$$

$$- r \frac{(\Delta g - K_o)}{(K_1 \cos^2 \alpha)^2} \left[\cos^2 \alpha \, dK_1 - 2K_1 \cos \alpha \sin \alpha \, d\alpha \right] \quad 3.5\text{-}11$$

$$\frac{dg}{g} = \frac{dr}{r} + \frac{d\Delta g - dK_o}{(\Delta g - K_o)} - \frac{dK_1}{K_1} + \frac{2 \sin \alpha \, d\alpha}{\cos \alpha}$$

The functional form in which these errors appear show the weights of the component errors on the desired measure for the system in which these elements are assembled. The weights associated with the errors are shown parenthetically in the following five relations.

$$\partial g = \left(\frac{g}{r}\right) \partial r \quad 3.5\text{-}12a$$

$$\partial g = \left(\frac{g}{\Delta g - K_o}\right) \partial \Delta g \quad 3.5\text{-}12b$$

$$\partial g = \left(\frac{g}{\Delta g - K_o}\right) \partial K_o \quad 3.5\text{-}12c$$

$$\partial g = \left(\frac{g}{K_1}\right) \partial K_1 \quad 3.5\text{-}12d$$

$$\partial g = (2g \tan \alpha) \partial \alpha = 2g (\partial \alpha)^2 \quad 3.5\text{-}12e$$

for small angles.

Each error is assumed to be random, zero mean and statistically independent. Therefore, the statistical combination of the five errors given in the above equation (3.5-12) is a vector sum rather than an algebraic one.

$$\frac{dg}{g} = \sqrt{\left(\frac{dr}{r}\right)^2 + \left(\frac{d\Delta g}{\Delta g - K_o}\right)^2 + \left(\frac{dK_o}{\Delta g - K_o}\right)^2 + \left(\frac{dK_1}{K_1}\right)^2 + (2\tan\alpha\, d\alpha)^2} \qquad 3.5\text{-}13$$

This is the r.m.s. system error based on r.m.s. values of the individual errors. An error budget can be established, for example, by letting each error source contribute equal amounts to the system error. Equations (3.5-12) show that unit errors in the individual parameters weigh differently against the system. Thus the budgeted errors should be inversely proportional to these weights in order that the parameter errors have the same effect on the system.

There are five sources of error noted in this analysis. If the system error is dg and the contribution of each source is ∂g, the orthogonal relationship among the independent error sources means that

$$dg = \sqrt{5}\ \partial g \qquad 3.5\text{-}14$$

As a basis for numerical evaluation let us assume that dg should be determined to within 0.02percent of its true value (within 32 milligals on the Moon). Then, the error budgeted to each error source is $\partial g = 9 \times 10^{-5} g$.

Consider that the measuring system is in a nominally circular orbit about 100 miles above the moon's surface. Then, the expected parameter values are

r	=	1180 miles (100 miles altitude)
g	=	136 gals (at altitude) = 8.5×10^{-4} mi/sec^2
α	=	0 (attitude controlled)
K_1	=	1 (normalized)
K_o	$\ll$	Δg (by design and calibration techniques)
Δg	=	$K_1\, g/r = 7.2 \times 10^{-7}/\text{sec}^2$

Then, from the nominal values and budgeted levels the tolerable maximum errors for the five error sources are:

$$\partial r = 1180 \times 9 \quad 10^{-5} = 0.11 \text{ mile}$$

$$\partial \Delta g = 7.2 \times 10^{-} \times 9 \times 10^{-5} = 6.5 \times 10^{-11}/\text{sec}^2$$

$$\partial K_o = 6.5 \times 10^{-11}/\text{sec}^2$$

$$\partial K_1 = 9 \times 10^{-5}$$

$$\partial \alpha = (4.5 \times 10^{-5})^{1/2} = 6.7 \times 10^{-3} \text{ radians} = 0.38 \text{ degrees}$$

To summarize the results of the above error analysis, the magnitudes of the tolerable errors indicate that:

1) The gradiometer should be so designed that the bias is zero theoretically.

2) Alignment of gradiometer to within $1/3^{\circ}$ of the local vertical is required.

3) The accuracy requirement on the gradiometer is 1/10,000 in this example.

It has been assumed that the gradiometer should be designed for a linear output so as not to introduce non-linear error sources. It should be noted that by taking data over many orbits and correlating the results, the accuracy of the desired measurement will improve as the square root of the number of repetitions.

3.6 Design of the VSG

In the following section we summarize the major choices and relationships employed in the design of a Vibrating String Gradiometer, as described in section 3.4.1. Note that we have drawn heavily on past experience with accelerometer designs for BSD, ASD, and NASA to assure the reasonableness of the selections. It is concluded that a VSG can be made to meet the requirements of lunar anomaly mapping as determined earlier in this report.

3.6.1 String Characteristics

All strings are made of the same material, namely beryllium-copper. The density of this metal is 0.3 lb/in^3.

The strings will have a minimum cross sectional area consistent with fabrication and handling in order to attain the highest operating frequency. Experience indicates A = 0.005 x 0.001 square inches is reasonable.

The initial design for the end strings will be based on Arma experience with the shortest string manufactured, again for high nominal operating frequency and high scale factor for the gradiometer. $L_1 = L_3 = 0.375$ inches.

Further, minimizing the end strings, minimizes the overall instrument length.

The center string shall be as long as possible, to obtain the greatest gradient information. Considering handling and assembly problems L_2 is 12 inches.

The nominal operating frequencies of the three strings are approximately 3200 cps for the end strings and 100 cps for the center string.

3.6.2 Proof Mass Characteristics

The proof mass is made as large as possible without overstressing the strings or the cross supports. This includes the restriction that the maximum change in tension M (a + e) is small compared to T_o in order to operate in the linear range of the tension versus length relation.

To achieve this T_o in the selected string is chosen as 10 grams force. This is the least T_o used in a VSA to date. Secondly the proof is chosen as 1.2 kilograms of mass. As an extreme of environment, leading to conservative design, the tension change is evaluated for the instrument situated 1000 feet from the zero-g ambient point, the center of gravity of the satellite, at 1200×10^{-4} or 0.1 gram force change. Thus the choice of the mass is such that even under these extremes the stress is well below the elastic limits of Be Cu and the change is small compared with the nominal tension.

For testing purposes in the 1g environment, however, a separable section of the proof mass will be used. This test mass is about 6 gms. and will be unlocked from the caged proof mass for the test.

The matching requirements of the two proof masses in the gradiometer is important in keeping the bias output of the instrument well below the level of the nominal output. Theoretically the bias is zero. However, in the presence of a proof mass differential dM the output of the unit will be proportional to

$$Me + dM\,(a + e)$$

where a is the ambient acceleration. For an altitude of 100 miles above the Moon, for example, $e = 1.4 \times 10^{-6}$ ft/sec^2 for an instrument one foot long and $a = 1.4 \times 10^{-6}$ ft/sec^2 for each foot radially from the cg of the satellite orbiting the Moon. Assuming the gradiometer were 100 feet radially away from the satellite cg it is necessary that

$$dM \ll \frac{Me}{101\ e}$$

Thus, conservatively, for a perturbation of less than 0.01 percent dM should be less than $M \times 10^{-6}$ or 0.0012 grams. The masses can be calibrated by "weighing" them on a test vibrating string since VSA experience has shown that an acceleration (force) resolution of this order of magnitude is feasible.

3.6.3 Cross Coupling Restraints

With regard to cross coupling of energy between strings, the structural support members will be designed with internal damping and variable cross section so as to attenuate any energy which could be transmitted from one end string to the other. Such cross coupling tends to reduce the difference frequency of the two strings with the consequent output error.

Cross support tapes constrain the proof masses in the two directions perpendicular to the sensitive axis of the instrument. Design analysis shows that the stiffness of the proposed cross supports, in the axial direction will be only one-three hundredth of the stiffness of the center vibrating string, the latter being L_1/L_2 as stiff as the end strings. Thus the cross supports do not interfere to any extent, with the sensitivity of the instrument. In the directions normal to the sensitive axis these same cross supports are 25×10^3 times as stiff and will prevent change in the orientation of the sensitive axis due to ambient cross-acceleration in the orbiting vehicle.

3.6.4 Magnet Assemblies

The VSG requires a permanent magnet assembly with each string. The end-string chosen is one whose length is equal to that of an Arma VSA. The same magnet used in the VSA is appropriate for the VSG. Its length is approximately 80 percent of the string length.

For the center string, the magnet length will be only a small portion of the string length. This is feasible since very little energy is required to drive the high Q strings. The magnet assembly for the center string may be made up of several sections placed along the string or one magnet at the center. This choice will be optimized during the final design.

3.6.5 Beaching

The instrument is protected against the high g environments prior to being set in orbit by the use of automatic beaching (caging) of the full proof mass at a point below the elastic limit of the sensitive elements.

3.6.6 Accuracy of the VSG

Generally speaking the measured output of the VSG is a sum of frequencies squared. It refers to a bias term K_o an acceleration e (gradient times length) times a scale factor K_1 and a residual R due to higher order terms. Analytically it is of the form

$$f^2 = K_o + K_1 e + R \qquad 3.6\text{-}1$$

By virtue of design and calibration techniques K_o and R are negligible and the error associated with the determination of the true value of e is due to the errors in measured f and calibration of K_1

$$\frac{\partial e}{\partial f} = \frac{1}{K_1}(2f) \text{ leads to } \frac{\partial e}{e} = 2\frac{\partial f}{f} \qquad 3.6\text{-}2$$

$$\frac{\partial e}{\partial K_1} = \frac{-1}{K_1^{\,2}} f^2 \text{ leads to } \frac{\partial e}{e} = \frac{-\partial K_1}{K_1} \qquad 3.6\text{-}3$$

The instrument error is the square root of the sum of squares of these independent errors. Both $\partial f/f$ and $\partial K_1/K_1$ are obtainable in the order of 10^{-5} or better. Then the resultant uncertainty is $\sqrt{5} \times 10^{-5} = 2.2 \times 10^{-5}$. This can be reduced by longer counting periods.

3.6.7 Temperature Sensitivity

The main effects due to an increase in temperature are as follows:

a. The outside aluminum structure of the instrument increases its length due to its thermal coefficient of expansion. This elongation causes an increase in the tape tensions. Increased tape tensions cause increased tape frequencies.

b. The inside beryllium copper structure increases its length due to its thermal coefficient of expansion. This expansion causes a decrease in the tape tensions. Decreased tape tensions cause decreased tape frequencies.

By arranging these effects to be self compensating, it is possible to design the instrument to be relatively insensitive to steady state temperature changes.

Through the device of temperature control, and through the use of a drift model in the data processing computations these effects can be further reduced to admit extended intervals between recalibration operations.

3.6.8 Summary of Performance of the VSG

Threshold Output	10^{-13} g/ft
Resolution	1/100,000 based on 1 minute counting; Low noise.
Range	10^{-13} to 10^{-6} g/ft
Caging Level	10^{-4} g/ft
Accuracy (de/e)	2.2×10^{-5} as per para. 3.6.6
Size	Length 16 1/2", Diameter 4 1/2"
Weight	20.5 lbs

3.6.9 Summary of Physical Design

In summary design studies and past experience with similar devices show that the proposed VSG is feasible. It will be necessary to extend some of the fabrication techniques, particularly those for matching strings and masses because of the low levels of acceleration and high accuracy requirements.

Since temperature variation is a problem in any sensitive instrument, it may be desirable to employ a thermostatically controlled constant temperature environment.

The computational and data processing functions indicated for the VSG output are but a small part of the computer requirements for the geophysical interpretation of the data so collected. It is recommended that the individual string frequencies be telemetered to an Earth-based station and the data processed at a convenient computer facility, along with the geophysical computations.

3.7 Instrument Testing and Calibration

The vibrating string gradiometer has been designed to operate in a very low ambient acceleration compared to the one g environment on the ground. It is necessary, however, to test the unit prior to actual use in orbit to give assurance that the instrument is in operating condition. Once in orbit the equipment will operate as planned. Periodically during the course of the satellite's lifetime, the VSG will be recalibrated automatically for bias and scale factor. This will enhance the accuracy of the accumulated data.

3.7.1 Prelaunch Testing

It is considered appropriate to test the gradiometer prior to launch to the fullest extent possible even though the ground environment is recognized to be far beyond the extreme limits of the instrument. This is done primarily to check the functional operation of the instrument.

(Note: It is possible to manufacture the equipment in a caged configuration and release the proof masses only after the satellite is in orbit. All testing would then be done on the satellite.)

The gradiometer can be operated on the ground in a manner analogous to its operation in orbit by employing test masses. These are attached to the strings but detached from the caged proof masses during test. Design studies have indicated that test masses of about one-two hundredth of the proof mass is practical. Then tests for bias, scale factor linearity and cross acceleration effects may be carried out. The gradiometer, in test condition, (proof masses caged, test masses released) is mounted on a precision rotary head.

When the VSG is mounted normal to the local vertical, the frequencies of the two end strings should be identical. The difference in frequency of the two end strings is compared to that difference when the unit is rotated 180° about the local vertical, i.e. with the sensitive axis still horizontal. One half the sum of these differences is the bias, since the rotation takes account of the possible initial misalignment.

Scale factor and linearity in the acceleration measuring mode are determined by testing the output of the instrument at several points in the one g gravity field. Here, the gradiometer is rotated about the horizontal normal to its sensitive axis. The sine of the angle from horizontal is the portion of the gravity field being measured.

For calibrating the gradiometer function, the instrument is placed on a horizontally rotating platform (centrifuge) and the output determined as a function of angular velocity and position.

The effect of cross acceleration on the gradiometer output is determined by noting the variation in output as the gradiometer is rotated about its sensitive axis while this axis is held horizontal. The instrument's cross support system is the primary mechanism by which cross acceleration effects are minimized.

3.7.2 In-Orbit Calibration

The prelaunch tests assure that the instrument and its associated elements are functionally operative. Since the actual proof masses are not used in the one g environment, calibration is required in orbit. There are several possible techniques which may be employed to determine the bias and scale factors of the instrument.

In orbit, the test masses (which are secured to the sensitive strings) are locked to the proof masses and the combination will be uncaged. Assuming that the gradiometer is mounted in its own gimbaling system so that its orientation can be modified automatically, the gradiometer will be rotated about its own cg at various angular rates. The change in output as a function of angular rate gives the scale.

The bias output would be determined in a manner analogous to that outlined for the prelaunch test. It is important to note that the bias may also be isolated in geophysical data processing.

A second possible method for calibrating the gradiometer in orbit employs the effect of a known mass, on board the satellite, on the gradiometer output. Consider a "calibration mass" suspended along the sensitive axis of the gradiometer on the side opposite that of the earth (i. e. gravitating body). The presence of this mass in close proximity to the instrument should be sufficient to offset the effect of the earth's gravity. The distance between the test mass and instrument when the output of the latter is zero represents the scale factor (after the bias has been taken into account). Further, by noting the outputs as a function of distance the input-output function is determined.

There is a monitoring technique which can be used without any special constraints on the gradiometer and which is effective all during the normal operating time of the instrument. This relates the maximum and minimum outputs of the unit to the period of the satellite in an eccentric orbit.

These outputs are

$$\Delta g_a = K \frac{\mu}{r_a^{\,3}} \quad \text{at apogee} \qquad 3.7\text{-}1$$

$$\Delta g_p = K \frac{\mu}{r_p^{\,3}} \quad \text{at perigee} \qquad 3.7\text{-}2$$

The time interval between successive perigee passages, which can be measured inboard, is related to the orbit by

$$T = 2\pi \sqrt{a^3/\mu} \qquad 3.7\text{-}3$$

where a = semi-major axis = 1/2 $(r_a + r_p)$ 3.7-4

The relative distances a^3/μ and r^3/μ are thus related by

$$\frac{a}{\mu^{1/3}} = \frac{1}{2}\left(\frac{r_a}{\mu^{1/3}} + \frac{r_p}{\mu^{1/3}}\right) \qquad 3.7\text{-}5$$

or

$$\left(\frac{T}{2\pi}\right)^{2/3} = \frac{1}{2}\left[\left(\frac{K}{\Delta g_a}\right)^{1/3} + \left(\frac{K}{\Delta g_p}\right)^{1/3}\right] \qquad 3.7\text{-}6$$

This latter data can be extracted from the system and the equality checked each orbit period.

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