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THE EFFECT OF THE ROUGH AIR-SEA INTERFACE ON VERY-LOW-FREQUENCY AND EXTREMELY-LOW-FREQUENCY PROPAGATION

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ABSTRACT

In the search for extremely reliable electromagnetic communication to submerged submarines, the question arose, "What is the effect of the roughness and irregularity of the sea surface on the propagation of electromagnetic waves?" The purpose of this investigation is to obtain an engineering understanding of the effect of the rough air-sea interface on electromagnetic signals used in communication to submerged submarines.

The frequency of the electromagnetic wave is restricted to the ELF or VLF range. In the initial part of the investigation, the sea surface is assumed to be a two-dimensional (constant in one variable or direction) sinusoidal surface; later a doubly (three-dimensional) sinusoidal surface is considered. The source of electromagnetic energy is assumed to be a plane wave with arbitrary direction of propagation and polarization.

The fields on the air side of the sea surface are computed with the aid of the assumptions that the sea is a perfect electric conductor and that the sea surface is only slightly rough (i.e., the maximum slope of the sea surface is much less than 1). The integral equations governing the tangential magnetic fields are formulated and solved. These solutions show a variation of the tangential magnetic field (of the order of 2 db. from the flat surface case) depending

on polarization and direction of propagation of the incident plane wave.

The fields in the sea are computed by assuming the tangential magnetic field is continuous through the air-sea interface. The method used in these calculations is a numerical one based on finite differences.

Both from the numerical solutions and a heuristic theory of propagation in the sea, it is seen that the perturbation of the fields caused by the roughness of the sea surface decays rapidly with depth if the sea wave wavelength is less than or the order of magnitude of the skin depth of the sea at the frequency considered; if the sea wave wavelength is many orders of magnitude larger than the skin depth, there is little decay (at the depths considered) of the perturbation, so that the phase and amplitude of the fields in the sea vary with the height of the sea vertically above them.

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PREFACE

This report is concerned with a problem that arises in the theory of communication to submarines. In the search for extremely reliable radio communication to submerged submarines, the question arose, "What is the effect of the roughness and irregularity of the sea surface on the propagation of radio waves?" Within the content of the submarine communication problem this report will attempt to answer that question.

The effect that nonuniform or rough surfaces have on electromagnetic propagation is not completely known. The interaction of such surfaces with incident electromagnetic energy is a particularly difficult problem. No attempt to solve the above problem (where the rough surface is taken to be the sea surface) for all frequencies and classes of surfaces appears feasible at present. However, for restricted frequency intervals and classes of surfaces, detailed solutions may be obtained.

The radio wave propagation problem associated with long range communication to submerged antennas is necessarily concerned with "low frequencies." This follows in part from consideration of the attenuation of an electromagnetic wave as it propagates through a conducting medium. For propagation through sea water, the attenuation is approximately proportional to $e^{-d/\delta}$, where d is the depth below the sea surface and δ is the skin depth of the sea water at a frequency of the radio wave ($\delta \sim f^{-1/2}$ for the frequencies considered here). The above implies the need only to consider a restricted frequency range for the submarine communication problem.

With this in mind, a study of the electromagnetic fields caused by incident VLF and ELF[1] plane waves on the rough air-sea interface has been made and is presented in this report. Main attention is given to the fields in the sea, somewhat near the air-sea interface (i.e., within twenty-five meters or so of the sea surface), as these are the electromagnetic fields presently in use in communication systems. In the VLF range, δ for sea water is a few meters; this implies the fields at a depth of tens of meters (a few skin depths) are orders of magnitude less than the fields at the sea surface. For moderate sea states, the sea wave heights are the same order of magnitude as the skin depth in the VLF range; the above implies the radio signal is greatly changed as it propagates downward through the rough sea, as compared to the signal under the flat sea surface condition. For radio waves in the ELF range,

[1]VLF (very low frequency) is usually taken to be the range from 3 to 30 kc/s; ELF (extremely low frequency) is from about 1 cps to 3 kc/s. (James R. Wait, Electromagnetic Waves in Stratified Media [Pergamon Press, New York, 1962], p. 1).

the skin depth is the order of ten to one hundred meters so that the effect of sea roughness will be somewhat less than in the VLF range.[2]

It is clear then that an effect of the rough air-sea interface is to change or distort the radio signal as it propagates to a submerged antenna. That is, the rough air-sea interface has the effect of introducing noise into the radio signal. One of the major problems in communication systems is to preserve as good a signal to noise ratio as necessary for detection of the signal. In some cases, as possibly (under certain conditions) the one considered here, the noise properties of the communication channel are determined primarily by the propagation properties of the time-varying signal path used in the system; that is, the noise created by the time-varying path is greater than the other noise created in the system; e.g., atmospheric noise, and the noise created by the time-varying path is the limiting factor in the communication system. Part of the propagation path to a submerged antenna is through the time-varying rough air-sea interface. One of the objectives of this study is to gain an engineering understanding of the distortion of the electromagnetic signal by the rough air-sea interface, so that ways of alleviating this condition may be found.

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[2]In sea water at a depth of ten meters the attenuation of an electromagnetic wave at 18.6 kc/s is approximately 47 db., while the attenuation at 3 cps is approximately .6 db. A curve of skin depth vs. frequency is given on page 55.

There are two major difficulties in the solution of the electromagnetic wave sea surface interaction problem. The first is the solution of the electromagnetic boundary value problem for a particular, completely specified rough surface. In what follows, as is true in most discussions of boundary value problems involving rough surfaces, only an approximate solution is obtained for the particular rough surface considered.

The second, and in many respects a more difficult problem, is the mathematical description of the sea surface.[3] The description of the sea surface is statistical, and this implies that the solution to the "sea surface-radio wave" problem would be given in statistical terms. However, as discussed later in this study, statistical results such as average field strengths are not very meaningful for this problem.

In the initial part of the following investigation, the sea surface will be assumed to be two-dimensional (constant in one variable or direction). The "basically spherical" earth is replaced by a "basically flat" earth. This approximation is made often in "low frequency" propagation problems, particularly when only local fields are considered.[4]

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[3]The mathematical descriptions of the sea surface are considered in Appendix A.

[4]Anderson, W. L., "The Fields of Electric Dipoles in Sea Water -- The Earth-Air-Ionosphere Problem," Technical Report EE-88, Engineering Experiment Station, University of New Mexico, Albuquerque, N. M., May, 1963, p. 3.

The theoretical foundation for the two-dimensional problem is presented in Chapter 1, starting directly from Maxwell's equations (a set of vector partial differential equations, boundary and/or interface conditions, and source conditions). It is shown that for the problem considered, Maxwell's equations may be approximated by a linear and time invariant operator, in which case it is convenient to consider only monochromatic electromagnetic fields. The concept of vector potentials is then given, along with a brief outline of their theory for the monochromatic case. In the two-dimensional problem the vector electromagnetic boundary value problem may be reduced to a set of scalar boundary value problems by use of the vector potentials. The scalars used in this reduction are the rectangular components of the vector potentials.

The formulation of the scalar boundary value problem in terms of integral equations is given. The starting point in this formulation is Green's theorem, involving two arbitrary functions. One of the functions is restricted until it is the desired solution of the boundary value problem; the second function is chosen to facilitate interpretation of the mathematical formulation in physical terms. An unfortunate result of considering a plane wave as the source term and an infinite rough plane as the scatterer is that Sommerfeld's radiation condition is not sufficient to render the solution unique.

To render the problem "well-set," a detailed discussion of the boundary value problem in terms of integral equations is given. The plane wave source condition is obtained as a limit of the usual source condition with a finite source. Also, because the virtual sources which are assumed to exist on the scattering surface are of unbounded extent as the scattering surface is unbounded, the radiation condition is imposed again by way of a limiting process. With the above mathematical formulation, the problem is then "well-set" and the solution unique.

To compute the fields in the air and on the sea surface, the sea is assumed to be a perfect electric conductor. Later, when considering the fields in the sea, the electrical properties of the sea are assumed to be:[5]

i) ϵ_r = 81--relative permittivity,

ii) σ = 4 mhos/m--electric conductivity of the sea,

iii) μ_r = 1--relative permeability.

The above assumption of a surface impedance of zero is a usual approximation made in discussing the rough surface problem;[6] its validity is discussed somewhat later in this

[5]Stratton, J. A., Electromagnetic Theory, McGraw-Hill Book Company, Inc., New York, 1941, p. 606.

"Sea Water" in McGraw-Hill Encyclopedia of Science & Technology, Vol. 12 (Mc-Graw-Hill, New York, 1960), p. 106.

[6]Much of the work on "electromagnetic rough surface" problems uses the assumption that the surface impedance is zero; however, this assumption is not usually verified. (See Lerner and Max, "Very Low Frequency and Low Frequency Fields Propagating Near and Into a Rough Sea," a paper

report. The two-dimensional vector electromagnetic problem may then be reduced to a set of uncoupled scalar problems for the rectangular components of the vector potentials. The problem subdivides into two parts, depending on the polarization of the incident electromagnetic wave. For different polarizations, different vector potentials are used in the formulation of the integral equations (i.e., scalar boundary value problems) and this is reflected in the different boundary conditions applied to the scalar field considered.

As stated above, the interaction of electromagnetic waves and rough surfaces still remains an unsolved problem. The theoretical treatment of "rough surface problems" was begun by Rayleigh in his classic Theory of Sound.[7] A review of major theoretical investigations is given in Appendix B. An extensive bibliography may be found in Lysanov's review work[8] on Bechmann and Spizzichino's monograph on scattering

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presented to the URSI Spring 1963 Meeting; R. E. Hiatt, T. B. A. Senior, and V. H. Weston, "Surface Roughness and Impedance Boundary Conditions," in "Studies in Radar Cross Section XL," Ann Arbor, Michigan, The University of Michigan Research Institute, July 1960, an unpublished report; S. P. Morgan, "Effect of Surface Roughness on Eddy Current Losses at Microwave Frequencies," Journal of Applied Physics, Vol. 20, 1949, p. 352.)

[7]Rayleigh, J. W. S., The Theory of Sound, Vol. II, Dover Publications, New York, 1945, pp. 89-96.

[8]Lysanov, Y. P., "Theory of the Scattering of Waves at Periodically Uneven Surfaces," Soviet Physics Acoustics, Vol. 4, No. 1 (Jan.--March, 1958), pp. 1-8.

of waves.[9] It may be stated that there exists no theoretical solution for the "rough surface problem." However, under certain assumptions about the rough surface, there have been developed methods for approximate calculation of the fields. In the problem considered, the sea surface is assumed to be sinusoidal which is a particular realization of the sea surface (for brief mathematical discussion of the sea surface, see Appendix A), with wavelength L much less than λ, the wavelength of the electromagnetic wave in free space. The technique used in the calculation of the vector potential on the sea surface is an "integral equation" type method. The physical parameters or constants of the sea surface and electromagnetic wave are such as to permit accurate calculation of the vector potential and hence the electromagnetic fields by this method.

The solution of the integral equations for the fields or potential in the air, but on the sea surface, is considered in Chapter Two. In the TM case,[10] the approximate method of degenerate kernels is used. Because of the relative magnitude of the physical parameters involved, the integral

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[9] Beckmann, Peter and Andre Spizzichino, The Scattering of Electromagnetic Waves from Rough Surfaces, Pergamon Press, New York, 1963, pp. 476-491.

[10] TM--transverse magnetic (the magnetic field is perpendicular to a fixed direction); TE--transverse electric (the electric field is perpendicular to a fixed direction). This notation is explained in Harrington, Time-Harmonic Electromagnetic Fields, (McGraw-Hill Book Company, Inc., New York, 1961), p. 219, and is used in Section 1.2.

equations with the degenerate kernel accurately approximate the complete integral equation. (This result is due to Lysanov[11] and Meecham.[12]) The approximate integral equation can be solved by classical Fourier methods. The solution shows a change or perturbation in the "worst case" for the physical parameters considered of about thirty-five per cent from the flat interface case.[13] This is in basic agreement with Lerner and Max,[14] who obtain a similar result by a completely different method. The above result is also shown to be independent of incident angle (except glancing angle, which is not directly considered). Lerner and Max considered only glancing angles.

In the TE case, the vector potential and the magnetic field are unperturbed, which again is in agreement with Lerner and Max, Wait[15] and Morgan.[16] Both Wait and Morgan

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[11] Lysanov, Y. P., "An Approximate Solution of the Problem of Scattering of Sound Waves from an Irregular Surface," Soviet Physics Acoustics, Vol. 2, 1956, p. 190.

[12] Meecham, W. C., "Fourier Transform Method for the Treatment of the Problem of Reflections of Radiation from Irregular Surfaces," J. Acoust. Soc. Amer., Vol. 28 (May, 1956), p. 370.

[13] The tangential magnetic field has a variation of approximately 2.8 db. compared to its constant value in the flat interface case.

[14] Lerner, R. M. and J. Max, op. cit., p. 19.

[15] Wait, J. R., "The Calculation of the Field in a Homogeneous Conductor with a Wavy Interface," Proc. IRE, Vol. 47, No. 6 (June, 1960), p. 1155.

[16] Morgan, S. P., op. cit., p. 353.

assumed this result based on physical principles. Wait assumed the field was unperturbed in both cases, TE and TM; Morgan, however, solved the TM case, just as Lerner and Max, by conformal mapping of the static ($\omega = 0$, where ω is the radian frequency of the radio wave) problem. It is interesting that Morgan was considering losses in "rough wave guides" in the microwave frequency range, which shows that the above results depend only on the ratio of relative physical parameters, basically the ratio of L to δ, assuming $\lambda >> L$, and not necessarily on their absolute values.

To consider the fields in the sea, the integral equations for the vector potential in the sea but on the sea surface is given. In the VLF range for the physical parameters considered, this integral equation yields an "impedance type" relation for the vector potential much like Leontovich's impedance boundary condition for the fields.[17] However, as this relationship applies to any wave function, it also may be applied directly to the electromagnetic fields. The major phenomenon leading to this result is the great attenuation of radio waves in sea water. In the ELF range the attenuation is not as great in terms of physical distances and the "local impedance conditions" need not hold.

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[17] Leontovich, M. A., "Approximate Boundary Conditions," Investigations on Radio Wave Propagation, Part II, Moscow: Printing House of Academy of Sciences, 1948, pp. 5-12.

By use of the impedance boundary condition the effect of the finite conductivity of the sea on the fields may be estimated. The solution originally obtained (under the assumption of infinite conductivity) is seen to approximately satisfy the "complete" integral equation (where φ and φ_n are assumed to be related by the impedance boundary condition -- φ being the solution to the boundary value problem under consideration and φ_n the normal derivative of φ on the surface) which, implying no basic change, is necessary in the solutions.

The next section of Chapter Two is concerned with the use of quasi-stationary kernels and solutions. It is shown that φ_s, the "scattered field," may be considered a quasi-stationary field, while φ_i, the incident field, and φ_r, the reflected field (the field reflected if the rough sea surface were assumed to be flat), may not. The use of stationary kernel is valid in the computation of φ_s, but not for φ_i and φ_r.

Now knowing the fields on the surface, we wish to obtain the fields in the sea. This is then a Dirichlet type boundary value problem for the rectangular components of the fields. In some cases the complete field may be generated by the use of only one component of the field and its derivatives. In this case, we may again simply use a scalar component of the vector potential. In general, the vector potential cannot be used to go through the rough interface correctly as there are too many requirements on the potential at a rough interface. In this case, each component of a field vector may be calculated separately.

In Chapter Three, the method used to compute the fields in the sea is given. First, a general discussion of classical methods, separation of variables, is given along with the results to be expected from such considerations.

Basically, the above method implies that if the variation of the electromagnetic field with respect to the x variable is "too great," the wave will be attenuated as it travels downward in the y-direction. In order to see if a particular "mode" will be attenuated significantly more than the n = 0 mode (plane wave propagating approximately in the y-direction, with propagation constant γ), a crude breakpoint is chosen. This is if

$$L \leq \sqrt{2}\ \pi n \delta$$

the mode will be attenuated significantly more than the n = 0 mode. L is the sea wave wavelength for the sinusoidal sea wave considered, and n is the index of the mode. The larger the n is, the greater the variation of the field in the x-direction and the greater the attenuation. The above implies that the asymptotic fields in the sea (i.e., the far fields) tend to a plane wave propagating basically in the y-direction with propagation constant γ. Asymptotically the major perturbation is the n = 1 mode. This result is independent of the shape of the sea surface; that is, Wait's conclusion[18] with respect to this result is correct, but the

[18] Wait, loc. cit.

asymptotic perturbation does not have the same shape as the sea surface; the latter is true only for sinusoidally shaped surfaces.

For the fields near the sea surface, if the lower order modes are not attenuated much more than the n = 0 mode, there should be little difference between the fields predicted by Wait's approximations and the actual fields. If the lower order modes are attenuated, there will be a great difference and the field will "rapidly" become approximately a plane wave. Practically, the large sea waves have wavelengths so long that for much of the ELF and all of the VLF range, the lower order modes are "unattenuated" and Wait's prediction is "relatively accurate." The smaller sea waves superimposed on the large ones may have small enough wavelengths so that the field perturbations caused by them will be greatly changed by attenuation. However, the perturbations caused by the smaller waves are relatively small and their effect on the total field is further reduced by their additional attenuation (over the attenuation of the n = 0 mode).

The above theory gives a good basic understanding of the effect of the roughness of the sea surface on electromagnetic propagation used in communication to submerged antennas. However, to place a more precise meaning on the term "little change" in the fields, the numerical solution to the propagation problem in the sea was obtained. When the numerical solutions are compared with theory, a good understanding of the propagation problem results.

The method used for numerical solution of the Dirichlet boundary value problem is the "method of lines." This is a modification of the classical separation of variables. One of the reasons it was chosen is its close correlation with the physical processes involved in the problem. A finite difference approximation is used in the x-direction, basically in the direction along the surface, to determine the "propagation constant" in that direction. This "propagation constant" should be somewhat smaller than the "propagation constant" in the y-direction. The partial differential equations then become a differential equation in y which is basically normal to the surface with the "propagation constant" determined. The differential equation is then solved. Here again the rough surface causes problems. Below the surface, that is, below a plane tangent to the surface at the lowest point (trough) of the surface, the radiation condition may be applied and the computation is straightforward. However, above this the radiation condition does not apply and the solution is by iteration.

The results of the calculations are given in Chapter Four, along with interpretation of the calculations. The results agree with what is intuitively expected by use of the classical arguments given above.

In part two of this study, the three-dimensional electromagnetic wave rough sea surface interaction problem is considered. The method used for the three-dimensional problem parallels that used in the two-dimensional problem; for this reason the development given in part two is brief and refers to the parallel development used in part one. The mathematical model of the rough sea surface is a three-dimensional extension of the model used in the two-dimensional problem.

The vector potentials are no longer useful and the problem is formulated directly in terms of the electromagnetic fields themselves. The Stratton-Chu equations are used to represent the solutions of Maxwell's Partial Differential Equations. The boundary conditions (assuming the sea surface is a perfect electric conductor) are placed in the Stratton-Chu equations yielding the integral equations (or vector integral equation) to be solved.

In Chapter 7, the integral equations are solved. The kernel of the integral equations is assumed to be the static ($\omega = 0$) kernel and the kernel is further approximated by a simpler kernel, permitting solution by the Fourier Method, as was done in part one. The solution yields the tangential magnetic field on the sea surface which, as the surface becomes constant in one variable, tends to the solution obtained in the two-dimensional problem. No numerical solutions are obtained in the sea as the propagation properties of the fields in the sea are basically unchanged for the two or three-dimensional problem.

1.0 THE BASIC THEORY FOR THE TWO-DIMENSIONAL PROBLEM

1.1 Introduction

The purpose of this chapter is to present the basic theory, assumptions, restrictions, and approximations involved in the treatment to be given below of the question "What is the effect of the roughness and irregularity of the sea surface on the propagation of VLF and ELF electromagnetic waves?"

The starting point in the mathematical description of macroscopic electromagnetic phenomenon is Maxwell's equations.[1] Maxwell's equations are a set of partial differential equations, boundary and/or interface conditions which the electromagnetic fields must satisfy. A general review of Maxwell's equations is presented. Then the electromagnetic fields are restricted to be monochromatic along with the assumption that the sea surface is stationary. In a later chapter, after the boundary value problem has been solved for a stationary sea surface, the effect of the motion of the sea surface is considered.

The vector potential method is introducted, and it is shown that some electromagnetic boundary value problems may be formulated in terms of the vector potential.

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[1] The concern here is only with "large-scale" phenomena and in the past it has been confirmed that solution of Maxwell's equations does represent the actual measurable quantities. (See Stratton, op. cit., p. vii and Harrington, op. cit., p. 1)

A mathematical model of the physical situation involving the rough sea surface is given, along with a brief comment on the validity of the assumptions and approximations involved in the construction of this model. Later, some of these assumptions are again reviewed to give a further estimation of the accuracy of the solution.

Using as a basis the mathematical model just described, the electromagnetic boundary value problem is formulated in terms of integral equations. Starting with Green's theorem for two relatively arbitrary functions, restrictions are placed on one of the functions so that it represents the solution of the boundary value problem considered. The other is an auxiliary function arbitrarily chosen as the "free-space Green's function" which permits immediate interpretation of the integrals in terms of physical processes. The two-dimensional electromagnetic boundary value problem is formulated in terms of scalar components of the vector potentials, yielding the integral equations to be solved in the next chapter.

1.2 Review of Maxwell's Equations and Monochromatic Fields

In a source-free region, the set of partial differential equations included in Maxwell's equations[2] is

$$\left.\begin{array}{lr} \operatorname{curl} \vec{e} = -\dfrac{\partial}{\partial t}\vec{b} & \text{(a)} \\ \operatorname{curl} \vec{h} = \dfrac{\partial}{\partial t}\vec{d} + \vec{j} & \text{(b)} \\ \operatorname{div} \vec{d} = \rho & \text{(c)} \\ \operatorname{div} \vec{b} = 0 & \text{(d)} \end{array}\right\} \qquad (1.2.1)$$

[2] In rationalized m.k.s.c. units which are used throughout the remainder of this paper unless specifically stated otherwise.

The constitutive relations which characterize the electromagnetic properties of the medium in which the electromagnetic phenomena occur are

$$\vec{b} = \mu\vec{h} \qquad (a)$$
$$\vec{d} = \epsilon\vec{e} \qquad (b) \qquad (1.2.2)$$
$$\vec{j} = \sigma\vec{e} \qquad (c)\ .$$

For the problem considered here, it is sufficient to assume μ, ϵ, σ are scalars and are constant with respect to variations in position and electromagnetic fields (i.e., the medium is linear, homogeneous, and isotropic, which is assumed to hold throughout the remainder). The changes in media will be reflected in the application of boundary or interface conditions. Also, for the moment, it will be assumed μ, ϵ, σ may be considered time invariant, and the problem solved for this "static" condition and later correction is made for the time variation. Under the above assumptions and the assumption (see below) that the boundary or interface conditions are linear and time invariant, Maxwell's equations become a linear and time invariant operator.

In the case where Maxwell's equations are linear and time invariant, it is convenient to use monochromatic fields. Let all the electromagnetic fields have a time variation of the form $\cos(\omega t + \theta)$, where ω is the radian frequency for all the fields. The vector time functions can then be obtained from the complex vectors by the relation

$$\vec{e}(\bar{r},t) = \mathrm{Re}\left\{\vec{E}(\bar{r},i\omega)e^{i\omega t}\right\} \qquad (1.2.3)$$

where $\bar{r} = (x,y,z)$ -- the ordered triple of the rectangular coordinates of position, i.e., $\vec{e}(\bar{r},t) \equiv \vec{e}(x,y,z,t)$. Similar equations hold for the other electromagnetic fields.

For monochromatic electromagnetic fields, in a linear, homogeneous, isotropic, source-free region, the (complex) partial differential equations that the electromagnetic fields must satisfy become

$$\begin{aligned} &\text{curl } \vec{E} = -Z\vec{H} && \text{(a)} \\ &\text{curl } \vec{H} = Y\vec{E} && \text{(b)} \\ &\text{div } \vec{E} = 0 \qquad \text{div } \vec{H} = 0 && \text{(c)} \end{aligned} \tag{1.2.4}$$

where

$$Z = i\omega\mu \qquad Y = \sigma + i\omega\epsilon \ . \tag{1.2.5}$$

Another set of requirements the electromagnetic fields must satisfy are the boundary or interface conditions. These reflect the electromagnetic properties of the different media. It can be shown from the set of partial differential equations,[3] that at the interface between two different media

i) tangential electric and magnetic fields are continuous

$$\hat{n}\times[\vec{E}^+(\bar{r}) - \vec{E}^-(\bar{r})] = 0$$

$$\hat{n}\times[\vec{H}^+(\bar{r}) - \vec{H}^-(\bar{r})] = 0 \tag{1.2.6}$$

where

$$\vec{E}^+(\bar{r}) = \lim_{\substack{\bar{r}\to S \\ \bar{r}\in V^+}} \vec{E}(\bar{r})$$

S -- surface (interface) bounding volume V^+

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[3]This approach that the boundary conditions are derivable from the partial differential equations is taken by H. Bremmer in his "Propagation of Electromagnetic Waves," in Handbuch der Physik Band XVI Elektrische, Felder und Wellen (Berlin: Springer, 1958, pp. 424-425).

$\hat{n}$ -- outward normal to S (pointing out of V^+)[4].

Similarily

$$\vec{E}^-(\bar{r}) = \lim_{\substack{\bar{r}\to S \\ \bar{r}\in V^-}} \vec{E}(\bar{r})$$

ii) normal electric and magnetic displacements

$$\hat{n} \cdot [\vec{D}^+(\bar{r}) - \vec{D}^-(\bar{r})] = 0$$

$$\hat{n} \cdot [\vec{B}^+(\bar{r}) - \vec{B}^-(\bar{r})] = 0 \quad . \tag{1.2.7}$$

Equations (1.2.6) and (1.2.7) are the interface conditions. As stated above, the interface conditions are linear and time invariant.

There are also sets of boundary conditions; they may be applied only under somewhat idealized conditions (for Maxwell's equations the interface conditions always apply). A convenient though somewhat idealized model of some real materials is that of a perfect electric conductor.[5] A discussion of the boundary conditions at a perfect electric conductor is given by Stratton;[6] they are limiting cases of the interface conditions.

i) tangential electric field is zero.

$$\hat{n} \times \vec{E}^+ = 0 \tag{1.2.8}$$

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[4]For the definition of outward normal, see J. W. Gibbs (The Scientific Papers of J. W. Gibbs, Vol. Two (New York: Dover, 1961), p. 32).

[5]A perfect electric conductor is a material in which the electric field is zero.

[6]Stratton, op. cit. pp. 483-484. In this case, it is not assumed that the surface current is zero; however, this is not a physically realizable problem.

ii) normal magnetic field is zero.

$$\hat{n} \cdot \vec{H}^{+} = 0 \ . \tag{1.2.9}$$

The above set of boundary conditions is considerably more convenient than the general interface conditions of equations (1.2.6) and (1.2.7).

Unfortunately, the above sets of conditions (partial differential equations and interface conditions) may still not be sufficient to completely determine the electromagnetic fields. A "boundary" condition may be necessary in the "far field" if the medium is unbounded.[7] That is to say, even though in the "far field" there may be no change in the medium, it may still be necessary to restrict the electromagnetic fields by placing added requirements on them. Intuitively, if the sources and virtual sources are bounded in extent, the fields far from these sources (i.e., far fields) must be "outward traveling waves." In fact, it can be shown in the far field that the electromagnetic waves are approximately outward traveling plane waves.

Mathematically, Sommerfeld's radiation condition[8] (for two-dimensional space) is

$$\lim_{\rho\to\infty} \sqrt{\rho}\ \left\{\frac{\partial}{\partial\rho}\varphi + ik\right\} = 0 \tag{1.2.10}$$

where $\rho = \sqrt{x^2 + y^2}$

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[7] Ibid, pp. 485-486.

[8] Sommerfeld, Arnold, Partial Differential Equations in Physics (New York: Academic Press, 1964), pp. 189-190.

Jones, D. S., The Theory of Electromagnetism, (Oxford: Pergamon Press, 1964), p. 93.

and (1.2.10) holds uniformly in direction. Equation (1.2.10) holds for a solution to the scalar wave equation and therefore holds for each rectangular component of the vector electromagnetic fields. The Sommerfeld radiation condition suffices when the sources (both real and virtual) are of finite extent. Unfortunately, in the problem considered here, both the sources and virtual sources are not of finite extent. In this somewhat more complex problem, the equations are obtained as limiting cases (the computations are given below in the section 1.4). Toward this end, a second, somewhat more general "radiation condition" is used. It is assumed that the medium has some losses ($\sigma > 0$; $k = k_1 - ik_2$; $k_1, k_2 \geq 0$); later the limit as σ tends to zero is taken to obtain the solution for a lossless medium.[9] The last requirement on the fields is that they must satisfy the source condition. This requirement is considered in detail in the section on integral equations. The problem of satisfying the partial differential equations, interface conditions, source conditions, and possibly some form of the radiation condition (i.e., finding fields that satisfy Maxwell's equations) is a mathematically well-set problem, the solution of which is unique.

As the use of monochromatic fields simplified the electromagnetic boundary value problem, so the use of vector potentials in many cases further simplifies the boundary value problem. The great power of the vector potential method

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[9]This view is taken by Baker and Copson, (Baker, B. B., and E. T. Copson, Mathematical Theory of Huygen's Principle, Second Edition (Oxford: Clarendon Press, 1953), p. 154.

(along with dividing the electromagnetic fields into TE and TM modes) is given by Schelkunoff[10] although the method is considerably older.[11] Mathematically, the starting point for obtaining the vector potentials is the equations for linear, homogeneous, isotropic and time invariant media (which is assumed below).

$$\operatorname{div} \vec{E} = 0 \rightarrow \vec{E} = \operatorname{curl} \vec{F} \quad \operatorname{div} \vec{H} = 0 \rightarrow \vec{H} = \operatorname{curl} \vec{A} \tag{1.2.11}$$

The general realtion, in a source-free region,[12] is:

$$\begin{aligned} \vec{E} &= -\operatorname{curl} \vec{F} + \frac{1}{Y} \operatorname{curl} \operatorname{curl} \vec{A} \\ \vec{H} &= \operatorname{curl} \vec{A} + \frac{1}{Z} \operatorname{curl} \operatorname{curl} \vec{F} \end{aligned} \tag{1.2.12}$$

However, an additional problem involved in the use of the vector potentials is the interface condition. In some cases the simplicity gained by having to consider only a single scalar component of the vector potential may be lost when the interface conditions are applied. In general, interface conditions are difficult to apply to potential functions (this can be done only in certain coordinate systems and only if the interface is a surface generated by one of the variables equal to a constant); even in the many problems where the interface can

[10] Schelkunoff, S. A., Electromagnetic Waves (New York, D. Van Nostrand, 1943), pp. 127-129.

[11] Browich, T. S., "Electromagnetic Waves," Phil. Mag., Vol. 38, 1919, pp. 144-164.

[12] Harrington, Op. cit., p. 129.

be satisfied, new components of the vector potential are needed to satisfy these conditions.[13]

In the two-dimensional problem, the vector potentials are used to compute the fields on the sea surface which, for the frequency considered, is assumed to be a perfect electric conductor. There is no cross-coupling between modes (TE and TM) and a single scalar field φ may be used to generate the complete electromagnetic fields of the mode. This can be seen from the equations given below which may be obtained from equation (1.2.12).

TM Case (TM to z)[14]

$$E_x = \frac{1}{Y}\frac{\partial^2}{\partial x \partial z}\varphi \qquad H_x = \frac{\partial \varphi}{\partial y}$$

$$E_y = \frac{1}{Y}\frac{2}{\partial y \partial z}\varphi \qquad H_y = -\frac{\partial \varphi}{\partial x}$$

$$E_z = \frac{1}{Y}\left(\frac{\partial^2}{\partial z^2} + k^2\right)\varphi \qquad H_z = 0 \qquad (1.2.13)$$

where

$$(\nabla^2 + k^2)\,\varphi = 0$$

$$k = -\sqrt{YZ}$$

in particular $k_o = \frac{\omega}{c}$

[13] Sommerfeld, op. cit., pp. 246-65.

[14] Harrington, op. cit., p. 129. In the simplest terms TM to z means $H_z = 0$, TE to z means $E_z = 0$.

TE Case (TE to z)

$$E_x = -\frac{\partial\varphi}{\partial y} \qquad H_x = \frac{1}{Z}\frac{\partial^2}{\partial x\partial z}\varphi$$

$$E_y = -\frac{\partial\varphi}{\partial x} \qquad H_y = \frac{1}{Z}\frac{\partial^2}{\partial y\partial z}\varphi$$

$$E_z = 0 \qquad H_z = \frac{1}{Z}\left(\frac{\partial^2}{\partial z^2} + k^2\right)\varphi \qquad (1.2.14)$$

where

$$(\nabla^2 + k^2)\,\varphi = 0.$$

The solutions to Maxwell's equation may then be generated by considering solutions of:

$$(\nabla^2 + k^2)\,\varphi = 0$$

with the appropriate boundary and source conditions.

1.3 Physical Model for the Two-Dimensional Problem

In this section a more detailed description of the two-dimensional problem is given. The problem considered may be stated as determining the electromagnetic fields in the sea when a VLF or ELF electromagnetic plane wave is incident on the sea surface. We shall restrict ourselves to the computation of the electromagnetic fields in the sea relatively near the sea surface (within about twenty-five meters or seventy-five feet) along with a discussion of the asymptotic behavior of the fields far from the sea surface. The electromagnetic fields near the interface are the fields of greatest interest in communication systems involving submerged antennas.

The first consideration in setting up the model for the above electromagnetic boundary value problem is to define or at least characterize the term "sea surface" mathematically. A discussion of the sea surface is given in Appendix A.

As the mathematical model of the electromagnetic boundary value problem with the rough sea surface it will be assumed here that:

1. The "basically spherical" earth may be replaced by a "basically flat" earth. This is an often made approximation in "low frequency" propagation problems and it reduces the complicated spherical geometry to a plane surface geometry. In the problem considered, only the local fields are of interest, and the above approximation is quite good;[15] however, it does introduce some mathematical complexity as now the virtual sources which represent the scattering effect of the sea surface are no longer bounded in extent.

2. We shall take as a deterministic mathematical description of the sea surface

$$y(x,z) = A \cos k_s x = \xi(x) \tag{1.3.1}$$

This is a surface that varies sinusoidally in the x-direction and is constant in the z-direction (see Figure 1). As discussed in Appendix A, this is a particular realization of a random process. Consideration of the statistical significances of this fact is postponed until Chapter 7. The

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[15]Anderson, loc. cit.

relative magnitudes of the physical parameters is of great importance in the solution of the integral equations governing the fields on the air side of the sea surface. The relationships are[16]

$k_s A \le 1/7$ usually $k_s A \ll 1$

$L \ll 1$ and $k_s \ll 1$

$\lambda \gg L$ and $k_o A \lll 1$.

Where L is wave length of sea wave,

$k_s = \frac{2\pi}{L}$ -- wave number of the sea surface

$k_o = \frac{2\pi}{\lambda}$ -- wave number of the radio wave in the air

λ is wave length of the radio wave in air.[17] We note that the assumed shape of the wave is time invariant (i.e., the sea surface is represented as a standing wave). In practice, sea waves are actually traveling waves that even change their shape with time. However, the velocity of the electromagnetic waves in air is so much greater than the sea wave velocity that as far as electromagnetic fields are concerned, the sea waves can be considered stationary. However, the velocity of electro-

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[16]See Appendix A for these relations.

[17]For the problem considered here, typical values of the parameters are: $10^8 \ge \lambda \ge 10^4$ meters

$10^3 \ge L \ge 10$ meters

$25 \ge A \ge 0$ meters

magnetic radiation is considerably less in sea than in air, and as a result the effect of the sea wave velocity is more pronounced in the sea than in the air. In Section 2.9, the effect of the sea wave velocity on electromagnetic fields in the sea is considered.

3. The source of the electromagnetic fields is a plane wave incident on the sea surface. One of the reasons for such an assumption is that in the theory of ELF and VLF radio wave propagation, plane waves play a major role. In both ray theory for VLF and mode theory for ELF, a general feature is the local fields are represented approximately as a sum of plane waves.[18]

The incident plane wave propagating in an arbitrary direction above the rough sea surface may be resolved into two component plane waves; one propagating in the direction in which the surface varies and the other propagating in the direction in which the surface is constant. A second resolution of the problem is made on the basis of the polarization of the incident field (see Figure 1). To solve completely the plane wave two-dimensional surface problem at a fixed frequency and for a fixed or given surface, four boundary value problems must be solved with arbitrary angles of incident for the source waves.

[18] The sky wave in ray theory is represented as a sum of rays which are approximately plane waves [see H. Bremmer, Terrestrial Radio Waves (New York: Elsevier, 1949), p. 89]. In mode theory, the modes are TE or TM plane waves. The ground wave is also usually represented as a plane wave.

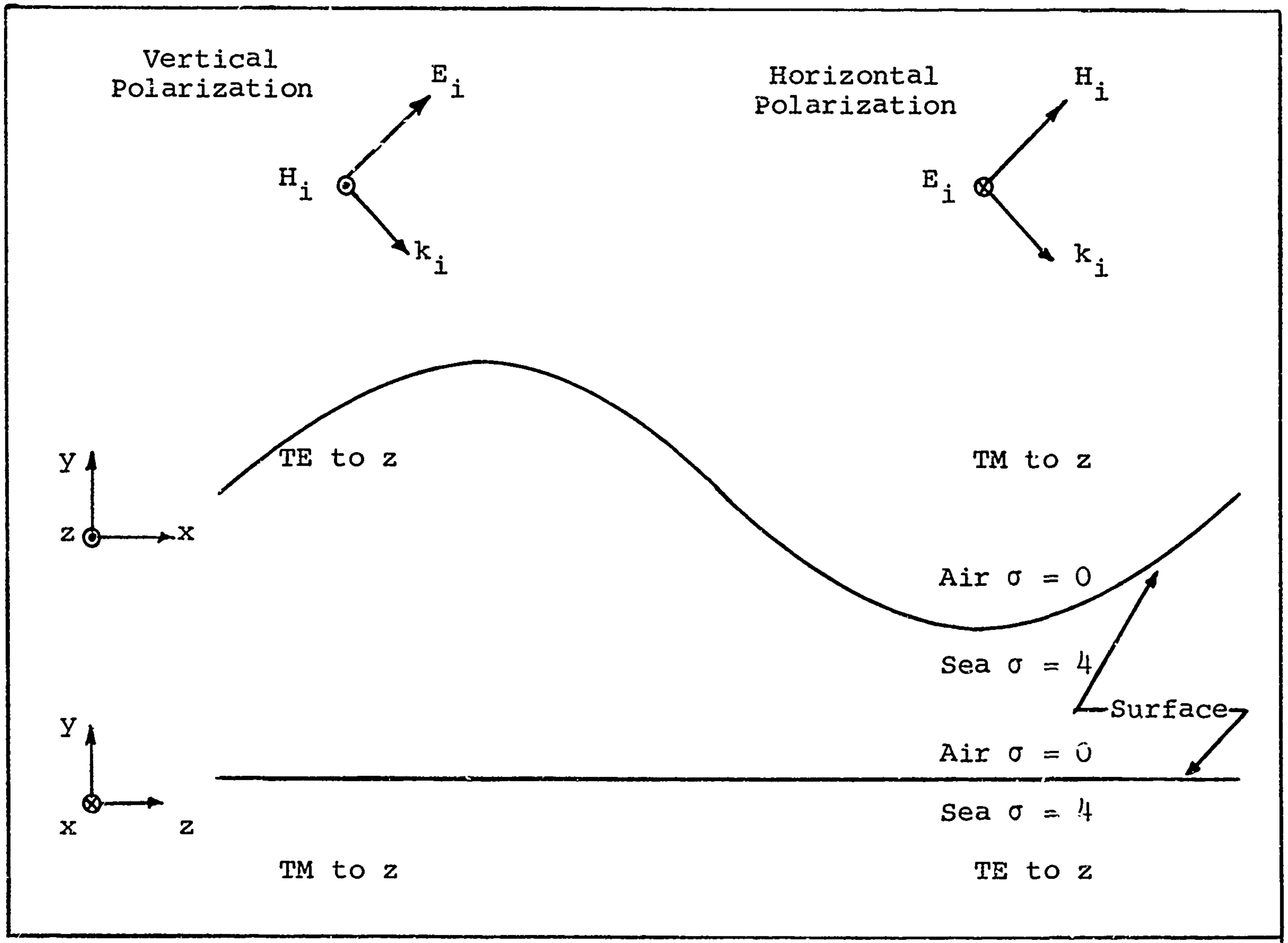

Figure 1.1. Diagram of Incident Plane Electromagnetic Waves.

It is to be noted that the use of plane waves as the sources introduces some mathematical complexity over the simple line source. The source condition for the plane wave source is computed as a limit of the source condition of a line source.

4. To compute the fields on the air side of the sea surface, the sea is assumed to be a perfect electric conductor. The sea is a very good conductor at the frequencies considered here.[19] The assumption that the surface impedance of the sea is negligible does not greatly effect the tangential magnetic fields which are used to compute the fields in the sea. Without this assumption the use of vector potentials would be greatly limited and a somewhat more complex problem would result. The justification of letting the surface impedance be negligible is considered later.

After computing the fields on the air side of the sea surface, the fields in the sea are considered on the basis that the tangential magnetic fields are continuous. The electromagnetic properties of the sea are assumed to be:

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[19] $\eta_o = 377$ -- "impedance" of air.

$$\eta_c = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}} \text{ -- "impedance" of sea.}$$

$$\approx \sqrt{\frac{j\omega\mu}{\sigma}} = (1+j)\ (.17) \quad f = 30 \text{ kc/s} \quad \sigma = 4 \quad \mu = \mu_o$$

as $\eta_c \sim \sqrt{f}$

$$|\eta_c| <<< \eta_o \qquad f \le 30 \text{ kc/s}$$

$$\sigma = 4 \left[\text{mhos/meter}\right]$$

$$\mu = 4\pi 10^{-7}\left[\text{henry/meter}\right]$$

$$\epsilon = \frac{9 \cdot 10^{-9}}{4\pi}\left[\text{farad/meter}\right]$$

Using assumptions one through four, the physical problem of the electromagnetic wave-rough sea surface interaction may be given a rigorous mathematical formulation.

1.4 Formulation of the Electromagnetic Boundary Value Problem in Terms of Integral Equations

In the past few decades there have appeared numerous works in which important results relating to the solutions of boundary value problems were obtained by techniques involving the use of integral equations.[20] Integral equations have been used extensively in the theoretical analysis of boundary value problems (e.g., proofs of theorems of existence, uniqueness, etc.).[21] However, integral equation techniques have only more recently been applied with success to the practical solution of

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[20] Mikhlin, S. G., Linear Integral Equations (Delhi: Hindustan Publishing Corp., 1960), pp. 145-213.

__________, Integral Equations (New York: Pergamon Press, 1957), pp. 137-333.

Courant, R., Methods of Mathematical Physics, Vol. II, Partial Differential Equations (New York: Interscience, 1962), pp. 240-320.

[21] See Mikhlin and Courant references directly above (footnote 21).

boundary value problems (e.g., the work on static elasticity).[22] These recent successes in the use of integral equation techniques have led to their application to problems which cannot be solved readily by other methods.

The same story basically holds true in electromagnetic theory. For electromagnetic fields, the usual integral formulation is in terms of the well-known formula of Stratton-Chu (integral representation of the solution to Maxwell's equations in terms of the boundary values).[23] However, only recently has the Stratton-Chu formula and the integral formulation of the reciprocity theorem[24] been used for the practical solution of electromagnetic boundary value problems.

One of the major advantages of the integral equation approach is that the boundary and/or interface conditions along with the source conditions are immediately considered at the beginning of the problem. Also, because the integral equations represent actual processes involved in the physical problem,

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[22]Muskbelishvilli, N. I., Singular Integral Equations (Netherlands, Groningen: Erven P. Noordhoff, 1953).

__________, Some Basic Problems in the Mathematical Theory of Elasticity (Netherlands, Groningen: Erven P. Noordhoff, 1953).

[23]Stratton, op. cit., pp. 464-468.

[24]Godziwski, Z., "The Surface Impedance Concept and the Structure of Radio Waves Over Real Earth (IEE, 1961).

Feinberg, E. L., "Propagation of Radio Waves Along an Inhomogeneous Surface," Nuovo Cinento, Series 10, Vol. 11, no. 1 Suppl., 1959, p. 66.

Harrington, op. cit., pp. 116-120, 317-338, 340-365.

the integral equations permit direct interpretation in terms of virtual sources.[25] The use of the concept of virtual sources is very useful in the mathematical formulation of the electromagnetic boundary value problem.[26] Further, by interpreting the integral equations in terms of actual physical processes, the problem of obtaining valid approximations is considerably reduced. Generally it is easier to see valid approximations in physical terms rather than in mathematical terms.

In a very practical vein, the integral equation method involves a reduction of "dimensionality," that is, a two-dimensional boundary value problem can be represented as an integral over a one-dimensional space; similarly, a three-dimensional boundary value problem leads to an integral in two dimensions. From the area of pure mathematical analysis, though it has great practical implications, integral operators are considerably easier to handle than differential operators, particularly when approximations are necessary.

The above reasons of course are not necessarily sufficient to cause all electromagnetic boundary value problems

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[25] Stratton, op. cit., p. 467.

Baker, B. B. and E. T. Copson, op. cit., p. 114.

[26] Waterman, P. C., "Scattering of Electromagnetic Waves by Conducting Surfaces," Wilmington, Mass.: Research and Advanced Development Division Avco corporation, Dec. 1962, an unpublished report.

to be formulated and solved in terms of integral equations. A major disadvantage of the integral formulation is that often to compute the fields at any point an integration is necessary. If only the far fields are needed, asymptotic expansion of the integral may be used to obtain the fields in closed form which does exhibit directly the variation of the field with respect to position. In the near field, an integration of an integral with a very complex kernel is needed for each point at which the field is to be computed; this is very laborious even with high speed computers.

For the rough surface problem considered, the integral methods of formulation are relatively convenient. The classical methods applied to differential operators (such as separation of variables) is extremely inconvenient, particularly in that the unknowns in this case are not susceptible to a physical interpretation that permits approximation. However, some general qualitative results can be obtained by arguments based on the separation of variables.

The usual starting point for the study of electromagnetic boundary value problems by use of integral equations is Green's theorem.[27] In two-dimensional space, Green's theorem takes the form

$$\int_S \left\{\varphi(\bar{\rho})\ \nabla^2\psi(\bar{\rho}) - \psi(\bar{\rho})\ \nabla^2\varphi(\bar{\rho})\right\}ds = \int_C \left\{\varphi(\bar{\rho})\psi_n(\bar{\rho}) - \varphi_n(\bar{\rho})\psi(\bar{\rho})\right\}d\ell \qquad (1.4.1)$$

[27] Kaplan, Wilfred, Advanced Calculus (Reading, Mass.: Addison-Wesley, 1953), p. 275.

where

$\bar{\rho} = (x,y)$ -- ordered pair of rectangular position coordinates in two-dimensional space.

$\varphi_n(\bar{\rho}) = \hat{n} \cdot \text{grad}\ \varphi(\bar{\rho})$ -- normal derivative.

φ, ψ are two arbitrary scalar functions of position with some requirements on their derivatives.

Surface S is bounded by a closed curve C.

The function ψ will be used as an auxiliary function and restrictions will be placed on φ such that it will be the solution of a scalar boundary value problem.[28] As a first step in the above process, let

$$(\nabla^2 + k^2)\ \psi(\bar{\rho}, \bar{\rho}') = -\ \delta(\vec{\rho}-\vec{\rho}')$$

$$(\nabla^2 + k^2)\ \varphi(\bar{\rho}, \bar{\rho}'') = -\ \epsilon_\varphi \delta(\vec{\rho}-\vec{\rho}'') \qquad (1.4.2)^{29}$$

where

$\vec{\rho} = x\ \hat{x} + y\ \hat{y}$ -- the position vector; i.e., φ and ψ satisfy the Helmholtz equation.

Placing restrictions (1.4.2) into (1.4.1)

$$\left\{\begin{array}{ll} \varphi(\bar{\rho}',\bar{\rho}'') & \bar{\rho}'\epsilon S \\ 0 & \rho' \not\epsilon (SUC) \end{array}\right\} = \left\{\begin{array}{ll} \epsilon_\varphi \psi(\bar{\rho}'',\bar{\rho}') & \bar{\rho}''\epsilon S \\ 0 & \bar{\rho}'' \not\epsilon (SUC) \end{array}\right\}$$

$$+ \int_C \left[\psi(\bar{\rho},\bar{\rho}')\ \varphi_n(\bar{\rho},\bar{\rho}'') - \varphi(\bar{\rho},\bar{\rho}'')\ \psi_n(\bar{\rho},\bar{\rho}')\right] d\ell \qquad (1.4.3)$$

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[28] The solution of Maxwell's equations (i.e., the electromagnetic fields) is obtained from the solution of the scalar boundary value problem.

[29] The $\delta(\vec{\rho}-\vec{\rho}')$ symbol represents a source term. The use of this notation is given in Friedman, Principles and Techniques of Applied Mathematics (New York: John Wiley, 1956), pp. 134-186, where the mathematical properties of the "δ - function" are given.

where

SUC denotes the union of the sets S and C (i.e., the points belonging to either S or C or both);

$\bar{\rho}\epsilon S$ denotes $\bar{\rho}$ is an element of S (i.e., $\bar{\rho}$ is a point in S.

Now to form an integral equation, let $\bar{\rho}' \to C$. The $\int_C$ is a "singular" or "discontinuous" integral. In Appendix C it is shown that

$$\lim_{\substack{\bar{\rho}'\to C\\ \bar{\rho}'\epsilon S\\ \bar{\rho}''\epsilon S}} \int_C \psi_n(\bar{\rho},\bar{\rho}')\varphi(\bar{\rho},\bar{\rho}'')d\ell = \int_{\substack{C\\ \bar{\rho}'\epsilon C\\ \bar{\rho}''\epsilon S}} \psi_n(\bar{\rho},\bar{\rho}')\varphi(\bar{\rho},\bar{\rho}'')d\ell - \frac{1}{2}\varphi(\bar{\rho}',\bar{\rho}'') \tag{1.4.4}$$

$$\lim_{\substack{\bar{\rho}''\to C\\ \bar{\rho}''\epsilon S\\ \bar{\rho}'\epsilon S}} \int_C \varphi_n(\bar{\rho},\bar{\rho}'')\psi(\bar{\rho},\bar{\rho}')d\ell = \int_{\substack{C\\ \bar{\rho}''\epsilon C\\ \bar{\rho}'\epsilon S}} \varphi_n(\bar{\rho},\bar{\rho}'')\psi(\bar{\rho},\bar{\rho}')d\ell - \frac{1}{2}\epsilon_\varphi\psi(\bar{\rho}',\bar{\rho}''). \tag{1.4.5}$$

Applying (1.4.4) to (1.4.3)

$$\varphi(\bar{\rho}',\bar{\rho}'') = 2\Big[\epsilon_\varphi\psi(\bar{\rho}'',\bar{\rho}') + \int_C\Big\{\varphi_n(\bar{\rho},\bar{\rho}'')\psi(\bar{\rho},\bar{\rho}') - \psi_n(\bar{\rho},\bar{\rho}')\varphi(\bar{\rho},\bar{\rho}'')\Big\}d\ell\Big] \tag{1.4.6}$$

$\bar{\rho},\bar{\rho}'\epsilon C \quad \bar{\rho}''\epsilon S$.

Equation (1.4.6) is the basic integral equation used below in the mathematical formulation of the electromagnetic boundary value problem. It holds only for closed curve C.

Equation (1.4.6) may be rewritten with $\psi(\bar{\rho},\bar{\rho}') = G(\bar{\rho},\bar{\rho}')$ (this is a restriction of the general equation), where $G(\bar{\rho},\bar{\rho}')$ is the "free-space Green's function."

$$G(\bar{\rho},\bar{\rho}') = \frac{1}{4} H_0^{(2)}(k|\vec{\rho}-\vec{\rho}'|) \tag{1.4.7}$$

$$\varphi(\bar{\rho}',\bar{\rho}'') = 2\Big[\epsilon_\varphi G(\bar{\rho}'',\bar{\rho}') + \int_C\Big\{\varphi_n(\bar{\rho},\bar{\rho}'')G(\bar{\rho},\bar{\rho}') - G_n(\bar{\rho},\bar{\rho}')\varphi(\bar{\rho},\bar{\rho}'')\Big\}d\ell\Big] \tag{1.4.8}$$

φ given by (1.4.8) satisfied the Helmholtz equation (1.4.2); a further requirement of the electromagnetic fields is that they must satisfy a source condition. In (1.4.8) the source condition is represented by the term with ϵ_φ as a factor. ϵ_φ is the strength of the source, $\epsilon_\varphi G(\bar{\rho}', \bar{\rho}'')$ represents the incident field, i.e., the field at $\bar{\rho}'$ due to source ϵ_φ at $\bar{\rho}''$ if no scatterer were present $\left\{\varphi_i(\bar{\rho}', \bar{\rho}'') = \epsilon_\varphi G(\bar{\rho}'', \bar{\rho}')\right\}$. The field represented by the integral then has the physical interpretation as the effect of the scatterer. The scattering term is immediately interpreted in terms of virtual sources on the scattering surface (or curve).

For the boundary value problem considered here the scattering surface (or curve) is of unbounded extent. The closed curve C representing the scatterer is composed of two parts:

i) $\bar{C}$ -- curve (surface) which is the mathematical representation of the sea surface,

ii) $C\infty$ -- the infinite semicircle (hemisphere) which closes $\bar{C}$.

For purposes of analysis, the field φ under consideration is divided into three parts.

$$\varphi = \varphi_i + \varphi_r + \varphi_s \tag{1.4.9}$$

where

φ_i -- the incident field (a plane wave in the problem considered here; however, the source condition for the plane wave will be obtained as a limit of source conditions for finite sources, i.e., the incident field considered just below is a cylindrical wave resulting from a finite line source).

φ_r -- the reflected field (a plane wave in the problem considered here. It is the reflected wave if the scatterer were a plane, i.e., the specularly reflected field basically).

φ_s -- the diffused scattered field (which represents the effect of the roughness of the scattering surface). It satisfies the Sommerfeld radiation condition.

The integral equation (1.4.8) will now be applied to each term of the sum on the right-hand side of (1.4.9). For the incident field φ_i

$$\begin{aligned}\varphi_i(\bar{\rho}',\bar{\rho}'') &= 2\Big[\varphi_i(\bar{\rho}',\bar{\rho}'') + \int_C \Big\{\varphi_{in}(\bar{\rho},\bar{\rho}'')G(\bar{\rho},\bar{\rho}') \\ &\qquad - G_n(\bar{\rho},\bar{\rho}')\varphi_i(\bar{\rho},\bar{\rho}'')\Big\}d\ell\Big] \qquad \text{(a)} \\ &= 2\Big[\varphi_i(\bar{\rho},\bar{\rho}'') + \int_{\bar{C}} \Big\{\varphi_{in}(\bar{\rho},\bar{\rho}'')G(\bar{\rho},\bar{\rho}') \\ &\qquad - G_n(\bar{\rho},\bar{\rho}')\varphi_i(\bar{\rho},\bar{\rho}'')\Big\}d\ell\Big] \qquad \text{(b)}\end{aligned} \tag{1.4.10}$$

as

$$\int_{C\infty} \Big[\varphi_{in}(\bar{\rho},\bar{\rho}'')G(\bar{\rho},\bar{\rho}') - G_n(\bar{\rho},\bar{\rho}')\varphi_i(\bar{\rho},\bar{\rho}'')\Big\}d\ell\Big] = 0 \tag{1.4.11}$$

by Sommerfeld's radiation condition.

(1.4.11) need only hold if the sources are of finite extent. To obtain the equivalent integrals (i.e., the integral representing the incident field) to (1.4.10 and 1.4.11) when the sources are unbounded in extent (i.e., the source or incident field is a plane wave), a limiting process is necessary.

To start the limiting process, let

$$\varphi_i(\bar{\rho}',\bar{\rho}'') = \sqrt{\frac{\pi k \xi}{2i}}\, e^{ik\xi} H_0^{(2)}\left\{k(|\vec{\rho}\,' - \hat{k}_i \xi|)\right\} \qquad (1.4.12)$$

$$\vec{\rho}\,'' = \hat{k}_i \xi$$

where $\hat{k}_i$ is to be the unit vector in the direction of propagation of the incident plane wave, and ξ is a parameter used in the limiting process.

Equation (1.4.12) represents the field caused by a line source at $\vec{\rho}\,''$ of strength $\sqrt{8i\pi k\xi}\, e^{ik\xi}$. As intuitively expected, the source point distance from the observation point must become unbounded (tend toward infinity) and the source strength also becomes unbounded to obtain a plane wave at the observation point. Both the above occur as $\xi \to \infty$.

Applying the limit $\xi \to \infty$ to equation (1.4.11) for the φ_i given by equation (1.4.12)

$$0 = \lim_{\xi\to\infty} \int_{C_\infty} \left\{\varphi_{in}(\bar{\rho},\bar{\rho}'')G(k|\vec{\rho}-\vec{\rho}\,'|) - G_n(k|\vec{\rho}-\vec{\rho}\,'|)\varphi_i(\bar{\rho},\bar{\rho}'')\right\}d\ell \qquad (1.4.13)$$

$$= \int_{C_\infty} \left\{-i\hat{k}_i\cdot\hat{n}e^{+ik\hat{k}_i\cdot\vec{\rho}\,'}G(k|\vec{\rho}-\vec{\rho}\,'|) - G_n(k|\vec{\rho}-\vec{\rho}\,'|)e^{+ik\hat{k}_i\cdot\vec{\rho}}\right\}d\ell \qquad (1.4.14)$$

as

$$\lim_{\xi\to\infty} \varphi_i(\bar{\rho}') = e^{+ik\hat{k}_i\cdot\vec{\rho}\,'}$$

a plane wave.

$$\lim_{\xi\to\infty} \varphi_{in'}(\bar{\rho}') = i\hat{n}'\cdot\hat{k}_i k e^{+ik\hat{k}_i\cdot\vec{\rho}\,'}$$

Equation (1.4.14) represents the relation

$$0 = \int_{C_\infty} \left\{\varphi_i(\bar{\rho})G_n(\bar{\rho},\bar{\rho}') - \varphi_{in}(\bar{\rho})G(\bar{\rho},\bar{\rho}')\right\}d\ell \qquad (1.4.15)$$

which implies

$$0 = \varphi_i(\bar{\rho}') + 2\int_{\bar{C}} \left\{\varphi_i(\bar{\rho})G_n(\bar{\rho},\bar{\rho}') - \varphi_{in}(\bar{\rho})G(\bar{\rho},\bar{\rho}')\right\}d\ell \qquad (1.4.16)$$

where φ_i is a plane wave.

Then (1.4.10) holds even if the source is unbounded in extent, i.e., φ_i is a plane wave.

For the reflected field φ_r

$$\varphi_r(\bar\rho) = 2\int_C \left\{\varphi_{rn'}(\bar\rho')G(\bar\rho,\bar\rho') - G_{n'}(\bar\rho,\bar\rho')\varphi_r(\bar\rho')\right\}d\ell' \quad \text{(a)}$$

$$= 2\int_{\bar C} \left\{\varphi_{rn'}(\bar\rho')G(\bar\rho,\bar\rho') - G_{n'}(\bar\rho,\bar\rho')\varphi_r(\bar\rho')\right\}d\ell' \quad \text{(b)}$$

(1.4.17)

as

$$\int_{C\infty} \left\{\varphi_{rn'}(\bar\rho')G(\bar\rho,\bar\rho') - G_{n'}(\bar\rho,\bar\rho')\varphi_r(\bar\rho')\right\}d\ell' = 0 \qquad (1.4.18)$$

by second radiation condition.

For the scattered field φ_s

$$\varphi_s(\bar\rho) = 2\int_C \left\{\varphi_{sn'}(\bar\rho')G(\bar\rho,\bar\rho') - G_{n'}(\bar\rho,\bar\rho')\varphi_s(\bar\rho')\right\}d\ell' \quad \text{(a)}$$

$$= 2\int_{\bar C} \left\{\varphi_{sn'}(\bar\rho')G(\bar\rho,\bar\rho') - G_{n'}(\bar\rho,\bar\rho')\varphi_s(\bar\rho')\right\}d\ell' \quad \text{(b)}$$

(1.4.19)

as

$$\int_{C\infty} \left\{\varphi_{sn'}(\bar\rho')G(\bar\rho,\bar\rho') - G_{n'}(\bar\rho,\bar\rho')\varphi_s(\bar\rho')\right\}d\ell' = 0 \qquad (1.4.20)$$

by Sommerfeld's radiation condition.

Adding equations (1.4.16), (1.4.17), and (1.4.19),

$$\varphi_r(\bar\rho) + \varphi_s(\bar\rho) = \varphi_i(\bar\rho) + 2\int_{\bar C} \left\{\varphi_{n'}(\bar\rho')G(\bar\rho,\bar\rho') - G_{n'}(\bar\rho,\bar\rho')\varphi(\bar\rho')\right\}d\ell' \quad \text{(a)}$$

$$\varphi(\bar\rho) = 2\left\{\varphi_i(\bar\rho) + \int_{\bar C} \left\{\varphi_{n'}(\bar\rho')G(\bar\rho,\bar\rho') - G_{n'}(\bar\rho,\bar\rho')\varphi(\bar\rho')\right\}d\ell\right\}. \quad \text{(b)}$$

(1.4.21)

Equation (1.4.21) is the integral equation for the field φ such that

1. φ satisfies the Helmholtz equation (1.4.2) with $\epsilon_\varphi = 0$.
2. φ satisfies the source condition (i.e., the incident

field is a plane wave).

3. φ satisfies the necessary radiations (as seen above).

To compute the mathematical formulation of the boundary value problem, it remains only to apply the necessary boundary or interface conditions required of φ on the interface C. While the complete boundary value problem may be formulated by setting up the integral equations on each side of the interface and applying the interface conditions; it is very convenient to assume the lower half-space bounded by $\overline{C}$ is a perfect conductor. The effect of this approximation is given below. The above assumption changes the interface condition to a boundary condition. It is also convenient to consider the field divided into two parts:

1. TE to z.
2. TM to z.

The TM to z part may be generated by a single component of the vector potential $\vec{A}$. For the basic equations, see Harrington[30] or Section 2.3. The boundary condition is $\varphi = 0$. This yields $E_{tan} = 0$ and $H_{normal} = 0$. φ_n is related to the tangential H field.

The TE to z part may be generated by a single component of the vector potential $\vec{F}$. The boundary condition is $\varphi_n = 0$; this again implies $E_{tan} = H_{normal} = 0$.

The boundary value problem may then be formulated in terms of a single scalar.

The TM case, the integ l equation is:

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[30] Harrington, op. cit., p. 129.

$$\int_{\overline{C}} \varphi_{n'}(\bar{\rho}')G(k_o|\vec{\rho}-\vec{\rho}'|)d\ell' = \varphi_i(\bar{\rho}) \ . \qquad (1.4.22)$$

The TE case, the integral equation is:

$$\varphi(\bar{\rho}) = 2\varphi_i(\bar{\rho}) + 2\int G_{n'}(k_o|\vec{\rho}-\vec{\rho}'|\varphi(\bar{\rho}')d\ell'. \qquad (1.4.23)$$

(1.4.22) and (1.4.23) are the integral equations to be solved in the next chapter.

2.0 SOLUTION OF THE INTEGRAL EQUATIONS

2.1 Introduction.

In this chapter, integral equations (1.4.22) and (1.4.23) will be solved. From these solutions, the fields on the air-side of the sea surface can be found. Also considered are the integral equations in the sea. From the approximate solution of the integral equations in the sea comes an impedance type relation between φ and φ_n which holds in a restricted frequency range and for a restricted class of surfaces. Using this impedance relation, a simplified integral representation in the sea is given. Then a brief comparison of the solutions obtained here with those obtained by other methods is presented (a more detailed discussion of other methods used to solve "rough surface" problems is given in Appendix B). The following sections are devoted to the question of the validity of some of the assumptions used (such as, if the sea can be considered a perfect electric conductor in determining the fields in the air). Lastly considered is the effect of the motion of the sea surface on the fields in the sea. Wait's[1] approximate solution is used in this discussion, as the use of the more accurate numerical solution would only complicate the discussion without adding any new information.

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[1]Wait, loc. cit.

A recent review work[2] considering the practical solutions of integral equations gives as two of the nine or so available methods

1. Methods of finite differences and sums.

2. Method of degenerate kernel.

Although numerical computations using finite difference methods have recently been shown to be a useful tool in the solution of integral equations, such computations proved to be impractical for the solution of integral equations considered here. The main reasons for the difficulty were:

1. The curve $\overline{C}$ is unbounded. The integral equations solved by the finite difference method have had a finite interval for the range of integration.

2. The kernel of the integral equations does not decrease rapidly enough with distance from the source point to allow useful approximations (i.e., cutting off the range of integration to a small interval about the source point). The computations therefore could not be made sufficiently detailed to yield a useful solution.

[2] Walther, A., "General Report on the Numerical Treatment of Integral and Integro-Differential Equations," Symposium on the Numerical Treatment of Ordinary Differential Equations, Integral and Integro-Differential Equations -- Proceedings of the Rome Symposium, 1960, Organized by the PICC (Basel, Berkhauser, 1960), p. 649.

For the solution of one of the integral equations (1.4.22), however, a modified form of the method of degenerate kernel was found to be the most practical. The general solution, using the degenerate kernel method, is to expand (and approximate) the kernel of the integral equation as follows:[3]

$$K(x,\xi) \approx \sum_{i=1}^{N} u_i(x)v_i(\xi) \tag{2.1.1}$$

$$y(x) = t(x) + \lambda \int K(x,\xi)y(\xi)d\xi \tag{2.1.2}$$

(2.1.2) is the integral equation to be solved.

$$y(x) = t(x) + \sum_{i=1}^{N} c_i u_i(x) \tag{2.1.3}$$

(2.1.3) is the solution of integral equation (2.1.2) with the approximation (2.1.1) where c_i is the solution of a system of N linear equations. In the problem considered below, this approximation is modified and takes the form

$$K(x,\xi) \approx K(x-\xi). \tag{2.1.4}$$

With the aid of this approximation, the resulting integral equations may be solved by classical methods (using Fourier Series).[4] The approximation is valid because of the relative values of the physical parameters of the problem.

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[3] *Ibid*, p. 654.

[4] Morse, Philip M. and Herman Feshbach, *Methods of Theoretical Physics*, McGraw-Hill, New York, 1953, Part 1, pp. 960-962.

The solution of (1.4.23) is approximately the unperturbed φ, as is easily verified.

The solution to the integral equations in the sea is based on the high rate of attenuation for electromagnetic waves in sea water.

2.2 TM Case.

The integral equation to be solved in the TM case is

$$\int_C \varphi_{n'}(\bar{\rho}')\, G(k_o|\vec{\rho}-\vec{\rho}'|)d\ell' = \varphi_i(\bar{\rho}) , \qquad (2.2.1)$$

where: k_o - wave number of the electromagnetic wave (in air),

$G(k|x|) = \frac{i}{4} H_o^{(2)}(k|x|)$ - the free space (two-dimensional) Green's function.

It can be shown that if

$$(\xi')^2_{max} << 1 \qquad (k_o\xi)_{max} << 1$$

where: $\xi(x)$ is the equation of the sea surface,

$\xi'(x)$ is the derivative of ξ (i.e., the slope of the sea surface) with respect to x.

Then

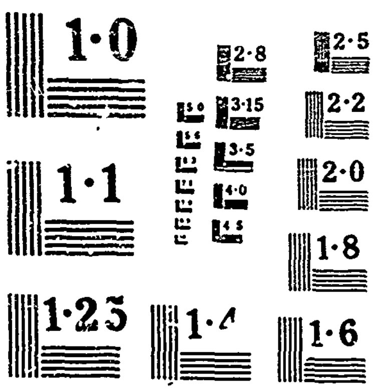
$$\int_C \varphi_{n'}(\bar{\rho}')\, G(k_o|\vec{\rho}-\vec{\rho}'|)d\ell' \approx \int_C \varphi_{n'}(\bar{\rho}')G_f(k_o|\vec{\rho}-\vec{\rho}'|)d\ell' , \qquad (2.2.2)$$

where:

$$G(k_o|\vec{\rho}-\vec{\rho}'|) = \frac{i}{4} H_o^{(2)}(k_o\sqrt{(x-x')^2 + (\xi(x) - \xi(x'))^2})$$

$$G_f(k_o|\vec{\rho}-\vec{\rho}'|) = \frac{i}{4} H_o^{(2)}(k_o|x-x'|) .$$

CONT.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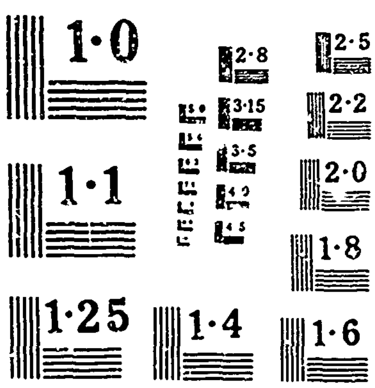
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The argument leading to this result is presented in the discussion on the three-dimensional problem.[5]

The integral equation then becomes

$$\int_C \varphi_{n'}(\bar{\rho}')\, G_f(k_o|\vec{\rho}-\vec{\rho}'|)d\ell' = \varphi_i(\bar{\rho}) \ . \qquad (2.2.3)$$

This equation may then be solved by Fourier Series. To solve the integral equations for $\varphi_{n'}$, let

$$\varphi_n(\bar{\rho}) = \sum_{n=-\infty}^{\infty} a_n e^{-ik_n x} \quad e^{-ik_{xi}x} = \sum_{n=-\infty}^{\infty} a_n e^{-ink_s x} \quad e^{-ik_{xi}x} \qquad (2.2.4)$$

$$\varphi_i(\bar{\rho}) = \sum_{m=-\infty}^{\infty} b_m e^{-ik_m x} \quad e^{-ik_{xi}x} = \sum_{m=-\infty}^{\infty} b_m e^{-imk_s x} \quad e^{-ik_{xi}x} \qquad (2.2.5)$$

where $k_s = 2\pi/L$

$\bar{\rho}\epsilon C$.

The validity of equations (2.2.4) and (2.2.5) is established by considering the motivation behind such expansions. The factor $e^{-ik_{xi}x}$ is present because the incident field (source) has such a factor. The sum expresses the fact that the periodic surface implies a periodic virtual source (besides the factor $e^{-ik_{xi}x}$) which in turn implies a periodic field. The expansions (2.2.4 and 2.2.5) are the most general mathematical statement of this periodicity of the fields.

---

[5]See page 121. This argument is given for the two-dimensional problem by Meecham, loc. cit., and Lysanov, loc. cit.

It should be noted that

$$\int_C G_f(k_o|\vec{\rho}-\vec{\rho}'|)\, e^{-i(nk_s + k_{xi})x'}\,dx' = e^{-ik_{xi}x}\left(\frac{1}{2\sqrt{(nk_s+k_{xi})^2 - k_o^2}}\right). \qquad (2.2.6)^6$$

Then substituting equations (2.2.4) and (2.2.5) into the integral equation (2.2.3) and using formula (2.2.6), the relations between the sets of a_n's and b_n's may be found by equating the coefficients of like terms in the series.

---

[6] Equation (2.2.6) is obtained from the relations given above.

$$\int_{-\infty}^{\infty} H_o^{(2)}(k_o|x|)e^{-ikx}\,dx = 2\int_o^{\infty} H_o^{(2)}(kx)\cos kx\,dx$$

$$\int_o^{\infty} J_o(k_o x)\cos kx\,dx = \begin{cases} \dfrac{1}{\sqrt{k_o^2 - k^2}} & 0 < k < k_o \\ \infty & k = k_o \\ 0 & k_o < k < \infty \end{cases}$$

(Bateman Manuscript Project, Vol. I, McGraw-Hill, New York, 1954; p. 43, equation No. 1.)

$$\int_o^{\infty} N_o(k_o x)\cos kx\,dx = \begin{cases} 0 & 0 < k < k_o \\ \dfrac{-1}{\sqrt{k^2 - k_o^2}} & k_o < k < \infty \end{cases}$$

(Bateman Manuscript Project, Vol. I, McGraw-Hill, New York, 1954; p. 47, equation No. 28.)

Therefore, equation (2.2.6) is obtained

The relation between the a_n's and b_n's is

$$a_n = b_n \left(2\sqrt{(nk_s + k_{xi})^2 - k_o^2}\right) \qquad \text{(a)}$$

$$a_o = 2ik_{yi}b_o$$

$$a_{-n} = a_n \approx +2nk_s b_n$$

$$\varphi_i = e^{-ik_{xi}x}\left(e^{ik_{yi}\xi(x)}\right) \qquad \xi(x) = A\cos(k_s x)$$

$$b_o = 1 - \frac{1}{2}(k_{yi}A)^2 + \dots \approx 1$$

$$b_1 = b_{-1} = \frac{ik_{yi}A}{2} - i\frac{1}{12}(k_{yi}A)^3 + \dots \approx ik_{yi}\frac{A}{2}$$

$$b_{-n} = b_n \approx \frac{1}{2}(k_{yi}A)^n$$

$$a_o \simeq 2ik_{yi} \qquad \text{(b)} \qquad (2.2.7)$$

$$a_{-1} = a_1 \simeq ik_s\, k_{yi}A \qquad \text{(c)}$$

$a_n \simeq n(k_{yi}A)^n \quad n > 1$ which may be neglected because of the factors $(k_{yi}A) \lll 1$.

$$\int_{-\infty}^{\infty} H_o^{(2)}(k_o|x|)e^{-ikx}\,dx = \begin{cases} \dfrac{-2i}{\sqrt{k^2 - k_o^2}} & k_o \neq k \\ \infty & k_o = k \end{cases}$$

It is assumed that $k_{xi} + nk_s \neq k_o$ for any integer n (note this assumes $k_{xi} \neq k_o$ for $n = 0$). This assumption is useful in avoiding the question of the convergence of the integrals above. The physical phenomena occurring when $k_s = k_{xi}$ leads to a "resonance" in the fields. This "resonance" is a physical occurrence; however, in some mathematical formulations the fields become infinite, which is not a physical possibility.

$$\varphi_n \simeq 2ik_{yi}(1 + k_sA \cos k_sx)e^{-ik_{xi}x} = 2\varphi_{ni}(x)_f \left\{1 + k_s\xi(x)\right\} \quad (2.2.8)$$

where $\varphi_{ni}(x)_f$ is the normal derivative of φ_i on the flat surface. The perturbation compared to the flat interface is $2k_sA$ which is [illegible] than about thirty per cent.[7]

2.3 TE Case.

In this case, the integral equation becomes:

$$\varphi(\bar{\rho}) = 2\varphi_i(\bar{\rho}) + 2\int_C \varphi(\bar{\rho}')G_n(k|\vec{\rho}-\vec{\rho}'|)d\ell' \quad (2.3.1)$$

In the flat interface case, the solution is

$$\varphi(\bar{\rho}) = 2\varphi_i(\bar{\rho}) \; . \quad (2.3.2)$$

Using the static kernel (Green's function) and placing the above approximation (equation 2.3.2) into the integral, the integral becomes zero. Therefore, no correction is necessary. The justification of the use of the static kernel is considered in Section (2.8). In this case, the solution is

$$\varphi(\bar{\rho}) = 2\varphi_i(\bar{\rho}) \; . \quad (2.3.3)$$

This is equivalent to the condition $H_{tan} = 2(H_{tan})i$.

---

[7] As $H_{tan} = \varphi_n$, the perturbation (maximum derivation) in H_{tan} is $2k_sA$ or about 28% for $k_sA \leq 1/7$.

2.4 Integral Equations in the Sea.

For the fields in the sea, the integral equation (1.4.21) still holds only with $\varphi_i = 0$.

$$\varphi(\bar{\rho}) = 2\int_C \left\{\varphi_{n'}(\bar{\rho}')\, G(k_c|\vec{\rho}-\vec{\rho}'|) - \varphi(\bar{\rho}')G_{n'}(k_c|\vec{\rho}-\vec{\rho}'|)\right\} d\ell' \tag{2.4.1}$$

where $k_c = \frac{1+i}{\delta}$ - the complex wave numbe_ in the sea for electromagnetic waves.

In the VLF range, for usual sea water ($\sigma = 4$, $\mu = \mu_o$), $|k_c| \leq .3$. This implies G and G_n tend rapidly to zero for $|\vec{\rho}-\vec{\rho}'| > 0$. In the order of 10 meters G and G_n become negligible.[8] If φ and φ_n are relatively constant on the sea surface (i.e., φ and φ_n are relatively constant for distances of the order of ten meters or so (3δ) along the sea surface) and the sea surface is relatively constant over these distances; then:

---

[8] To estimate the accuracy of neglecting G and G_n after a distance of about $\lambda_c/2$ from the "source point" the asymptotic approximations to G and G_n will be used (the errors in the asymptotic approximations are sufficiently small to be neglected here).

$$G \sim \frac{i}{4}\sqrt{\frac{2i}{\pi k_c \rho}}\; e^{-ik_c\rho}$$

$$G_n \sim \frac{i}{4}\sqrt{\frac{2i}{\pi k_c \rho}}\;(-ik_c)\, e^{-ik_c\rho}\; \rho_n$$

$$\int_o^\infty G(k_c x)dx = \frac{i}{2k_c}$$

$$\int_C \varphi_{n'}(\bar{\rho}')\ G\ (k_c|\vec{\rho}-\vec{\rho}'|)d\ell' \approx \frac{i}{2k_c}\ \varphi_n(\bar{\rho}) \tag{2.4.2}$$

$$\int_C \varphi(\bar{\rho}')\ G_{n'}\ (k_c|\vec{\rho}-\vec{\rho}'|)d\ell' \approx \varphi(\bar{\rho})\ \frac{i}{2k_c R} \tag{2.4.3}$$

$$\left|\int_{\lambda_c/2}^{\infty} G(k_c x)\ dx\right| \le \frac{1}{2|k_c|}\int_{\sqrt{2\pi}}^{\infty} \Phi(x)dx = \frac{1}{2|k_c|}\ (.0062)$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\ e^{-x^2/2}$$ – normal distribution

$$\left|\int_{\lambda_c/2}^{\infty} G(k_c x)dx\right| \le c_1\left|\int_0^{\infty} G(k_c x)dx\right| \quad c_1 = .003$$

$$\int_0^{\infty} G_n(k_c x)dx \approx \frac{1}{2k_c R}$$

$$\left|\int_0^{\infty} G_n(k_c x)dx\right| << 1$$

$$\left|\int_{\lambda_c/2}^{\infty} G_n(k_c x)dx\right| \le \int_{\sqrt{2\pi}}^{\infty} \Phi(x)dx \le .006 << 1$$

$$\left|\int_{\lambda_c/2}^{\infty} G_n(k_c x)dx\right| \le C_2\ \left|\int_0^{\infty} G_n(k_c x)dx\right|$$

$C_2 = .01|k_c|R.$

For many cases C_2 may be the order of 1 or greater; this occurs as $R \to \infty$. While the important contribution to the integral no longer occurs near the "source point" the result is the same; that is

$$\left|\int_{-\infty}^{\infty} \varphi(\bar{\rho}')\ G_{n'}(k_c|\vec{\rho}-\vec{\rho}'|)\ d\ell'\right| << 1$$

which is all that is required for the impedance boundary condition.

where R - radius of the curve.[9]

For $|k_c|R >> 1$ which holds in the VLF range and for the sea states considered:

$$\varphi = \frac{i}{k_c}\,\varphi_n \;. \tag{2.4.4}$$

This is an impedance type relation between φ and φ_n on the sea surface.

---

[9] To estimate $\int_{-\infty}^{\infty}\varphi(\bar{\rho}')\,G_{n'}(k_c|\vec{\rho}-\vec{\rho}'|)\,d\ell'$, again $\varphi(\bar{\rho}')$ is assumed constant over the meaningful range of integration. For small $|\vec{\rho}-\vec{\rho}'|$ (i.e. $\frac{|\vec{\rho}-\vec{\rho}'|}{\lambda_c} << 1$) the static Green's function may be used.

$$G_{n'}(k_c\rho) \approx \hat{n}'\cdot\text{grad}\left\{\frac{1}{2\pi}\,\ell n\,\rho\right\} = \frac{1}{2\pi}\,\frac{1}{\rho}\,\rho_n$$

$$\rho_n = \frac{(-\xi'\hat{x} + \hat{y})}{\sqrt{1+(\xi')^2}} \cdot \frac{(x\hat{x} + \xi\hat{y})}{\sqrt{x^2+\xi^2}}$$

then for $\rho \approx 0$

$$G_{n'}(k_c\rho) \approx \frac{1}{2\pi}\;\frac{-\xi' x + \xi}{\sqrt{1+(\xi')^2}\,(x^2+\xi^2)}$$

The Taylor series expansions of $\xi(x)$ may be used for x small

$$G_{n'}(k_c\rho) \approx \frac{1}{2\pi}\;\frac{\frac{1}{2}\xi'' x^2}{\sqrt{1+(\xi')^2}\;\;x^2(1+(\xi')^2)} = \frac{1}{2\pi}\;\frac{1}{2R}$$

$$R = \frac{(1+(\xi')^2)^{3/2}}{\xi''}$$ - radius of curvature of the curve (surface).

As the frequency decreases into the lower ELF range, $|k_c|$ decreases and for some sea surfaces the relation $|k_c| R \gg 1$ no longer holds. The impedance relationship between φ and φ_n, however, may still be an excellent approximation.

This relationship (2.4.4) is a Leontovich type of boundary condition,[10] and may be expressed in terms of the

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Using the above result and the assumption ξ'' is relatively constant over the meaningful range of integration (i.e., $L \gg \lambda_c$) for the problem considered here:

$$\int_{-\infty}^{\infty} \varphi(\bar{\rho}') \, G_{n'} \, (k_c|\vec{\rho}-\vec{\rho}|) d\ell = \varphi(\bar{\rho}) \frac{i}{4} \, 2 \int_0^{\infty} H_1^{(2)}(k_c \rho) k_c \rho_n \, d\ell$$

$$\approx \varphi(\bar{\rho}) \frac{i}{2} \frac{k_c}{R} \int_0^{\infty} x \, H_1^{(2)}(k_c x) \, dx$$

$$\approx \varphi(\bar{\rho}) \frac{i}{2R} k_c \frac{1}{k_c^2}$$

$$\approx \varphi(\bar{\rho}) \frac{i}{2k_c R}$$

So that if ξ'' and φ are relative constants over a distance of $\lambda_c/2$ or so along the surface, then:

$$\int_{-\infty}^{\infty} \varphi(\bar{\rho}') \, G_{n'} \, (k_c|\vec{\rho}-\vec{\rho}'|) d\ell' \approx \varphi(\bar{\rho}) \frac{i}{2k_c R}$$

[10] Leontovich, loc. cit.

Brekhovskikh, Leonid M., Waves in Layered Media, translated by David Liberman, Academic Press, New York, 1960, pp. 14-15.

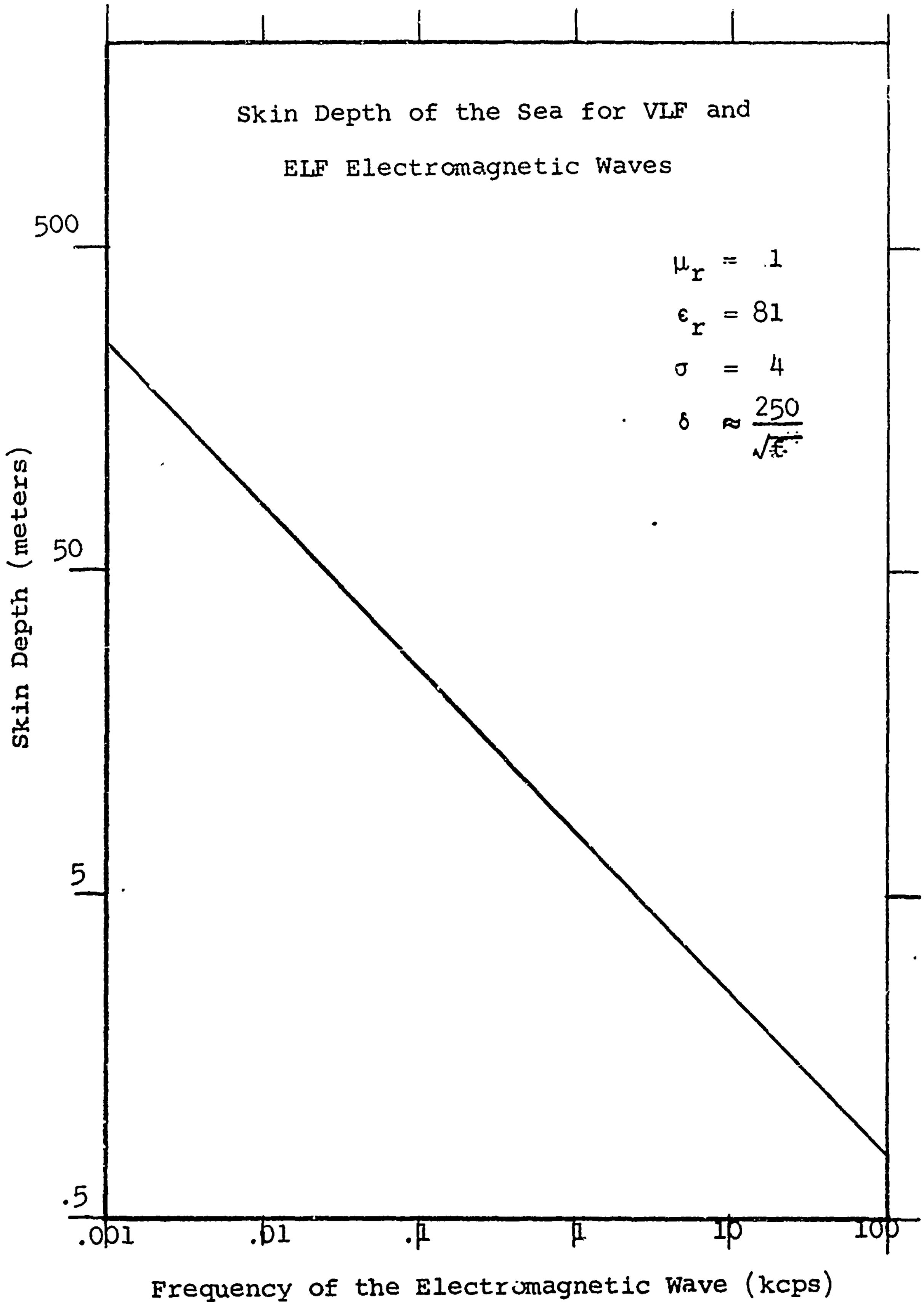

Figure 2.1. Skin Depth of the Sea

fields as:

$$E_{tan} = \eta_c \, H_{tan} \, . \tag{2.4.5}$$

This is true for the fields in the sea; however, as the tangential fields are continuous at an interface (2.4.5), it also holds for the fields in the air.

2.5 Simplified Integral Representation in the Sea.

The usual integral representation of a wave function (with $\varphi_i = 0$) is

$$\varphi(\bar{\rho}') = \int_C \left\{ G(k_c|\vec{\rho}-\vec{\rho}'|)\, \varphi_n(\bar{\rho}) - \varphi(\bar{\rho})\, G(k_c|\vec{\rho}-\vec{\rho}'|) \right\} d\ell \tag{2.5.1}$$

which includes terms representing virtual sources of two types:

$\varphi_1(\bar{\rho}') = \int_C \left\{ G(k_c|\vec{\rho}-\vec{\rho}'|)\, \varphi_n(\bar{\rho}) \right\} d\ell$ represents the fields of a "single layer" source, (2.5.2)

$\varphi_2(\bar{\rho}') = \int_C \left\{ G_n(k_c|\vec{\rho}-\vec{\rho}'|)\ \ \varphi(\bar{\rho}) \right\} d\ell$ represents the fields of a "double layer" source (dipole). (2.5.3)

In electromagnetic problems, these types of sources are the electric current and the magnetic current sources.[11] It would be of interest if a simplified integral representation of the wave function could be found, particularly if the fields were to be evaluated by numerical calculation of the

[11] Stratton, op. cit., p. 467.

integral. The fields of a "double layer" are particularly troublesome, because the calculation of the normal derivatives involves use of the derivative of the equation for the curve C. If there exists some reason to believe the fields are relatively insensitive to changes in the surface, a simplified representation would be possible; however, if the field depended greatly on the shape of the surface, no appreciable simplification would be possible. In the problem considered, the fact that the electromagnetic wave is attenuated rapidly in sea leads to the belief that a simplified representation should be possible.

Consider representing $\varphi(\bar{\rho}')$ using sources of a single layer only,

$$\varphi(\bar{\rho}') = \int_C f(\bar{\rho})\, G(k_c|\vec{\rho}-\vec{\rho}'|)d\ell \; . \tag{2.5.4}$$

If, in the sea, f can be considered reasonably constant the $\int_C$ is given approximately by

$$\varphi(\bar{\rho}) \approx \frac{i}{2k_c} f(\bar{\rho}) \tag{2.5.5}$$

or

$$f(\bar{\rho}) = \frac{2k_c}{i} \varphi(\bar{\rho}) \; . \tag{2.5.6}$$

Equation (2.5.6) yields the function f needed such that the representation of equation (2.5.4) tends to correct boundary value on $\bar{C}$. However, this does not mean (2.5.4) is a correct representation of the solution to the boundary value problem. $\varphi_n(\bar{\rho}')$ has yet to be considered.

$$\varphi_n(\bar{\rho}') = \int_{\bar{C}} f(\bar{\rho})\, G_n(k_c|\vec{\rho}-\vec{\rho}'|)d\ell$$

$$\approx \frac{1}{2} f(\bar{\rho}') = \frac{1}{2} \left(\frac{2k_c}{i}\right) \varphi(\bar{\rho}') = \frac{k_c}{i} \varphi(\bar{\rho}') \qquad (2.5.7)$$

Equation (2.5.7) represents the impedance boundary condition of Section 2.4,

then

$$\varphi(\bar{\rho}') = \int_{\bar{C}} \frac{2k_c}{i} \varphi(\bar{\rho})\, G(k_c|\vec{\rho}-\vec{\rho}'|)d\ell. \qquad (2.5.8)$$

Equation (2.5.8) is an accurate solution to the boundary value problem in the sea only under restricted conditions; the conditions are basically those assumed in Section 2.4.

2.6 Comparison with Other Theories.

The basic results from the solution of the integral equations in the air are:

TM Case

$$\varphi_n(x) \approx 2ik_{yi}(1 + k_s A \cos k_s x)e^{-ik_{xi}x} = 2\varphi_{ni}(x)_f(1 + k_s\xi(x)). \qquad (2.6.1)$$

TE Case

$$\varphi(x) \approx 2e^{-i(k_{xi}x + k_{yi}\xi)} \approx 2\varphi_i(x)_f. \qquad (2.6.2)$$

In the TM Case, Lerner and Max and Morgan use conformal mapping of the static ($\omega = 0$) problem to obtain the tangential magnetic field; the justification of such a procedure is considered in Section 2.8. The results of Lerner and Max basically agree with the result given above. The agreement is not complete, however, as Lerner and Max consider only grazing incidence, and the above solution includes all except grazing incidence. The tangential magnetic field is obtained from equation (2.6.1) by simply setting $H_{tan} = \varphi_n$. Morgan's physical model includes $\xi' \pm\infty$, which is excluded from the model of the problem used here. Morgan states, however, that there will be a change in the fields in the TM case. Wait assumes that the fields on the surface are relatively unchanged. This is of course approximately true even in the TM case (at least as far as the order of magnitude of the fields is concerned) for the usual set of physical parameters considered here.

In the TE case, Wait and Morgan again assume on physical grounds that the fields are unchanged. Lerner and Max obtain a similar result by solution of the static problem.

Another general method used to solve the "rough surface" problem is the perturbation theory (see Appendix B). Winter[12] used the simplest form of this technique to obtain the fields on the sea surface. The method used by Winter is applied below to the scalar functions to learn if the results agree.

> "Since the roughness scales of the sea surface are extremely small compared with the wavelength, the electromagnetic fields in free space are scarcely altered by the surface irregularities.

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[12] Winter, D. F., "Low Frequency Radio Propagation into a Moderately Rough Sea," Radio Propagation--Section D, Journal of Research, National Bureau of Standards, Vol. 67, no. 5, Sept.-Oct. 1963, p. 551.

Hence, the derivatives of $\varphi(z)$ can be calculated from the solution to the smooth sea problem with little error . . ."

This, as stated in the appendix, is not necessarily true. The above quotation is probably true in the far field; however, there is some question as to its correctness in the near field.

TM Case:

$$\varphi(x,o) = 0 \qquad \varphi(x,y) = 2i \sin k_{yi}y\, e^{-ik_{xi}x}$$

$$\varphi_y(x,\xi) = \varphi_y(x,o) + \varphi_{yy}(x,o)\xi + \frac{1}{2}\varphi_{yyy}(x,o)\xi^2 + \ldots$$

$$= (2ik_{yi} - 0 - 2ik_{yi}\frac{1}{2}(k_{yi}\xi)^2 + \ldots)\, e^{-ik_{xi}x}$$

$$\approx 2ik_{yi}e^{-ik_{xi}x}$$

$$\varphi_x(x,\xi) = \varphi_x(x,o) + \varphi_{xy}(x,o)\xi + \frac{1}{2}\varphi_{xyy}(x,o)\xi^2 + \ldots$$

$$= (0 + 2i(-ik_{xi})k_{yi}\xi + 0 + \ldots)\, e^{-ik_{xi}x}$$

$$\approx 2ik_{yi}e^{-ik_{xi}x}(-ik_{xi}\xi)$$

$$\varphi_n(x,\xi) \approx 2ik_{yi}(1 + k_{xi}\xi\ \xi_x)e^{-ik_{xi}x} \approx 2ik_{yi}e^{-ik_{xi}x}$$

$$\not\approx 2ik_{yi}(1 + k_s\xi)\, e^{-ik_{xi}x}.$$

The perturbation method yields, as assumed, an "unperturbed field;" unfortunately, this is not necessarily correct in this case.

Perturbation theory: $H_y \approx 2ik_y\xi_x(-ik_x\xi)$

Results above: $H_y \approx 2ik_y\xi_x$

TE Case:

$$\varphi_y(x,o) = 0 \qquad \varphi(x,y) = 2\cos k_{yi}y\, e^{-ik_{xi}x}$$

$$\begin{aligned}\varphi(x,y) &= \varphi(x,o) + \varphi_y(x,o)\,\xi + \tfrac{1}{2}\,\varphi_{yy}(x,o)\xi^2 + \ldots\\ &= (2 + 0 + \tfrac{1}{2}(-k_{yi}^2)\xi^2 + \ldots)\, e^{-ik_{xi}x}\\ &\approx 2\, e^{-ik_{xi}x}\end{aligned}$$

which is the unperturbed field and which agrees with the result obtained in this investigation.

2.7 Estimates of the Effect of the Finite Conductivity of the Sea on the Fields in the Air.

As was noted in the section describing the physical model (1.3), the fields on the sea surface have been computed under the assumption that the sea was a perfect electrical conductor (i.e., surface impedance was zero). In this section, the effect of finite conductivity on the solutions is considered. The impedance boundary condition obtained in Section 2.4 may be used to estimate the effect of the finite conductivity of the sea water (and therefore non-zero surface impedance) on the solutions.

The integral equation that the wave functions must satisfy is

$$\varphi(\bar{\rho}) = 2\varphi_i(\bar{\rho}) + 2\int_C \Big\{\varphi(\bar{\rho}')\, G_{n'}(k_o|\vec{\rho}-\vec{\rho}'|) - \varphi_{n'}(\bar{\rho}')\, G(k_o|\vec{\rho}-\vec{\rho}'|)\Big\} d\ell' \qquad (2.7.1)$$

In the TM case, φ represents a rectangular component of the vector potential $\vec{A}$ and H_{tan} is directly proportional to φ_n. The boundary condition $\varphi = 0$ on $\overline{C}$ was applied and the solution

$$\varphi_n(x) \approx 2\, ik_{yi}(1 + k_s\, A \cos k_s x) \tag{2.7.2}$$

was obtained.

Applying the impedance boundary condition to equation (2.7.2) implies that

$$\varphi_{(x)} \approx \frac{2k_{yi}}{k_c} (1 + k_s\, A \cos k_s x) \tag{2.7.3}$$

then

$$\int_C \varphi(\bar{\rho}\,') \, G_{n'}(|\vec{\rho}-\vec{\rho}\,'|)d\ell' \approx \frac{1}{2} \left(\frac{k_{yi}}{k_c}\right) (k_s A)^2 \tag{2.7.4}$$

where $G_{n'}(|\vec{\rho}-\vec{\rho}\,'|) = \frac{1}{2\pi} \ln|\vec{\rho}-\vec{\rho}\,'|$, the static two-dimensional Green's function (the approximation of the dynamic Green's function by the static Green's function is considered in Section 2.8).

As

$$\left| \int_C \varphi G_{n'} d\ell' \right| << |\varphi_i| \tag{2.7.5}$$

$$|\varphi| << |\varphi_i| \tag{2.7.6}$$

the original approximation of $\varphi = 0$ in $\overline{C}$ yields an accurate integral equation and the original solution (2.7.2) is verified. In this case there is little change due to the introduction of the non-zero surface impedance.

In the TE case, φ represents a rectangular component of the vector potential $\vec{F}$. φ_n was assumed to be zero on $\overline{C}$ and the solution

$$\varphi(x) \approx 2\varphi_i(x) \tag{2.7.7}$$

was obtained. The impedance boundary condition gives

$$\varphi_n \approx \frac{k_c}{i} \varphi \tag{2.7.8}$$

Again using the static kernel,

$$\int \varphi_n \, G d\ell \approx \left(\frac{k_c}{i}\right) 2\left\{2 \frac{1}{2} ik_y\right\} = -ik_y k_c \tag{2.7.9}$$

$$\left| \int \varphi_n \, G d\ell \right| << 1 \tag{2.7.10}$$

Again, the solution is basically unchanged from the zero surface impedance case. There is, however, a small correction term in the TE case.

2.8 The Justification of the Use of the Static Kernel in Some Integral Equations.

In general, electromagnetic fields are generated by and support nonstationary currents, that is, currents that oscillate at such frequencies as to make the interaction between current elements in different parts of space significantly affected by the finiteness of the velocity of propagation of electromagnetic effects. The quasi-stationary state is a special case of the general nonstationary state in which the velocity of propagation may be treated as being infinite. In the quasi-stationary state the currents oscillate slowly enough so that the approximation that all significant interactions between currents are effectively instantaneous is accurate. As instantaneous interaction between two separate elements is the same as the continuing or constant interaction in the stationary state (in the stationary or steady state, the electromagnetic fields and currents do not vary with time; therefore, the interactions are unaffected by the finite velocity of electromagnetic interactions). For monochromatic fields of radian frequency ω, the condition for the quasi-stationary state is that:[13]

$$\omega \ell_{max} << V \rightarrow \ell_{max} << \frac{\lambda}{2\pi} \tag{2.8.1}$$

where

V -- velocity of propagation of electromagnetic radiation in the medium ($3 \cdot 10^8$ meters/sec. in free space).

ℓ_{max} -- maximum distance between currents which significantly interact.

For the electromagnetic fields in the air, it is not clear that an ℓ_{max} satisfying the requirements (2.8.1) can be found. In fact, it is clear for φ_i and φ_r the quasi-stationary state does not hold as:

$$\int_{-\infty}^{\infty} H_o^{(2)}\left(k_o|x-x'|\right) e^{-i(nk_s + k_x)x'} dx' = \frac{2e^{-ik_x x}}{\sqrt{(nk_s + k_x)^2 - k_o^2}} \tag{2.8.2}$$

[13] King, R. W. P., "Quasi-Stationary and Nonstationary Currents in Electric Circuits," in Handbuch der Physics, Band XVI, Elektrische Felder und Wellen (Berlin, Springer, 1958), p. 165.

φ_i and φ_r are involved in the $n = 0$ term in which k_o may not be neglected, which clearly implies that for these fields the quasi-stationary approximation does not hold.

However, the scattered fields φ_s are quasi-stationary. The approximation involved may be seen in the integral equation (2.8.2) ($k_s >> k_o$ so that $\sqrt{(nk_s + k_x)^2 - k_o^2} \approx nk_s$) for terms $|n| > 0$.

The more accurate general result that the solution of $\nabla^2\varphi_s = 0$ yields an accurate solution to $(\nabla^2 + k_o^2)\varphi_s = 0$ may be seen from the boundary value problem.

$$(\nabla^2 + k_o^2)\,\varphi_s(x,y) = 0$$

$$\varphi_s(x,y) = \varphi_s(x + L,y) \qquad \text{boundary conditions} \qquad (2.8.3)$$

$$\frac{\partial^2}{\partial x^2}\varphi_s(x,y) = -\left(\frac{2n\pi}{L}\right)^2 \varphi_s(x,y)$$

$$\frac{\partial^2}{\partial y^2}\varphi_s(x,y) = \left\{\left(\frac{2n\pi}{L}\right)^2 - \left(\frac{2\pi}{\lambda}\right)^2\right\}\varphi_s(x,y) \qquad (a)$$

$$= \left(\frac{2n\pi}{L}\right)^2 \left(1 - \left(\frac{L}{n\lambda}\right)^2\right)\varphi_s \qquad (b) \qquad (2.8.4)$$

$$\text{if } L << \lambda \; \frac{\partial^2}{\partial y^2}\varphi_s \approx \left(\frac{2n\pi}{L}\right)^2 \varphi_s \quad \text{or}$$

$$\nabla^2\varphi_s = 0 \qquad (2.8.5)$$

then solutions of $\nabla^2\varphi_s = 0$ closely approximate the solutions of $(\nabla^2 + k_o^2)\varphi_s = 0$, as long as $L << \lambda$. As $\nabla^2\varphi_s = 0$, by the use of Green's theorem:

$$\varphi_s(\bar\rho) = \int_C \{\varphi_s(\bar\rho')\,G_{n'}(\bar\rho,\bar\rho') - \varphi_{sn'}(\bar\rho')\,G(\bar\rho,\bar\rho')\}\,d\ell' \qquad (2.8.6)$$

where

$$G(\bar{\rho},\bar{\rho}') = \frac{1}{2\pi} \ln |\vec{\rho}-\vec{\rho}'|$$

is the "static" Green's function.

As shown above, the stationary state should be used only to compute the "static" part of the field (i.e., φ_s), and the φ_i and φ_r should be removed before using static approximations.

The use of periodic boundary conditions played a major role in determining that φ_s could be computed accurately by use of quasi-stationary equations. It therefore seems wise to investigate the correctness of the assumption of a periodic surface. For $L << \lambda$, the periodic assumption would not seem to effect the fields in any major way (i.e., a small change in the surface some distance from where the fields are computed will have little effect on the fields). In general, the above statement is not true; particularly when $\lambda \approx L$, the periodic assumption may lead to a great change in the fields.

2.9 The Effect of Motion of the Sea Surface.

It was previously assumed that the sea surface was stationary. This approximation seemed to be reasonable, because the velocity of the electromagnetic wave in air is many times the velocity of the sea wave. However, even with the great difference in velocities, there still is some effect on the electromagnetic fields in the air due to the motion of the sea surface. This effect is greatly multiplied in the sea as the velocity of the electromagnetic radiation is much less in the sea than in the air. However, it is sufficiently accurate to solve the boundary value problem for a stationary surface and then assume the computed field moves with a velocity related to the velocity of the surface.

In many cases, o a stationary observer in the sea, it may appear that the sea surface is flat, but that the source of the electromagnetic wave is approximately moving up and down in the sea. If the wave were to propagate into the sea as $e^{-\gamma D}$, where D is the depth below the sea surface, this analysis would be correct; however, as will be shown later, this is not exactly true. The difficulty with the general case (i.e., using the actual solution to the boundary value problem) is the "equivalent velocity of propagation" can be obtained only numerically, so no general result can be stated.

If it is assumed that the fields propagate approximately as $e^{-\gamma D}$, then from the usual doppler theory

$$f_o = \frac{f}{1 - v/c} \qquad (2.9.1)$$

where

- f is the frequency of the electromagnetic wave
- f_o the observed frequency (to a stationary observer in the sea)
- v the assumed phase velocity of the electromagnetic field as a whole (the equivalent velocity that the

sources of the field would have) and carries a plus sign if the source is approaching the observer and negative sign if the source is moving away from the observer

c phase velocity of electromagnetic radiation in the sea.

v may be obtained as follows:

The velocity that an equivalent source (that is, a source that would yield the same fields), moving up and down in the sea, is given by:

$$v = \frac{d}{dt} D = \frac{d}{dt} A \cos (k_s v_s t) = - A k_s v_s \sin (k_s v_s t) \quad (2.9.2)$$

$$|v| \leq k_s A \; v_s$$

$$c = f\lambda_c = f(2\pi\delta_c) = f\frac{1600}{\sqrt{f}} = 1600 \sqrt{f} = 5 \cdot 10^4 \sqrt{f_{kc}} \quad (2.9.3)$$

For f = 1 kc/s

$c = 5 \cdot 10^4$ meters/sec., a decrease by a factor of approximately 10^4 from the phase velocity in free space.

$v_s \leq 30$ meters/sec., even for high sea states.

$|v| \leq \frac{1}{7} 30 = 4.3$ meters/sec.

$$999.9 \leq f_o \leq 1000.1 \text{ cps.} \quad (2.9.4)$$

This amount of doppler shift may not a[illegible]r to be too large; and this is a "worst case" calculation. The "largeness" of this effect is due to the shortening of the wave length in the sea.

3.0 COMPUTATION OF THE FIELDS IN THE SEA.

3.1 Introduction.

In the two-dimensional case the problem of calculating the fields in the sea reduces to the solution of a Dirichlet type boundary value problem.[1] Unfortunately, again because of the rough sea surface, the classical method of separation of variables can not be directly applied.[2] However, once the fields are below the lowest point of the rough sea surface, separation of variables can be used directly to compute the fields. The method of separation of variables is used below to obtain some interesting and general results.

To compute the fields in the sea, the integral representation of the wave functions may be used. The integrals would then be numerically evaluated to obtain the fields. The major reasons why such an approach was not taken are:

1. As the fields near the surface were to be computed, the kernel of the integral would have to be evaluated for small, intermediate and later large arguments. The evaluation of the Hankel function of complex argument is a somewhat involved though straightforward problem, involving large amounts of computation.

---

[1] A Dirichlet boundary value problem is to find φ such that

$$(\nabla^2 + k^2)\ \varphi(\bar{\rho}) = 0$$

$$\lim_{\substack{\bar{\rho}\to C \\ \bar{\rho}\in V}} \varphi(\bar{\rho}) = f(\bar{\rho}) \text{ a given "relatively" arbitrary function.}$$

[2] See Appendix B.

2. The amount of computation is greatly increased by the fact that the integral must be evaluated for each point at which the field is computed, or at least for points in the near field(i.e., near the sea surface).

Basically, for the above reasons, even though the numerical evaluation of the integral representation has been effectively used previously,[3] the method was not used here.

There still remains a wide selection of methods that may be used to compute the fields in the sea. The most general of these is the method of finite differences as used to obtain a solution to certain boundary value problems. The finite difference method is a widely used technique for the numerical solution of boundary value problems.[4]

Briefly, this method consists of replacing the partial differential equation with a partial difference equation. This approximation involves an error, usually called discretization

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[3]Banaugh, Robert P., "Scattering of Acoustic and Elastic Waves by Surfaces of Arbitrary Shape," (Ph. D. Thesis, University of Wisconsin), 1962.

Mei, Kenneth Kwan-hsiang, "Scattering of Radio Waves by Rectangular Cylinders," (Ph. D. Thesis, University of Wisconsin), 1963.

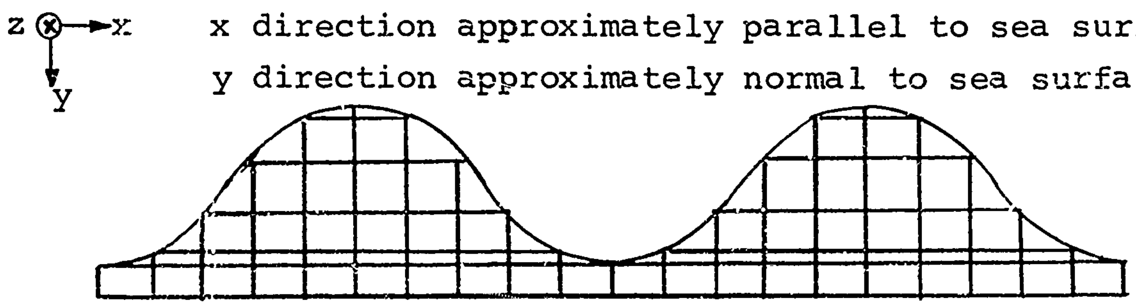
Lerner and Max, loc. cit.

[4]Collatz, L., The Numerical Treatment of Differential Equations, (Berlin: Springer, 1960).

Kantorovich, L. V., and V. I. Krylov, Approximate Methods of Higher Analysis, (New York: Interscience, 1953).

Forsythe, G. E., and W. R. Wasow, Finite-Difference Methods for Partial Differential Equations, (New York: John Wiley and Sons, 1960).

error. The problem then becomes one of solving a set of difference equations. Because of the accuracy needed, the number of equations may become quite large, on the order of 100 or more. The technique usually used to solve this set of equations is by interation procedures, though it should be noted that 100 x 100 matrices may be directly inverted on computers available today.[5] In what follows the general results obtainable by the classical method of separation of variables are first discussed. This approach should give some insight useful in considering the computation of the fields in the sea. The technique used, the so-called "method of lines," is a modification of the finite difference method. The partial differential equation is approximated by a difference differential equation. The finite difference approximation is used in the direction approximately parallel to the rough surface to obtain the "propagation" or separation constant to be used in the differential equation which characterizes the fields in a direction approximately normal to the sea surface.



x direction approximately parallel to sea surface.
y direction approximately normal to sea surface.

Figure 3.1. Grids for Computing the Fields in the Sea.

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[5] Banaugh, op. cit., p. 18.

3.2 Classical Separation of Variables.

The boundary value problem for the fields in the sea may be stated as to find φ (a rectangular component of the field vectors) such that it satisfies the Helmholtz equation

$$(\nabla^2 + k^2)\ \varphi = 0 \tag{3.2.1}$$

and

$$\varphi(x,y) = \varphi(x + L,y)\ e^{ik_{xi}L} \approx \varphi(x + L,y) \tag{3.2.2}$$

Equation (3.2.2) is a periodic boundary condition (i.e., it requires φ to be periodic with a period L within a constant factor $e^{ik_{xi}L}$). This follows directly from the physical problem, as discussed in section 2.2. The factor $e^{-ik_{xi}L}$ may be neglected as $k_oL << 1$.

φ must satisfy the radiation condition in the (+y) direction below the lowest point (trough) of the sea surface. Above the trough of the sea wave, both "inward" and "outward" waves may exist.[6] As the sea is a "highly" conducting medium, the second radiation condition holds and the fields decay exponentially.

φ must take on the correct boundary value on surface $\xi(x)$

$$\lim_{\substack{(x,y) \to C \\ (x,y) \in S}} \varphi(x,y) = f(x) \tag{3.2.3}$$

---

[6] See Appendix B.

Separation of variables may be applied to the boundary value problem in the region below the lowest point (trough) of the sea surface; this is possible there because the radiation condition applies. In the region above the trough of the sea surface but below the crest separation of variables applies; however, there are too many constants to determine by classical methods.[7] It is nevertheless possible to obtain very general qualitative results by this method.

From the boundary condition

$$\varphi(x,y) = \varphi(x + L,y) \tag{3.2.4}$$

$$\varphi(x,y) = \sum_{n=-\infty}^{\infty} \left[a_n^+ e^{i\left(\sqrt{k_c^2 - (nk_s + k_{xi})^2}\right) y} + a_n^- e^{-i\left(\sqrt{k_c^2 - (nk_s + k_{xi})^2}\right) y}\right] e^{-i(nk_s + k_{xi})x} \tag{3.2.5}$$

The qualitative results follow from a discussion of the properties of k_y. Since k_{xi} and k_{yi} are negligible compared to k_c and k_s:

$$k_y \approx \sqrt{k_c^2 - (nk_s)^2} \qquad n > 0 \qquad \text{(a)}$$

$$k_y \approx k_c \qquad n = 0 \qquad \text{(b)} \tag{3.2.6}$$

[7] See Appendix B.

For $n = 0$, $k_y \approx k_c$; the field is approximately constant in the x-direction resulting in a plane wave propagating approximately straight down into the sea as in the flat interface case.

If the field is not approximately a constant in the x-direction, the propagation constant becomes

$$\gamma_y = \sqrt{\gamma^2 - \gamma_x^2} = \sqrt{\gamma^2 + k_x^2}$$

$$= \sqrt{\frac{2i}{\delta^2} - \left(\frac{2\pi n}{L}\right)^2} \qquad (3.2.7)$$

As the term added to γ^2 increases in magnitude, $\mathrm{Re}\left[\gamma_y\right]$ increases and the particular mode characterized by this propagation constant decreases more rapidly with depth than the $n = 0$ mode. This implies that as $y \to \infty$ the electromagnetic fields become a plane wave with propagation constant γ.[8] Asymptotically, $(y \to \infty)$ the major perturbation in the $n = 1$ mode; this is independent of the shape of the sea surface. The major perturbation near the sea surface will depend on the shape of the sea surface, and will be attenuated with depth, the fields tending towards a plane wave propagating downward (the $n = 0$ mode).

Intuitively, a reasonable breakpoint in these types of propagation is when $\mathrm{Re}\left[\gamma_y\right]$ for $n = 1$ is $1.85\ \mathrm{Re}\gamma = \frac{1.85}{\delta}$.

[8]As $\mathrm{Re}\left[\gamma_y\right] > \mathrm{Re}\left[\gamma\right]$ the modes $n > 0$ are attenuated more than the $n = 0$ mode (plane wave). Asymptotically $(y \to \infty)$ the higher order modes become zero more rapidly than the $n = 0$ mode (i.e., the ratio of the amplitude of the higher order modes to the $n = 0$ mode approaches zero asymptotically).

$$\gamma_y = \sqrt{\frac{2i}{\delta^2} + \left(\frac{2a}{\delta^2}\right)} = \frac{\sqrt{2}}{\delta}\sqrt{a + i}$$

$$\mathrm{Re}\left[\gamma_y\right] = \frac{1.85}{\delta} \rightarrow a = 1 \rightarrow L \leq \sqrt{2}\,\pi n \delta \qquad (3.2.8)$$

For $L \leq$ then $n = 1$ mode is considerably more attenuated as it propagates down into the sea than the $n = 0$ mode and the higher order modes are even more rapidly attenuated. A qualitative idea of what this means is given in the chart below:

f = 3 cps	δ = 145 meters	$\bar{L}$ = 650 meters
3 kc/s	4.6	20
18.6 kc/s	1.85	8
30 kc/s	1.45	6.5

For $L = \bar{L}$ the added attenuation of the lower order modes should "rapidly" bring the field into a plane wave. In the VLF range, for $L = 10\bar{L}$, the effect of the added attenuation on the mode solution should be evident, even at the depths considered. If $L \leq 100\bar{L}$, at the depths considered in this report little decay in the perturbation due to the rough sea surface would be expected.

3.3 Method of lines

In one method of numerical solution of partial differential equations (the method of lines), one of the variables, say x, is discretized, while the other variable y is left continuous. After the finite difference approximations are substituted for the x derivatives, the partial differential equations become a coupled system of ordinary differential equations, i.e., difference differential equations.[9]

Usually when the method of lines is used on an automatic digital computer, the problem is discretized in the y-direction also as a finite difference method is used to solve the system of ordinary differential equations.[10] In the modification of the method of lines used below, the problem is discretized in the y-direction; however, the differerce differential equations are solved by a method closely akin to separation of variables and Euler's method.[11] In this problem, as is usual in the method of lines, the number of subdivisions in the y-direction greatly exceeds the number of such divisions in the x-direction, i.e., the discretization distance in the x-direction is larger than the discretization distance in the y-direc tion. The problem is then concerned with a rectangular net with relatively long rectangles. This is very useful in satisfying the boundary condition at the sea surface (which is a slightly rough surface, i.e., almost flat surface with relatively large variation in x, resulting in only small variations in $y = \xi(x)$).

The boundary value problem was stated mathematically in in equations (3.2.1), (3.2.2) and (3.2.3).

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[9] Forsythe, op. cit., p. 178.

[10] Ibid.

[11] Scarborough, Numerical Mathematical Analysis, fourth ed., (Baltimore: Johns Hopkins Press, 1963).

As shown in the preceding section, some general qualitative results may be stated, based on the relative values of physical parameters. Using the numerical results obtained by the use of the method given in this section, quantitative values will be placed on the terms "relatively little change," etc., by comparison of the theory to solutions of some representative problems.

The method used discretizes the x variable by replacing the x-derivatives by a finite-difference approximation

$$\gamma_x^2\,(x,y) = \frac{1}{\varphi(x,y)}\ \text{D.O.}\ \varphi(x,y) + E \qquad (3.3.1)$$

where D.O. is a difference operator and E an error term. The D.O. depends on y, and will be given explicitly along with the error term E later. The partial differential equation then becomes a set of ordinary differential equations; the coupling of the equations is through the γ_x^2 term. Letting

$\varphi(x,y) = X(x)\ Y(y)$ as in the separation of variables method

$$\left\{\frac{d^2}{dy^2} + (\gamma^2 - \gamma_x^2)\right\} Y(y) = 0 \qquad (3.3.2)$$

$$\gamma_y^2 = \gamma^2 - \gamma_x^2 . \qquad (3.3.3)$$

Intuitively, because the sea surface is only slightly rough and because of the relative values of the physical parameters of the problem, the propagation should be basically in the y-direction (approximately normal to the sea surface), so that:

$$|\gamma| >> |\gamma_x| \rightarrow \gamma_y \approx \gamma. \qquad (3.3.4)$$

Equation (3.3.4) holds true asymptotically (i.e., $y \rightarrow \infty$), and in most cases is approximately true near the sea surface. For the physical parameters used in the examples given here, the correction should be relatively small.

The solution to the ordinary differential equation

$$\left(\frac{d^2}{dy^2} + \gamma_y^2\right) Y(y) = 0 \qquad (3.3.5)$$

is well-known to be:

$$Y(y) = C_1 e^{-\gamma_y y} + C_2 e^{\gamma_y y} . \qquad (3.3.6)$$

In the computer program, it proved convenient to choose γ_y such that $\mathrm{Re}\left[\gamma_y\right] < 0$, as the propagation is in the +y direction; the radiation condition then takes the form $C_1 = 0$.

Below the lowest point (trough) of the sea surface, the radiation condition must be applied, in which case the method of separation of variables could be used in its classical form. However, to obtain numerical results, the use of the classical method of separation of variables does not prove convenient.

The boundary value problem is best solved numerically, even in this case, where the separation of variables could be used. The problem is a marching type problem.[12] If $\varphi(x,y)$ is known at the lattice points for $y = y_1$, $\varphi(x,y_1 + \delta y)$ can be computed from the equations

$$\gamma_x^2 = \frac{1}{\varphi(x,y_1)} \text{ D.O. } \varphi(x,y_1) \qquad \text{(a)}$$

$$\gamma_y = \sqrt{\gamma^2 - \gamma_x^2} \qquad \text{(b)} \qquad (3.3.7)$$

$$\varphi(x,y_1 + \delta y) = \varphi(x,y_1)\, e^{\gamma_y(\delta y)} \qquad \text{(c)}$$

This process is repeated until the depth y, the greatest depth at which the fields are to be computed, is attained.

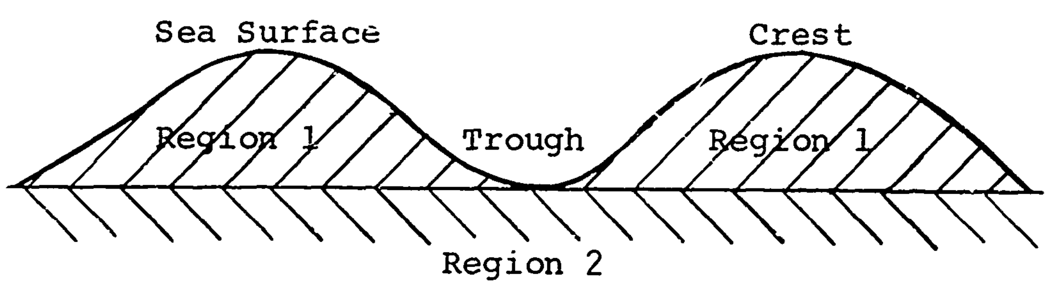

Figure 3.2 Regions of Solution.

To compute the fields between the crest of the surface and the trough of the surface, the problem is considerably more complicated. The radiation condition does

[12]A marching type boundary value problem is a problem in which the knowledge of boundary conditions may be used directly to compute numerically the field near the boundary; this may, in turn, be used to extend the field further in a step-by-step fashion.

not apply in this region and the method of separation of variables, at least its classical form, cannot be used.

The method used is, of course, based on the same difference-differential equations assuming some value is given to $\varphi(x,y)$ for all x,y in the region considered. To compute a new set of values for $\varphi(x,y)$, let

$$
\left.\begin{aligned}
&\gamma_y = \sqrt{\gamma^2 - \gamma_x^2} && \text{(a)}\\
&\gamma_x^2 = \frac{1}{\varphi(x,y)}\ \text{D.O.}\ \varphi(x,y) && \text{(b)}\\
&\varphi(x,y - \delta y) = C_1 + C_2 && \text{(c)}\\
&\varphi(x,y + \delta y) = C_1 e^{-\gamma_y 2\delta y} + C_2 e^{\gamma_y 2\delta y} && \text{(d)}
\end{aligned}\right\} \quad (3.3.8)
$$

The above two equations are solved for C_1 and C_2; then a new value of $\varphi(x,y)$ is computed:

$$\varphi(x,y) = C_1 e^{-\gamma_y \delta y} + C_2 e^{\gamma_y \delta y} \qquad (3.3.9)$$

This process is repeated for each x,y in the region considered until the process has been repeated for the whole region. The process is stopped when there is very little difference between the old values of φ and the newly computed values of φ.

3.4 Error Estimates.

When using numerical methods, the question of error estimates is of primary importance. If no reasonable error bounds can be found the method used and the results obtained may be useless. In this section, the error estimates for the numerical process described in the previous section are given. Because a large digital computer was used carrying many extra significant figures, round off error may be neglected. The major error term then arises from the discretization process.

As different finite difference approximations were used in the different regions of the problem, two sets of estimates are given.

Region 1. (Between the crest and trough of the sea wave.)

$$\text{D.O. } \varphi(x,y) = \frac{\varphi(x + h_x, y) + \varphi(x - h_x, y) - 2\varphi(x,y)}{h_x^2} \qquad (3.4.1)$$

where h_x is the discretion constant in the x variable

$$\varphi_{xx}(x,y) = \text{D.O. } \varphi(x,y) + E$$

where E is an error term

$$E \approx \frac{1}{12} \varphi_{xxxx}(x,y)\, h_x^2$$

if h_x is sufficiently small and φ smooth enough (it is assumed this is true in the remainder of this section; this is easily verified)

$$\epsilon_1 \le \left|\frac{E}{\varphi_{xx}}\right| \le \frac{1}{12} \left|\frac{\varphi_{xxxx}}{\varphi_{xx}}\right| h_x^2 \approx \frac{1}{12} (|\gamma_x| h_x)^2 = \frac{1}{6}\left(\frac{h_x}{\delta_x}\right)^2$$

where, γ_x - the effective propagation constant for propagation in the x-direction.

$$\delta_x \approx \left|\frac{\sqrt{2}}{\gamma_x}\right|$$

ϵ_1 -- a relative error term

Let $\tilde{\gamma}_x^2$ -- the computed value of γ_x^2

$$\gamma_x^2 = \tilde{\gamma}_x^2 + \epsilon_{\gamma_x}^{\ 2}$$

$$\epsilon_{\gamma_x}^{\ 2} \le \frac{1}{6}\left(\frac{h_x}{\delta_x}\right)^2$$

$\epsilon_{\gamma_x}^{\ 2}$ -- the relative error in γ_x^2

From the above, it is clear that the less the variation of the field in the x-direction, the larger h_x may be made and still have sufficient accuracy in the calculations. However, the main question of accuracy involves γ_y and, of course, $\varphi(x,y)$ not γ_x^2.

As $|\gamma|^2 > |\gamma_x^2|$ under most sets of conditions for the problem considered here,

$$\gamma_y = \sqrt{\gamma^2 - \gamma_x^2} = \gamma\sqrt{1 - \left(\frac{\gamma_x}{\gamma}\right)^2} \approx \gamma\left[1 - \frac{1}{2}\left(\frac{\gamma_x}{\gamma}\right)^2 + \ldots\right]$$

where γ_y -- the effective propagation constant for propagation in the y-direction

$$\delta_y \approx \left|\frac{\sqrt{2}}{\gamma_y}\right|$$

$$\epsilon_{\gamma_y} \le \frac{1}{2} \frac{\epsilon_{\gamma_x}^2}{\gamma^2} |\gamma_x^2|$$

$$\le \frac{1}{20} \epsilon_{\gamma_x}^2 \quad \text{under most conditions}$$

For $h_x < 4\delta_x$

$$\epsilon_{\gamma_x}^2 \le .01 \text{ and } \epsilon_{\gamma_y} \le .0001 \ .$$

The final error estimates involve the field $\varphi(x,y)$ which of course is the estimate desired.

$$\epsilon_\varphi = |e^{\gamma_y h_y}\{e^{-\gamma_y(1 - \epsilon_{\gamma_y})h_y} - e^{-\gamma_y h_y}\}| \approx |\gamma_y| \epsilon_{\gamma_y} h_y$$

$$\epsilon_\varphi \le \frac{2}{\delta_y} \epsilon_{\gamma_y} h_y \ .$$

For the condition above, if $h_y < 5\delta_y$

$$\epsilon_\varphi \le .01 \ .$$

Under most conditions, ϵ_φ is considerably less.

Region 2. (Below the bottom of the sea wave.)

$$\text{D.O. } \varphi(x,y) = \frac{\{\varphi(x + 2h_x, y) + \varphi(x - 2h_x, y)\}}{12h_x^2} + \frac{+ 16\{\varphi(x + h_x, y) + \varphi(x - h_x, y)\} - 30\varphi(x,y)}{12h_x^2} \tag{3.4.2}$$

$$\varphi_{xx} = \text{D.O. } \varphi(x,y) + E$$

with $E \approx \frac{1}{90} \varphi_{xxxxxx} h_x^4$

For $h_x < 2\delta_x$

$$\epsilon_{\gamma_x}^2 \le .003$$

$$\epsilon_{\gamma_y} \le .0003 \ .$$

For $h_y < 50\delta_y$

$$\epsilon_\varphi \le .015.$$

ϵ_φ -- relative error in the field.

In this region, the first derivative is also calculated. This allows taking the curl of the field vectors.

$$\text{D.O. } \varphi = \frac{\varphi(x + 3h_x, y) - \varphi(x - 3h_x, y)}{60h_x}$$

$$\frac{- 9\{\varphi(x + 2h_x, y) - \varphi(x - 2h_x, y)\} + 45\{\varphi(x + h_x, y) - \varphi(x - h_x, y)\}}{60h_x} \quad (3.4.3)$$

$$\varphi_x = \text{D.O. } \varphi + E$$

$$E \approx \frac{1}{30}\varphi_{xxxxx} h_x^4$$

$$\epsilon_{D_x} \le \frac{1}{30} \cdot \left|\frac{\varphi_{xxxxx}}{\varphi_x}\right| h_x^4 \le \frac{1}{7.5}\left(\frac{h_x}{\delta_x}\right)^4$$

For $h_x < 2\delta_x$

$$\epsilon_{D_x} \le .01.$$

Similarly, for D_y

$$\epsilon_{D_y} \le \frac{1}{7.5}\left(\frac{h_y}{\delta_y}\right)^4$$

For $h_y < 2\delta_y$

$$\epsilon_{D_y} \le .01.$$

Based on these rough error estimates, all calculations are sufficiently accurate (within 1%). Nearly all calculations are more accurate, as the conditions given above are the extremes of those encountered. It should be noted, however, based on the calculations made, the error in each computation usually took on the maximum value computed for it (i.e., the error was very nearly equal to its bound given above).

3.5 Some Conclusions and Verifications Based on the Computer Calculation

In the next chapter the numerical results of the computer calculations are given; the errors and the verification of approximations made in the computer program are briefly discussed in light of the experience of the computer runs.

The first question considered in numerical solution of differential equations is: how fine must the grid be made to obtain accurate results? This is usually determined in the finite difference calculations by subdividing the finite difference interval until no changes occur in the solution. The "subdividing" method was used in the computer calculations. The accuracy required was three significant places. This method was applied to both intervals in the x and in the y variables. The effect of interval change in the x variable is somewhat small as long as the finite difference formulas are reasonably accurate. However, the changes in the interval in the y variable could greatly effect the numerical result. This sensitivity occurred only in problems where the lower order modes were decaying rapidly. The reason for this phenomenon is that if a large interval is chosen in the y variable and if the decay rate is rapid and not "corrected" in the large interval, it causes the field to "overshoot". This effect may be cumulative, in which case the computed field rapidly becomes zero, or if the interval is small enough the computed field appears to "hunt" after reaching equilibrium (i.e., plane wave). In either case, the errors are easily seen in the computed values.

An assumption that is verified in the computed results is that $|\gamma_x| << |\gamma_y|$, even where relatively rapid decay of lower order modes occur ($|\gamma_x|$ is largest there, $|\gamma_x| < 10\ |\gamma_y|$, however it should be noted that this is for $k_s A < 1/7$, if $k_s A$ was not restricted in any way, the above conclusion would not hold).

It would seem reasonable, that the computer program would accurately compute the field, as the program is based directly on the physical processes involved in the propagation of the wave. However, it is also clear, that if the original assumptions and restrictions on the parameters considered do not hold, the program may yield inaccurate results.

4.0 SELECTED RESULTS OF COMPUTER CALCULATIONS.

4.1 Introduction.

The purpose of this chapter is to present numerical calculations of the electromagnetic fields in the sea. From these calculations, a correlation between what the theory predicts for the basic mode of propagation in the sea (see Section 3.2) and the computed electromagnetic fields in the sea may be found. Selected results are presented, and these should be sufficient to see the basic propagation pattern in the sea.

As stated previously, the sea surface was assumed to be described by the equation $\xi(x) = A \cos(k_s x)$. For the calculation given in this chapter, A was chosen such that the maximum slope of the wave was 1/7.[1] These calculations should then bound the electromagnetic fields in the sea produced in an actual physical situation (i.e., the difference in propagation patterns caused by the changes in the fields due to the rough surface is maximum).

First, to see the correlation between the "propagation theory" and the actual propagation effects, a set of curves presenting the numerical solution of the scalar boundary value problem at the lowest level (trough) of the sea wave is given.

Secondly, a set of results presenting the electromagnetic fields in the sea at different depths is presented. These

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[1]See Appendix A.

fields are normalized to see the deviation from a plane wave.

Lastly, some comments on the numerical results are given.

4.2 Scalar Fields at the Lowest Level (Trough) of the Sea Wave

The symbol α is used to connote attenuation (in db):

α_1 -- Relative ratio of the field at the point considered to the field on the sea surface vertically above it in the TE case.

α_2 -- Relative ratio of the field at the point considered to the field on the sea surface vertically above it in the TM case; this is basically to present the propagation effects.

α_3 -- The "corrected" value of α_2, that is, the relative ratio of the field at the point considered to a fixed normalization value; this is to account for the difference in the field on the sea surface in the TM case (then α_1 and α_3 on the same scale).

α_c -- The computed relative ratio of the field at the point considered to the field on the sea surface vertically above it assuming the wave propagates straight downward with the propagation constant of sea water at the frequency of the electromagnetic wave and that the field on the sea surface is a constant (the same normalization constant used to compute α_1 and α_3).

β is used to connote phase shift (in degrees):

β_1 -- Relative phase of the field at the point considered to the field on the sea surface vertically above it in the TE case.

β_2 -- Relative phase of the field at the point considered to the field on the sea surface vertically above it in the TM case.

β_c -- The computed relative phase of the field at the point considered to the field on the sea surface vertically above it, assuming the wave propagates straight downward with the propagation constant of sea water at the frequency of the radio wave and that the field on the sea surface has a constant (zero) phase.

The curves presented are for:

$f = 18.6$ kc/s

A = .3 meters	L = 12 meters
A = 1.5	L = 64
A = 4.3	L = 180

$f = 3$ cps

A = 15	L = 640

For sea water at 18.6 kc/s, $\delta = 1.85$ meters; therefore based on the "propagation theory," the fields under the surface with L = 12 should show a marked decrease in the perturbations caused by the rough sea surface and indeed this is the case. For L ≥ 64, there should be little decrease in the perturbation and the calculated change is very small as can be seen on the graphs, even for the 4.3 meter waves.

For sea water at 3 cps, $\delta = 145$ meters and some decrease in the perturbation for the 15 meter wave is noted.

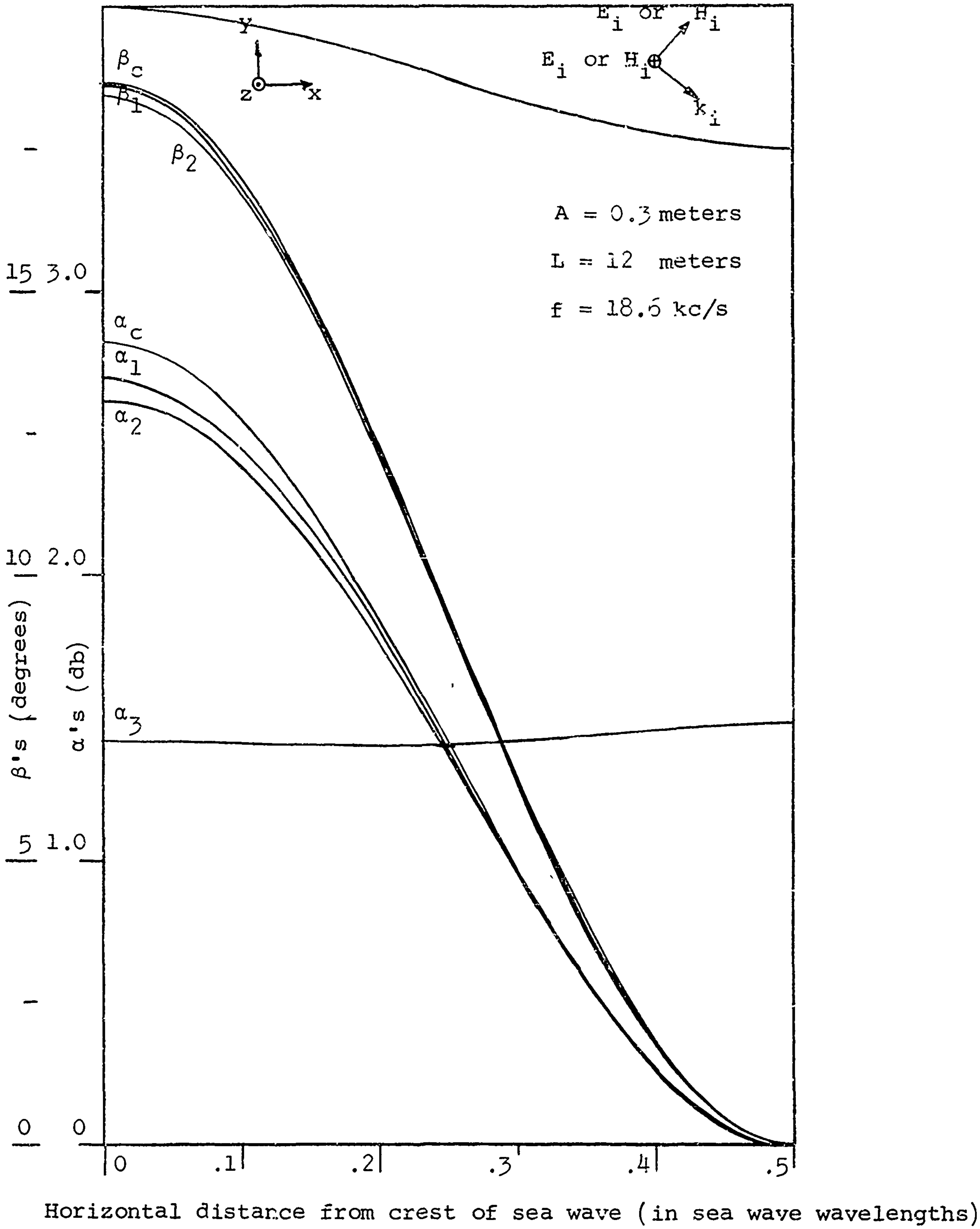
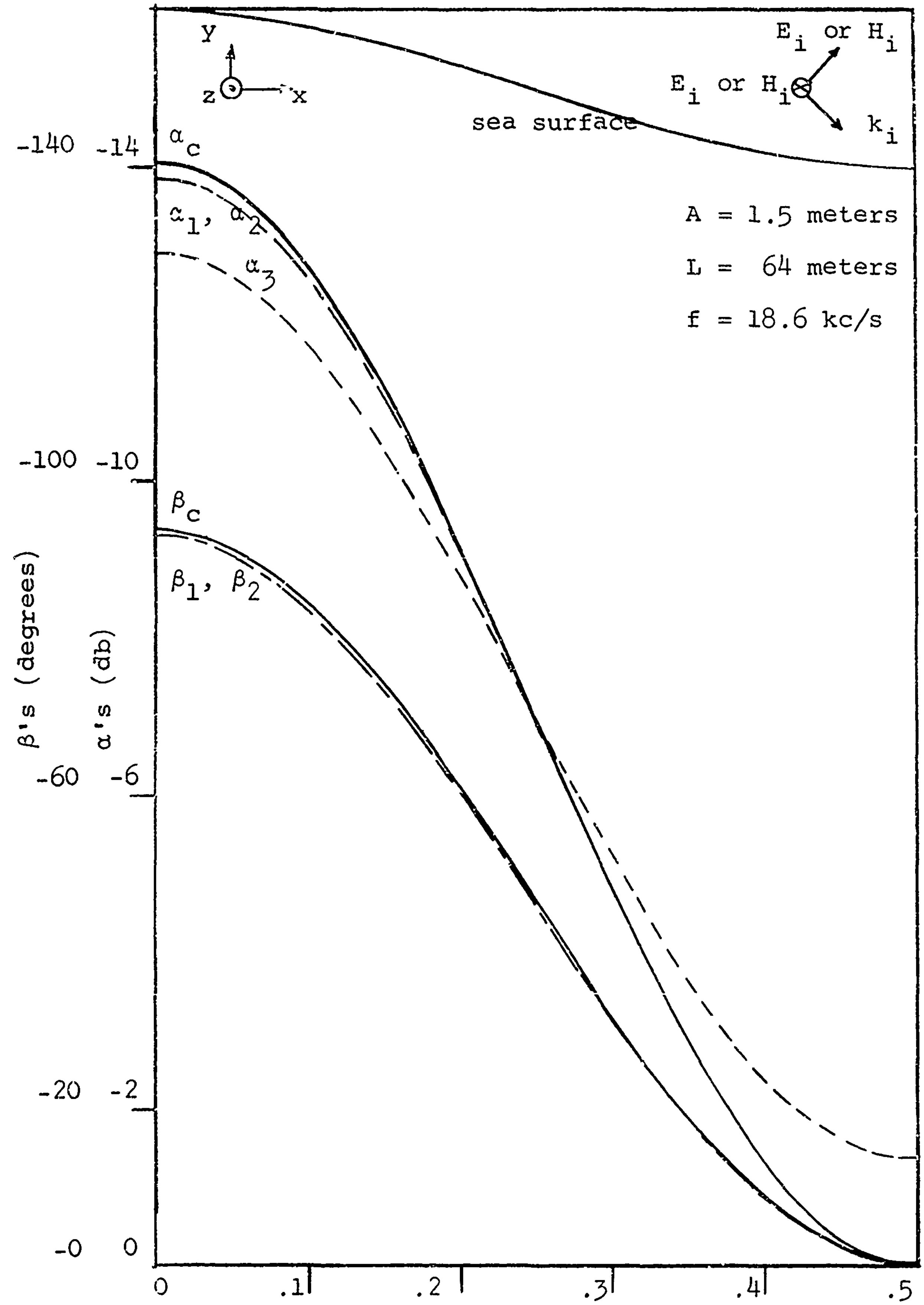


Figure 4.1, Scalar Field at the Trough Level of the Sea Wave



Horizontal distance from crest of sea wave (in sea wave wavelengths)

Figure 4.2, Scalar Field at the Trough Level of the Sea Wave

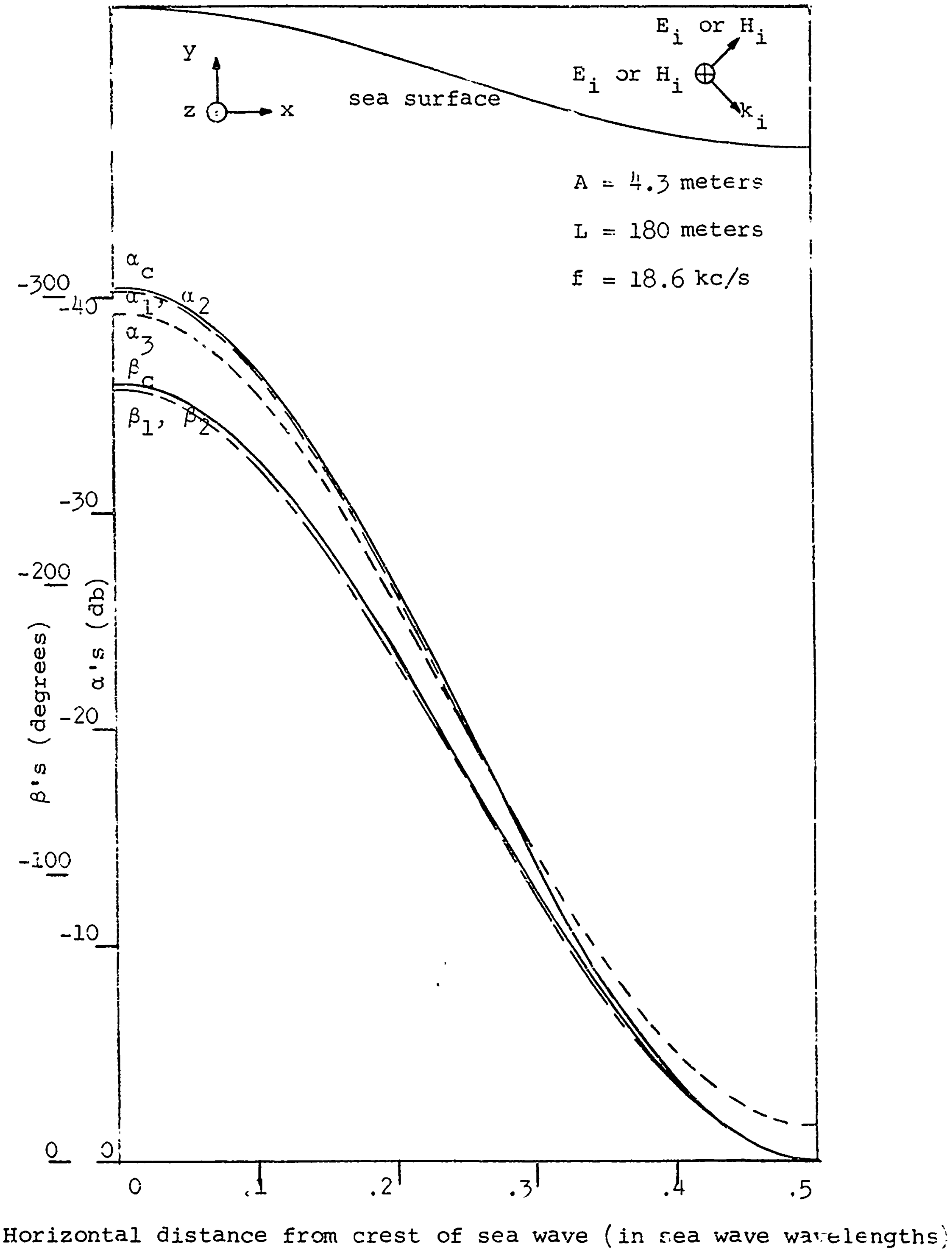
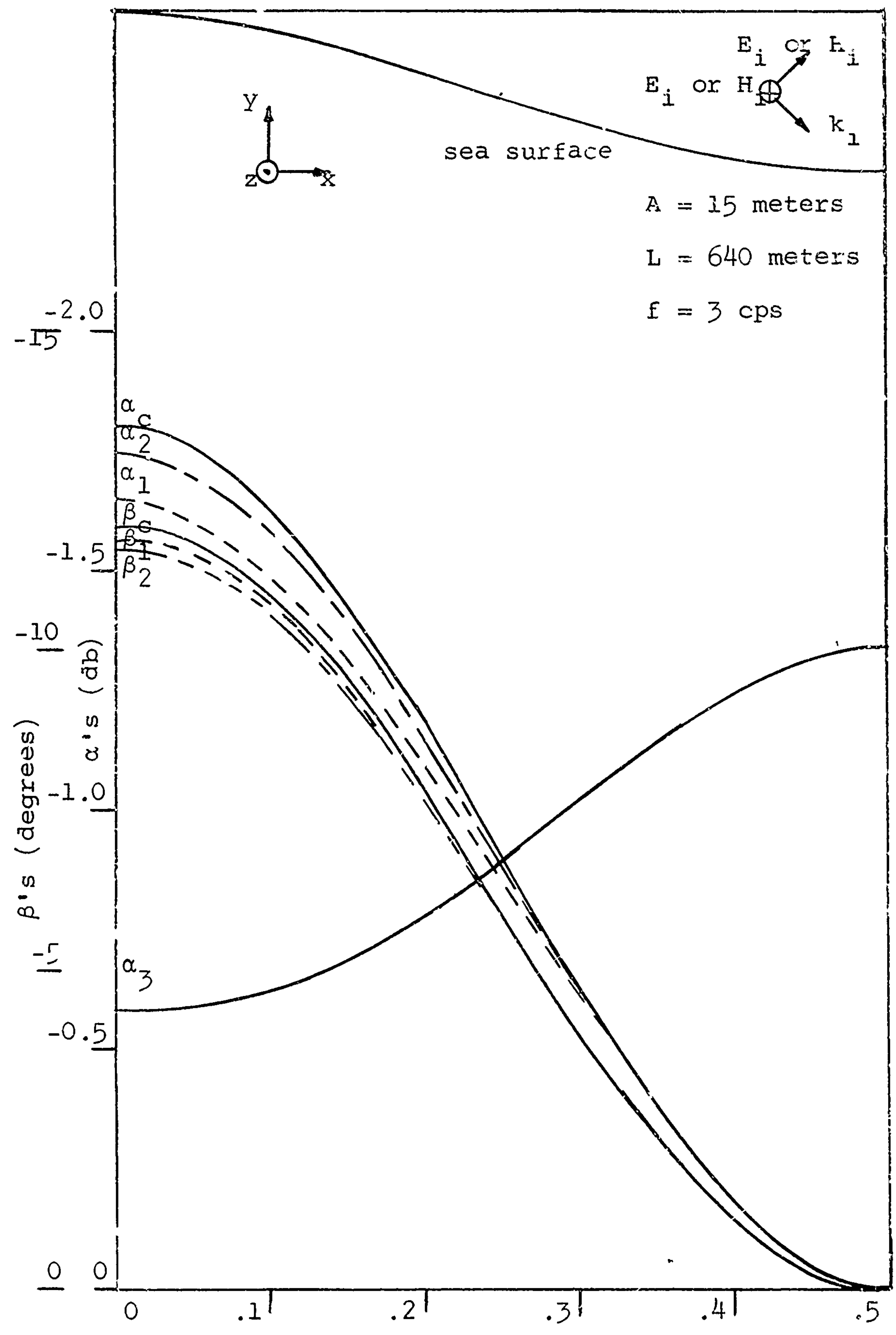


Figure 4.3, Scalar Field at the Trough Level of the Sea Wave



Horizontal distance from crest of sea wave (in sea wave wavelengths)

Figure 4.4, Scalar Field at the Trough Level of the Sea Wave

4.3 The Electromagnetic Fields in the Sea.

Due to the vector nature of the electromagnetic fields a set of curves is necessary to describe the electromagnetic fields at a fixed depth. The description of the fields is presented in terms of E_z or H_z depending on the polarization and direction of propagation of the incident field.

Again, α is used to connote attenuation (in db).

α_1 -- Relative ratio of the field (in this case E_z or H_z) at the point considered to a normalization value.

α_c -- The computed relative ratio of the field at the point considered to the field on the sea surface vertically above it assuming the wave propagates straight downward with the propagation constant of sea water at the frequency of the radio wave and that the field on the sea surface is a constant.

β again is used to connote phase shift (in degrees).

β_1 -- Relative phase of the field at the point considered to the field on the sea surface vertically above it in the TE case.

β_c -- The computed relative phase of the field at the point considered to the field on the sea surface vertically above it, assuming the wave propagates straight downward with the propagation constant of sea water at the frequency of the radio wave and that the field on the sea surface has a constant (zero) phase.

A second set of curves is presented to represent the other field components. These are normalized by the factor

$\eta_c^{\pm 1}$ (plus sign if E_z was considered; negative sign if H_z).

$$\bar{\alpha} = \begin{cases} 20 \log \left| \dfrac{\eta_c H_x}{E_z} \right| \\ 20 \log \left| \dfrac{E_x}{\eta_c H_z} \right| \end{cases}$$

$$\bar{\bar{\alpha}} = \begin{cases} 20 \log \left| \dfrac{\eta_c H_y}{E_z} \right| \\ 20 \log \left| \dfrac{E_x}{\eta_c H_z} \right| \end{cases}$$

$$\bar{\beta} = \begin{cases} \text{phase}(H_x) + 45^\circ - \text{phase } E_z \\ \text{phase}(E_x) - 45^\circ - \text{phase } H_z \end{cases}$$

$$\bar{\bar{\beta}} = \begin{cases} \text{phase}(H_y) + 45^\circ - \text{phase } E_z - 90^\circ \\ \text{phase}(E_x) - 45^\circ - \text{phase } H_z - 90^\circ \end{cases}$$

where phase H_x is the phase of H_x.
$\bar{\alpha}$ denotes the deviation of the ratio of the horizontal fields from the plane wave (flat interface) case (in db). $\bar{\bar{\alpha}}$ denotes the ratio of the vertical field to the horizontal field (in db).

Similarly, $\bar{\beta}$ and $\bar{\bar{\beta}}$ are measures of the phase deviation from the plane wave case for the horizontal and vertical fields, respectively.

On each set of curves in this section a complete description of the sea surface (A and L) is given, along with the frequency, polarization and direction of propagation of the incident electromagnetic wave. The depth D (below the trough) at which the fields are computed is also given on each set of curves.

The curves presented in this section are all for the direction of propagation of the incident plane wave in the x-y plane. The sets of curves differ with respect to polarization. To obtain the fields due to an incident plane wave with propagation vector in the y-z plane the "dual" of the solutions is taken (i.e., for a vertically-polarized wave, the curve for the horizontally-polarized wave must be used). Rather than belabor the discussion, a very simple procedure will be given in Section 4.4 for using the curves to compute the horizontal fields in the sea.

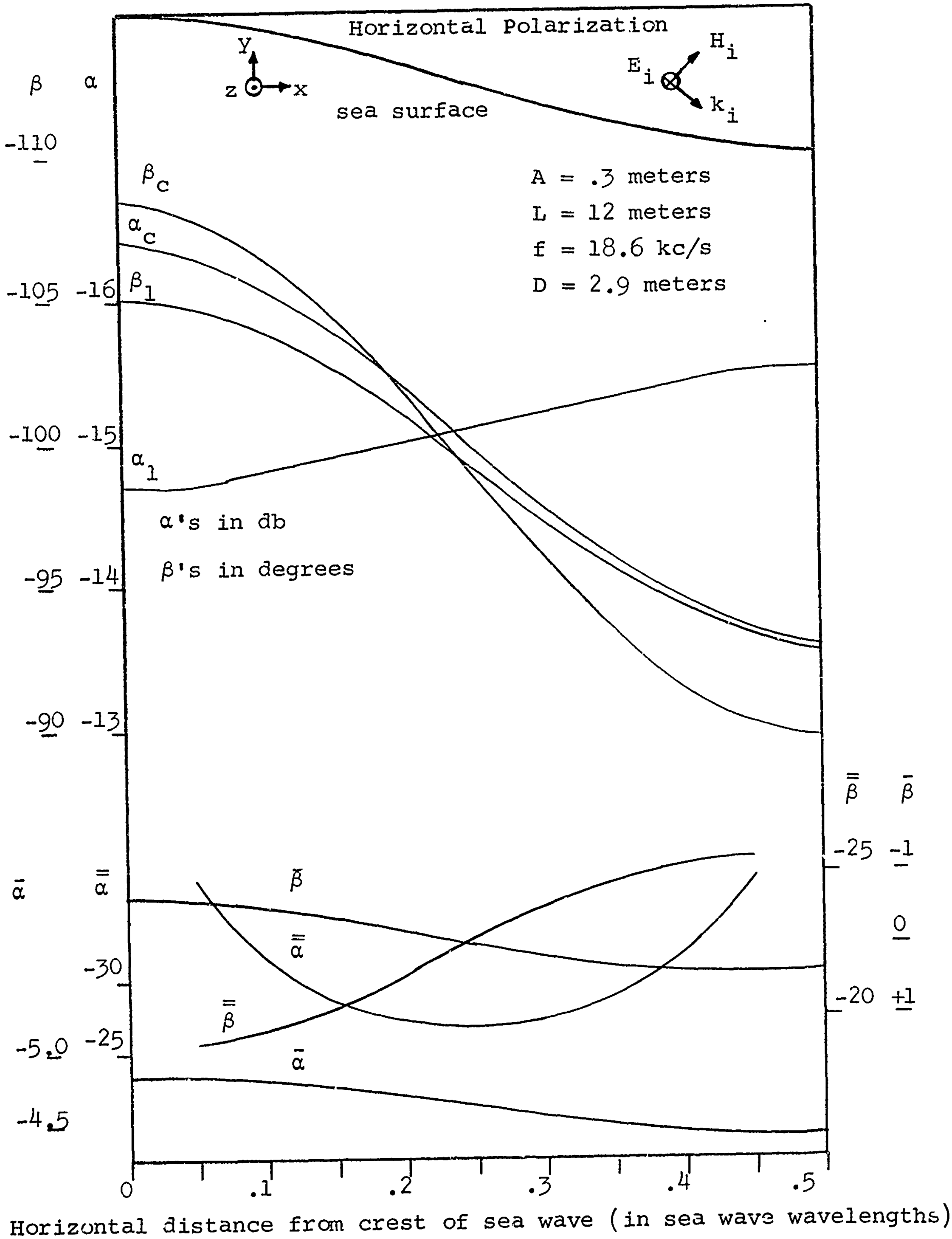
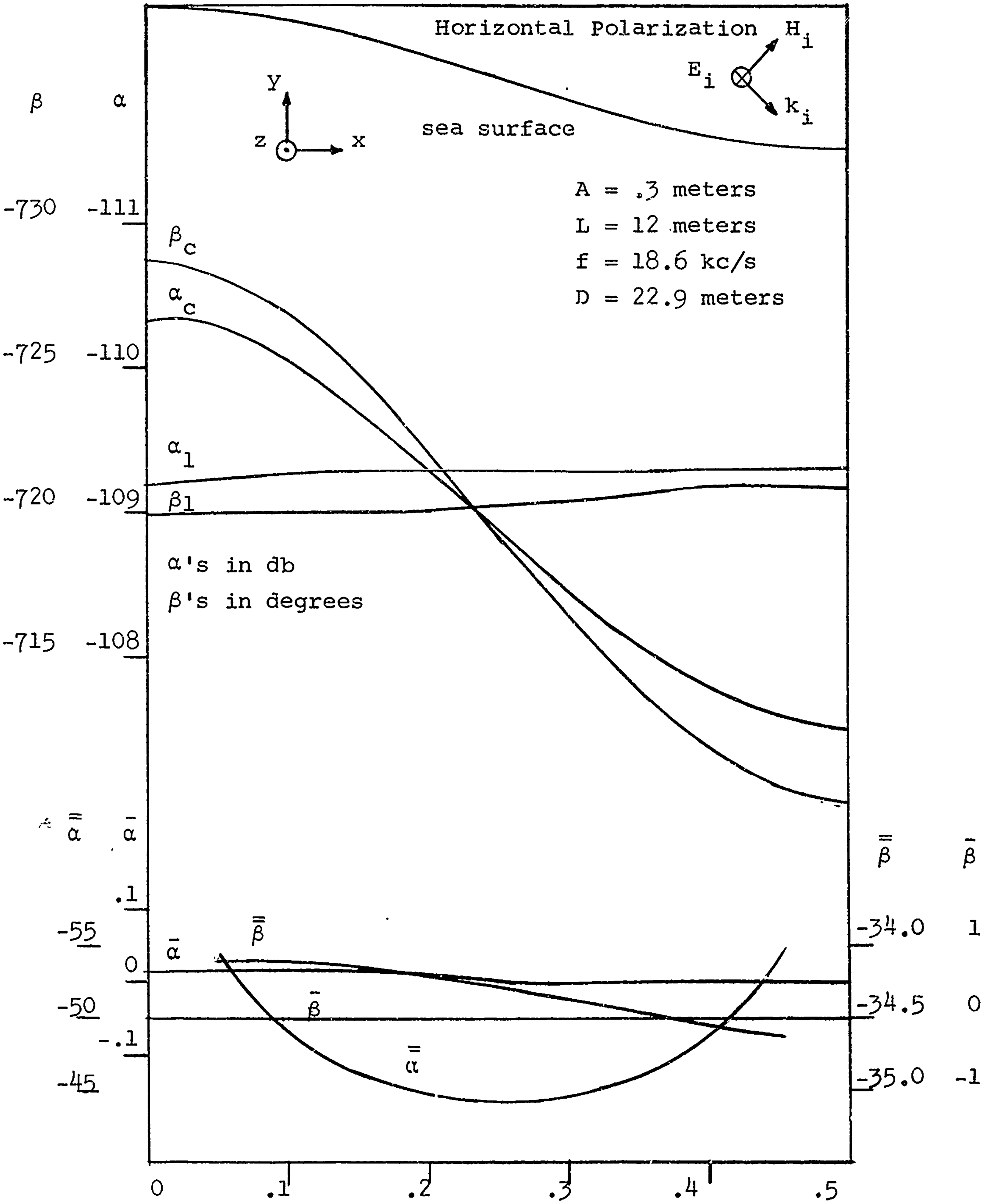
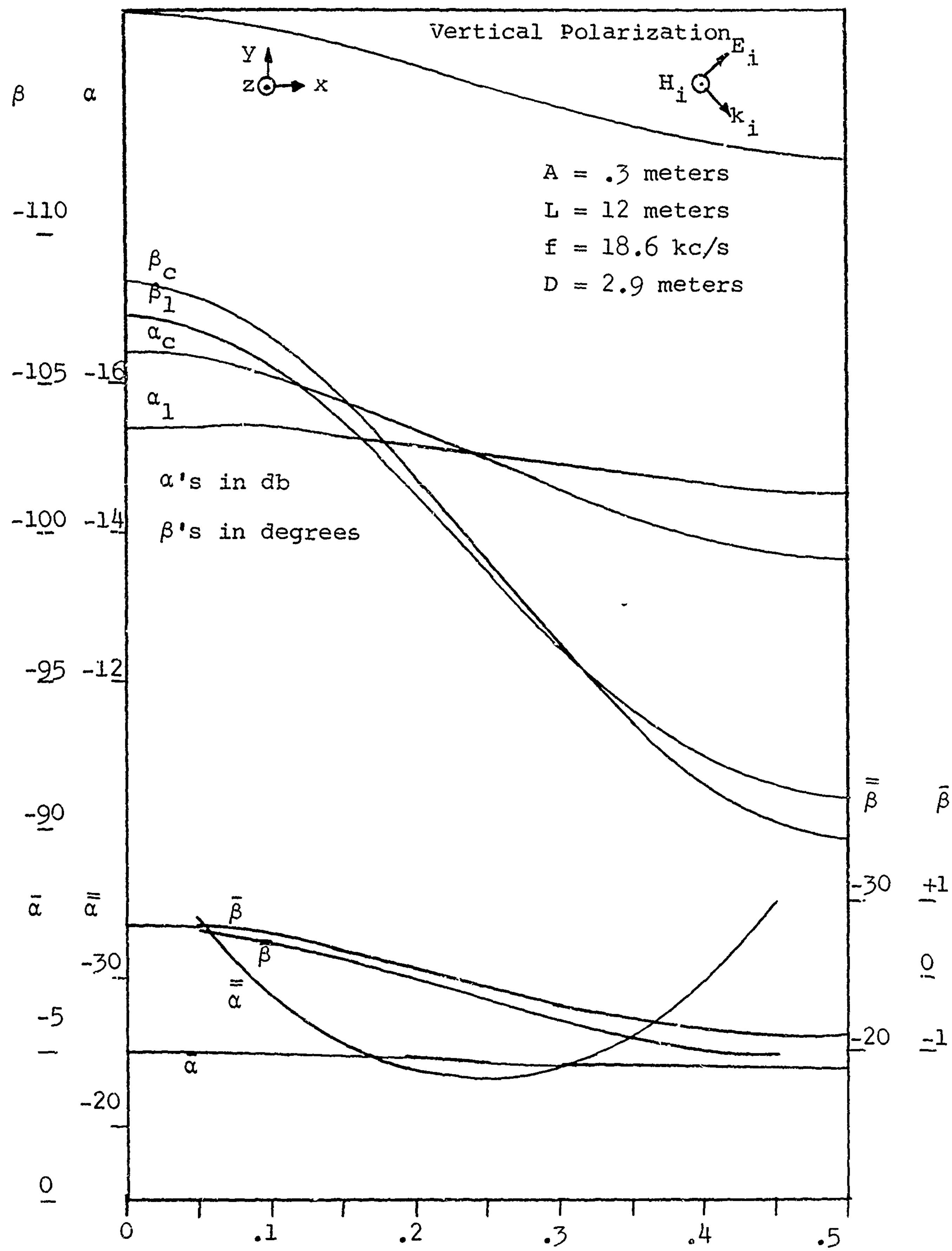


Figure 4.5 Electromagnetic Fields in the Sea.



Horizontal distance from crest of sea wave (in sea wave wavelengths)

Figure 4.6. Electromagnetic Fields in the Sea.



Horizontal distance from crest of sea wave (in sea wave wavelengths)

Figure 4.9. Electromagnetic Fields in the Sea.

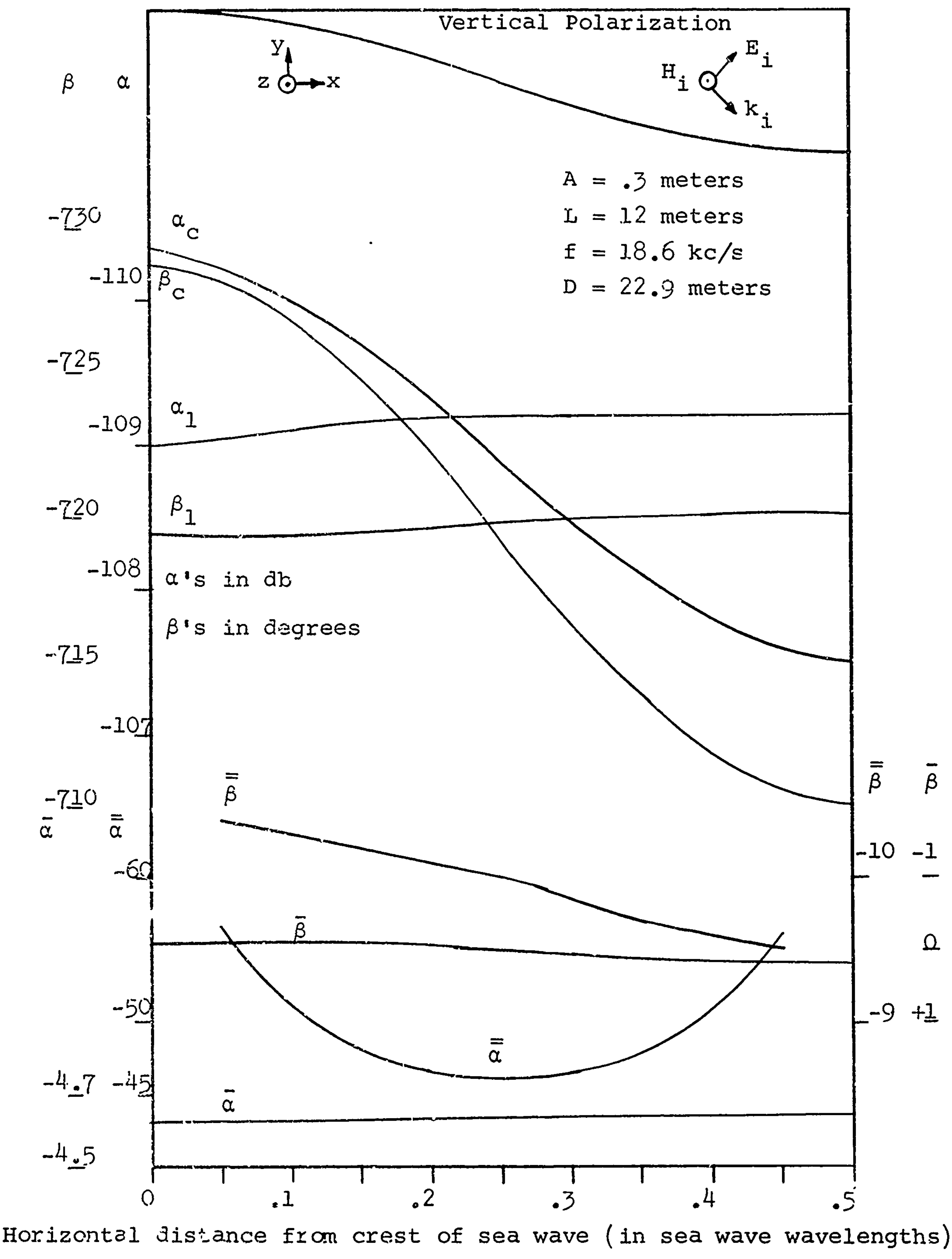
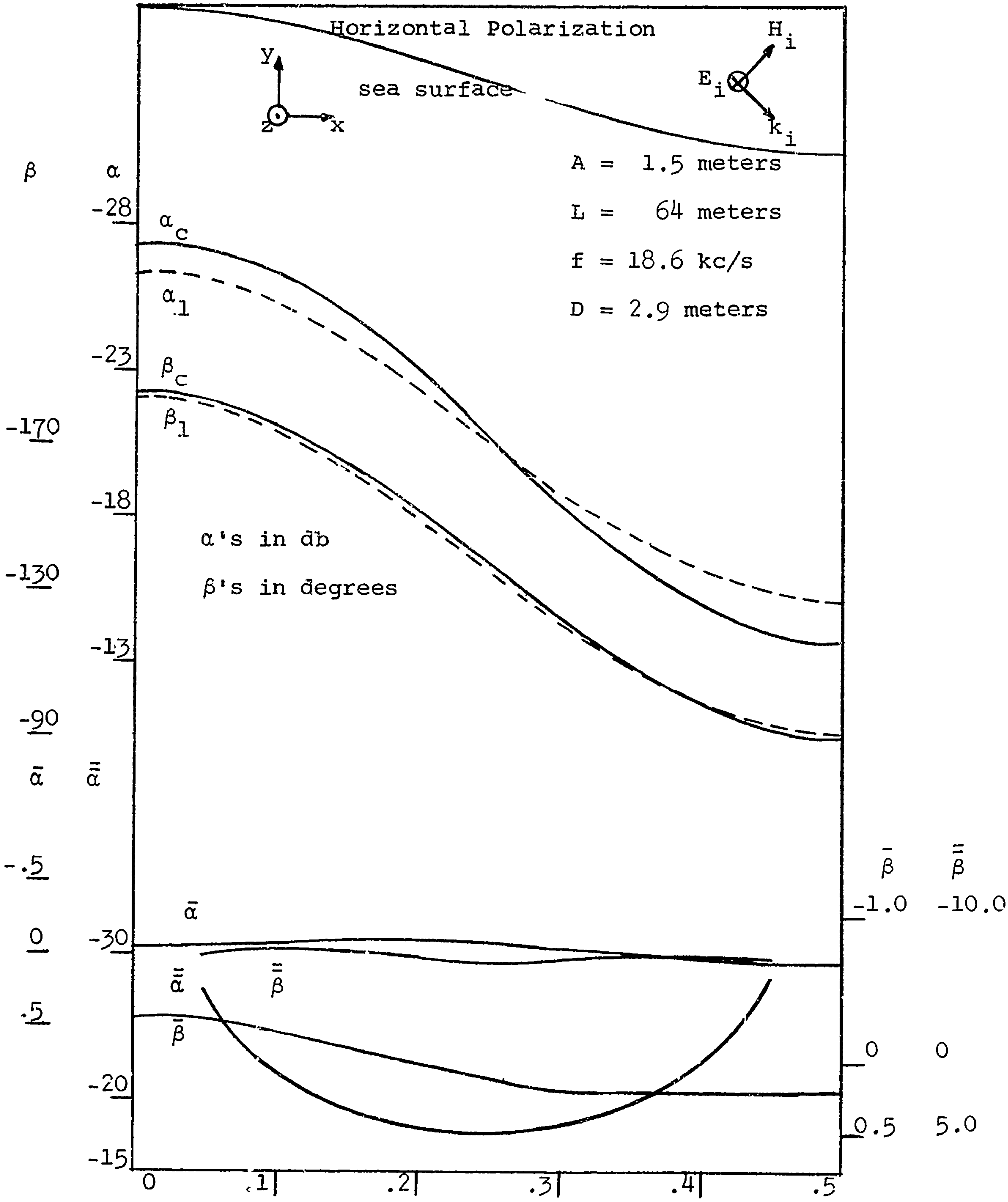
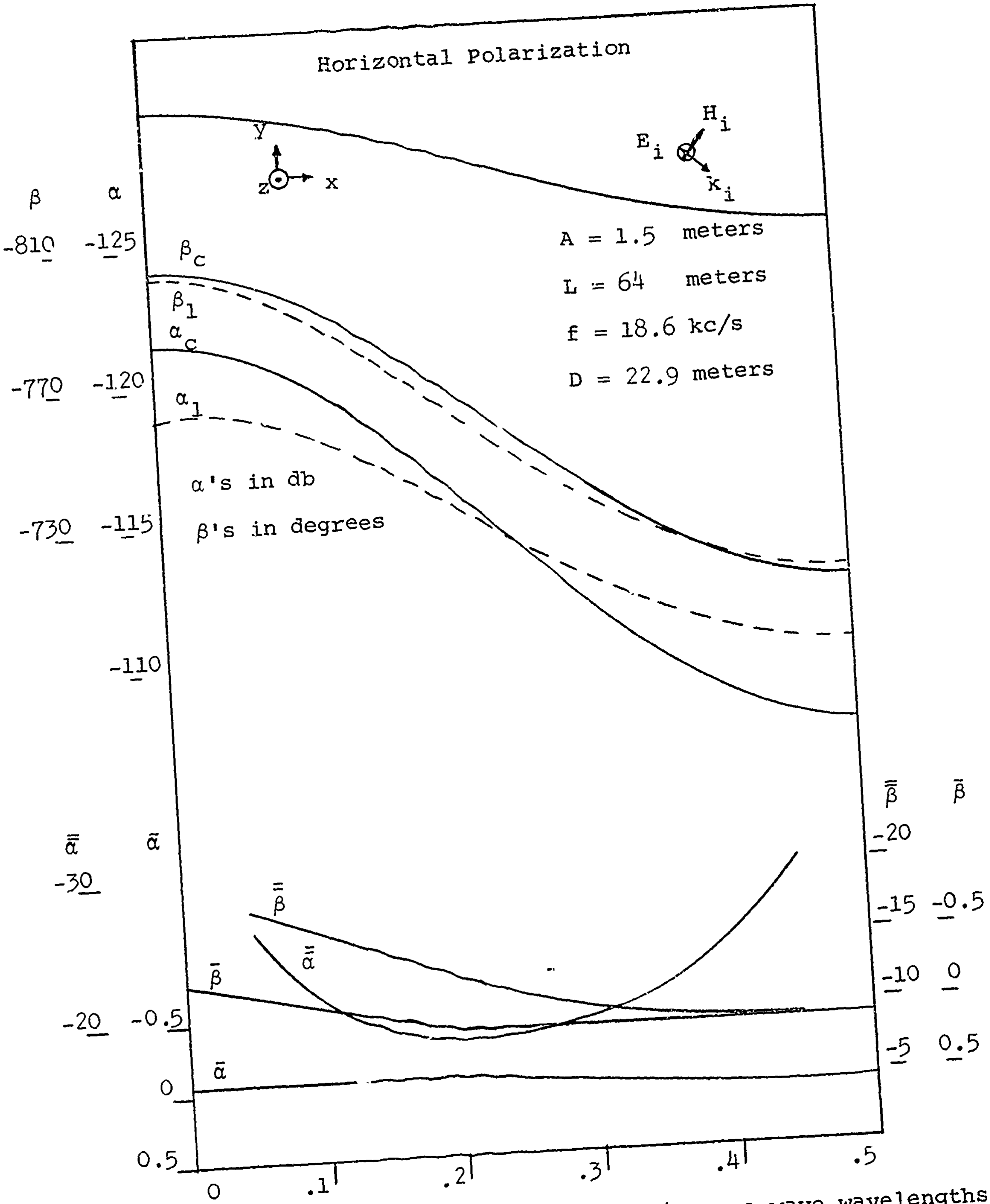


Figure 4.8. Electromagnetic Fields in the Sea.



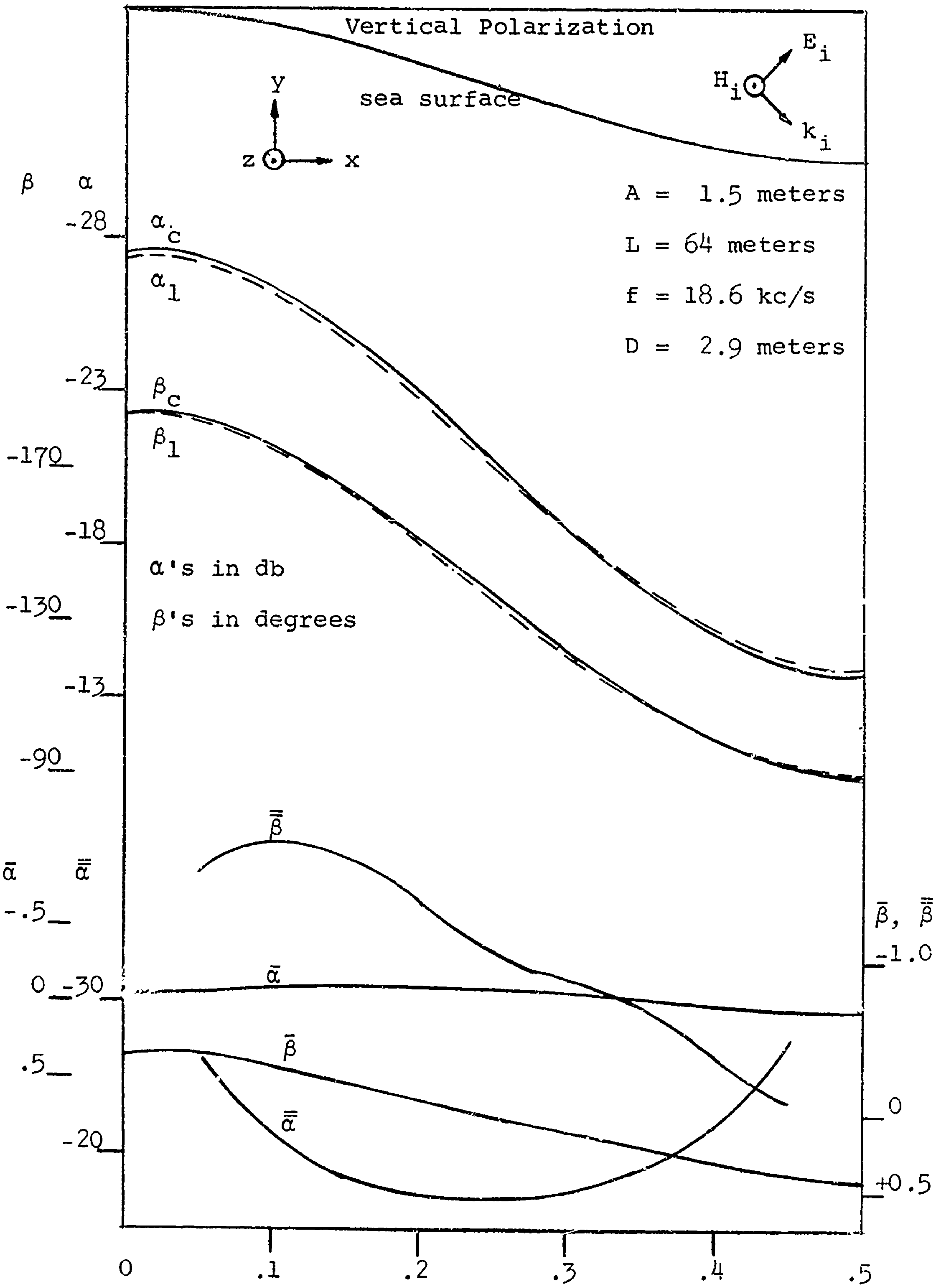
Horizontal distance from crest of sea wave (in sea wave wavelengths)

Figure L.9, Electromagnetic Fields in the Sea



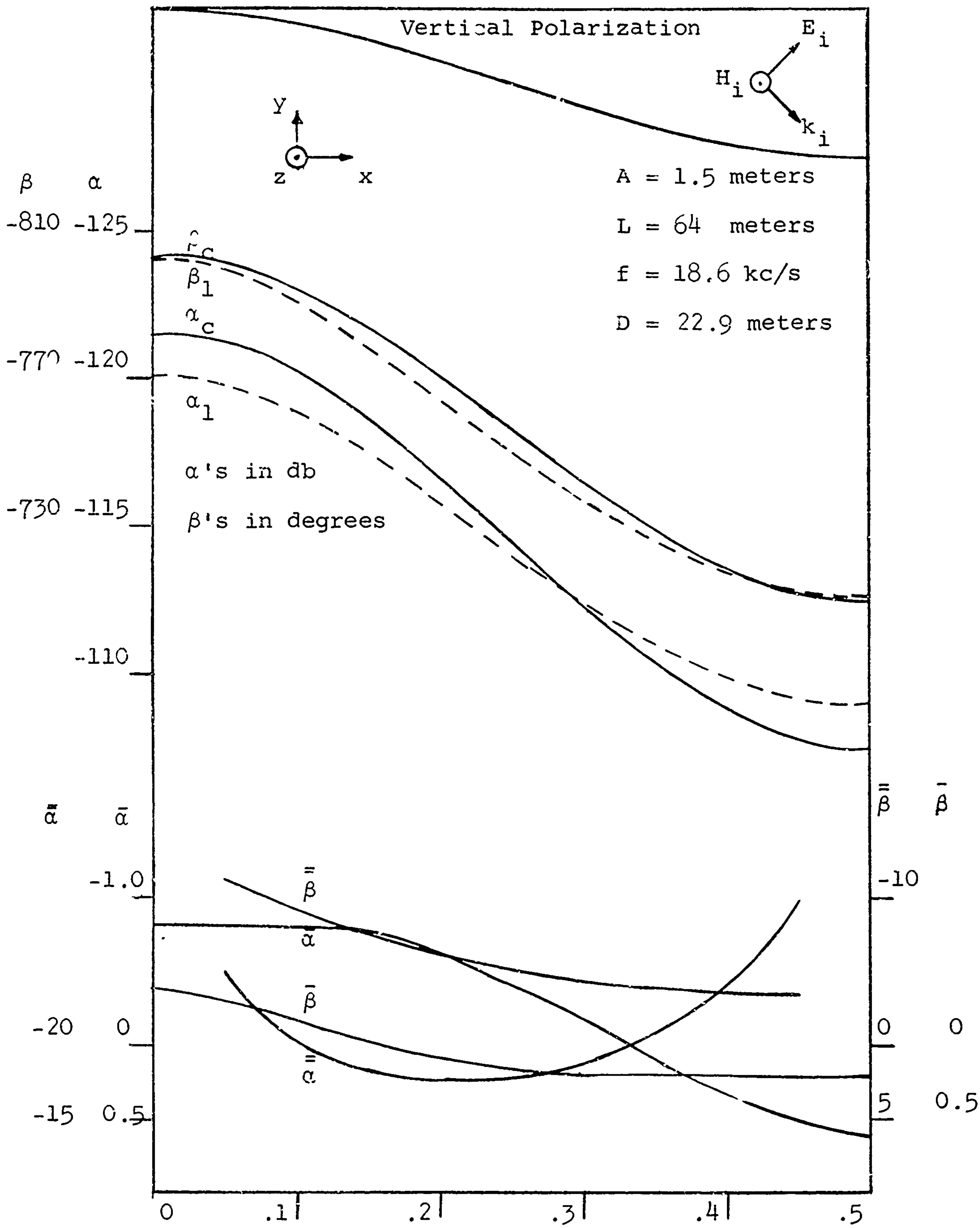
Horizontal distance from crest of sea wave (in sea wave wavelengths)

Figure 4.10, Electromagnetic Fields in the Sea



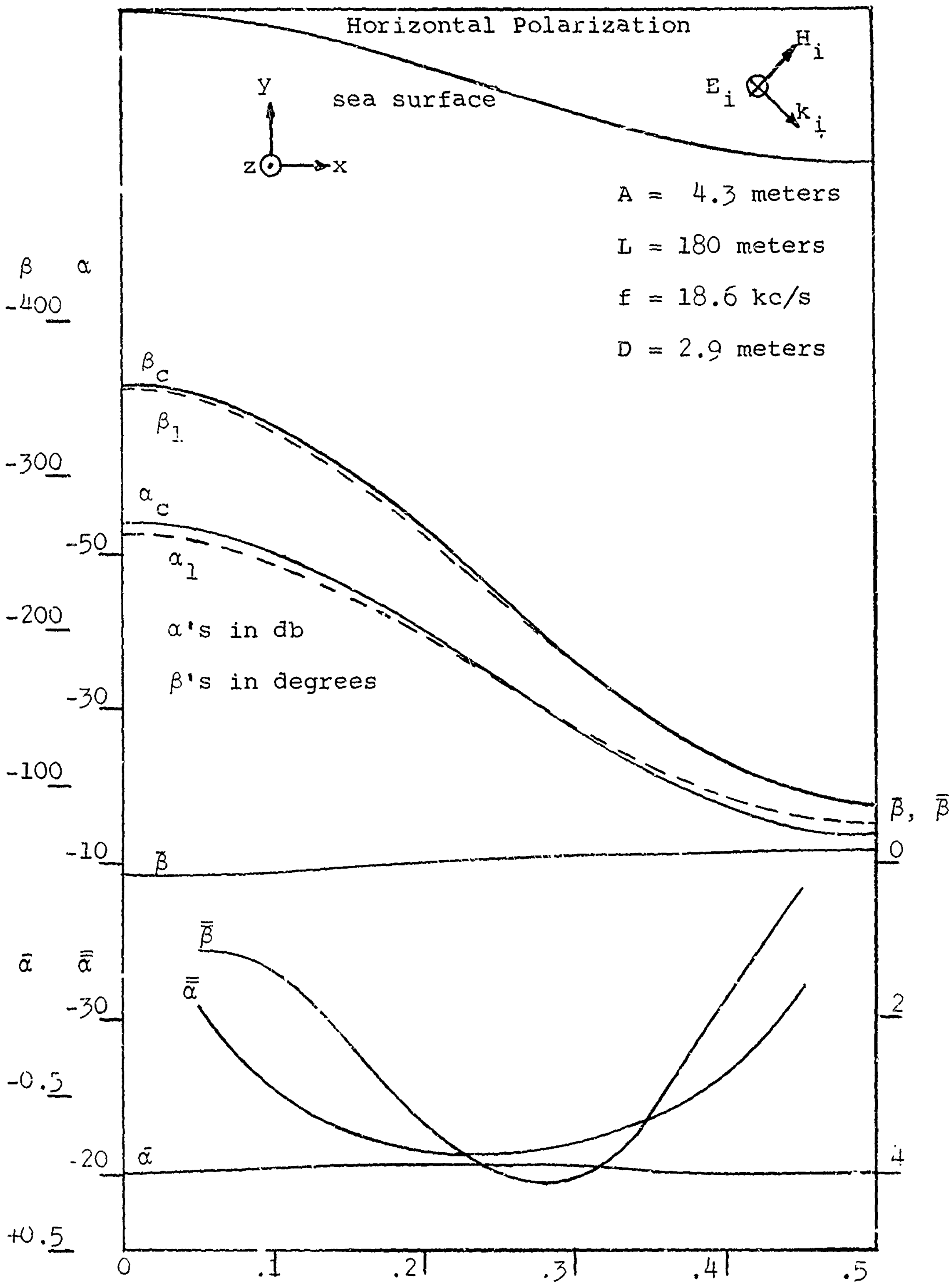
Horizontal distance from crest of sea wave (in sea wave wavelengths)

Figure 4.11, Electromagnetic Fields in the Sea



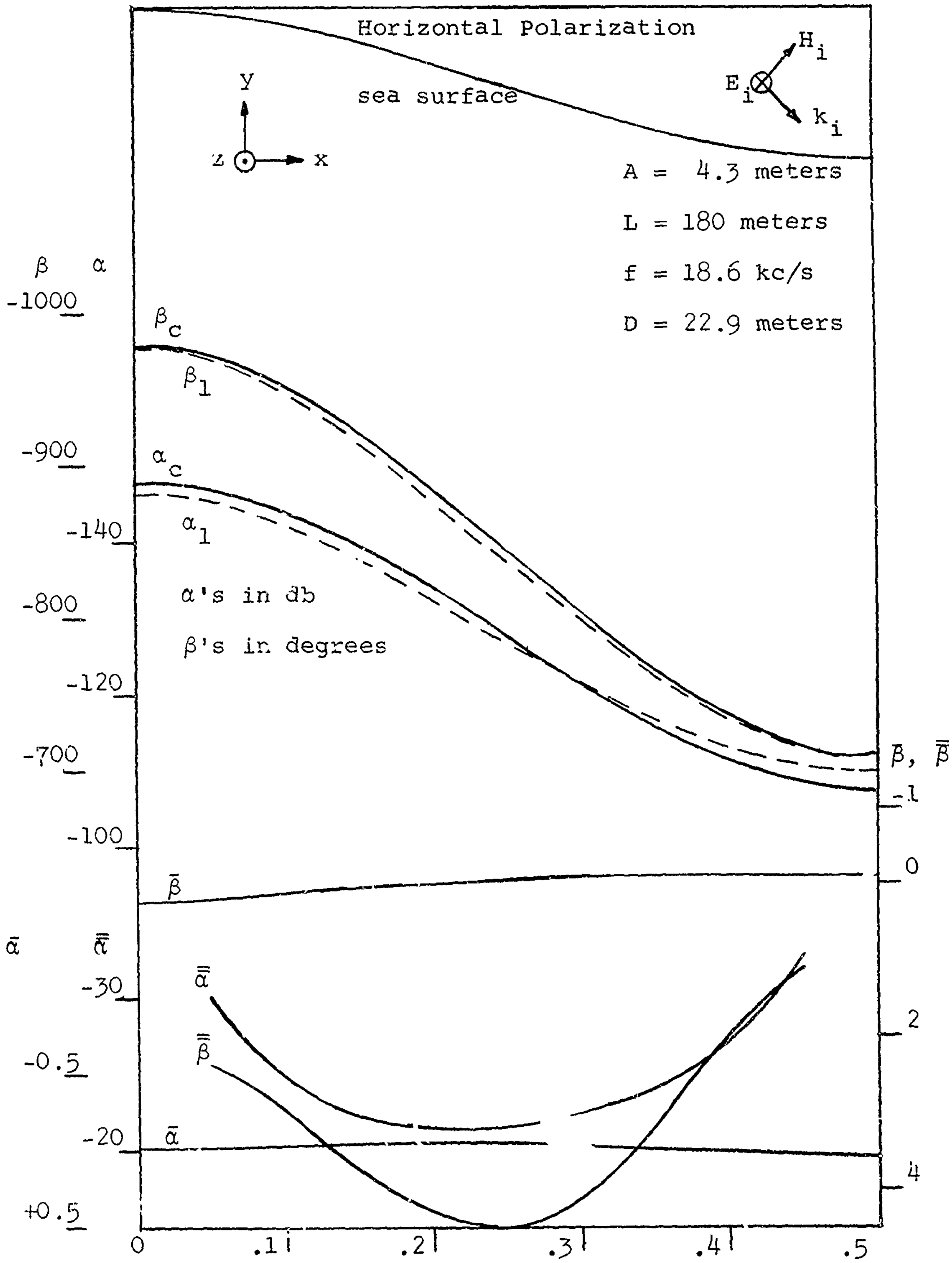
Horizontal distance from crest of sea wave (in sea wave wavelengths)

Figure 4.12, Electromagnetic Fields in the Sea



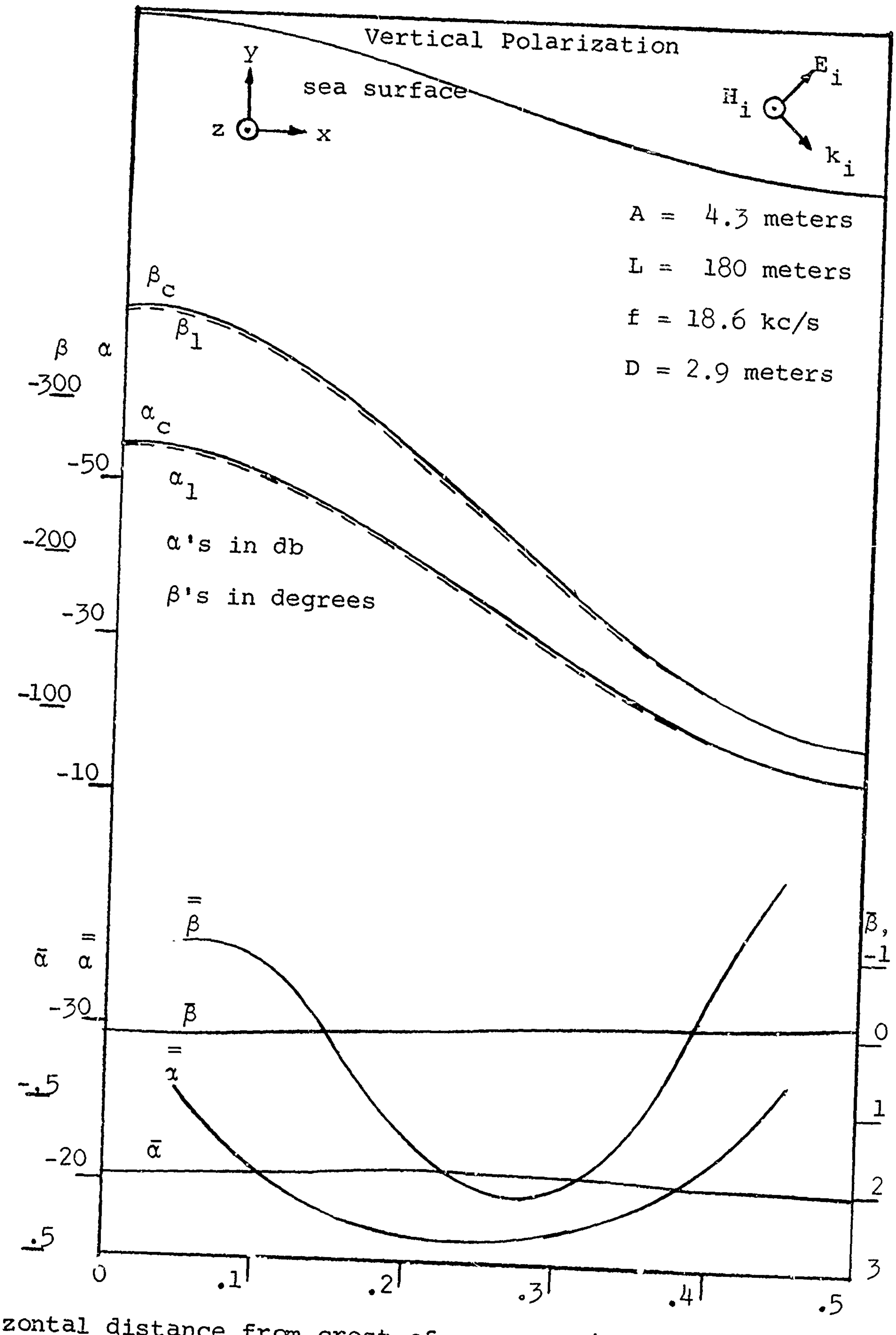
Horizontal distance from crest of sea wave (in sea wave wavelengths)

Figure 4.13. Electromagnetic Fields in the Sea.



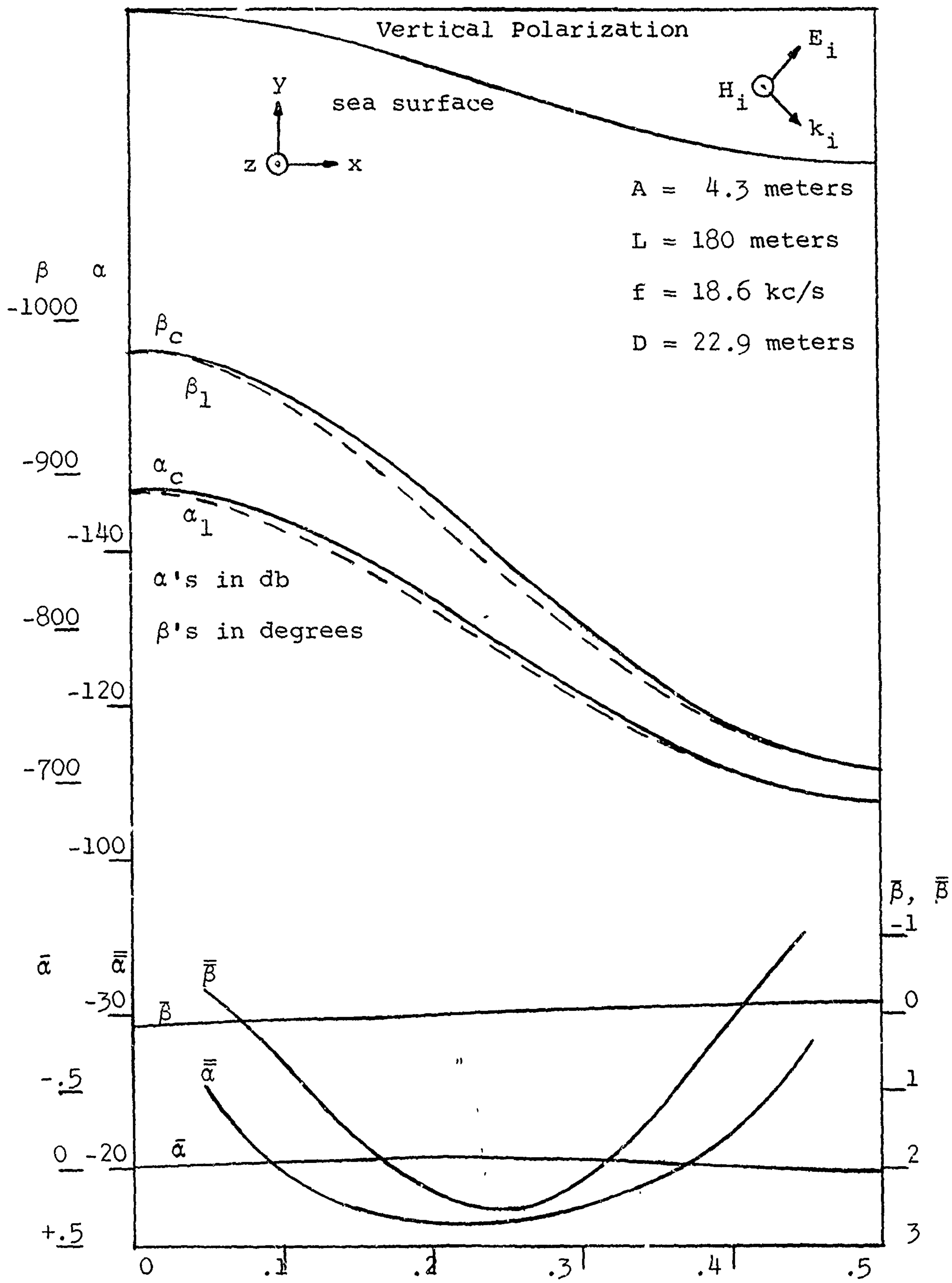
Horizontal distance from crest of sea wave (in sea wave wavelengths)

Figure 4.14. Electromagnetic Fields in the Sea.



Horizontal distance from crest of sea wave (in sea wave wavelengths)

Figure 4.15. Electromagnetic Fields in the Sea.



Horizontal distance from crest of sea wave (in sea wave wavelengths)

Figure 4.16. Electromagnetic Fields in the Sea.

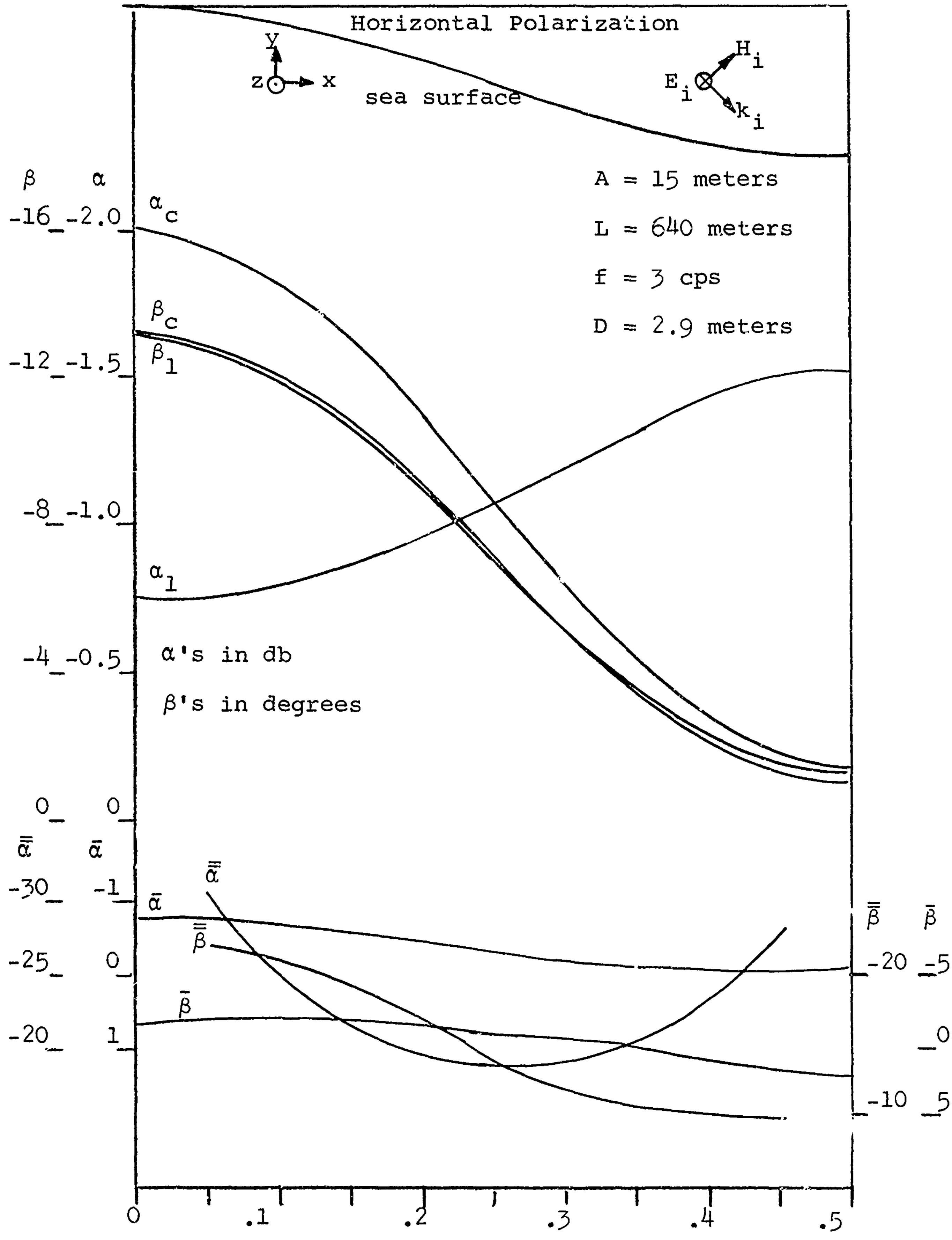


Figure 4.17. Electromagnetic Fields in the Sea.

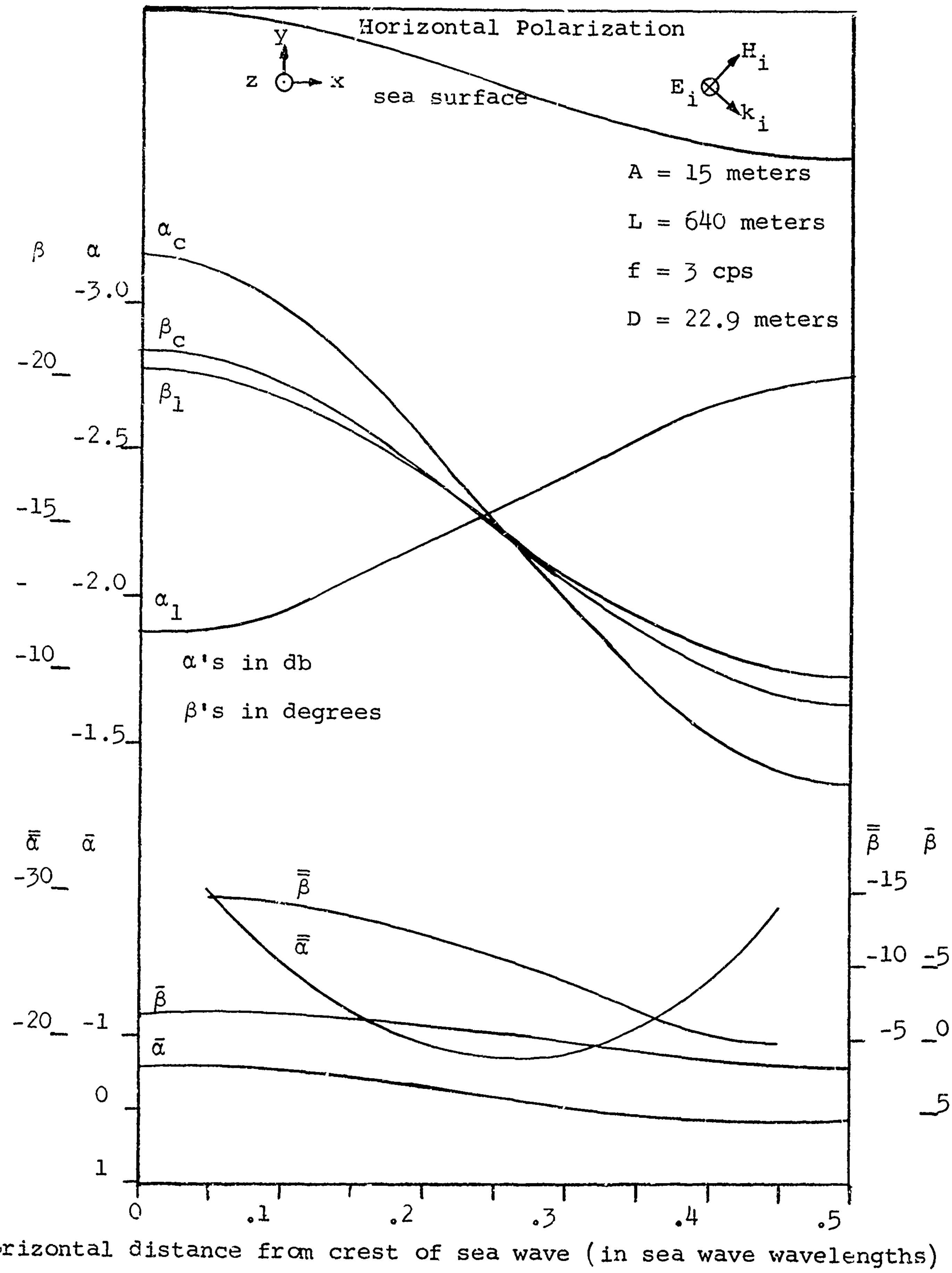
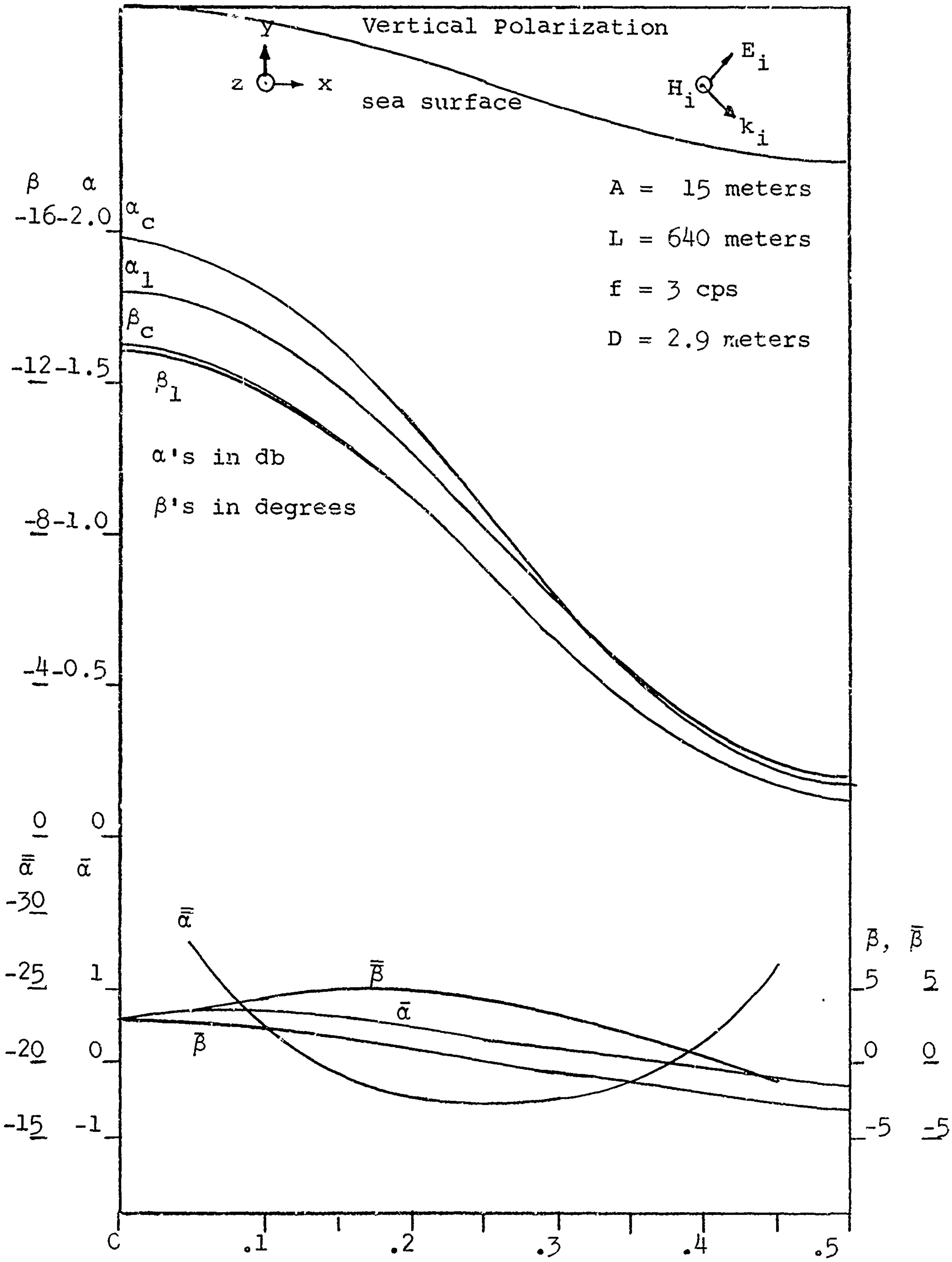
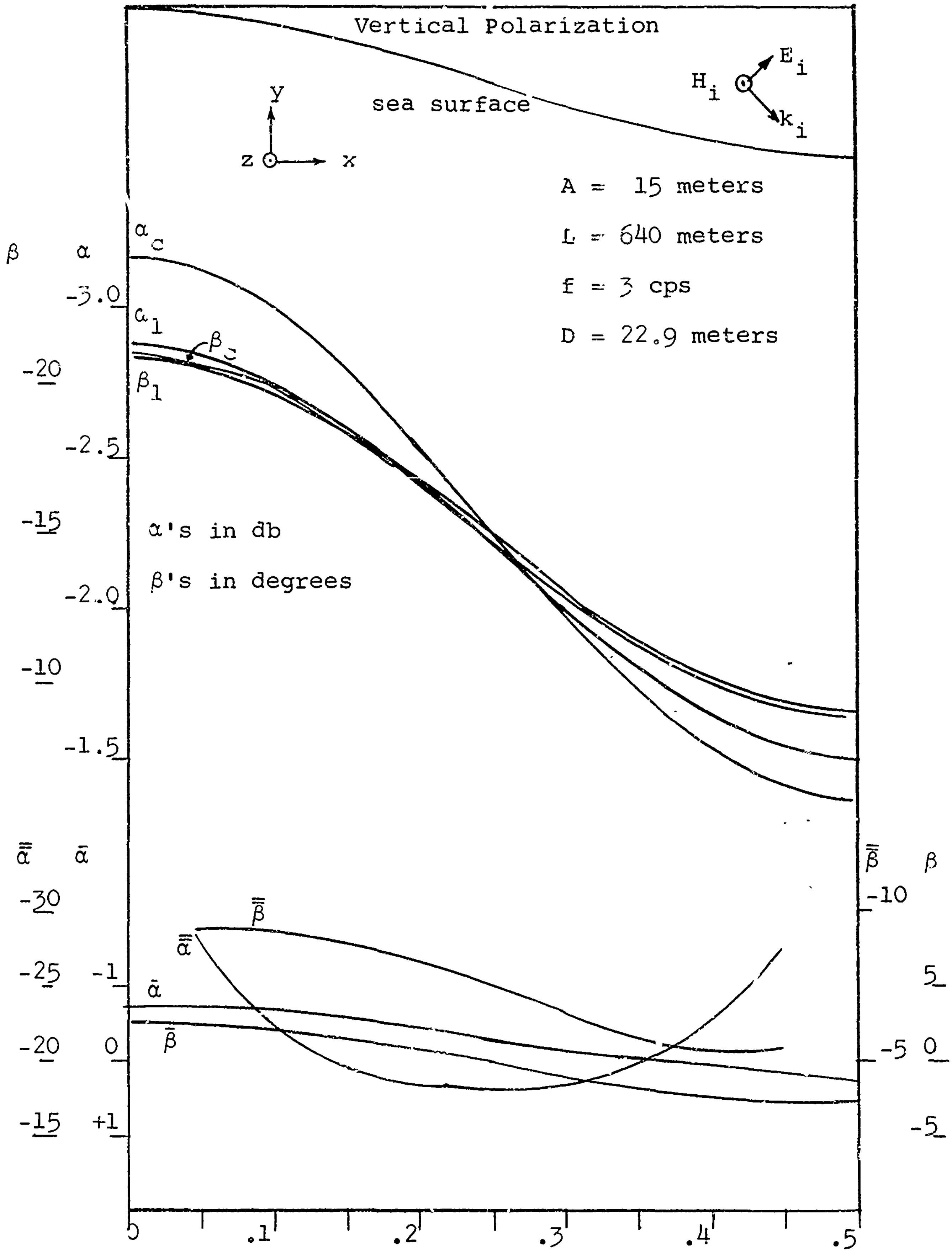


Figure 4.18. Ele tromagnetic Fields in the Sea.



Horizontal distance from crest of sea wave (in sea wave wavelengths)

Figure 4.19. Electromagnetic Fields in the Sea.



Horizontal distance from crest of sea wave (in sea wave wavelengths)

Figure 4.20. Electromagnetic Fields in the Sea.

4.4 Interpretation of Numerical Calculations.

From the curves of Section 4.2, there is a clear indication even at these relatively small depths that the basic theory of propagation (Section 3.2) is qualitatively correct. The field for L = 12 meters at f = 18.6 kc/s (L = 12, $\sqrt{2}\pi\delta = 8$) and L = 640 meters at f = 3 cps. (L = 640, $\sqrt{2}\pi\delta = 650$) shows the decay of the higher order modes. The other curves (L = 64, 180, 640 at f = 18.6 kc/s, $\sqrt{2}\pi\delta = 8$) show the perturbation caused by the roughness of the sea surface is "relatively unchanged." The small changes (decaying of the perturbation) that do occur, occur slowly and are only beginning to become evident.

The curves of Section 4.3, which describe the electromagnetic fields present the picture of a plane wave propagating downward. As $Y_Y \approx Y$, the relationships for the horizontal fields are basically that of a plane wave (i.e., $E_h = \eta_c H_h$); the deviation from this condition is the order of tenth of a db and one degree at the depth considered. As $|Y_x| << |Y|$, the vertical fields should be relatively small; they usually are at least 20 db below the horizontal fields.

The practical interest is then in computing the horizontal fields, the vertical fields being so small. As the horizontal fields are basically related by η_c (neglecting a small factor) the electric field may be computed from the magnetic field and vice versa. A very simple procedure for computing the fields in the sea from the curves of Section 4.3 is:

1. Calculate the tangential field assuming a flat sea surface (the tangential electric field, for example).

2. E_z at depth D is related to the E_z computed on the flat surface by the α_1 and β_1 for the horizontal polarized wave given in Section 4.3 (this is independent of the polarization of the actual incident wave).

3. Similarly, E_x at depth D is related to the E_x on the flat surface by the α_1 and β_1 for the vertically polarized wave.

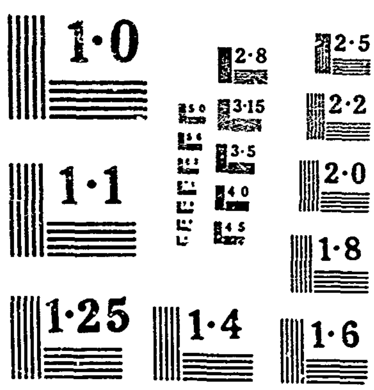
5.0 THE BASIC THEORY FOR THE THREE-DIMENSIONAL PROBLEM

5.1 Introduction to the Three-Dimensional Problem.

It would seem clear that the three-dimensional electromagnetic boundary value problem will be considerably more complex and complicated than the two-dimensional electromagnetic boundary value problem treated previously. Comparing the two problems, some striking differences and similarities appear.

In the two-dimensional problem, the TE and TM modes can be completely decoupled. With this convenience, the two-dimensional electromagnetic boundary value problem can be reduced to a set of considerably simpler uncoupled scalar boundary value problems. In the three-dimensional electromagnetic problem, the TE and TM modes can no longer be decoupled. Coupling occurs through the application of boundary conditions on the rough surface. The three-dimensional problem must then be solved "all at once". However, the coupling between the components of the field is "weak". The dominant magnetic field component is virtually independent of the other field components. Therefore, the dominant component of the magnetic field may be computed by assuming the other magnetic field components are zero. The field pattern is basically one of a "dominantly" TM mode or at least it can be considered as such.

CONT.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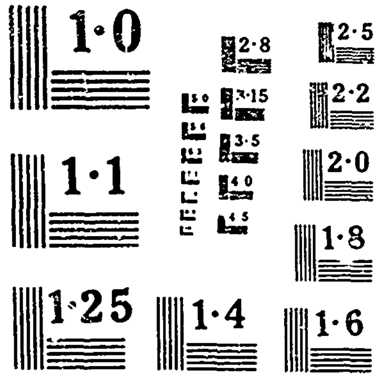
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A difference in the mathematical formulation of the two problems is that the use of vector potentials, which proved quite useful in simplifying the two-dimensional electromagnetic problem, no longer provides a clear means of simplification in the three-dimensional problem. The major reason for this is the effect of the boundary conditions at a rough surface.

Other difficulties in the three-dimensional electromagnetic problem are:

i. The solution is now represented as an integral over a two-dimensional space, so that two integrations are necessary, not just one as previously needed for the two-dimensional case.

ii. The solution is a vector and a function of two space variables making the solution more complicated and therefore somewhat more difficult to interpret.

As a result of the above difficulties, relatively little work has been done on the three-dimensional problem.[1] The work done usually involves the use of a perturbation technique which, as previously pointed out, may not be "accurate" in the near field.[2]

The basic approach is based on integral equation techniques. Many of the assumptions used in the three-dimensional problem were discussed in the two-dimensional problem and therefore a

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[1] Lysanov, Y. P., "Theory of the Scattering of Waves at Periodically Uneven Surfaces," Soviet Physics Acoustics, Vol. 4, no. 1 (Jan.-March, 1958), pp. 1-8.

[2] Hiatt, loc. cit.
Winter, loc. cit.

somewhat abbreviated discussion is given below.

The mathematical description of the physical model is basically an extension of the two-dimensional model to three dimensions. It is assumed that:

1. The "basically spherical" earth is replaced by a "basically flat" earth.

2. The sea surface $\xi(x,y)$ is given by:

$$\xi(x,y) = A \cos k_{sx}x \cos k_{sy}y$$

3. The source of electromagnetic energy is a plane wave incident on the rough surface.

4. For the computation of the electromagnetic fields in the air but on the sea surface, the sea is assumed to be a perfect electric conductor ($\sigma = \infty$, $z_s = 0$).

5.2 Formulation of the Electromagnetic Boundary Value Problem in Terms of Integral Equations

A convenient starting point in formulating the integral equations for the three-dimensional problem is the vector form of the Helmholtz's formula.

$$\vec{E}(\bar{r}) = \int_S \left\{\vec{E}(\bar{r}')\, G_{n'}(k_0|\vec{r}-\vec{r}'|) - \vec{E}_{n'}(\bar{r}')\, G(k_0|\vec{r}-\vec{r}'|)\right\}ds' \tag{5.2.1}$$

$$\vec{H}(\bar{r}) = \int_S \left\{\vec{H}(\bar{r}')\, G_{n'}(k_0|\vec{r}-\vec{r}'|) - \vec{H}_{n'}(\bar{r}')\, G(k_0|\vec{r}-\vec{r}'|)\right\}ds' \tag{5.2.2}$$

where

S is closed surface bounding volume V

$\bar{r}' \in S$

$\bar{r} \in V$

$\hat{n}$ -- outward normal as previously defined.

$\vec{E}$ and $\vec{H}$ are given in terms of their rectangular components.

Again dividing the surface S into two parts, $\bar{S}$ and S_∞, $\bar{S}$ represents the rough sea surface and S_∞ "infinite" hemisphere.

Following the method used in Section 1.4, equations (5.2.1) and (5.2.2) become

$$\vec{E}(\bar{r}) = \vec{E}_i(\bar{r}) + \int_{\bar{S}} \left\{ \vec{E}(\bar{r}')\, G_{n'}(k_o|\vec{r}-\vec{r}'|) - \vec{E}_{n'}(\bar{r}')\, G(k_o|\vec{r}-\vec{r}'|) \right\} ds' \tag{5.2.3}$$

$$\vec{H}(\bar{r}) = \vec{H}_i(\bar{r}) + \int_{\bar{S}} \left\{ \vec{H}(\bar{r}')\, G_{n'}(k_o|\vec{r}-\vec{r}'|) - \vec{H}_{n'}(\bar{r}')\, G(k_o|\vec{r}-\vec{r}'|) \right\} ds' \tag{5.2.4}$$

where $\bar{r} \in V$, $r' \in \bar{S}$.

Equations (5.2.3) and (5.2.4) may be shown to be equivalent to the Stratton-Chu type equations.

$$\vec{E} = \vec{E}_i - \int_{\bar{S}} \left[Z(\hat{n}' \times \vec{H})G + (\hat{n}' \times \vec{E}) \times \text{grad } G + (\hat{n}' \cdot \vec{E}) \text{ grad } G \right] ds \tag{5.2.5}$$

$$\vec{H} = \vec{H}_i + \int_{\bar{S}} \left[Y(\hat{n}' \times \vec{E})G - (\hat{n}' \times \vec{H}) \times \text{grad } G - (\hat{n}' \cdot \vec{H}) \text{ grad } G \right] ds \tag{5.2.6}$$

The $\int_{\bar{S}}$ are "singular" integrals (as were the $\int_{\bar{C}}$ in the two-dimensional problem), the details are available in Appendix C. Applying the results of Appendix C to (5.2.5) and (5.2.6)

$$\vec{E} = 2\left[\vec{E}_i - \int_{\bar{S}} \left\{ Z(\hat{n}' \times \vec{H})G + (\hat{n}' \times \vec{E}) \times \text{grad } G + (\hat{n}' \cdot \vec{H}) \text{grad} G \right\} ds \right] \tag{5.2.7}$$

$$\vec{H} = 2\left[\vec{H}_i + \int_{\bar{S}} \left\{ Y(\hat{n}' \times \vec{E})G - (\hat{n}' \times \vec{H}) \times \text{grad } G + (\hat{n}' \cdot \vec{H}) \text{grad} G \right\} ds \right] \tag{5.2.8}$$

where $\bar{r}$ and $\bar{r}' \in \bar{S}$.

The $\vec{E}$ and $\vec{H}$ given by (5.2.7) and (5.2.8) satisfy the vector partial differential equations, radiation and source conditions of Maxwell's equations; there remains only the boundary condition on $\bar{S}$.

The boundary condition at a perfect electric conductor are:

i) the tangential electric field is zero: $\hat{n} \times \vec{E}^{+} = 0$ (5.2.9)

ii) the normal magnetic field is zero: $\hat{n} \cdot \vec{H}^{+} = 0$ (5.2.10)

where the + sign on $\vec{E}^{+}$ and $\vec{H}^{+}$ denote a limiting process (see Section 1.2).

Applying (5.2.9) and (5.2.10) to (5.2.7) and (5.2.8), the following integral equations are obtained:

$$\vec{E} = 2\left[\vec{E}_i - \int_{\bar{S}} \left(Z(\hat{n}' \times \vec{H})\, G + (\hat{n} \cdot \vec{E})\, \text{grad } G \right) ds \right] \tag{5.2.11}$$

$$\vec{H} = 2\left[\vec{H}_i + \int_{\bar{S}} - \left((\hat{n}' \times \vec{H}) \times \text{grad } G \right) ds \right] \tag{5.2.12}$$

6.0 THE SOLUTION OF THE INTEGRAL EQUATIONS

6.1 Introduction

In this chapter, integral equation (5.2.12) will be solved; this solution gives the tangential magnetic field on the air-side of the sea surface. The technique used is similar to that used in the two-dimensional problem. The integral equation in the sea again yields an impedance type boundary condition on the sea surface.

6.2 The Integral Equations in the Air

To compute the tangential magnetic field on the sea surface, equation (5.2.12) is considered (note that the electric field is absent from this equation, yielding an integral equation in only the magnetic field).

$$\vec{H}(\bar{r}) = 2\left[\vec{H}_i(\bar{r}) - \int_S \left\{(\hat{n}' \times \vec{H}(\bar{r}')) \times \text{grad } G(k_o|\vec{r}-\vec{r}'|)\right\} ds'\right] \tag{5.2.12}$$

where

$$G(k_o|\vec{r}-\vec{r}'|) = \frac{e^{-ik_o|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|} \quad .$$

For the scattered field k_o equal to zero (i.e., $\omega = 0$ and the Green's Function becomes the static or stationary Green's Function) is a valid approximation (see Section 2.8). In this case, the integrals that must be evaluated (in 5.2.12) take the form:

$$I = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f(x,y)}{[x^2 + y^2 + \xi^2(x,y)]^{3/2}} \, dxdy \qquad (6.2.1)$$

Following the method used for the two-dimensional problem (6.2.1) is approximated by

$$\bar{I} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f(x,y)}{(x^2 + y^2)^{3/2}} \, dxdy \qquad (6.2.2)$$

$$\approx I \; .$$

This approximation is valid as:

$$\left| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) \left\{ \frac{1}{(x^2 + y^2 + \xi^2)^{3/2}} - \frac{1}{(x^2 + y^2)^{3/2}} \right\} dxdy \right|$$

$$\leq C \left| \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f(x,y)}{(x^2 + y^2 + \xi^2)^{3/2}} \quad dxdy \right|$$

$$C \leq 1 \pm 3/2 \sup(\xi_x^2, \xi_y^2)$$

$$\sup\,(\xi_x^2, \xi_y^2) \leq 1/50$$

$$.97 \leq C \leq 1.03 \quad .$$

To simplify the computations, the problem is subdivided into two parts depending on the polarization of the incident wave.

1. Vertical polarization. Without loss of generality it may be assumed that the incident magnetic field is in the x direction and that the propagation vector $\vec{k}_i$ is in the y-z plane. The incident field is a TM to y electromagnetic wave. As the currents flowing on the sea surface produce a magnetic field mainly in the x-z plane, the y component of the magnetic field is small and the total wave remains

basically TM to y. In this case, the integral equation (5.2.12) "simplifies" to

$$\begin{Bmatrix} H_x \\ H_y \\ H_z \end{Bmatrix} = 2 \begin{Bmatrix} H_i \\ 0 \\ 0 \end{Bmatrix} +$$

$$2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \begin{Bmatrix} -G_z(n_zH_x-n_xH_z) + G_y(-n_yH_x+n_xH_y) \\ G_z(-n_zH_y+n_yH_z) - G_x(-n_yH_x+n_xH_y) \\ -G_y(-n_yH_y+n_yH_z) + G_x(n_zH_x-n_xH_z) \end{Bmatrix} dS' \tag{6.2.3}$$

where

n_x, n_y, n_z are the rectangular components of the outward normal.

$$n'_x = \frac{\xi_x(\bar{r}\;)}{\sqrt{1+\xi_x^2(\bar{r}')+\xi_y^2(\bar{r}')}} \approx \xi_x(\bar{r}')$$

similar equations hold for n_y and n_z .

G_x, G_y, G_z are the rectangular components of grad G.

$$G_x = \frac{(x-x')}{\left\{(x-x')^2+(y-y')^2+(\xi(x,y)-\xi(x',y'))^2\right\}^{3/2}}$$

similar equations hold for G_y and G_z.

Following the method used previously, let

$$\vec{H} = \sum_{m,n} \begin{Bmatrix} h_{x,m,n} \\ h_{y,m,n} \\ h_{z,m,n} \end{Bmatrix} e^{-i(k_{sx}mx + k_{sy}ny)}\, e^{-ik_{yi}y} \tag{6.2.4}$$

$$\vec{H}_i = \sum_{m,n} \begin{Bmatrix} h_{xi,m,n} \\ h_{yi,m,n} \\ h_{zi,m,n} \end{Bmatrix} e^{-i(k_{sx}mx + k_{xy}ny)} e^{-ik_{yi}y} \quad (6.2.5)$$

$$\xi(x,y) = A \cos k_{sx}x \cos k_{sy}y = \frac{A}{4} \{e^{i(k_{sx}x + k_{sy}y)} +$$

$$e^{i(k_{sx}x-k_{sy}y)} + e^{i(-k_{sx}x + k_{sy}y)} + e^{i(-k_{sx}x - k_{sy}y)}\} \quad (6.2.6)$$

Placing equations (6.2.4), (6.2.5) and (6.2.6) into (6.2.3) with the use of the formula

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{(x-x')}{((x-x')^2+(y-y')^2)^{3/2}} e^{-i(k_{sx}x'+k_{xy}y')}dx'dy'$$

$$= \frac{2i\pi\, k_{sx}}{\sqrt{k_{sx}^2+k_{xy}^2}} e^{-i(k_{sx}x+k_{sy}y)} \quad (6.2.7)^1$$

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[1]To obtain equation (6.2.7) consider

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{(x-x')}{((x-x')^2+(y-y')^2)^{3/2}} e^{-i(k_{sx}x'+k_{sy}y')}dx'dy'$$

$$= e^{-i(k_{sx}x+k_{sy}y)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{x'e^{i(k_{sx}x'+k_{sy}y')}}{((x')^2+(y')^2)^{3/2}} dx'dy'$$

$$\int_{-\infty}^{\infty} \frac{x'e^{ik_{sx}x'}}{((x')^2+(y')^2)^{3/2}} dx' = 2i\int_{0}^{\infty} \frac{x' \sin k_{sx}x'}{((x')^2+(y')^2)^{3/2}} dx'$$

$$= 2i\left[\frac{1}{2}\frac{\sqrt{\pi}}{\Gamma(3/2)} k_{sx} K_o (k_{sy}y')\right] \qquad \mathrm{Re}[y'] > 0$$

(Bateman Manuscript Project, Vol. I, McGraw-Hill, New York, 1954; p. 69, equation No. 11.)

$$\text{for } \vec{H}_i = \begin{Bmatrix} e^{-ik_{zi}\xi(x,y)} \\ 0 \\ 0 \end{Bmatrix} e^{-ik_{yi}y}$$

the following results are obtained.

$$H_x \approx 2\left\{1 + \frac{k_{sx}{}^2 A}{\sqrt{k_{sx}^2 + k_{sy}^2}} \cos k_{sx}x \cos k_{sy}y\right\} e^{-ik_{yi}y} \quad \text{(a)}$$

$$(6.2.8)$$

$$H_y \approx 2\left\{\frac{k_{sx}k_{sy}A}{\sqrt{k_{sx}^2 + k_{sy}^2}} \sin k_{sx}x \sin k_{sy}y\right\} e^{-ik_{yi}y} \quad \text{(b)}$$

$$H_z \approx (-2k_{sx}A \sin k_{sx}x \cos k_{xy}y)\, e^{-ik_{yi}y} \quad \text{(c)}$$

Noting the $\int_0^\infty \frac{x' \sin k_{sx}x'}{((x')^2 + (y')^2)^{3/2}}\, dx'$ is an even function of y' for real y'

$$\int_{-\infty}^{\infty} \frac{x' e^{ik_{sx}x'}}{((x')^2 + (y')^2)^{3/2}}\, dx' = 2ik_{sx}\, K_o(k_{sx}|y'|)$$

$$y' - \text{real} \quad y' \neq 0$$

where K_o is a modified Bessel function.

$$\int_{-\infty}^{\infty} K_o(k_{sx}|y'|)e^{ik_{sy}y'}dy' = 2\int_0^\infty K_o(k_{sx}y') \cos k_{sy}y'\, dy'$$

$$= 2\left[\frac{\pi}{2} \frac{1}{\sqrt{(k_{sx}^2) + (k_{sy}^2)}}\right]$$

(Bateman Manuscript Project, Vol. I, McGraw-Hill, New York, 1954; p. 49, equation No. 40.)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{(x-x')}{((x-x')^2 + (y-y')^2)^{3/2}}\, e^{-ik_{sx}x'} e^{-ik_{sy}y'}\, dx'dy'$$

$$= \frac{2i\pi k_{sx}}{\sqrt{(k_{sx})^2 + (k_{sy})^2}}\, e^{-i(k_{sx}x + k_{sy}y)} \quad .$$

2. Horizontal polarization.

Again without loss of generality, it may be assumed that the incident magnetic field is in the x-z plane and the propagation is in the x-z plane. In this case the incident field is TM to y and the total field is basically TM to y. The integral equation becomes

$$\begin{Bmatrix} H_x \\ H_y \\ H_z \end{Bmatrix} = 2 \begin{Bmatrix} H_{xi} \\ 0 \\ H_{zi} \end{Bmatrix} +$$

$$2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \begin{Bmatrix} -G_z(n_zH_x-n_xH_z) + G_y(-n_yH_x+n_xH_y) \\ G_z(-n_zH_y+n_yH_z) - G_x(-n_yH_x+n_xH_y) \\ -G_y(-n_zH_y+n_yH_z) + G_x(n_zH_x-n_xH_z) \end{Bmatrix} dS' \tag{6.2.9}$$

$$\vec{H}_i = \begin{Bmatrix} \frac{k_{zi}}{k_o}\, e^{-ik_{zi}\xi(x,y)} \\ 0 \\ \frac{k_{xi}}{k_o}\, e^{-ik_{zi}\xi(x,y)} \end{Bmatrix} e^{-ik_{xi}x} \tag{6.2.10}$$

Placing (6.2.5), (6.2.6), (6.2.7) and (6.2.11) into (6.2.10) the following results are obtained:

$$H_x \approx 2\frac{k_{zi}}{k_o}\left\{1 + \frac{k_{sx}^2 A}{\sqrt{k_{sx}^2 + k_{sy}^2}} \cos k_{sx}x \cos k_{sy}y\right\} e^{-ik_{yi}y} \quad \text{(a)}$$

$$H_y \approx 2\frac{k_{zi}}{k_o}\left\{\frac{k_{sx}k_{sy}A}{\sqrt{k_{sx}^2 + k_{sy}^2}} \sin k_{sx}x \sin k_{sy}y\right\} e^{-ik_{yi}y} \quad \text{(b)} \qquad (6.2.11)$$

$$H_z \approx -2\frac{k_{zi}}{k_o}\left\{k_{sx} A \cos k_{xy}y \sin k_{sx}x)\right\} e^{-ik_{yi}y} \quad \text{(c)}$$

The tangential magnetic field is basically the same for horizontal and vertical polarization (neglecting the factor $\frac{k_{zi}}{k_o}$); the basic difference is in the direction of the fields (relative to the direction of propagation).

Vertical polarization:

tangential magnetic field ≈

$$2\left\{1 + \frac{k_{sx}^2 A}{\sqrt{k_{sx}^2 + k_{sy}^2}} \cos k_{sx}x \cos k_{sy}y\right\} e^{-ik_{yi}y} \quad (6.2.12)$$

Horizontal polarization:

tangential magnetic field ≈

$$2\frac{k_{zi}}{k_o}\left\{1 + \frac{k_{sx}^2 A}{\sqrt{k_{sx}^2 + k_{sy}^2}} \cos k_{sx}x \cos k_{sy}y\right\} e^{-ik_{xi}x} \quad (6.2.13)$$

At this point there is an obvious need to check these results with the results of the two-dimensional problem. There are two cases:

1. $k_{sy} = 0,\ k_{sx} \pm 0$

   H_x perpendicular to the direction of surface variation

   $H_{tan} = \varphi_n$ (two-dimensional case)

   $\approx 2(1 + k_{sx} A \cos k_{sx}x)e^{-ik_{yi}y}$

which agrees with the results for the limiting case ($k_{sy}=0$) in equation (6.2.12).

2. $k_{sx} = 0,\ k_{sy} \neq 0$

   H_x parallel to the direction of surface variation

   $H_{tan} \approx 2\varphi_i$

   ≈ 2

which again agrees with the three-dimensional result.

A major difference caused by polarization is in the factor $\frac{k_{zi}}{k_o}$ present in the horizontally polarized case. This factor is present in the flat surface or unperturbed case.

As the basic modes of propagation in the sea are the same for the two or three-dimensional problem, no numerical computations are given in the three-dimensional case.

6.3 The Integral Equations in the Sea.

The same integral equations (5.2.3) and (5.2.4) hold for the electromagnetic fields in the sea as well as in the air; in the sea, $\vec{E}_i$ and $\vec{H}_i$ are equal to zero and $k = k_c$. The development of this section parallels that of Section 2.4, only is somewhat more complicated because of the vector nature of the fields and the fact that the integrals are now over a two-dimensional space. However, the scalar problem in the sea may be treated first.

The integral equation for the scalar wave function $\varphi(\bar{r})$ in terms of its boundary values is

$$\varphi(\bar{r}) = 2\int_{\bar{S}} \left\{\varphi(\bar{r}')\, G_{n'}(k_c|\vec{r}-\vec{r}'|) - \varphi_{n'}(\bar{r}')G(k_c|\vec{r}-\vec{r}'|)\right\} ds' \tag{6.3.1}$$

for $\bar{r}$ and $\bar{r}' \epsilon \bar{S}$.

Using the same assumptions as in Section 3.4

$$\int_{\bar{S}} \varphi_{n'}(\bar{r}')\, G(k_c|\vec{r}-\vec{r}'|)ds' \approx \varphi_n(\bar{r}) \int_0^{2\pi} d\Phi \int_0^{\infty} \frac{e^{-ik_c\rho}}{4\pi\rho}\, \rho d\rho$$
$$\approx \frac{\varphi_n(\bar{r})}{2ik_c} \qquad \bar{r}\,\epsilon\bar{S} \tag{6.3.2}$$

$$\int_{\bar{S}} \varphi(\bar{r}')\ G_{n'}(k_c|\vec{r}-\vec{r}'|)ds' \approx$$

$$\varphi(\bar{r}) \int_0^{2\pi} d\Phi \int_0^{\infty} \frac{1}{4\pi}(-ik_c - \frac{1}{\rho}) \frac{e^{-ik_c\rho}}{\rho}\ \rho_{n'}\rho d\rho$$

$$\approx \varphi(\bar{r}) \frac{1}{2} \frac{1}{2\bar{R}} \int_0^{\infty} (-ik_c - \frac{1}{\rho})\ \rho e^{-ik_c\rho} d_\rho \qquad (6.3.3)^1$$

$$\approx \varphi(\bar{r}) (\frac{-1}{2ik_c\bar{R}}) \qquad (6.3.4)$$

1

$$\frac{\rho_n}{\rho} = \frac{-(x-x')\xi_x - (y-y')\xi_y + \xi(x,y) - \xi(x',y')}{(x-x')^2 + (y-y')^2 + (\xi(x,y) - \xi(x',y'))^2} .$$

For x-x' and y-y' sufficiently small

$$\xi(x,y) = \xi(x',y') + \xi_x(x-x') + \xi_y(y-y')$$
$$+ \frac{1}{2}\left\{\xi_{xx}(x-x')^2 + \xi_{xy}(x-x')(y-y') + \xi_{yy}(y-y')^2\right\}$$
$$+ \ldots$$

$$\frac{\rho_n}{\rho} \leq \frac{1}{2} (\frac{1}{R_x} + \frac{1}{R_{xy}} + \frac{1}{R_y}) = \frac{1}{2} \frac{1}{\bar{R}}$$

where

R_x - radius of curvature of the section of the surface (i.e. curve) in the x plane.

R_{xy} - radius of curvature of the section of the surface (i.e. curve) in the x=y plane.

For $|k_c|\bar{R} << 1$, the "impedance" type boundary condition is again encountered:

$$\varphi(\bar{r}) \approx 2\left\{\frac{-\varphi_n(\bar{r})}{2ik_c}\right\} = i\frac{\varphi_n(\bar{r})}{k_c} \qquad \bar{r}\,\varepsilon\,\bar{S} \qquad (6.3.5)$$

or

$$\vec{E}(\bar{r}) \approx i\frac{\vec{E}_n(\bar{r})}{k_c} \qquad \bar{r}\,\varepsilon\,\bar{S} \qquad (6.3.6)$$

Equation (6.3.5) appears to be a direct "carry over" from the twc-dimensional problem where the same equation was derived. In the three-dimensional problem, however, equation (6.3.6) does not directly imply $E_{tan} = \eta_c\, H_{tan}$ as

$$\hat{n}x\vec{H} = \hat{n}x\frac{1}{Z}\text{ curl }\vec{E} \neq \frac{\partial}{\partial n}\vec{E}_{tan}\ .^{2}$$

Then, to obtain the impedance type relationship between the electromagnetic quantities directly, the Stratton-Chu equations are used. Only one of the two vector Stratton-Chu integral equations is necessary.

$$\vec{H}(\bar{r}) = 2\int_{\bar{S}} \Big\{Y(\hat{n}'x\vec{E}(\bar{r}'))\; G(k_c|\vec{r}-\vec{r}'|) \;-$$

$$(\hat{n}'xH(\bar{r}'))\text{ x grad }G(k_c|\vec{r}-\vec{r}'|) - (\hat{n}'\cdot H(r'))\text{grad }G\Big\}ds' \qquad (6.3.7)$$

$$(\hat{n}x\vec{H})\text{ x grad }G = (\hat{n}\cdot\text{grad }G)\vec{H} - (\text{grad }G\cdot\vec{H})\hat{n}\ . \qquad (6.3.8)$$

The relation obtained directly involves only the tangential fields, so the second term on the right-hand side of equation (6.3.8) may be neglected. Also, using the boundary condition at a perfect electric conductor $\hat{n}\cdot\vec{H} = 0$ so the third term on the right-hand side of equation (6.3.7) is taken to be zero.

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[2]This follows from $(\text{curl} - \frac{\partial}{\partial n})\vec{E} \neq 0$.

$$\vec{H}(\bar{r}) = 2\int_{\bar{S}} \Big\{ Y(\hat{n}x\vec{E}(\bar{r}')) \; G(k_c|\vec{r}-\vec{r}'|) -$$

$$(\hat{n}'\cdot\text{grad } G(k_c|\vec{r}-\vec{r}'|)\vec{H}(\bar{r}')\Big\} \, ds' \qquad (6.3.9)$$

$$\int_{\bar{S}} (\hat{n}'x\vec{E}(\bar{r}')) \; G(k_c|\vec{r}-\vec{r}'|) \; ds' \approx \hat{n}x\vec{E}(\bar{r}) \int_{\bar{S}} G(k_c|\vec{r}-\vec{r}'|) \; ds'$$

$$\approx \hat{n}x\vec{E}(\bar{r}) \, \frac{1}{2ik_c} \qquad (6.3.10)$$

$$\int_{\bar{S}} (\hat{n}'\cdot\text{grad } G(k_c|\vec{r}-\vec{r}'|) \; \vec{H}(\bar{r}')ds' \approx \vec{H}(\bar{r}) \int_{\bar{S}} G_{n'}(k_c|\vec{r}-\vec{r}'|)ds'$$

$$\approx -\frac{\vec{H}(\bar{r})}{2ik_c\bar{R}} \, . \qquad (6.3.11)$$

Then from placing equations (6.3.10) and (6.3.11) into (6.3.9)

$$H_{tan} \left\{1 - \frac{1}{ik_c\bar{R}}\right\} \approx \frac{Y}{ik_c} \, E_{tan} \qquad (6.3.12)$$

for $|k_c|\bar{R} >> 1$ (see discussion Section 2.4) .

$$H_{tan} \approx \frac{Y}{ik_c} \, E_{tan} \approx \frac{1}{\eta_c} \, E_{tan} \qquad (6.3.13)$$

as

$$k_c = i\sqrt{YZ} \qquad \eta_c = \sqrt{Z/Y} \quad .$$

Equation (6.3.13) yields an estimate on the tangential electric field at a good electric conductor in terms of the tangential magnetic field there.

7.0 CONCLUSION

From the solutions given in Chapters 2 and 6 for the tangential magnetic field on the rough sea surface, the perturbation of this field due to the roughness of the sea surface is less than 3 db for most sea conditions. The perturbation depends both on the polarization of the incident field and the direction of propagation of the incident field.

From both the numerical solutions (Chapter 4) and the heuristic theory of propagation in the sea (Chapter 3), it is seen that the perturbation of the fields in the sea caused by the roughness of the sea surface decays rapidly with depth if the sea wave wavelength is less than or the order of magnitude of the skin depth of the sea at the frequency considered; if the sea wave wavelength is many orders of magnitude larger than the skin depth, there is little decay (at the depths considered) of the perturbation, so that the phase and amplitude of the fields in the sea vary directly with the height of the sea vertically above them. These results are in agreement with the work of Lerner and Max.[1]

[1] Lerner, R. M., and J. Max, "Very Low Frequency and Low Frequency Fields Propagating near and into a Rough Sea," a paper presented to the URSI Spring 1963 Meeting.

The above results are also in agreement with the work of Winter.[2] Winter used a statistical description of the electromagnetic fields in the sea, obtaining the basic conclusion that the fields in the sea are "on the average" greatly perturbed by the rough sea surface (from the curves of Chapter 4, the perturbation may be 40 db or greater). The statistical description of the electromagnetic fields in the sea, in this case, do not convey very much information. This is particularly true in light of the fact that the sea surface varies so slowly (the sea wave velocity is very small compared to the velocity of the electromagnetic wave in the sea), permitting the observer to follow the variation of the sea wave by observing the changes in the electromagnetic field.

For VLF signals, the perturbation of the fields in the sea due to large sea waves is relatively unchanged (within a few db) at depths of ten meters or so. The large perturbation in the signal due to the rough sea surface could then be "corrected" in part by monitoring the sea surface height above the receiving antenna.

No "correction" is necessary for the perturbation due to small sea waves as these perturbations are small (a few db). For ELF signals, even the perturbation due to large sea waves is small, so no "correction" is needed.

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[2]Winter, D. F., "Low-Frequency Radio Propagation into a Moderately Rough Sea," Radio Propagation, Vol. 67D, No. 5, Sept.-Oct. 1963, pp. 551-562.

APPENDIX A

GENERAL NATURE OF THE SEA SURFACE WITH A DISCUSSION OF THE MATHEMATICAL DESCRIPTION OF THE SEA SURFACE

In this appendix, the nature of the sea surface is discussed. The present state of knowledge about the sea surface is given; particular attention is given the statistical nature of the mathematical description of the sea surface. Also, at the end of the appendix is presented a brief list of useful formulas pertaining to the mathematical description of the sea surface and some tables giving pertinent data.

Even casual observation shows the great irregularity of the sea surface; no single wave retains its identity long; the period, form, etc., vary greatly even for consecutive waves. Indeed, the sea surface, in a rough sea, seems to vary almost randomly in both time and space.

The study of the sea surface can, for purposes of the discussion below, be placed into three general categories:

i. The study of the sea surface by classical hydrodynamics.

ii. The study of the sea surface by probabilistic methods. In this category only the linearized problem will be considered as this allows us to obtain general results. By assuming a linearized free-surface boundary condition, the problem becomes linear and the sea surface can then be described by a known random process (a stationary Gaussian process).

iii. Other theories which consider the non-linear effects of the sea surface, methods of generating the sea surface, etc. In this case the results obtained are far less useful for a description of the sea surface than the results of Category ii. However, such results place the limitations of the theory of Category ii in the correct perspective.

Before going into a more detailed study of the sea surface a few general comments are appropriate. First, the present theories assume a fixed meteorological condition. Then the description of the sea surface given by the theory holds as long as this assumption is approximately true. A theory based on the correlation of the changes in meteorological conditions with changes in the sea surface could be used but would be somewhat complex. This may, however, be necessary in some cases. Second, the presently known theories consider only the gross meteorological conditions (e.g., average wind velocity, average fetch, etc.). In practice, this is all that can be assumed without making the problem inordinately complex. Third, no theory at present is complete in the description of the sea surfaces. Under certain meteorological conditions one theory may be approximately correct; however, it fails when the meteorological conditions change. It should be noted that no theory now available gives the complete description of the detailed properties of the sea surface; only the gross features can be mathematically described. A brief discussion of the three general categories follows:

i. Classical Hydrodynamics. When the depth of the sea is large, a solution of the hydrodynamic equations which represent the sea surface is a trochcidal wave.[1] The parametric equation of the trochoid is:

$$x = r\theta - a \sin \theta$$
$$z(x) = r - a \cos \theta \qquad (A.1)$$

where

r, a -- fixed parameters for a given trochoid.

θ -- parameter that varies (i.e., generates the curve).

The trochoid is a two-dimensional wave which "could exist" in a swell.

A more general deterministic model (as opposed to a statistical model) of the sea state would include the irregularity of the sea; however, such a formulation is too complex for practical use. The deterministic model is considered below. Leaving out details, the following equation of a simple harmonic progressive wave is a solution to the linearized hydrodynamic equations:

$$\xi(x,y,t) = A \cos \left[\frac{2\pi}{L}(x \cos \theta + y \sin \theta) - \frac{2\pi}{T} t + \varepsilon\right] \qquad (A.2)$$

where

ξ -- sea surface

A -- amplitude of the simple harmonic progressive wave

[1]Lamb, H., Hydrodynamics, Dover Publications, Inc., New York, 1945, p. 423.

Kerr, D. E., Propagation of Short Radio Waves, McGraw-Hill, New York, 1951, p. 487.

T -- period of the simple harmonic progressive wave

L -- wavelength of the simple harmonic progressive wave

c -- velocity of the simple harmonic progressive wave

θ -- direction of propagation measured with respect ot the +x axis

ϵ -- phase at $x = y = t = 0$ (arbitrary)

If

$\vec{\rho} = x\hat{x} + y\hat{y}$ -- position vector in the horizontal plane

$\vec{k}_s = k_x\hat{x} + k_y\hat{y}$ -- propagation vector in the horizontal plane

$k_s = |\vec{k}_s| = \frac{\omega^2}{g} = \frac{2\pi}{L}$ -- wave number of the sea surface[2]

g -- acceleration due to gravity

$\omega = \frac{2\pi}{T}$ -- radian frequency of the wave

then

$$\xi(\bar{\rho},t) = A \cos (\vec{k}\cdot\vec{\rho} - \omega t + \epsilon) \qquad (A.3)$$

$\bar{\rho} = (x,y)$ in the ordered pair notation used previously.

If we assume the waves are progressing in the +x direction (i.e., $k_x \geq 0$) which is reasonable if the wind is in the +x direction, the general solution is[3]

$$\xi(\bar{\rho},t) = \int_{-\pi/2}^{\pi/2} \int_0^{\infty} \left[a(\omega,\theta)\cos(\vec{k}\cdot\vec{\rho}-\omega t) + b(\omega,\theta)\sin(\vec{k}\cdot\vec{\rho}-\omega t)\right]d\omega\, d\theta \qquad (A.4)$$

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[2] Longuet-Higgins, M. S., "The Directional Spectrum of Ocean Waves, and Processes of Wave Generation," Proceedings of the Royal Society, Vol 265, no. 1322, Jan. 30, 1962, p. 286.

[3] Pierson, W. J., Jr., "Wind Generated Gravity Waves," Advances in Geophysics, Vol. II, 1955, Academic Press, New York, p. 107.

where

$a(\omega,\theta)$ and $b(\omega,\theta)$ are the spectra of $\xi(\bar{\rho},t)$

If $\xi(0,y,t)$ is known, $a(\omega,t)$ and $b(\omega,t)$ can be obtained and from them $\xi(s,y,t)$. However, it is clear that the deterministic model of the sea surface as given above is not practical. As has been the case with many problems whose complexity defies deterministic solution, one next attempts to formulate the problem in probabilistic terms. We will now consider such an attempt.

ii. Probabilistic Description of the Sea Surface. In the past few years there has been an increased tendency to treat many natural phenomena as random processes. The main feature of such a process is an indeterminacy in the expected behavior of a single occurrence coupled with strong statistical properties for a large number of occurrences. The "indeterminacy" in the sea surface comes from the complexity of its mathematical description [e.g., finding $\xi(0,y,t)$].

Chronologically, experimental data first led to the assertion that the sea surface could be represented approximately as a stationary multivariate Gaussian process. From a theoretical view, it can be shown by using equation (A.3) with the assumption that the random variable $\varepsilon(\omega,\theta)$ has a uniform distribution, that the general solution for the sea surface is:[4]

$$\xi(\bar{\rho},t) = \int_0^\infty \int_{-\pi}^{\pi} \cos\,(\vec{k}\cdot\vec{\rho} - \omega t + \varepsilon(\omega\theta)\sqrt{A^2(\omega,\theta)d\omega\,d\theta} \qquad (A.5)$$

[4] Ibid., p. 122.

This expression is a multivariate Gaussian process where $A^2(\omega,\theta)$ is the energy density spectrum. This [as does (A.3)] assumes linearized equations and boundary conditions; if these assumptions hold, the surface is given by (A.5). However, as stated above, the experimental data implies that the surface is only approximately Gaussian, the error being due in large measure to the nonlinear effects neglected by this theory.

The same results can be given in the form:

$$\xi(\bar{\rho},t) = \int_{-\pi}^{\pi} \int_{0}^{\infty} A(\omega,\theta) \cos (\vec{K}\cdot\vec{\rho} - \omega t + \epsilon(\omega,\theta)) \, d\omega \, d\theta \qquad (A.6)$$

where

$$\frac{1}{2} A^2(\omega,\theta) = E(\omega,\theta) \text{ is the energy density spectrum.}$$

Using this Gaussian model of the sea surface, many of the general properties of the "sea state" have been calculated.[5] The basic question to be resolved is: Can the multivariate Gaussian process describe accurately the real sea surface? If not, from what standpoint is it deficient?

To answer this question, we consider a specific model of the sea surface (i.e., a given spectrum $E(\omega,\theta)$) and then compare

[5] Ibid., p. 93.

Longuet-Higgins, M. S., "The Statistical Analysis of a Random Moving Surface," Trans. Royal Society of London, Series A, Vol. 249, 1956-57, p. 321.

Longuet-Higgins, M. S., "Statistical Properties of an Isotropic Random Surface," Trans. Royal Society of London, Series A, Vol. 250, 1957-58, p. 157.

this random process with the known properties (experimental data) of the sea surface. Naturally, for a different spectrum the properties of the sea predicted by the random process will be different. The best known spectrum of the "sea surface" is a semi-empirical expression given by Neumann for a fully developed sea:[6]

$$E(\omega)\ d\omega = \frac{\pi}{2}\frac{c}{\omega^6}\ e^{-2g^2/\omega^2 v^2}\ d\omega \qquad (A.7)$$

where:

v is the velocity of the wind "generating" the "sea," and c is a constant.

The total energy E for this spectrum then becomes:

$$E = \int_0^\infty E(\omega)\ d\omega = \frac{c\ \pi^{3/2}}{2^{13/2} g^5}\ 3v^5 \qquad (A.8)$$

This seems to be in agreement with some experimental data if $c = 3.05\ m^2/sec^5$. To study in detail what this model predicts, we will consider the shape of the spectrum and some of its results.

Neumann's spectrum rises rapidly at $\omega = \frac{g}{1.6v}$ and has a maximum at $\omega = \sqrt{\frac{2}{3}}\ g/v$. For large ω, $\omega(\omega) \approx \omega^{-6}$ and for small ω, $E(\omega) \approx e^{-2g^2/\omega^2 v^2}$ (i.e., the exponential predominates), and therefore, there is very little energy in the low frequency (long wavelength) part of the spectrum.

Neumann's spectrum is for a fully developed sea only. In actuality, of course, the sea may not be fully developed. The

[6] Pierson, op. cit., p. 148.

growth of the sea waves depends upon the fetch (distance over which the wind blows), the duration (length of time the wind has been blowing) and naturally, the wind velocity. There are methods that consider the problem of duration-limited and fetch-limited seas (but only approximately).

From empirical data an approximate directed spectrum (which depends on the direction relative to the wind as well as on the duration and fetch of the wave) is:[7]

$$E(\omega,\theta) = \begin{cases} c\,\dfrac{e^{-2g^2/\omega^2 v^2}}{\omega^2}\cos^2\theta & \text{for } -\frac{\pi}{2} < \theta < \frac{\pi}{2}\ \ \omega_i < \omega < \infty \\ 0 & \text{otherwise} \end{cases} \tag{A.9}$$

where

θ -- polar angle with reference to the x axis (the wind is blowing in the +x direction).

ω_i - intersection radian frequency (a function of duration and fetch).

The above model is at best a good approximation to the sea surface under certain conditions. There are better models;[8] however, they are more complex concerning the directional part of the formula and other errors have been noted.

[7] Ibid., p. 155.

[8] Longuet-Higgins, M. S., "The Directional Spectrum of Ocean Waves, and Processes of Wave Generation," Proceedings of the Royal Society, Vol. 265, no. 1322, Jan. 30, 1962, p. 286.

Pierson, W. J., Jr. (Ed.), "The Directional Spectrum of a Wind Generated Sea as Determined from Data Obtained by the Stereo Wave Observation Project," New York University Meteorological Papers, Vol. 2, no. 6, June, 1960 (unpublished).

This is particularly true for high sea states where the sea may not be fully developed. Other empirical formulas are available to represent this case.[9] It is still not known conclusively how well the statistical results of this model hold for actual sea waves. This model of the sea surface contains some energy in the high frequency part of the spectrum and predicts the sea to be "completely covered by short wavelength ripples." Physically this seems reasonable even though the above spectrum is known to be deficient in the high frequency range. The error in the high frequency part of the spectrum is due to the fact that non-linear effects are prevalent at these frequencies; we consider this effect in the next section.

iii) The Generation of the Sea Surface and Non-Linear Effects.

Phillips[10] has described the generation of the sea surface by considering turbulent fluctuation of the air (wind) above the air-sea interface. At present this theory seems to be accepted as correct for the "original" formation of the sea surface. However, once the waves have been formed, their growth occurs through other mechanisms.

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[9] Pierson, W. J., Jr., "A Study of Wave Forecasting Methods and of the Height of a Fully Developed Sea," Deutsche Hydrographische Zeitschrift, Vol. 12, no. 6, 1959, p. 244.

[10] Phillips, O. M., "On the Generation of Waves by Turbulent Wind," Journal of Fluid Mechanics, Vol. 2, 1957, p. 417.

At present there are two principal theories on the growth of waves:

(a) Phillips' resonance theory.[11]

(b) Miles' sheer-flow instability theory.[12]

These two theories do not agree in many respects (e.g., Phillips' theory gives a "growth rate" proportional to time and Miles' gives a "growth rate" exponential with time). While both mechanisms play a role in the growth of waves, Phillips has recently given a description of the domains of dominance of each mechanism. These theories, while they do give some understanding of the sea surface, are unfortunately of no qualitative help at present in describing the sea surface.

We will consider only one other mechanism of energy transfer, the breaking of the waves. Under sufficiently high winds, the energy transfer from the breaking of waves reaches an equilibrium; then for that range of frequencies where the nonlinear (breaking) effects are important the energy spectrum is given by:

$E(\omega) = \alpha g^2 \omega^{-5}$ where $\alpha = 7.4 \; 10^{-3}$ (empirical constant).

The same basic conclusions (under basically the same restrictions) were obtained by Mikhailov,[13] who used the theory of

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[11] Phillips, O. M., "Resonance Phenomena in Gravity Waves," Proc. of Symposia in Applies Math., Vol. XIII, Amer. Math. Society, McGraw-Hill, New York, 1962, p. 91.

[12] Miles, J. W., "Generation of Surface Waves by Shear Flows," Proc. of Symposia in Applied Math., Vol XIII, Amer. Math. Society, McGraw-Hill, New York, 1962, p. 79.

[13] Mikhailov, V. I., "On the Theory of Scattering of Electromagnetic Waves on the Sea-Surface," Bulletin, Academy of Science, U.S.S.R., Geophysics Series, 1960, p. 818.

turbulence. There is close agreement between these theories [Phillips has $E(\omega) \approx \omega^{-5}$, Mikhailov cites experimental work which gives $E(\omega) \approx \omega^{-4}$ to ω^{-6}; both theories are within these bounds. Note Neumann in this range has $E(\omega) \approx \omega^{-6}$].

Tick[14] has considered the problem of non-linear effects from a perturbation point of view. He adds to the "linear (Gaussian) spectrum" a correction spectrum due to the non-linear effects. However, the statistics of the "non-linear" spectrum are not unknown.

All theories (as those above) which do not directly consider capillary waves (waves of very short wavelength) do not hold for frequencies in the "capillary" range. While much work on capillaries has been done, at present there is nothing available on correlating this theory with the overall sea surface. And there is no correct information available on the "capillary" range of the energy density spectrum. Work on radar return and light reflection from the sea surface has shown that the slopes of the sea surface are nearly Gaussian and that the sea surface curvature is highly non-Gaussian. The usual experimental data taken for description of the sea surface does not include frequencies in the "capillary" range.

[14] Tick, L. J., "A Non-Linear Random Model of Gravity Waves I," Journal Math. and Mech., Vol. 8, 1959, p. 643.

Pierson, W. J., Jr., "A Note on the Growth of the Spectrum of Wind-Generated Gravity Waves as Determined by Non-Linear Considerations," Journ. Geophysical Research, Vol. 64, no. 8, August 1959, p. 1007.

The review of the past work done on the mathematical description of the sea surface indicates the point of departure for the study of propagation of electromagnetic waves near the air-sea interface. The complexity of the air-sea interface alone is enough to make one resort to statistical analysis; however, the fact that both the linear model and the experimental data give a Gaussian process for the gross feature of the sea surface implies that a useful model would be a satistical one.

If one is willing to neglect the non-linear effects of the sea surface and use the Gaussian model, the Ergodic theorem[15] implies that the ensemble analysis of statistical properties is the same as space (or time) analysis of the statistical properties (note that this requires a stationary process, i.e., equilibrium state). There are questions as to the validity of the Gaussian model as regards the statistical properties of the sea surface. However, if one is only interested in the gross statistical features, then the Gaussian description may be adequate. However, such description will give only the "smooth average" shape of the sea without any details such as ripples or minor variations. The experimental data on sea surface variation is meagre; the processing of such data for meaningful results is long and costly; therefore, it is doubtful if an accurate description of the sea surface can be

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[15] Lee, Y. W., Statistical Theory of Communication, John Wiley, New York, 1960, p. 207.

readily obtained.[16] The shape of the surface of the sea depends upon past as well as upon present conditions; however, to a degree of approximation, past effects though still significant may be discounted. Also, by describing the sea condition by means of only a few average parameters, the sea surface will not be represented accurately in every detail, but from the practical approach, this may be all that can be done.

[16]For an excellent review of much of the spectral analyses of the sea surface, see Proceedings of a Conference on Ocean Wave Spectra (Englewood, N. J., Prentice-Hall, 1963).

Some Useful Formulae

For reference, some useful relations pertaining to the sea surface are given below.

If $\xi(\bar{r}) = A \cos(\vec{k}\cdot\vec{r} - \omega t)$ is a solution to the linearized hydrodynamic equations, then:

$$\omega^2 = gk_s \tanh k_s h$$

for $k_s h >> 1$ (large depth)

$$\omega^2 = gk_s \tag{1}$$

where

$$k_s = \frac{2\pi}{L}$$

$$\omega = \frac{2\pi}{T}$$

We then have for a single harmonic wave in deep water:

$v_p = \frac{L}{T} = \frac{\omega}{k} = \frac{g}{2\pi} T$ -- phase velocity of the sea wave.

$v_g = \frac{d\omega}{dk} = \frac{1}{2} v_p = \frac{g}{4\pi} T$ -- group velocity of the sea wave.

$$L = \frac{g}{2\pi} T^2$$

For relatively "regular" seas (after they have consolidated into a "regular" series of connected troughs and crests).

$\frac{H}{L}\left(\frac{\text{height}}{\text{length}}\right)$: $\frac{1}{12} < \frac{H}{L} < \frac{1}{35}$

(some references give $\frac{1}{7} < \frac{H}{L} < \frac{1}{20}$, where $\frac{1}{7}$ is the theoretical limit for stability (greater slopes will "break")).

For a swell: $\frac{1}{35} < \frac{H}{L} < \frac{1}{200}$.

Using Neumann's spectrum for a fully developed sea:[17]

$$E\ 2\ .242(\frac{v}{10})^5 \qquad v \text{ -- knots}$$

$$T_{ave} = .285\ v \qquad T \text{ -- sec.}$$

$$L_{ave} = \frac{2}{3}(5.12)\ T_{ave}^2 \qquad L \text{ -- ft.}$$

[17] Pierson, W. J., G. Neumann and R. W. James, Observing and Forecasting Ocean Waves, Hydrographic Office Publication no. 603, 1958, pp. 45, 47, 50.

APPENDIX B

REVIEW OF THE THEORY OF ROUGH SURFACE PROBLEMS

In this appendix, a review of the present theory of boundary value problems involving a rough surface is presented. Particular attention is given to the accuracy and applicability of the methods presented to the electromagnetic wave-rough surface interaction problem considered in the main sections of this report.

"In the mathematical sense the problem at hand is extremely complex, since it is impossible to use the method of separation of variables to obtain a solution of the wave equation which satisfies the boundary conditions specified on the uneven surface. The methods for obtaining an accurate solution of this problem in general form has (sic) not yet been found. However, there are a number of theoretical papers (which have appeared essentially during the last five to six years) in which a number of approximate methods for computing the field have been developed." The above is a quotation from a survey paper, "Theory of the Scattering of Waves at Periodically Uneven Surfaces," by Iu. P. Lysanov.[1] This paper treats for the most part the problem of a scalar field in the presence of a periodically uneven (rough) surface with "natural" boundary

[1] Lysanov, Iu. P., "Theory of the Scattering of Waves at Periodically Uneven Surfaces," Soviet Physics Acoustics, Vol. 4, No. 1, Jan.--March, 1958, p. 1.

conditions. When the boundary conditions of the actual problem are not the "natural" ones, the "natural" boundary conditions are used as an approximation. The "interface" boundary conditions are not considered. The problem treated is then the simplest problem considering a rough surface; even then this problem is not solvable using known techniques. Although work on this problem has continued since 1958, little advancement has been made and no basically new methods or theories have resulted. The investigation has generally been confined to refinement of techniques given in Lysanov's paper and experimental work.

Basically, Lysanov's approach will be followed. He considers six general classifications or techniques:

1. Rayleigh Method. This is the oldest approach and due to Rayleigh. Rayleigh assumed that the reflected (scattered) field could be represented as a sum of outward (from the surface) directed plane waves. However, this is only true for the field above the highest point of the surface. Lippmann showed that the assumption of Rayleigh was not quite correct. Unfortunately, he could not obtain an "accurate" solution to the problem by the use of his variational technique. Barantsev[2] has also obtained the

[2]Barantsev, R. G., "Plane Wave Scattering by a Double Periodic Surface of Arbitrary Shape," Soviet Physics Acoustics, Vol. 7, No. 2, Oct.-Dec., 1961, p. 123.

same result by use of Laplace transforms. In Barantsev's approach to the problem, as in Rayleigh's, an infinite number of algebraic equations must be solved. In the case where the irregularities are small (the maximum height of the surface wave is much smaller than a wave length of the radiation field), Rayleigh obtained a solution for the first mode (specular reflection). It agrees quite well with experimental data in the "far field;" however, near the interface many evanescent (surface) waves are present and an accurate solution is not practicable (even if Rayleigh's assumption was considered correct).[3]

2. The Method of Small Perturbations. The boundary conditions are specified on the uneven surface, $z = \xi(x)$, and then are transferred to the plane $z = 0$ by means of an expansion into power series with respect to ξ. It has been shown by Lysanov that this method leads to solutions identical to Rayleigh's results. This method has been used by Feinberg for solving the problem of propagation

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[3]La Casce, E. O., "Some Notes on the Reflection of Sound from a Rigid Corrugated Surface," Journal of Acoustical Society of America, Vol. 33, No. 12, Dec., 1961, p. 1772.

La Casce, E. O., B. D. McCombe, R. L. Thomas, "Measurements of Sound Reflection from a Rigid Corrugated Surface," Journal of Accoustical Society of America, Vol. 33, No. 12, Dec., 1961, p. 1768.

Senior, T. B. A., "The Scattering of Electromagnetic Waves by a Corrugated Sheet," Canadian Journal of Physics, Vol. 37, 1959, p. 787 (AD 238 810).

of electromagnetic waves along the earth's surface and more recently by Senior[4] who considered the effect of surface roughness on the reflection of electromagnetic waves. Senior is able to replace surface roughness by a change in the impedance at the boundary. The surface impedance is then a function of position. Hessel[5] and others have also considered the problem of varying surface impedance; Hessel seems to account for Wood's anomaly this way. However, this method is useful only in the far field. Bass[6] also uses the method of perturbations to obtain results parallel to Senior, and they again are valid only in the far field. It should be stressed here that the method of small perturbations may be in error in the near

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[4]Senior, T. B. A., "Impedance Boundary Conditions for Imperfectly Conducting Surfaces," Applied Science Research, Section B., Vol. 8, 1960, p. 418.

Senior, T. B. A., "Impedance Boundary Conditions for Statistically Rough Surfaces," Applied Science Research, Section B., Vol. 8, 1960, p. 437.

Hiatt, R. E., T. B. A. Senior, and V. H. Weston, "Studies in Radar Cross Sections XL," -- "Surface Roughness and Impedance Boundary Conditions," University of Michigan Research Institute, Ann Arbor, Michigan, July 1960 (unpublished).

[5]Hessel, A., A. A. Oliver, "On the Theory of Wood's Anomalies, in Progress Report No. 19," R. 452.1961, Polytechnic Institute of Brooklyn (unpublished), AD 256 809.

[6]Bass, F. G., V. G. Bocharov, "On the Theory of Scattering of Electromagnetic Waves from a Statistically Uneven Surface," Radiotekhnika Elektronika, Vol. 3, No. 2, 1958, p. 180.

field and is valid only in the far field.[7] Therefore, Winter,[8] who applied the method of small perturbation to compute the fields on the rough sea surface, could be in error in his calculations. The results obtained by Winter's method were compared with the results obtained in this report and there was a difference (see Section 2.6).

Lysanov[9] has extended the theory of small perturbations to include point sources.

3. The Method of L. M. Brekhovskikh. This could better be called the method of geometric optics. Basically, it uses Kirchhoff's principle (or approximations). For the frequency range we are interested in, the use of Kirchhoff's principle could well be in error. In any case, for the solution to the problem on the interface this method would assume away the problem. The problem becomes one of evaluation of an integral which cannot be done in closed form near the surface.

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[7]Bass, F. G., "On the Theory of Combination Scattering of Waves on a Rough Surface," Izvestia VUZ, Radiofizika, Vol. 4, No. 1, 1961, AD 262 417.

Lysanov, op. cit., p. 3.

[8]Winter, D. F., "Low Frequency Radio Propagation into a Moderately Rough Sea," Radio Propagation -- Section D, Journal of Research, National Bureau of Standards, Vol. 67D, No. 5, Sept.-Oct. 1963, p. 551.

[9]Lysanov, Iu. P., "On the Field of a Point Radiator in a Laminar-Inhomogeneous Medium Bounded by an Uneven Surface," Soviet Physics Acoustics, Vol. 7, No. 3, Jan.-March, 1962, p. 255.

4. The Integral Equation Method. The problem of the scalar boundary value problem with "natural" boundary conditions can, by use of Green's theorem, be formulated conveniently as an integral equation. This equation is exact; however, it cannot be solved without making some approximations. If the rough surface satisfies certain conditions (is not too rough), an approximate integral equation is obtained which can be solved. There is a difference in the field assumed on the surface by geometric optics (Kirchhoff's approximation) and the field on the surface obtained by the integral equation method. This has led Meecham[10] to question the use of Kirchhoff's approximation. This criticism could also be extended to the method of small perturbations in the near field. A variation of the integral equation method was used in the main part of this paper.

Recently, a paper solving the exact integral equation for a sinusoidal surface was presented;[11] the numerical results involved approximate solutions to an infinite set of algebraic equations.

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[10] Meecham, W., "On the Use of the Kirchhoff Approximation for the Solution of Reflection Problems," J. Rational Mech. Analysis, Vol. 5, 1956, p. 323.

[11] Uretsky, Jack L., "Reflection of a Plane Sound Wave from a Sinusoidal Surface," submitted as a Letter to the Editor, J. Acoust. Soc. Am., March 1963.

5. The Method of Images. The method of images can be used to investigate the fields in the presence of an uneven surface with a sufficiently simple shape; this has been done by Twersky for a perfectly reflecting plane covered with half-cylinders or hemispheres (with little interaction between scatterers). Biot[12] considered the perfect conduction plane covered by hemispheres with strong interaction (as there would be in the case of the sea surface). Twersky also has considered multiple scattering in a very general way. This method could be used to find the fields on the surface assumed to be a plane covered with hemispheres. However, to obtain a more realistic model of the sea surface would involve higher multi-pole expansions and does not seem practical (although Biot thought this method could be used for the air-sea interface problem). It does give a good idea of what is happening to the electromagnetic field in the air near the air-sea interface (Biot uses a static ($\omega = 0$) solution).

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[12] Biot, M. A., "Some New Aspects of the Reflection of Electromagnetic Waves on a Rough Surface," Journal Applied Physics, Vol. 28, December 1957, p. 1455.

Biot, M. A., "On the Reflection of Electromagnetic Waves on a Rough Surface," Journal Applied Physics, Vol. 29, June 1958, p. 998.

6. Method of Matching Fields. This method can be used only in the case where the rough surface is such that the space may be separated into regions in which the wave equation allows solution by the method of separation of variables (when the wave equation is written in an appropriate coordinate system). As this is not possible for the air-sea interface, we will not consider this method further.

Lysanov also references some experimental papers. Since 1958 other experimental papers have been published which seem to imply that Rayleigh's theory (including the second mode) is correct for the low frequency problem except near $\lambda_r = \lambda_w$ (λ_r, wavelength of radiations; λ_w, wavelength of the surface).[13] Since 1958 many papers on the statistical analysis of the reflection of sound from the rough sea surface have been published; these use either Rayleigh's assumption or Kirchhoff's approximation for the solution of the boundary value problem and as such are only useful in the far field.[14]

[13] La Casce, loc. cit.

[14] Proud, J. M., Jr., R. T. Beyer, and Paul Tamarkin, "Reflection of Sound from Randomly Rough Surfaces," Journal of Applied Physics, Vol. 31, No. 3, March 1960, p. 543.

Marsh, H. W., "Exact Solution of Wave Scattering by Irregular Surfaces," Journal of the Acoustical Society of America, Vol. 33, No. 3, March 1961, p. 330.

Marsh, H. W., M. Schulkin and S. G. Kneale, "Scattering of Underwater Sound by the Sea Surface," Journal of the Acoust. Soc. of Amer., Vol. 33, No. 3, March 1961, p. 334.

Beckmann, Petr, and Andre Spizzichino, The Scattering of Electromagnetic Waves from Rough Surfaces, Pergamon Press Ltd., London, 1963.

One of the few papers considering "interface" boundary conditions on an irregular surface was published by Wait[15] in 1959; this paper has been used to calculate the effect of the rough surface on the electromagnetic fields in the sea. Wait's basic approach is the work of Leontovich on approximate boundary conditions for a good conductor. Wait obtains the field on a plane in the sea, and solves the wave equation in the sea with this as a boundary condition. A short discussion of this work is given in a paper by Whalen.[16] Some comment on the accuracy of Wait's assumption is given in the main part of this paper. Other work directly considering the rough sea surface-VLF radio wave interaction are: Lerner and Max[17] and Winter.[18] Winter used the method of small perturbation to compute the fields on the sea surface and the stochastic Stratton-Chu integral equations to obtain statistical results for the fields in the sea. The use of the method of small perturbation is

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[15]Wait, J. R., "The Calculation of the Field in a Homogeneous Conductor with a Wavy Interface," Proc. IRE, Vol. 47, No. 6, June 1960, p. 1155.

[16]Whalen, J. L., "Measured Effects of Ocean Waves on the Phase and Amplitude of VLF Electromagnetic Radiation Received Below the Waves," USL Tech. Memorandum No. 941.1-67-61, 3 August 1961 (unpublished).

[17]Lerner, R. M. and J. Max, "Very Low Frequency and Low Frequency Fields Propagating Near and Into a Rough Sea," a paper presented to the URSI Spring 1963 Meeting.

[18]Winter, loc. cit.

discussed in Section 2 of this appendix and Winter's statistical results are considered in the conclusion chapter of the main part of this report. The basic method of Lerner and Max was outlined in Section 2.6 of this report, and their results are considered in the conclusion part of this report.

APPENDIX C

SINGULAR INTEGRALS

Although the results given in this appendix are available in the literature, it seems useful to make readily available to the reader the basic theory and results pertaining to "singular" integral equations. These results have been used in the body of the paper to obtain the integral equations of Sections 1.4 and 5.2.

The starting point in considering "singular" integrals is the concept of improper integral.[1] By "singular" integral is meant an integral whose value changes discontinuously (i.e., a "discontinuous integral").

The field of a single layer is:

$$F_s(\bar{r}) = \int_S \varphi(\bar{r}')\, G(k|\vec{r}-\vec{r}'|)dS' \tag{C.1}$$

$\bar{r} \in V_i \qquad \bar{r}' \in S$

V_i - interior volume bounded by closed surface S.

V_e - exterior volume bounded by closed surface S.

$\varphi(\bar{r})$ - strength of the single layer sources.

$F_s(\bar{r})$ - field of the single layer sources.

[1]Mikhlin, S. G., Integral Equations, Pergamon Press, London, 1957, p. 113.

$$\lim_{\bar{r}\to S} F_s(\bar{r}) = \lim_{\substack{\bar{r}\to S \\ N_\epsilon \to 0}} \left[\int_{S-N_\epsilon} \varphi(\bar{r}')\, G(k|\vec{r}-\vec{r}'|)dS' + \int_{N_\epsilon} \varphi(\bar{r}')\, G(k|\vec{r}-\vec{r}'|)\, dS' \right] \qquad (C.2)$$

N_ϵ - neighborhood of point $\bar{r}$ in S (in this neighborhood $G(k|\vec{r}-\vec{r}'|)$ becomes unbounded and the integral is improper).

$$\lim_{\substack{\bar{r}\to S \\ N_\epsilon \to 0}} \left[\int_{S-N_\epsilon} \varphi(\bar{r}')\, G(k|\vec{r}-\vec{r}'|)\, dS' \right] = \int_S \varphi(\bar{r}') G(k|\vec{r}-\vec{r}'|)dS'$$

$$\lim_{\substack{\bar{r}\to S \\ N_\epsilon \to 0}} \int_{N_\epsilon} \varphi(\bar{r}')\, G(k|\vec{r}-\vec{r}'|)\, dS' = \lim_{N_{\epsilon'}\to 0} \varphi(\bar{r}) \int_{N_{\epsilon'}} G(k\bar{r}')\, dS' =$$

$$\lim_{N_{\epsilon'}\to 0} \varphi(\bar{r}) \int_{N_{\epsilon'}} G(\bar{r}')\, dS' = 0$$

as G is an even function.

$N_{\epsilon'}$ - neighborhood of point $\bar{r}'$ in S.

$G(\bar{r})$ - static Green's function (the approximation $G(\bar{r}') \approx G(k\bar{r}')$ is valid for $k\bar{r}' \ll 1$), which holds in $N_{\epsilon'}$.

$$\lim_{\bar{r}\to S} F_s(\bar{r}) = \int_S \varphi(r')\, G(k|\vec{r}-\vec{r}'|)\, dS' \qquad (C.3)$$

$\bar{r} \in V_e$ or $\bar{r} \in V_i$ $\quad \bar{r}' \in S$

The field of a double layer is represented by a singular integral.

$$F_d(\bar{r}) = \int_S \varphi(\bar{r}') \frac{\partial}{\partial n'} G(k|\vec{r}-\vec{r}'|)\, dS' \qquad (C.4)$$

$\varphi(r)$ - strength of double layer sources.

$\hat{n}$ - outward normal to S.

$F_d(\bar{r})$ - field of double layer.

$$\lim_{\substack{\bar{r}\epsilon V_i \\ \bar{r}\to S \\ \bar{r}'\epsilon S}} F_d(\bar{r}) = \int_S \varphi(\bar{r}') \frac{\partial}{\partial n'} G(k|\vec{r}-\vec{r}'|)\, dS' + \lim_{N_{\epsilon'}\to 0} \varphi(\bar{r}) \int_{N_{\epsilon'}} \frac{\partial}{\partial n'} G(\bar{r}')\, dS'$$

$$\lim_{N_{\epsilon'}\to 0} \int_{N_{\epsilon'}} \frac{\partial}{\partial n'} G(\bar{r}')\, dS' = \lim_{N_{\epsilon'}\to 0} \int \frac{1}{4\pi}\left(\frac{\partial}{\partial n'} \frac{1}{r'}\right) dS' =$$

$$\lim_{N_{\epsilon'}\to 0} - \int_{N_{\epsilon'}} \frac{1}{4\pi}\, d\Omega = - \frac{\Omega}{4\pi} \qquad (c.5)^{2}$$

Ω - "solid angle" of surface S -- (for smooth surface $\Omega = 2\pi$)

$$\lim_{\substack{\bar{r}\epsilon V_i \\ \bar{r}\to S \\ \bar{r}'\to S}} F_d(\bar{r}) = \int_S \varphi(\bar{r}') \frac{\partial}{\partial n'} G(k|\vec{r}-\vec{r}'|)\, dS' - 1/2\, \varphi(\bar{r}) \qquad (c.6)^{3}$$

$$\lim_{\substack{\bar{r}\epsilon V_e \\ \bar{r}\to S \\ \bar{r}'\to S}} F_d(\bar{r}) = \int_S \varphi(\bar{r}') \frac{\partial}{\partial n'} G(k|\vec{r}-\vec{r}'|)\, dS' + 1/2\, \varphi(\bar{r}) \qquad (c.7)$$

[2] Courant, R. and D. Hilbert, Methods of Mathematical Physics, Vol. 2, Interscience, New York, 1962, p. 253.

[3] Mikhlin, S. G., Linear Integral Equations, Delhi-Hindustan, 1960, p. 156.

$$\lim_{\substack{\bar{r}\epsilon V_i \\ \bar{r}\to S}} \varphi(\bar{r}) = \lim_{\substack{\bar{r}\epsilon V_i \\ \bar{r}\to S}} \varphi_i(\bar{r}) + \lim_{\substack{\bar{r}\epsilon V_i \\ \bar{r}\to S \\ \bar{r}'\epsilon S}} \int_S \Big\{\varphi_{n'}(\bar{r}')G(k|\vec{r}-\vec{r}'|) - \varphi(\bar{r}')G_{n'}(k|\vec{r}-\vec{r}'|)\Big\}\, dS'$$

$$\varphi(\bar{r}) = \varphi_i(\bar{r}) + \int_S \Big\{\varphi_{n'}(\bar{r}')G(k|\vec{r}-\vec{r}'|) - \varphi(\bar{r}')\, G_{n'}(k|\vec{r}-\vec{r}'|)\Big\}dS' + \frac{1}{2}\varphi(\bar{r}) \quad \bar{r},\bar{r}'\epsilon S \tag{C.8}$$

by equations (C.3) and (C.6).

Equation (C.8) is the equation used in Sections 1.4 and 5.2.

The extension of these results to the coupled electromagnetic equations (Stratton-Chu) is straightforward.[4]

[4]Hönl, H., A. W. Maue, and K. Westpfahl, "Theorie of Beugung," in Handbuch Der Physik, Band XXV/1. (Berlin: Springer, 1961), p. 218.

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