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Launch Windows for Orbital Missions

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Prepared by A. H. MILSTEAD
Astrodynamics Department

Prepared for COMMANDER SPACE SYSTEMS DIVISION
UNITED STATES AIR FORCE
Inglewood, California

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ABSTRACT

A great many orbital missions involve launching a vehicle into a particular earth-referenced plane. The launch window is defined as the time span around the nominal launch time during which the vehicle may be launched and the target plane achieved within a specified additional ideal velocity budget. This paper presents analytical formulations for the launch window as functions of the additional ΔV budget and other parameters for fixed and for variable launch azimuths. The effect of launch azimuth constraints (e.g., for range safety) on the launch window is investigated and several related problems are discussed.

CONTENTS

FIGURES

I. INTRODUCTION

Many of the orbital missions currently under study by the civilian and military space agencies involve the launching of a vehicle from an earth-fixed site and the establishment of that vehicle in a particular plane (hereafter called the target plane) passing through the earth's center. Such a mission, for example, is the Gemini rendezvous mission, in which it is desired to launch the Gemini capsule into the orbit plane of the vehicle with which it expects to rendezvous. If the inclination of the target plane to the equator is greater than or equal to the launch site latitude, the vehicle may nominally be launched directly into the target plane at one of the times when the launch site passes through the target plane. However, if the vehicle is launched before or after the in-plane launch opportunity (e.g., a delay due to a count-down hold), an additional ideal velocity expenditure will be required to place the vehicle in the target plane. The time span about the in-plane launch opportunity during which the vehicle may be launched and the target plane achieved within a specified budget for the additional ideal velocity is called the launch window.

If the mission involves goals beyond simply entering a given plane (e.g., rendezvousing with another vehicle already established in the plane or burning out under certain sun lighting conditions, etc.), the definition of the launch window may be modified to include these goals. It is most often convenient (particularly in feasibility and preliminary design studies) to define the mission in several separate phases, of which the achievement of the target plane is the first. Once the launch vehicle is established in the target plane, the next phase (e.g., rendezvous, earth referenced positioning, orbital phasing for timed nodal crossings, etc.) is begun and, in general, involves only in-plane maneuvering. The ideal velocity requirements for the phases following the establishment of the launch vehicle in the target plane are frequently independent of where the target plane is entered and, therefore, have no effect on the launch window as defined above. In cases where the launch window is affected by the in-plane maneuvering, that effect can usually be calculated separately and superimposed on the launch-window as defined above.

This paper will consider only the establishment of the vehicle in the target plane, because this phase is common to many missions, rather than specializing to a particular mission and discussing in-plane maneuvers.

The development of equations will assume the following pattern: First, the relation between the vehicle's horizontal velocity component and the angle (plane change angle) through which it must be turned to achieve the target plane is established. It is then shown that, if the path of the vehicle from launch to burnout is assumed to lie in an inertial plane the minimum plane change angle (and therefore minimum required velocity for plane change) is achieved by launching in such a manner as to intercept the target plane after 90 degrees of angular travel.

Equations are developed, which give the minimum achievable plane change angle and the launch azimuth necessary to achieve that minimum plane change angle as functions of launch delay and constant quantities. An equation is also developed which gives the launch window itself as a function of the maximum allowable plane change angle, and several interesting cases and extensions of this formulation are discussed. Expressions for the launch window for the case of fixed launch azimuth are formulated, and, finally, a method for finding launch windows when the launch azimuth is constrained is presented.

II. ANALYSIS

It will be assumed that the target plane has a fixed inclination to the equator and that the line of nodes (line of intersection of the equator and target plane) moves in the equatorial plane at a uniform rate, ω_r, where ω_r

is positive in the same sense as the earth's rotation about its axis.[1] The path of the vehicle from lift-off from the launch site (assumed to lie in the northern hemisphere) to interception of the target plane will be assumed to lie in an inertial plane (hereafter called the launch plane) passing through the earth's center. It will further be assumed that it is desired to enter the target plane in an easterly sense, i. e., the launch azimuth is assumed to be between 0 and 180 deg.[2] The angle, α, which occurs between the launch plane and the target plane is indicated in Figure 1, and it is this angle α through which the horizontal component of the vehicle's velocity vector must be rotated in order to make the launch and target planes coincident. The vehicle's horizontal velocity component, V_H, is given by

$$V_H = V \cos \gamma \tag{1}$$

where V is the vehicle's inertial velocity and γ is the flight path angle of the vehicle's inertial velocity vector with respect to the local horizontal. If the vehicle is established in its orbit before reaching the target plane, the conservation of angular momentum gives

$$h = rV \cos \gamma = \text{a constant} \tag{2}$$

---

[1] If the target plane is inertially at rest, then ω_r is zero. If the target plane is a satellite orbit plane, then ω_r is given approximately by

$$\omega_r = -9.96\left(\frac{a_e}{a}\right)^{7/2} \frac{\cos i}{(1 - e^2)^2} \frac{\text{degrees}}{\text{mean solar day}}$$

where a_e is the earth's equatorial radius, a is the orbit's semi-major axis, e is the orbit's eccentricity, and i is the orbit's inclination to the earth's equator.

[2] The results of this paper can be extended to include consideration of launch azimuths between 180 and 360 degrees and southern hemisphere launch sites. These cases are excluded in the present development for the sake of clarity and brevity.

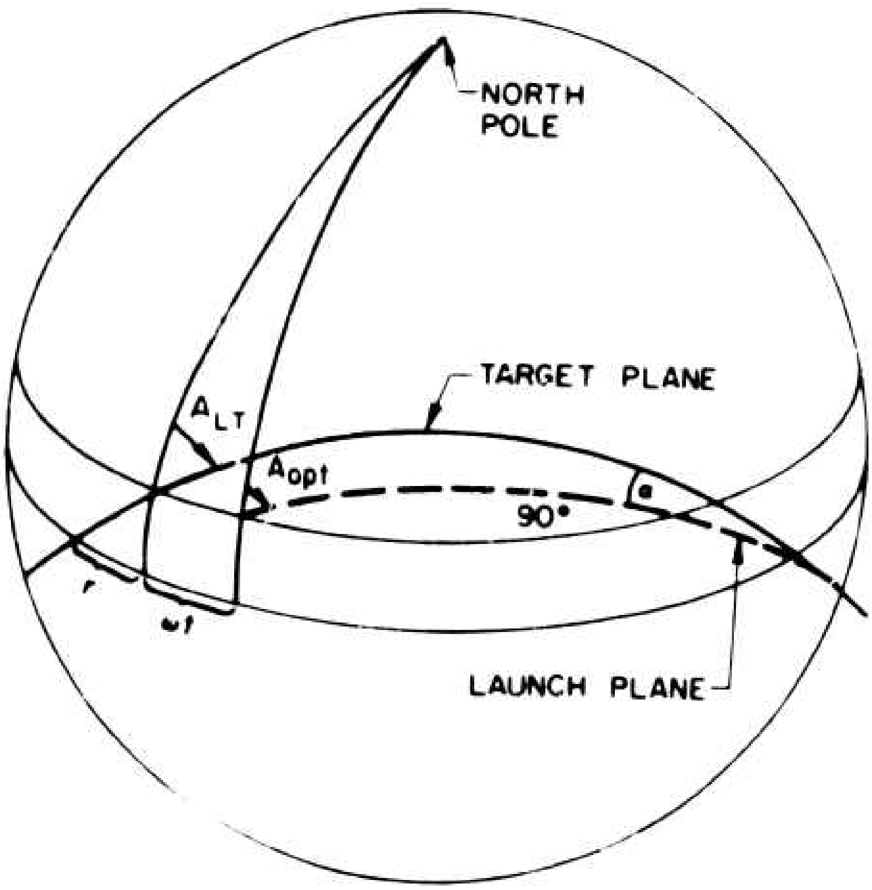

Figure 1. Launch Geometry

where h is the specific angular momentum of the vehicle in orbit and r is the radial distance from the earth's center. The horizontal velocity component may be expressed as a function of r only by combining Equations (1) and (2) to yield

$$V_H = \frac{h}{r} = \frac{r_p V_p}{r} = \frac{r_a V_a}{r} , \tag{3}$$

where subscripts p and a, respectively, denote perigee and apogee conditions. Obviously, if the orbit is circular, V_H is simply the orbital velocity.

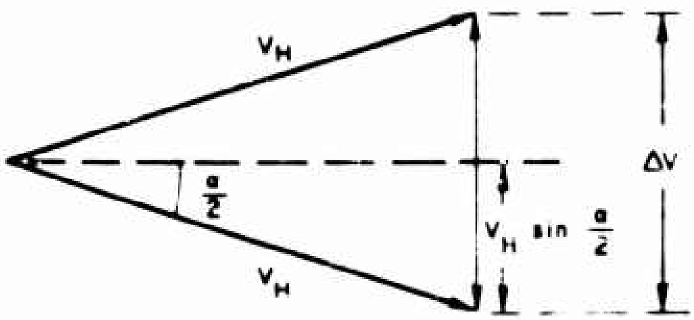

Figure 2 Velocity Requirements For Plane Change

It can be seen from Figure 2 that the velocity, ΔV, necessary to rotate the launch plane through an angle α without changing magnitude or flight path angle of the vehicle's velocity vector is

$$\Delta V = 2V_H \sin \frac{\alpha}{2} \tag{4}$$

Equation (4) shows that for a given V_H, the velocity requirement, ΔV, for the plane change is smallest when the plane change angle, α is a minimum. The condition for minimum α (and therefore for minimum ΔV) may be found by considering the projection of the launch and target planes onto an inertial sphere (concentric with the earth) at the time of launch, as shown in Figure 3.

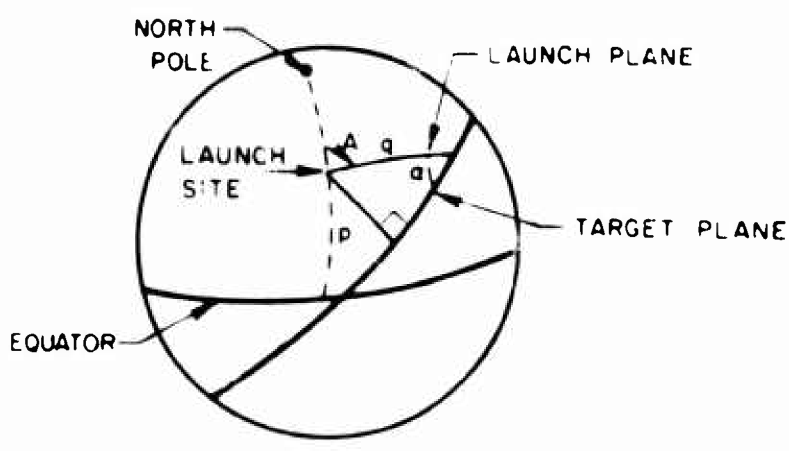

Figure 3. Geometry For Determining Optimum Launch Azimuth

The great circle arc q is the angular distance from the launch site to the point where the target plane is intercepted with an angle α. The arc p is the minimum great circle distance from the launch site to the target plane. Note that p is a function of the launch time only (or equivalently of the positions of the launch site and target plane at launch), while, for a given p, q may be varied by varying the launch azimuth, A. From Figure 3, the relationship between q and α for a given p is

$$\sin \alpha = \frac{\sin p}{\sin q} \tag{5}$$

The minimum value of α for the configuration shown is found by differentiating Equation (5) as follows:

$$\frac{d(\sin \alpha)}{dq} = - \frac{\sin p \cos q}{\sin^2 q} = 0$$

or

$$q = 90 \text{ degrees} \tag{6}$$

Equation (6) shows that the minimum plane change angle may be achieved by launching the vehicle in such a manner as to intercept the target plane 90 degree downrange from the launch site's position at the time of launch. Note the 90 degree downrange intercept condition implies that the horizontal component of the launch velocity vector is parallel to the target plane. It also follows from Equation (6) that the plane change angle, α, is equal to the arc p (the great circle distance of the launch site from the target plane at the time of launch).

A. Launch Window for Unconstrained Launch Azimuth

If the target and launch planes are projected onto a cylinder tangent to the earth at the equator, the picture on the unrolled cylinder will look somewhat like Figure 4. The time reference (t = 0) in Figure 4 is the northerly in-plane launch opportunity, i. e., the time when the launch site is in the target plane. The launch site moves eastward with respect to the target plane at a rate $\omega = \omega_E - \omega_r$, where ω_E is the earth's rotation rate in inertial space. The arc r is the distance measured along the earth's equator from the target plane's ascending node to the meridian of the target plane's crossing (on a northeasterly azimuth) of the launch site latitude, L, at which point the target plane's local azimuth is A_{LT}. The meridian containing the launch site at the time of launch, t, is $\omega t + r$ east of the target plane's ascending node, at which time the target plane's local latitude and azimuth are, respectively, L' and $A_L{}'$. The vehicle is launched with inertial

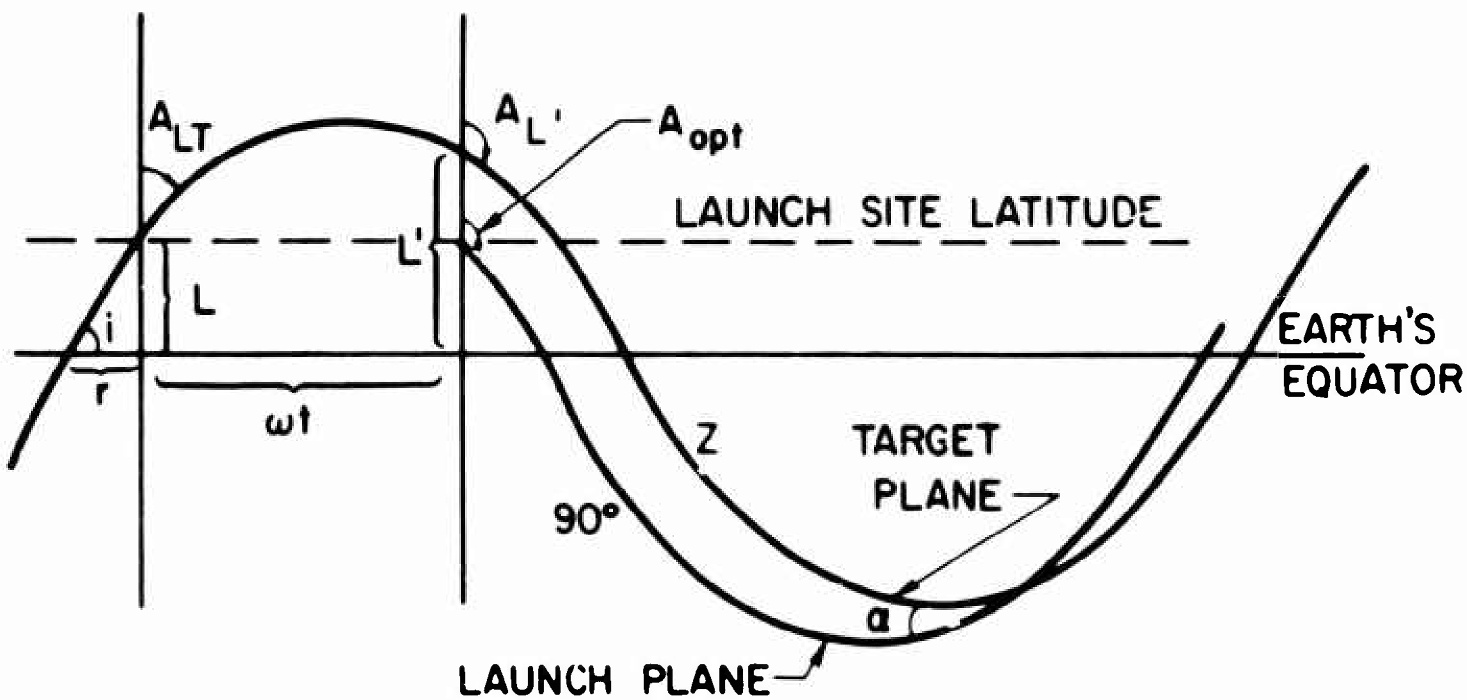

Figure 4. Launch Geometry for Optimum Launch Azimuth

azimuth A_{opt} to intercept the target plane 90 degrees downrange at an angle α. The arc of the target plane's projection from the launch site meridian to interception of the launch plane is designated as Z.

The first step in formulating the launch window (defined as the time span during which the maximum allowable plane change angle, α_m, is not exceeded)[3] is to express the plane change angle, α, as a function of time, t. This is accomplished by relating the respective parameters through spherical trigonometric relations applied to the spherical triangles represented in Figure 4. The quantities A_{LT} and r are independent of the launch delay, t, and may be expressed in terms of the launch site latitude and target plane inclination as:

$$\frac{\sin A_{LT}}{\sin r} = \frac{\sin i}{\sin L}$$

$$\cos i = \sin A_{LT} \cos L$$

or, solving for A_{LT} and r,

$$\sin A_{LT} = \frac{\cos i}{\cos L} \quad (0 \leq A_{LT} \leq 90 \text{ deg}) \tag{7}$$

$$\sin r = \frac{\tan L}{\tan i} \quad (0 \leq r \leq 90 \text{ deg}) \tag{8}$$

The plane change angle, α, may be expressed as

$$\sin \alpha = \sin (L' - L) \sin A_{L'} \tag{9}$$

---

[3] The plane change angle has a direct relation [Equation (4)] to the velocity requirement for plane change. Therefore this definition of launch window is equivalent to that given in the introduction.

and the local latitude, L', of the target plane is expressible in the following two forms (see Figure 4):

$$\left.\begin{aligned} \sin L' &= \frac{\sin i \sin (\omega t + r)}{\sin A'_L} \\ \cos L' &= \frac{\cos i}{\sin A'_L} \end{aligned}\right\} \qquad (10)$$

Expanding Equation (9) and substituting Equations (10) to eliminate L' and $A_{L'}$ yields the following expression for the plane change angle as a function of time:

$$\sin \alpha = \sin i \cos L \sin (\omega t + r) - \cos i \sin L \quad . \qquad (11)$$

Note that Equation (9) and, therefore, Equation (11), give positive values for a $\sin \alpha$, when L (the launch site latitude) is less than L' (the local target plane latitude) and negative values for $\sin \alpha$ when L is greater than L' (see Figure 4). This property will be used later to derive analytic expressions for the launch window.

Figure 5 shows the plane change angle, α (taken as positive), versus time for a launch site latitude of L = 28.34 deg (geocentric) and target orbit inclinations of i = 28.34, 29, 30, 31 and 32 deg. The nodal regression rate, ω_r, of the target planes is taken to be zero so that $\omega = \omega_E = 0.250684$ deg/min for this case. The value of r for each i is found from Equation (8). The values of ΔV corresponding to the plane change angles, α, in Figure 5 are shown on the right ordinate and were calculated from Equation (4), assuming a V_H of 25,580 fps.

The launch azimuth, A_{opt}, may be formulated as a function of time by noting from Figure 4 that

$$\cos 90 \text{ deg} = 0 = \cos (L' - L) \cos Z - \sin (L' - L) \sin Z \cos A_{L'} \quad . \qquad (12)$$

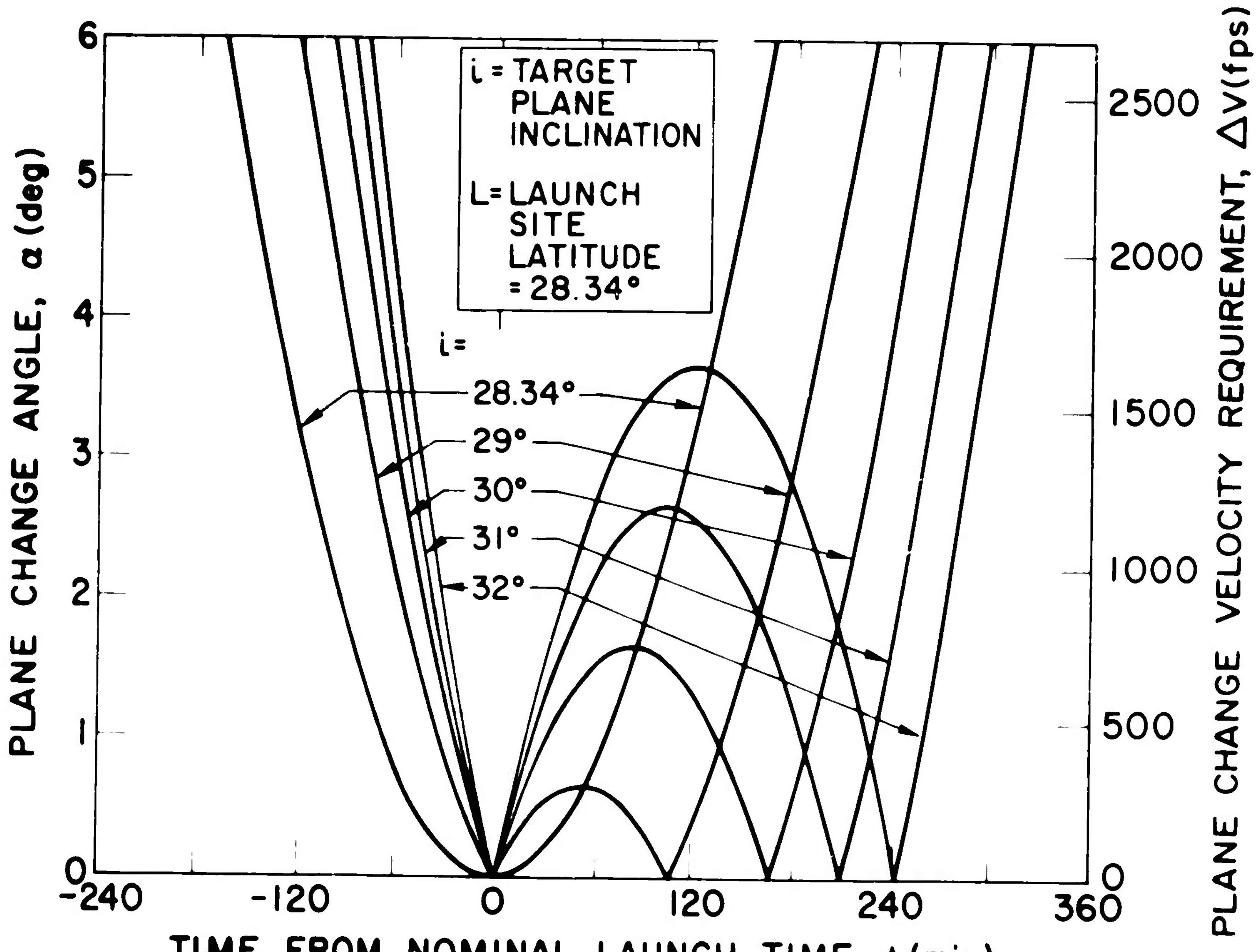

Figure 5. Launch Windowe for Variable Launch Azimuths

It also follows from Figure 4 that

$$\left.\begin{aligned} \cos Z &= \sin (L' - L) \cos A_{opt} \\ \sin Z &= \frac{\sin A_{opt}}{\sin A_{L'}} \end{aligned}\right\} \qquad (13)$$

Substituting Equations (13) into Equation (12) and combining the resulting equation with Equation (10) yields the expression for the launch azimuth, A_{opt}, which gives the minimum plane change angle, α.

$$\tan A_{opt} = \frac{\cos L \cos i + \sin L \sin i \sin (\omega t + r)}{\sin i \cos (\omega t + r)} \qquad (14)$$

$$(0 \leq A_{opt} \leq 180 \text{ deg})$$

Figure 6 shows the optimum launch azimuth as a function of time for the same conditions as in Figure 5.

Launch windows may be obtained from Figure 5 by measuring the time span, Δt, during which the ΔV (or α) capability for plane change is not exceeded. For example, if the maximum allowable plane change velocity increment is ΔV = 1000 fps (corresponding in this case to α_m = 2.23 deg) and the target plane inclination is 30 degrees, then the launch window from Figure 5 is Δt = 45 + 212 = 257 min. Note from Figure 5 that, if $i > (L + \alpha_m)$ (where α_m is the maximum plane change capability), the launch window is split into two parts. For example, if i = 32 degrees and α_m = 3 degrees, the launch window extends from -44 minutes to +70 minutes and from +172 minutes to +286 minutes yielding a total launch window of 228 minutes.

The development to follow will result in an equation which will give the launch window, Δt, as a function of the target plane inclination, i, the launch site latitude, L, and the maximum plane change capability, α_m. This formula

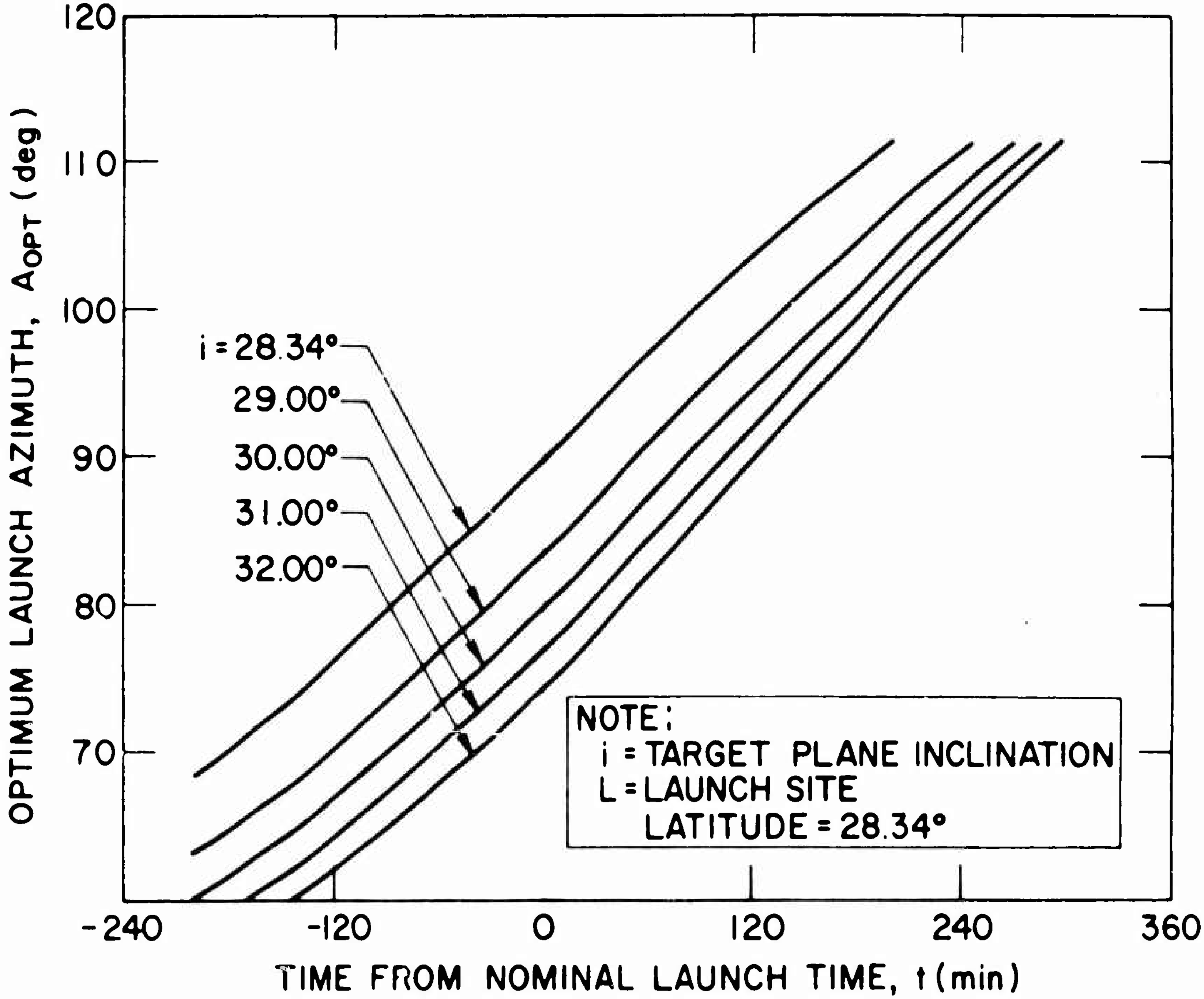

Figure 6. Optimum Launch Azimuth

will obviate the need for a graph like Figure 5 to obtain a launch window. The formulation will also eliminate the need to calculate the parameter r.

It was noted in Equation (11) above that sin α takes on positive or negative values depending on whether L′ is greater or less than L. If the plane change angle, α, is defined as a positive angle (e.g., as used in Figure 5) then Equation (11) must be rewritten as

$$\pm \sin \alpha = \sin i \cos L \sin (\omega t + r) - \cos i \sin L$$

or, equivalently,

$$\omega t + r = \sin^{-1}\left(\frac{\cos i \sin L \pm \sin \alpha}{\sin i \cos L}\right) \tag{15}$$

A typical time history curve of α (see Figure 5) may be described as having two "wings" on either side of a "hump" and as being symmetrical about the center of the hump. The center of the hump occurs at $\omega t + r = 90$ degrees (see Figure 4). Therefore, if the arcsine in Equation (15) is taken to be less than 90 degrees (quadrants I and IV), the corresponding time values will be to the left of the hump. The positive sign for sin α in Equation (15) corresponds to times on the hump, while a negative sign for sin α corresponds to time values on the wings. Note (from Figure 5) that for each time value there is one corresponding value of α (for given i and L) but that, in general, a particular value of α has four associated time values, i.e., one on each of the wings and two on the hump. For example, on the i = 32 degree curve in Figure 5, α = 3 degrees corresponds to times of -44, +70, +172 and +286 minutes. The time values associated with a given α may be distinguished in Equation (15) by the choice of sign for sin α and by the quadrant in which the arcsine is taken, as shown in the following Table (see Figure 5):

Quadrants of $\sin^{-1}\left(\frac{\cos i \sin L \pm \sin \alpha}{\sin i \cos L}\right)$	Sign of $\sin \alpha$	Time Value (See Figure 5)
I and IV	-	Left wing value
I and IV	+	Left hump value
II and III	+	Right hump value
II and III	-	Right wing value

It was noted above that the center of the hump occurs at $\omega t + r = 90$ degrees. This point obviously corresponds to a local maximum for α[4], and the value, α_{hc}, of α at the hump center [from Equation (11) with $\omega t + r = 90$ degrees] is

$$\alpha_{hc} = i - L \tag{16}$$

Therefore, for values of the maximum plane change angle $\alpha_m > i - L$, the launch window is continuous from the left wing to the right wing (see Figure 5), and, for $\alpha_m < i - L$, the launch window is divided into two equal parts by a portion of the hump.

For the continuous launch window ($\alpha_m > i - L$), the above table specifies negative signs for $\sin \alpha$ in Equation (15). Therefore, evaluating Equation (15) for the left and right wing values, respectively, and subtracting eliminates the quantity r.

$$(\omega t + r)_{RW} - (\omega t + r)_{LW} = \omega(\Delta t) = \sin^{-1}\left(\frac{\cos i \sin L - \sin \alpha_m}{\sin i \cos L}\right) \text{ (Quad II \& III)}$$

$$- \sin^{-1}\left(\frac{\cos i \sin L - \sin \alpha_m}{\sin i \cos L}\right) \text{ (Quad I \& IV)}$$

[4]This can be shown by differentiating Equation (11) with respect to time.

However, for any quantity χ,

$$\sin^{-1}\chi \text{ (Quad II \& III)} = 180 \text{ deg} - \sin^{-1}\chi \text{ (Quad I \& IV)}$$

Therefore

$$\frac{\omega(\Delta t)}{2} = 90 \text{ deg} - \sin^{-1}\left(\frac{\cos i \sin L - \sin a_m}{\sin i \cos L}\right) \text{ (Quad I \& IV)}$$

$$= \cos^{-1}\left(\frac{\cos i \sin L - \sin a_m}{\sin i \cos L}\right) \text{ (Quad I \& II)}$$

or

$$\cos \frac{\omega(\Delta t)}{2} = \frac{\cos i \sin L - \sin a_m}{\sin i \cos L} \qquad (17)$$

Equation (17) is a closed form expression for the launch window (Δt) for $a_m > i - L$.

Since the relation of the plane change angle to launch time is periodic, a launch window corresponding to [$\omega(\Delta t)$ = 360 degrees][5] may be considered infinite. Therefore, the plane change capability necessary for an infinite launch window may be obtained by letting $\omega(\Delta t)/2 = 180$ degrees in Equation (17). This substitution leads to

$$\sin a_m = \sin L \cos i + \sin I \cos L = \sin (L + i)$$

or

$$a_m = L + i \qquad (18)$$

[5]For $\omega_r = 0$, i. e., an inertial target plane, such a launch window is one sidereal day.

Equation (18) may also be derived by inspection of the geometry involved by noting that the maximum great circle distance ever encountered from the launch site to the target plane is i + L.

It was noted above that when $\alpha_m < i - L$, a portion of time is excluded from the launch window given by Equation (17). An expression for the excluded portion of time may be derived in a manner similar to the derivation of Equation (17) using Equation (15) and the table on page 15. The resulting expression is

$$\cos\left[\frac{\omega(\Delta t)_{exc}}{2}\right] = \frac{\cos i \sin L + \sin \alpha_m}{\sin i \cos L} \tag{19}$$

Although the above relations have been derived with the implicit assumption that i > L (the definition of the time reference becomes meaningless when i < L); Equation (17) may be used to obtain the launch window when i < L. However, when i < L and $\alpha_m < L - i$, Equation (17) will yield imaginary values for the launch window $[|\cos \omega(\Delta t)/2| > 1]$. The correct interpretation of this phenomenon is that no launch window exists for these conditions.

Equations (17) and (19) may be combined to give the following expression for the launch window:

$$\Delta t = \frac{2}{\omega}\left[\cos^{-1}\left(\frac{\cos i \sin L - \sin \alpha_m}{\sin i \cos L}\right) - \cos^{-1}\left(\frac{\cos i \sin L + \sin \alpha_m}{\sin i \cos L}\right)\right] \tag{20}$$

where arccosines with arguments larger in magnitude than unity are defined to be zero [e.g., the second arccosine in Equation (20) will be zero when $\alpha_m > i - L$, in which case Equation (20) reduces to Equation (17)].

Equation (20) may be used to determine the target plane inclination, i, which gives the maximum launch window for a fixed launch site latitude, L, and plane change capability, α_m. Obviously, if $\alpha_m \geq L$, an i of zero may be chosen to give an infinite launch window [see Equation (18)]. Therefore, the region of interest for this question is $0 < \alpha_m < L$. Equation (20) is continuous for $0 \le i \le 90$ deg, but $d(\Delta t)/di$ is discontinuous at $i = L - \alpha_m$ and at $i = L + \alpha_m$, at which points the right sided slope is infinite (see Figure 7). However, it can be shown that:

$$\text{(see Figure 7)} \quad \begin{cases} \dfrac{d(\Delta t)}{di} = 0 \text{ [and } \Delta t = 0 \text{ by the definition following Equation (20)] for } 0 < i < (L - \alpha_m) \\[2ex] \dfrac{d(\Delta t)}{di} > 0 \text{ for } (L - \alpha_m) < i < (L + \alpha_m) \\[2ex] \dfrac{d(\Delta t)}{di} < 0 \text{ for } (L + \alpha_m) < i < 90 \text{ deg} \\[2ex] \dfrac{d(\Delta t)}{di} = 0 \text{ for } i = 90 \text{ deg} \end{cases}$$

The correct interpretation of the above conditions is that the maximum launch window, Δt, (for $0 < \alpha_m < L$) occurs when $i = (L + \alpha_m)$, the minimum Δt ($= 0$) occurs for $0 < i < (L - \alpha_m)$, and a local minimum Δt occurs at $i = 90$ degrees (designating a polar target plane). An expression for the maximum launch window Δt_{max} may be obtained by substituting $i = (L + \alpha_m)$ into Equation (20) to give

$$\cos\left[\frac{\omega(\Delta t_{max})}{2}\right] = 2\,\frac{\tan L}{\tan(L + \alpha_m)} - 1 \qquad (21)$$

The launch window for polar target planes may be obtained by substituting $i = 90$ degrees into Equation (20) to give

$$\sin\left[\frac{\omega(\Delta t)}{4}\right] = \frac{\sin \alpha_m}{\cos L} \qquad (22)$$

Figure 7 is a plot of Equation (20) for $L = 28.34$, $\omega = 0.25063$ deg/min, and $\alpha_m = 6$ degrees. The dashed line in Figure 7 is the locus of maximum launch windows (when α_m and L are held constant and i is varied) obtained from Equation (21). Figure 8 shows the launch windows [Equation (20)] as a function of α for several values of i. The dashed line in Figure 8 is, as before, the locus of maximum launch windows

B. <u>Launch Window for Constrained Launch Azimuth</u>

Because of range safety considerations and/or because of some characteristic of the launch vehicle, the launch azimuth may be constrained to lie between certain values or to be fixed If the launch azimuth, A_L, is fixed, then the inclination, i', of the plane into which the vehicle is launched is fixed also. The relation connecting A_L and i' can be seen from Figure 9 to be

$$\cos i' = \cos L \sin A_L \tag{23}$$

The local azimuth, A_{LT}, of the target plane at the launch site latitude (on the northerly crossing) is similarly given by

$$\sin A_{LT} = \frac{\cos i}{\cos L} \quad (0 \leq A_{LT} \leq 90 \text{ deg}) \tag{24}$$

Let the time reference ($t = 0$) for the fixed azimuth case be defined as the time when the launch site passes through the target plane on the target plane's crossing (on the northeasterly azimuth) of the launch site latitude (This definition is consistent with the variable launch azimuth case and will allow fixed and variable launch azimuth results to be combined later) The geometry at $t = 0$ is shown in Figure 9, and the geometry for a later time, t, is shown in Figure 10

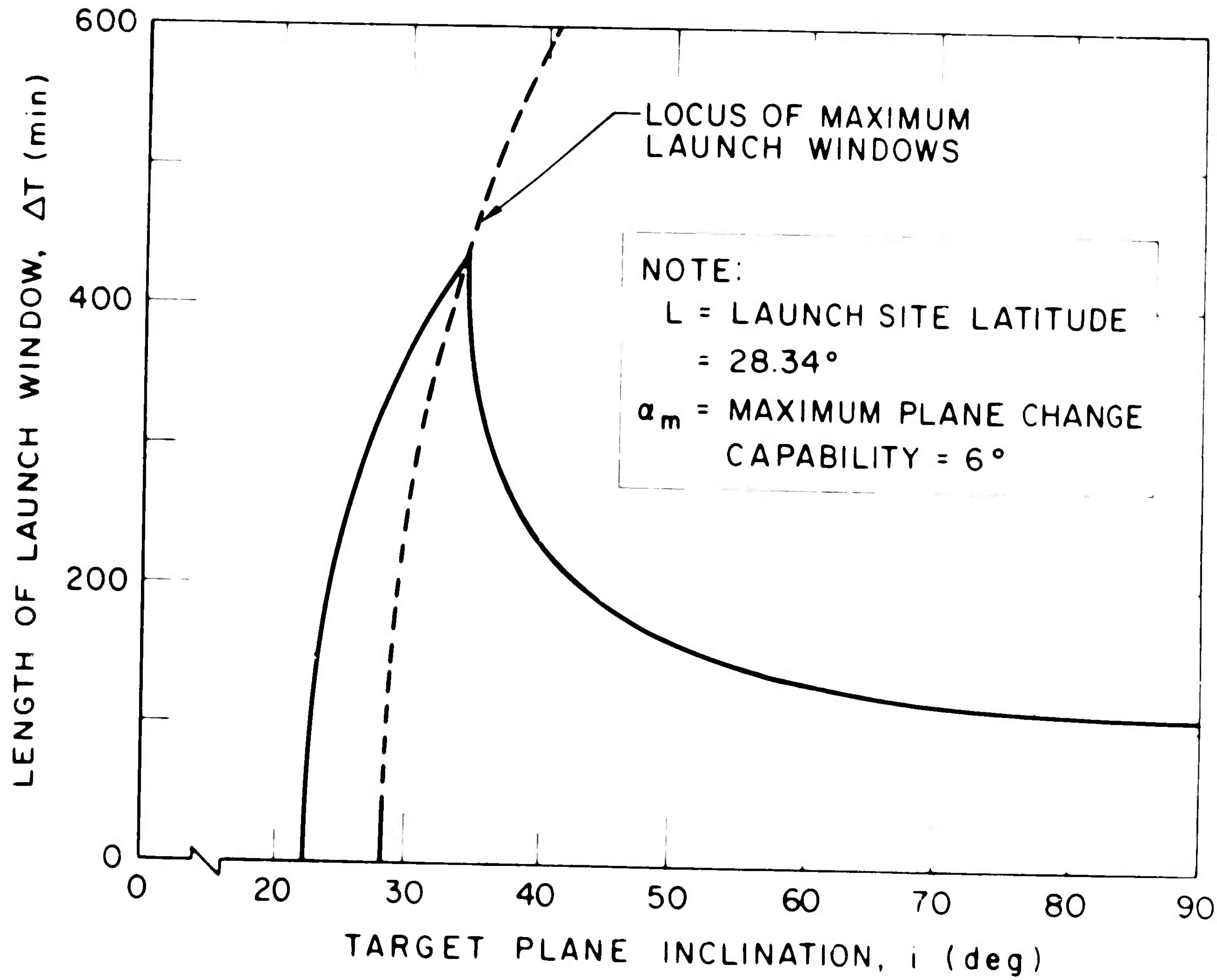

Figure 7. Launch Window as a Function of Target Plane Inclination

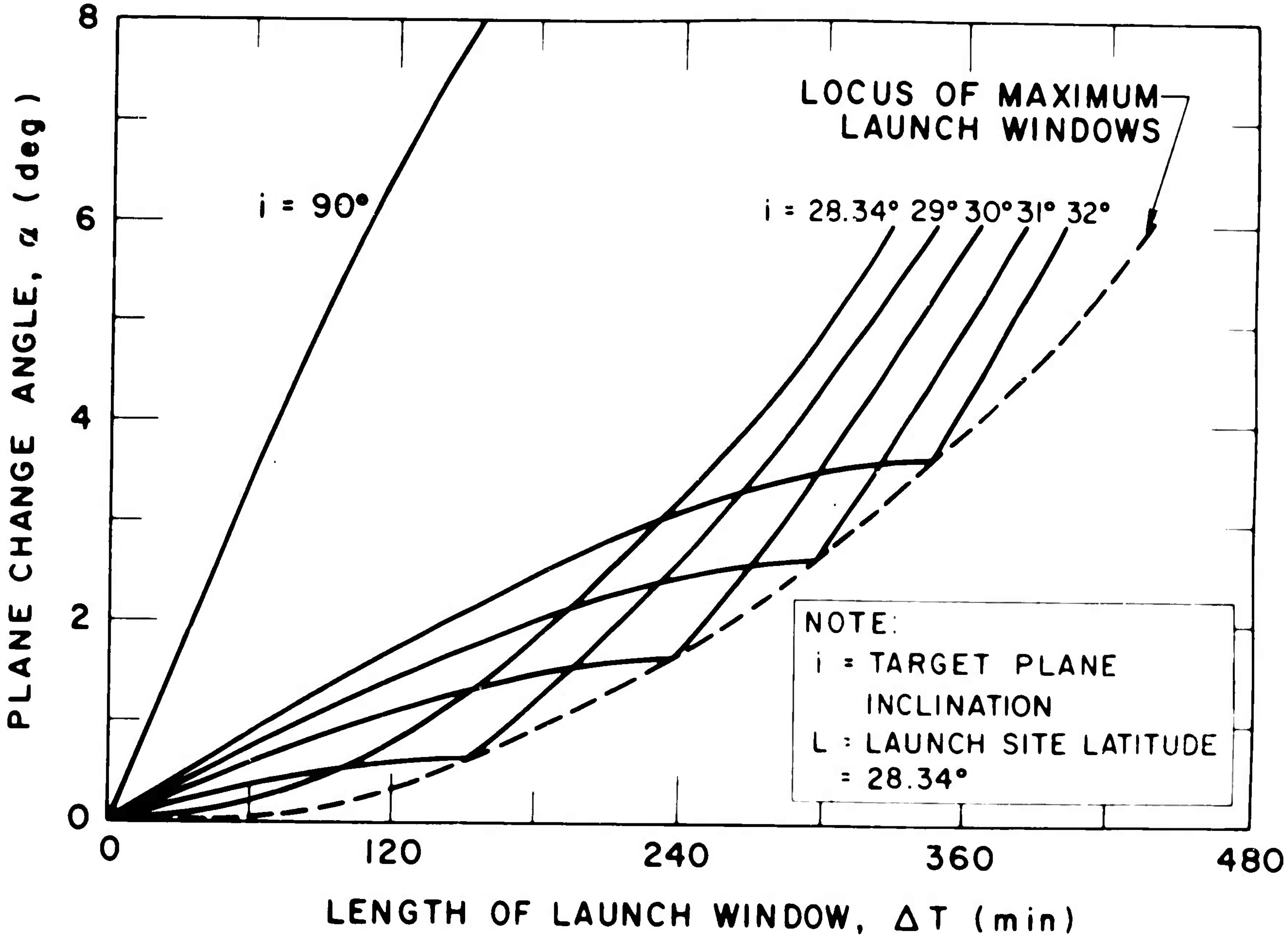

Figure 8 Launch Windows for Variable Azimuths

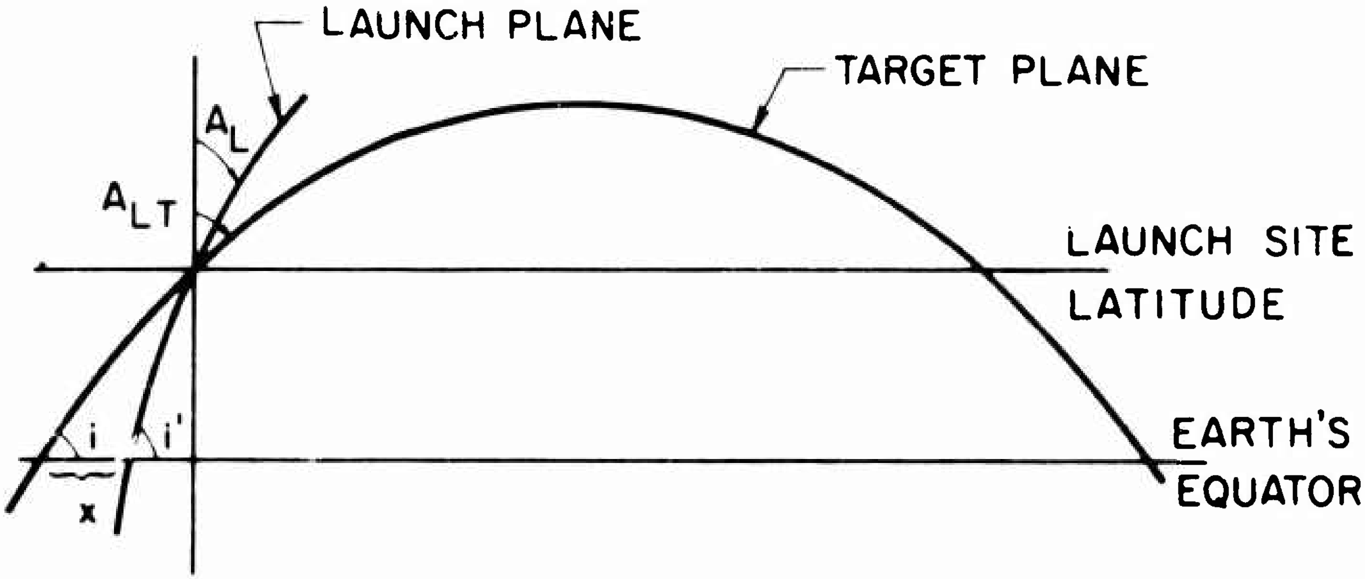

Figure 9. Geometry for Fixed Launch Azimuth at t = 0

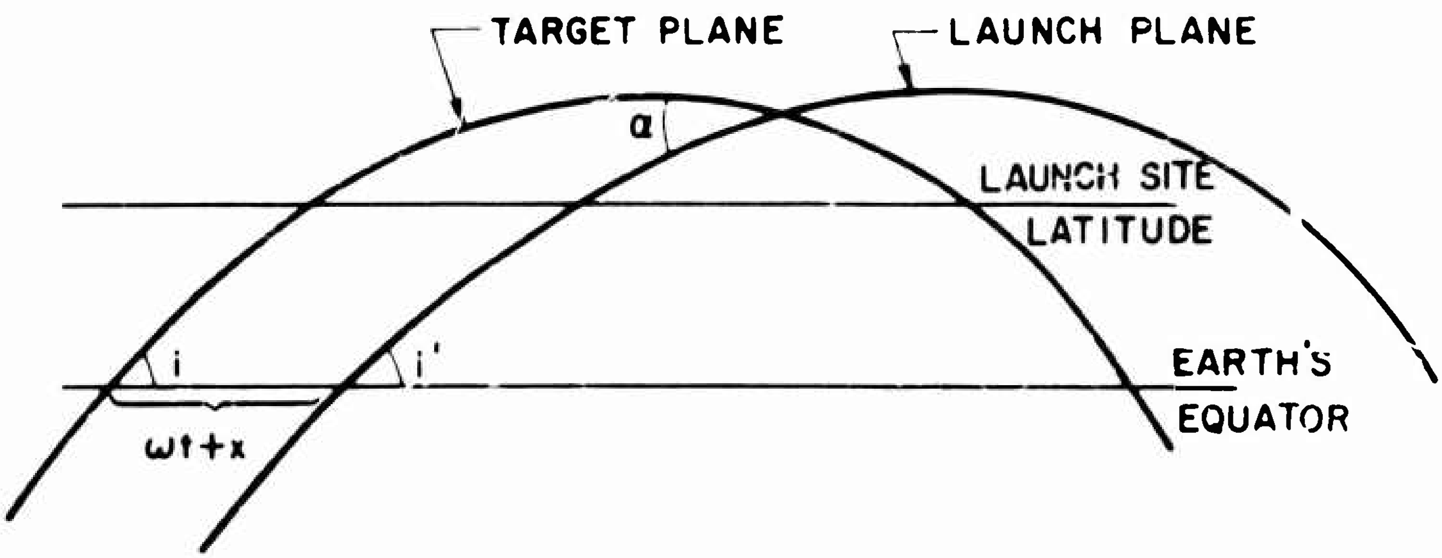

Figure 10. Geometry for Fixed Launch Azimuth at t > 0

The quantity x in Figure 9 is the difference in longitude of the ascending nodes of the two planes at t = 0. An expression from which x can be found may be written as

$$\cos (A_{LT} - A_L) = \cos i \cos i' + \sin i \sin i' \cos x$$

or,

$$\cos x = \frac{\cos (A_{LT} - A_L) - \cos i \cos i'}{\sin i \sin i'} \quad (-90 \text{ deg} \leq x \leq 90 \text{ deg}) \tag{25}$$

where x has the same sign as $(A_{LT} - A_L)$.

The expression for the plane change angle, α, may now be written (see Figure 10) as

$$\cos \alpha = \cos i \cos i' + \sin i \sin i' \cos (\omega t + x) \tag{26}$$

The condition for minimum plane change angle for the fixed azimuth case may be found by differentiating Equation (26) as follows:

$$\sin \alpha \frac{d\alpha}{d(\omega t)} = \sin i \sin i' \sin (\omega t + x) = 0 \tag{27}$$

The two solutions to Equation (27) are

$$\left.\begin{aligned} (\omega t + x)_1 &= 0 \text{ or } 360 \text{ deg} \\ &\text{and} \\ (\omega t + x)_2 &= 180 \text{ deg} \end{aligned}\right\} \tag{28}$$

It is apparent from examination of the second derivative of Equation (26) that the first solution above corresponds to the minimum and the second to the maximum values of α. Further, the minimum and maximum values of the plane change angle may be found by substituting the solutions (28) into (26) or by inspection to be

$$\left.\begin{aligned} \alpha_{min} &= |i' - i| \\ \alpha_{max} &= |i' + i| \end{aligned}\right\} \qquad (29)$$

It is also evident from Equation (28) and the geometry that, when the minimum or maximum plane change angle occurs, the intersection of the target and launch planes occurs on the equator. The minimum α occurs when the planes cross the equator at the same point in the same sense (northerly or southerly), and the maximum α occurs when the planes cross the equator at the same point in opposite senses.

Figure 11 shows the plane change angle as a function of time for a target plane inclination of $i = 30$ degrees, a launch site latitude of $L = 28.34$ degrees, and launch azimuths, A_L, of 70, 79.722, 90, 100.278, and 110 degrees. As before, the nodal regression rate, ω_r, is assumed to be zero. Note that the curves defined by $A_L = 70$ degrees and $A_L = 110$ degrees are identical. This is due to the symmetry of the launch azimuths about $A_L = 90$ degrees [Equation (23) shows that launch azimuths symmetrical about 90 degrees yield identical values of i' and therefore, from Equation (26), the same shape curves of α versus t]. The curves defined by $A_L = 79.722$ degrees and $A_L = 100.278$ degrees exhibit a similar property and in addition contain a point where $\alpha = 0$. These are the only two launch azimuths which yield a zero α for the particular i and L under consideration, because 79.722 degrees is the local target plane azimuth, A_{LT}, at the launch site latitude [see Equation (21)] and 100.278 degrees is 180 degrees - A_{LT}. Therefore, vehicles launched on these azimuths may at some time be launched directly into the target plane.

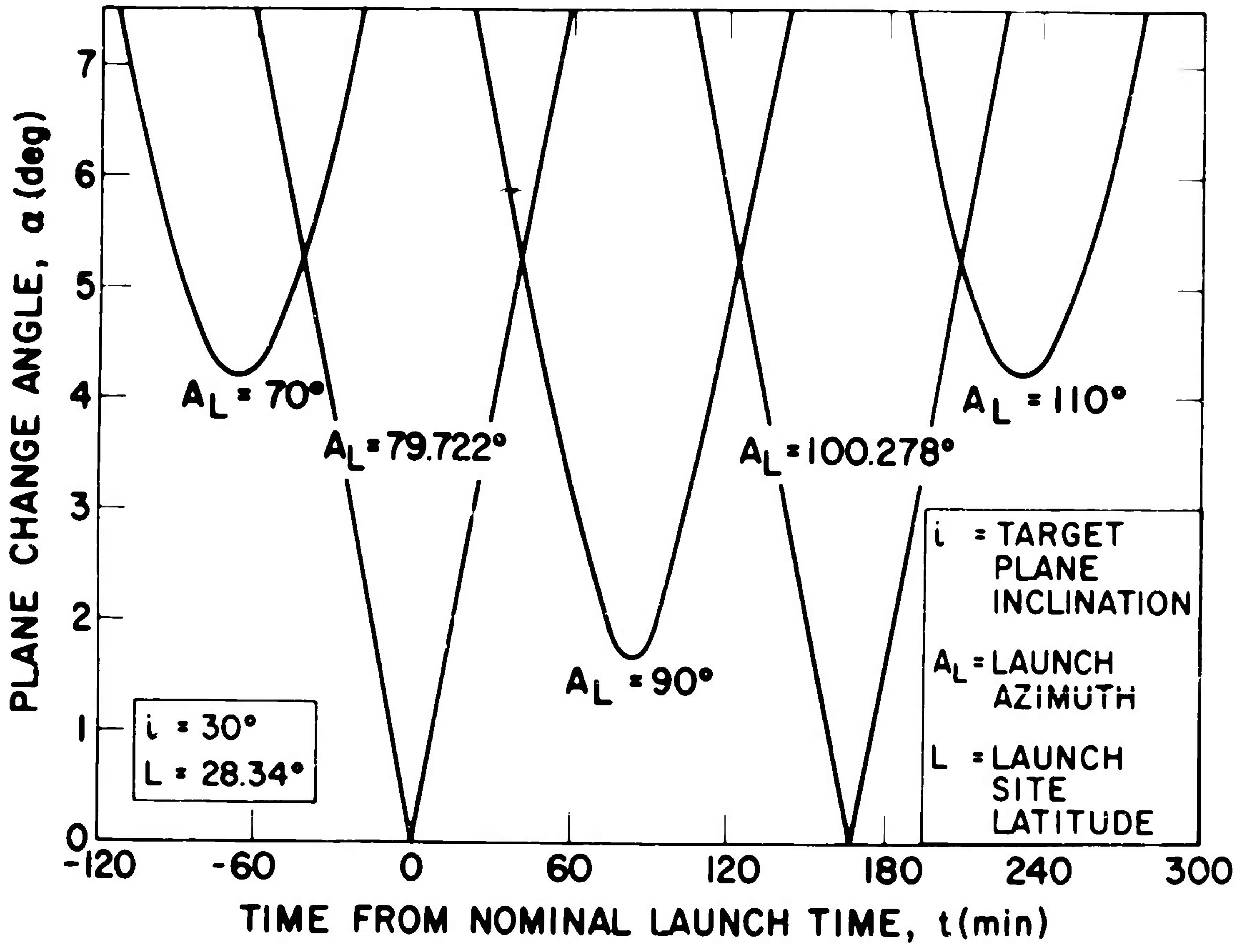

Figure 11 Launch Windows for Fixed Launch Azimuth

In many problems with a fixed launch azimuth constraint, the launch azimuth is chosen to be the azimuth which results in a direct launch into the target plane at the nominal launch time (i. e., $A_L = A_{LT}$). Such a condition results in the inclinations of the launch and target planes being identical ($i = i'$) and in their longitudinal separations at $t = 0$ being zero ($x = 0$). Therefore, for this case, Equation (26) reduces to

$$(\cos \alpha)_{(A_L - A_{LT})} = \cos^2 i + \sin^2 i \cos \omega t \tag{30}$$

It can be seen from Equation (26) (also see Figure 11) that the curve of α versus t is symmetrical about the value of t corresponding to the minimum α. Therefore, the time measured from the minimum α point [defined by $(\omega t + x) = 0$] to the time defined by $\alpha = \alpha_m$ (the maximum allowable plane change angle) is one half the launch window, Δt. Using this reasoning, Equation (26) may be written as

$$\cos \alpha_m = \cos i \cos i' + \sin i \sin i' \cos \frac{\omega \Delta t}{2}$$

or, solving for the launch window

$$\cos \frac{\omega \Delta t}{2} = \frac{\cos \alpha_m - \cos i \cos i'}{\sin i \sin i'} \tag{31}$$

where i' is obtained from Equation (23).

If the launch azimuth is not constrained to be constant but is constrained not to exceed a certain value, then the launch window is defined by a combination of variable and fixed azimuth formulations. Suppose, for example, that a southerly azimuth limit, A_S, exists. The time history of the plane change angle, α, is given by the variable azimuth formulation Equation (11), up until

A_S is reached and then by the fixed azimuth formulation, Equation (26), with $A_L = A_S$.

Figure 12 shows a typical launch window for several southerly azimuth limits. For this case, the launch site latitude is 28.34 degrees and the target plane inclination is 30 degrees. Figure 12 shows that for a plane change capability of 2 degrees, the launch window extends from -41 minutes to +208 minutes for a total launch window of 249 minutes with no launch azimuth constraints. However, if the launch azimuth is constrained to be no larger than 100 degrees, the window is reduced by 28 minutes to give a total of 221 minutes.

The time during an otherwise unconstrained launch window when the southerly launch azimuth limit is reached may be read from Figure 6 or, if such a graph is not available, may be calculated from Equation (14)[6].

For example, the 100 degree launch azimuth limit for the case sited above is reached 165 minutes after the northerly in-plane launch opportunity. Therefore, the time history of α during the launch window is given by Equation (11) up to $t = 165$ minutes and by Equation (26) after $t = 165$ minutes. Northerly launch azimuth limits may be handled in a manner similar to that outlined above for southerly launch azimuth limits. The launch window may then be measured from a plot similar to Figure 12 or found by calculating the beginning and end points of the launch window from Equation (11) and/or Equation (26), taking the difference, and excluding the "hump" portion [Equation (19)].

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[6] One method for solving Equation (14) would be to substitute the southerly azimuth limit, A_S, for A_{opt} in Equation (14) and re-arrange the equation in the form

$$\cos(\omega t + r) = \cot A_S \sin L [\cot L \cot i + \sin(\omega t + r)]$$

Assignment of some value between zero and unity to $\sin(\omega t + r)$ on the right side of the above equation then yields a value of $(\omega t + r)$ from the left side which is substituted into the right side of the equation. This iteration procedure is continued until a sufficiently accurate value of $(\omega t + r)$ is obtained.

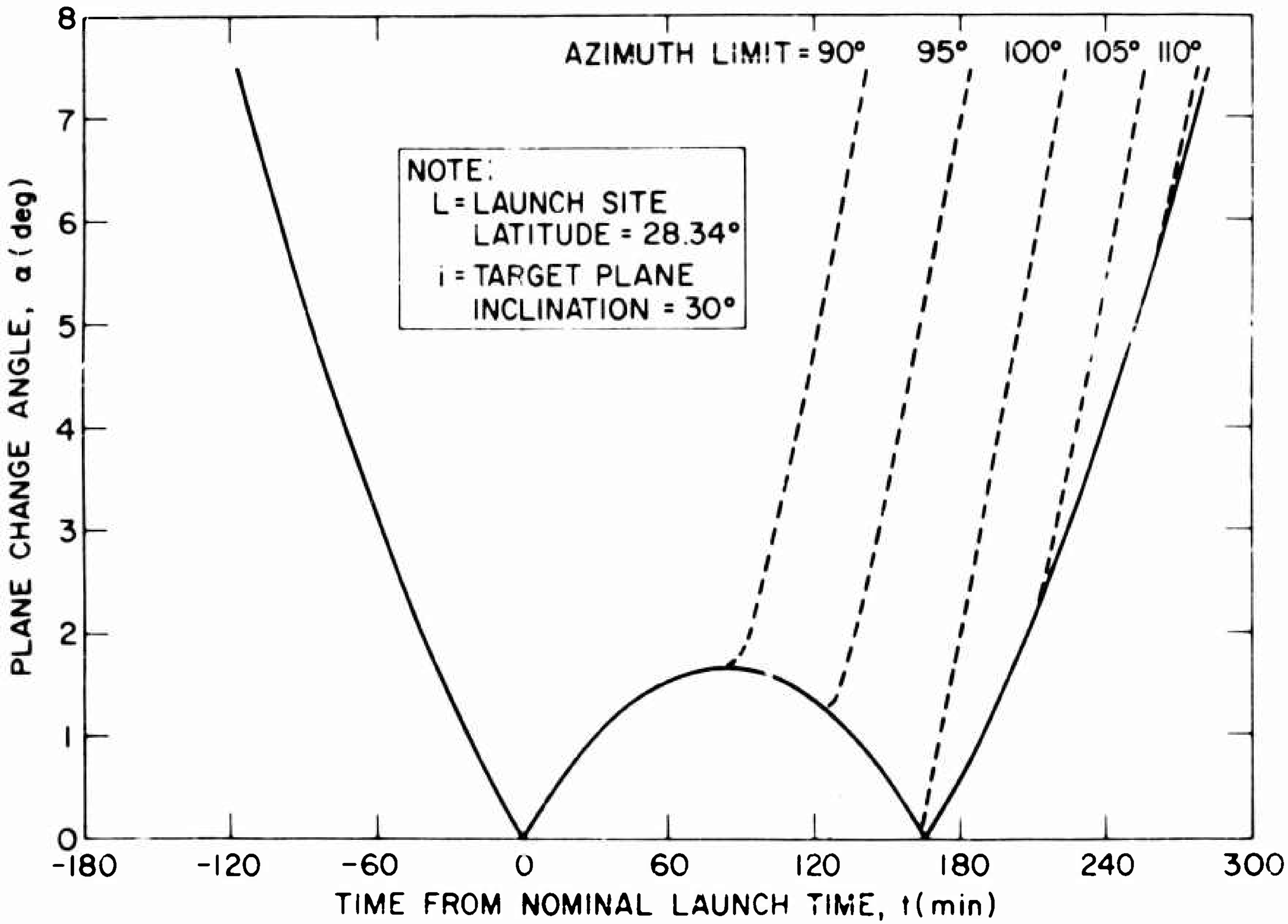

Figure 12. Launch Windows for Variable Launch Azimuth with Southerly Azimuth Limit

III. SUMMARY

The launch window has been defined as the time span during which a vehicle may be launched and established in a given inertial plane passing through the earth's center without exceeding a given maximum value, a_m, for the plane change angle (or, equivalently, the maximum ideal velocity budgeted to plane change). This paper has developed concise formulations for the plane-change angle as a function of launch delay only (assuming target plane inclination and launch site latitude fixed) for variable, optimized launch azimuth [Equation (11)] and for fixed launch azimuth [Equation (26)]. It has been shown that the optimum launch azimuth (i. e., yielding minimum plane change angle) is one which results in a 90-degree downrange intercept of the target plane, and the optimum launch azimuth has been formulated as a function of time only [Equation (14)]. Further development of the equations has led to concise formulations for the length of the launch window (assuming target plane inclination and launch site latitude fixed) corresponding to a given a_m (maximum plane change angle) for variable, optimized launch azimuth [Equation (20)] and for fixed launch azimuth [Equation (31)]. A method was then defined to obtain a time history of plane change angle for a combined fixed and variable launch azimuth case (e. g., a case where the launch azimuth is constrained to be within certain limits). Numerical examples and graphical illustrations were given for most of the cases and several miscellaneous formulations growing out of the development were presented.