PLASTIC DEFORMATION OF UNSTIFFENED AND STIFFENED RECTANGULAR PLATES

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LIST OF SYMBOLS

$\dot{w}$	lateral velocity of plate
I	impulse per unit mass applied to plate
dm	element of mass
T	initial kinetic energy imparted to plate
$\bar{\mu}$	mass per unit area of plate
dA	elemental area of plate
I_o	amplitude of the impulse'
$f(x, y)$	spatial distribution of the impulse
V	work done by the internal forces during the plate deformation
I_t	total impulse on plate
σ_i	stress factor proportional to octahedral shear stress
e_i	strain factor proportional to octahedral shear strain
σ_x, σ_y	tensile or compressive stress in x and y direction respectively
τ_{xy}	shear stress
ϵ_x, ϵ_y	tensile or compressive strain in x and y direction respectively
γ_{xy}	shear strain
x, y	rectangular coordinate
w	lateral deflection of plate
$\bar{z}$	distance from neutral surface of stiffened plate to midsurface of face
z_p	distance between midsurface of face and any point in face
σ_s	yield stress in pure tension
e_s	yield strain in pure tension
E	modulus of elasticity of plate material
$\omega(e_i)$	plastic function in stress-strain law
h	thickness of one facing of double bottom

$\bar{h}$	thickness of other facing of double bottom
$\bar{\bar{z}}$	distance from neutral plane to any point in face for second part of double bottom
a, b	length and width of plate
λ	a measure of the slope of the plastic portion of of the stress-strain curve of a linear hardening material
y_r	y coordinate of the rth stiffener
x_s	x coordinate of the sth stiffener
I_{A_r}	moment of inertia of the rth stiffener about its own neutral axis
w_0	amplitude of the deformation
ρ	mass density of the plate material
p_0	amplitude of the applied pressure
T_0	length of time that pulse is applied
β	shape factor for exponential pulse

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AND STIFFENED RECTANGULAR PLATES

J. Greenspon

Introduction

The slamming of ships in heavy seas has been a problem of great practical significance for a long time. Within the past ten years some measure of progress has been made in understanding this problem and in formulating recommendations how to design a ship to withstand slamming. A complete review of the status of this problem is given by Professor Schade (University of California) in his report of Committee #A "Slamming and Impact" of the International Ship Structures Congress. As far as the structural design of the ship is concerned there are several main problem areas to be considered. The first is the midship section. The main stresses resulting in the midship section due to slamming are those resulting from beam type action of the hull called primary stresses due to the slamming pulse which can be considered as acting on the forward 20% of the hull. The second problem is the deformations and stresses induced in large stiffened plate sections of the hull called secondary stresses in the slamming area. The third is the stresses and deformations in local unstiffened plate sections called tertiary stresses. A recent report [1*] considers the general method of computing the total dynamic elastic stress picture in a hull under slamming using the ideas of primary, secondary, and tertiary stresses mentioned above.** The plastic deformation of unstiffened plates under slamming impact has been considered briefly by Greenspon[2] and much more extensively by Nagai.[3,4] For design purposes one should attempt to stipulate the amount of secondary and tertiary deformation that can be allowed. Ther tertiary or unstiffened plate defination can result in washboarding of the local forward bottom of the ship and possibly in local failures. The secondary plastic deformation can result in washboarding of large sections of the forward bottom and is probably a more serious problem than the tertiary plastic deformations. The purpose of this report is to present a basic approach which can be used to compute the secondary plastic deformations and, as a special case, the tertiary deformations.

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* Superscripts refer to references listed at the back of the report.

** To the author's knowledge these ideas were first introduced by M. St. Denis, DTMB Report C-555

II. Theory

The approach to this problem mainly follows the work of two reports.[5,6] Let the work done by the internal forces of an elastic-plastic stiffened plate under impact be V. Let I be the impulse per unit mass applied to the plate. The impulse momentum relation for an elemental mass can be written

$$\dot{w}\,dm = I\,dm \tag{1}$$

where $\dot{w}$ is the lateral velocity imparted to the mass by the impulse (only lateral velocity $\dot{w}$ is being considered, $\dot{u}$ and $\dot{v}$ are being neglected)

Thus

$$\dot{w} = I \tag{2}$$

The kinetic energy imparted to the plate is

$$T = \int_A \frac{1}{2}\bar{\mu}\,\dot{w}^2\,dA = \frac{1}{2}\int_A \bar{\mu}\,I^2\,dA \tag{3}$$

Where $\bar{\mu}$ is the mass per unit area of the plate and dA is an elemental area. The impulse can vary over the surface of the plate, therefore write

$$I(x,y) = I_0\,f(x,y) \tag{4}$$

The kinetic energy becomes

$$T = \frac{1}{2}\bar{\mu}\,I_0^2 \int_A f^2(x,y)\,dA \tag{5}$$

Equating the initial kinetic energy to the energy of deformation the expression for the impulse per unit mass becomes

$$I_0 = \sqrt{V\,\frac{2}{\bar{\mu}\int_A f^2(x,y)\,dA}} \tag{6}$$

The total impulse on the plate will then be

$$I_t = \int_A \sqrt{V \frac{2\bar{\mu}}{\int_A f^2(x,y)\, dA}}\; f(x,y)\, dA \tag{7}$$

The work of deformation, V, per unit volume of an elastic plastic body can be written

$$V = \int_0^{e_i} \sigma_i\, de_i + \frac{K\theta^2}{2} \tag{8}$$

where
$$\sigma_i = \frac{\sqrt{2}}{2}\sqrt{(\sigma_x-\sigma_y)^2+(\sigma_y-\sigma_z)^2+(\sigma_z-\sigma_x)^2+6(\tau_{xy}^2+\tau_{yz}^2+\tau_{zx}^2)} \tag{9}$$

$$e_i = \frac{\sqrt{2}}{3}\sqrt{(\epsilon_x-\epsilon_y)^2+(\epsilon_y-\epsilon_z)^2+(\epsilon_z-\epsilon_x)^2+\frac{3}{2}(\gamma_{xy}^2+\gamma_{yz}^2+\gamma_{zx}^2)}$$

$$\theta = \epsilon_x + \epsilon_y + \epsilon_z$$

The curve of σ_i vs e_i describes the stress-strain law of the material as shown in Fig. 1.

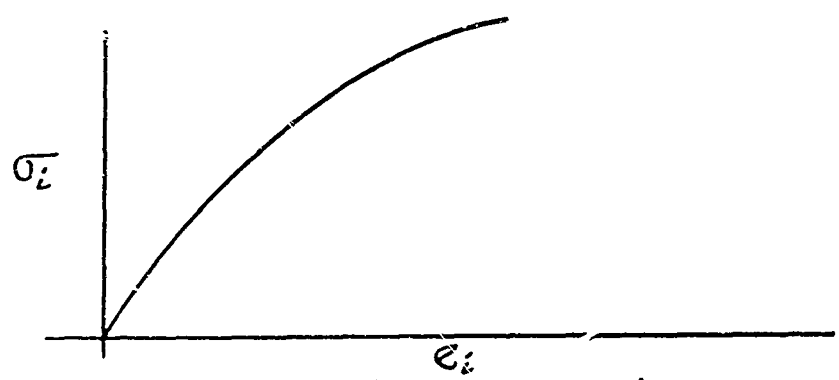


Fig. 1. General Stress-Strain Law

Assuming an incompressible material $\theta=0$ and considering only plate stresses,

$$V = \int_{V_0}\left[\int_0^{e_i} \sigma_i\, de_i\right] dV_0 \tag{10}$$

dV_0 = element of volume

$$\sigma_i = \sqrt{\sigma_x^2 - \sigma_x\sigma_y + \sigma_y^2 + 3\tau_{xy}^2}$$

$$e_i = \sqrt{\epsilon_x^2 + \epsilon_x\epsilon_y + \epsilon_y^2 + \frac{1}{4}\gamma_{xy}^2}$$

Consider only the lateral deflection, w, to be of significance and neglect u and v and their derivatives.

The strains ϵ_x, ϵ_y , and γ_{xy} then become[8]

$$\epsilon_x = \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - z\frac{\partial^2 w}{\partial x^2} \quad , \quad \epsilon_y = \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 - z\frac{\partial^2 w}{\partial x^2} \tag{11}$$

$$\gamma_{xy} = \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} - 2z\frac{\partial^2 w}{\partial x \partial y}$$

where z is the distance of any element of the plate from the neutral plane.

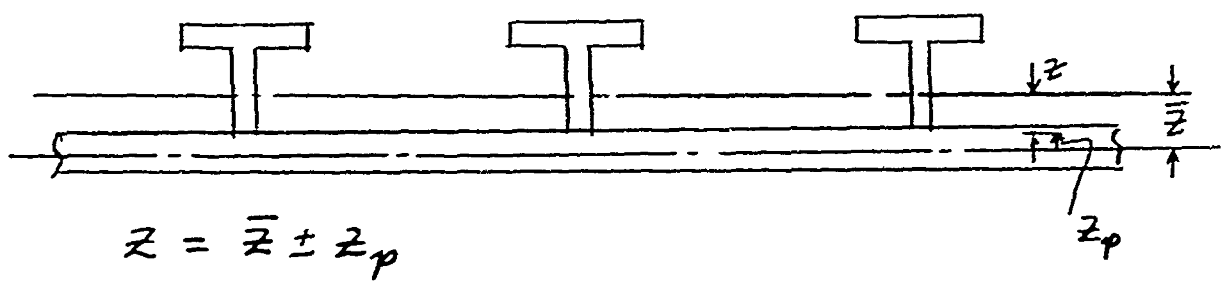


Fig. 2. Location of Neutral Plane and Elements

Further restrict the material to be an elastic-linear hardening material (although this is not much of a restriction) with the stress stain law shown in the figure below

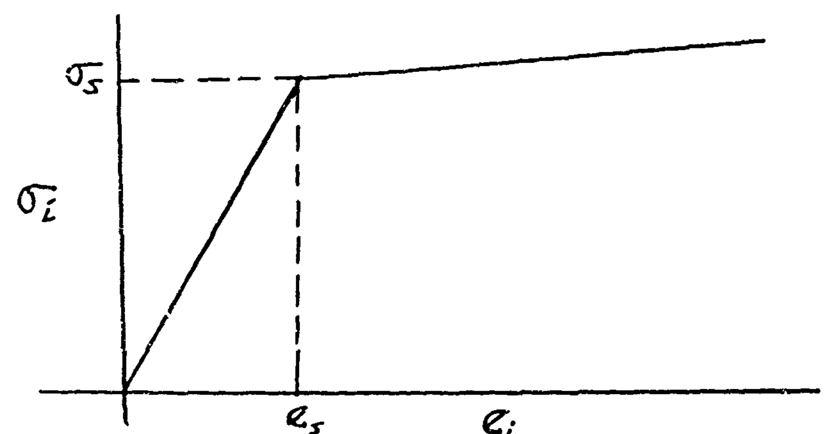


Fig. 3. Elastic-Linear-Hardening Law

The stresses can be written in terms of the strains as follows (assuming a Poisson's ratio of ½)

$$\sigma_x = \frac{4}{3}\frac{\sigma_i}{e_i}\left(\epsilon_x + \frac{1}{2}\epsilon_y\right)$$

$$\sigma_y = \frac{4}{3}\frac{\sigma_i}{e_i}\left(\epsilon_y + \frac{1}{2}\epsilon_x\right) \qquad (12)$$

$$\tau_{xy} = \frac{1}{3}\frac{\sigma_i}{e_i}\gamma_{xy}$$

Where the stress-strain law is

$$\frac{\sigma_i}{e_i} = E\left[1 - \omega(e_i)\right] \qquad (13)$$

where

$$\omega(e_i) = 0 \quad \text{for } e_i < e_s \qquad (\text{elastic}) \qquad (14)$$

$$\omega(e_i) = \lambda\left(1 - e_s/e_i\right) \qquad (\text{plastic})$$

$$\lambda = 1 - \frac{1}{E}\frac{d\sigma_i}{de_i}$$

Substituting the linear-hardening law into the expressions for the stresses and then into the relation for V_1 we obtain the work done by the internal forces on the plating of the stiffened plate.

$$V_1 = \int_0^a\int_0^b\int_{\bar{z}-\frac{h}{2}}^{\bar{z}+h/2}\left[\frac{Ee_i^2}{2}(1-\lambda) + E\lambda e_s e_i\right]dx\,dy\,dz_p - \int_0^a\int_0^b\int_{\bar{z}-\frac{h}{2}}^{\bar{z}+h/2}\frac{E\lambda e_s^2}{2}\,dx\,dy\,dz_p \qquad (15)$$

where $\bar{z}$ = distance from neutral plane to midsurface of face

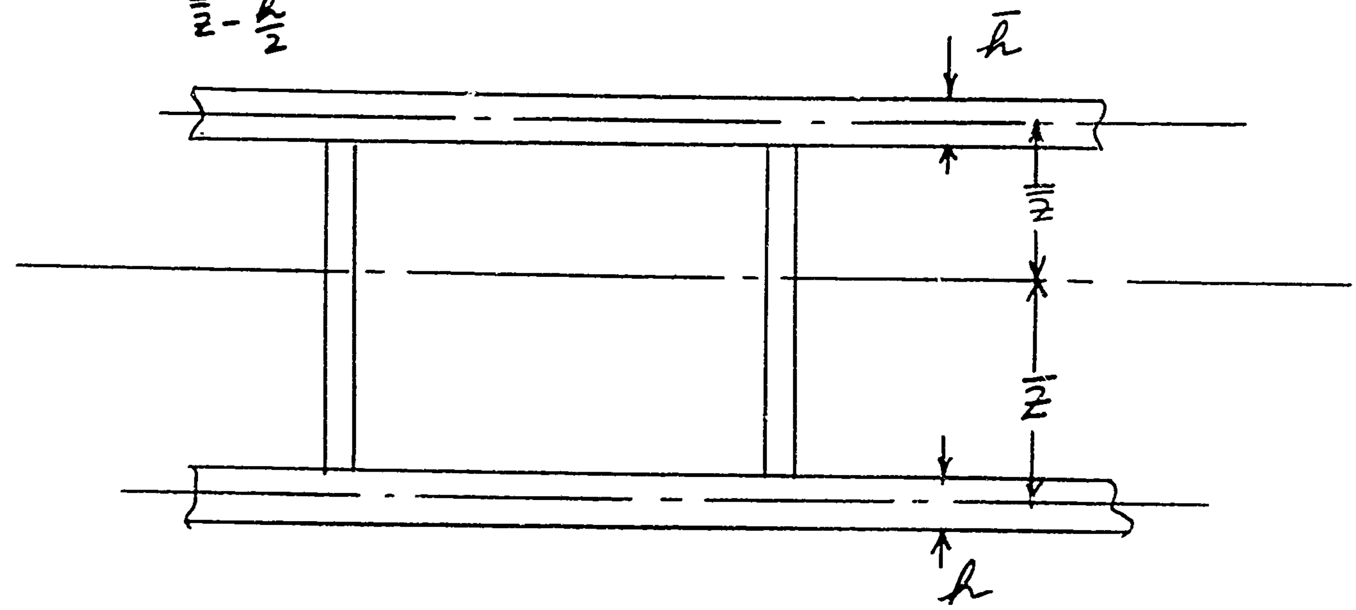
h = thickness of plate

a, b = length and width of plate

If there is a top plate then there is a similar expression V_2 that has to be added for this.

If $\bar{\bar{z}}$ = distance from neutral plane to center of other face

$$V_2 = \int_0^a \int_0^b \int_{\bar{\bar{z}} - \frac{\bar{h}}{2}}^{\bar{\bar{z}} + \bar{h}/2} \left\{ \text{Same expression as in } V_1 \right\} \tag{16}$$



Now substituting the expression for e_i

$$V_1 = \int_0^a \int_0^b \int_{\bar{z} - \frac{h}{2}}^{\bar{z} + h/2} \Big[\frac{E}{2}(1-\lambda)\, \frac{4}{3}\left(\epsilon_x^2 + \epsilon_x \epsilon_y + \epsilon_y^2 + \frac{1}{4}\gamma_{xy}^2\right) + \frac{E\lambda e_s}{\sqrt{3}}\, 2\sqrt{\epsilon_x^2 + \epsilon_x\epsilon_y + \epsilon_y^2 + \frac{1}{4}\gamma_{xy}^2}\, \Big] dx\, dy\, dz_p - \int_0^a \int_0^b \int_{\bar{z} - h/2}^{\bar{z} + \frac{h}{2}} \frac{E\lambda e_s^2}{2}\, dx\, dy\, dz_p \tag{17}$$

Substituting the expressions for the strains in terms of the deflections

$$V_1 = \int_0^a \int_0^b \Bigg\{ \frac{E(1-\lambda)}{2(1-\nu^2)} \left(\alpha z + \gamma \frac{z^2}{2} + \beta \frac{z^3}{3}\right)\Bigg|_{z = \bar{z} - \frac{h}{2}}^{z = \bar{z} + h/2} + \frac{2E\lambda e_s}{\sqrt{3}} \left[\frac{(2\beta z + \gamma)\sqrt{\alpha + z\gamma + z^2\beta}}{4\beta} + \frac{4\alpha\beta - \gamma^2}{8\beta}\, \frac{1}{\sqrt{\beta}} \sinh^{-1}\left(\frac{2\beta z + \gamma}{\sqrt{4\alpha\beta - \gamma^2}}\right)\right]_{z = \bar{z} - \frac{h}{2}}^{z = \bar{z} + \frac{h}{2}} - \int_0^a \int_0^b \frac{E\lambda e_s^2}{2}\, h\, dx\, dy \tag{18}$$

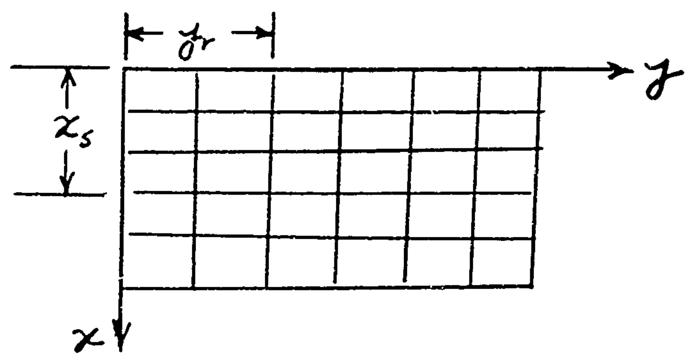
and a similar expression for V_2

Where $\alpha = \frac{1}{4}\left(\frac{\partial w}{\partial x}\right)^4 + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\left(\frac{\partial w}{\partial y}\right)^2 + \frac{1}{4}\left(\frac{\partial w}{\partial y}\right)^2$ (19)

$$\beta = \left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial^2 w}{\partial y^2}\right) + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2$$

$$\gamma = -\left(\frac{\partial w}{\partial x}\right)^2\left(\frac{\partial^2 w}{\partial x^2}\right) - \left(\frac{\partial w}{\partial y}\right)^2\left(\frac{\partial^2 w}{\partial y^2}\right) - \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\left(\frac{\partial^2 w}{\partial y^2}\right) - \frac{1}{2}\left(\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial w}{\partial y}\right)^2 - \frac{\partial^2 w}{\partial x \partial y}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}$$

For the stiffeners we use an analysis similar to that employed in Ref. 5



For a stiffener located at $y = y_r$

$$V_r = \int_0^a \int_{A_r} \left(\frac{E}{2}(1-\lambda)\epsilon_x^2 + E\lambda e_s e_x\right) dA_r\, dx - \int_0^a \int_{A_r} \frac{E\lambda e_s^2}{2} dA_r\, dx \tag{20}$$

where A_r indicates integration over the cross sectional area of the r^{th} stiffener

For a stiffener located at $x = x_s$

$$V_s = \int_0^b \int_{A_s} \left(\frac{E}{2}(1-\lambda)\epsilon_y^2 + E\lambda e_s \epsilon_y\right) dA_s\, dy - \int_0^b \int_{A_s} \frac{E\lambda e_s^2}{2} dA_s\, dy \tag{21}$$

Where A_s indicates integration over the cross sectional area of the s^{th} stiffener

Substituting the expressions for ϵ_x, ϵ_y

$$V_r = \int_0^a \int_{A_r} \left\{ \frac{E}{2}(1-\lambda) \left[\frac{1}{4}\left(\frac{\partial w}{\partial x}\right)^4 - \left(\frac{\partial w}{\partial x}\right)^2 z \frac{\partial^2 w}{\partial x^2} + z^2 \left(\frac{\partial^2 w}{\partial x^2}\right)^2 \right] + E\lambda e_s \left[\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - z \frac{\partial^2 w}{\partial x^2} \right] \right\} dA_r\, dx - \int_0^a \int_{A_r} \frac{E\lambda e_s^2}{2} dA_r\, dx \qquad (22)$$

Now $\int_{A_r} z^2 dA_r = I_{A_r}$ the moment of inertia of the stiffener about its own neutral axis

and $\int_{A_r} z\, dA_r = z_c A_r$ where z_c is the distance from the neutral plane of the stiffener to the centroid of the stiffener (equals zero)

so
$$V_r = \int_0^a \left\{ \frac{E}{2}(1-\lambda) A_r \frac{1}{4}\left(\frac{\partial w}{\partial x}\right)^4 + \frac{E}{2}(1-\lambda) I_{A_r} \left(\frac{\partial^2 w}{\partial x^2}\right)^2 + E\lambda e_s A_r \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \right\} dx - \frac{E\lambda e_s^2}{2} A_r a \qquad (23)$$

The terms containing A_r are the contribution from stretching and the one containing I_{A_r} is the contribution of bending.

similarly
$$V_s = \int_0^b \left\{ \frac{E}{2}(1-\lambda) A_s \frac{1}{4}\left(\frac{\partial w}{\partial y}\right)^4 + \frac{E}{2}(1-\lambda) I_{A_s} \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + E\lambda e_s A_s \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \right\} dy - \frac{E\lambda e_s^2}{2} A_s b \qquad (24)$$

III. Special Case of Uniform Plate

If we consider only the uniform plate and consider that the plastic deformation is large enough so that work done by membrane action is of primary significance then (neglecting the $\frac{E\lambda e_s^2}{2} \ell a b$ term for the time being)

$$V = \int_0^a \int_0^b \left\{ \frac{2}{3} E(1-\lambda) \ell \alpha + \frac{2\lambda E e_s}{\sqrt{3}} \ell \sqrt{\alpha} \right\} dx\, dy \qquad (25)$$

Now for a perfectly plastic material $\lambda = 1$ and

$$V = \int_0^a \int_0^b \frac{2\lambda E e_s}{\sqrt{3}} h \left[\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \right] dx\,dy \tag{26}$$

Now assume a deformation pattern

$$w = w_0 \sin\frac{\pi x}{a} \sin\frac{\pi y}{b} \tag{27}$$

then

$$V = \frac{2 E e_s h}{\sqrt{3}} w_0^2 \int_0^a \int_0^b \left[\frac{1}{2}\frac{\pi^2}{a^2}\cos^2\frac{\pi x}{a}\sin^2\frac{\pi y}{b} + \frac{1}{2}\frac{\pi^2}{b^2}\sin^2\frac{\pi x}{a}\cos^2\frac{\pi y}{b} \right] dx\,dy \tag{28}$$

If the impulse applied to the plate is uniformly distributed then the impulse per unit area can be written as

$$\bar{I} = I_0 \rho h = \sqrt{V \frac{2\rho h}{ab}} = w_0 \sqrt{\frac{\bar{K}\, 2\rho h\, \psi(a,b)}{ab}} \tag{29}$$

where

$$\psi(a,b) = \int_0^a \int_0^b \left[\frac{1}{2}\frac{\pi^2}{a^2}\cos^2\frac{\pi x}{a}\sin^2\frac{\pi y}{b} + \frac{1}{2}\frac{\pi^2}{b^2}\sin^2\frac{\pi x}{a}\cos^2\frac{\pi y}{b} \right] dx\,dy \tag{30}$$

$$\bar{K} = \frac{2 E e_s h}{\sqrt{3}}$$

It is found that (31)

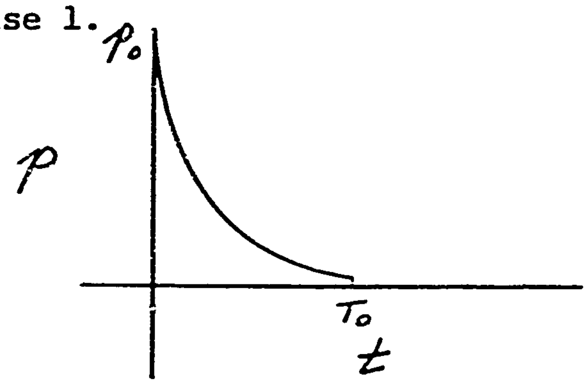
$$w_0 = \frac{\bar{I}}{\sqrt{\frac{\sigma_s h}{\sqrt{3}} \rho h \pi^2 \left[\frac{1}{a^2} + \frac{1}{b^2}\right]}}$$

σ_s = yield stress in pure tension

It is assumed that the plastic work goes in deforming the plate with permanent deformation w_0 and that the plastic deformations are very large compared to the elastic deformations.

Consider the two differently shaped pulses applied to the plate

Case 1.



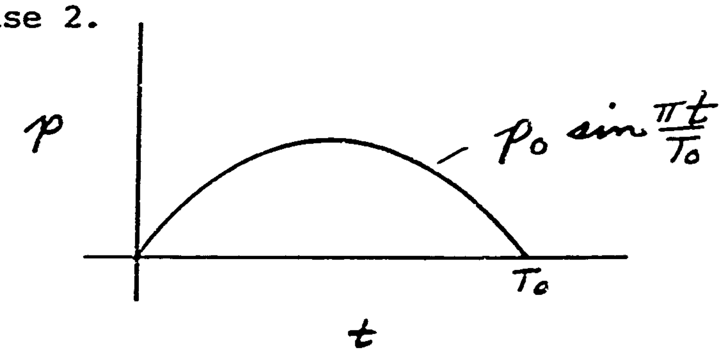
$$p = p_0 e^{-\beta t}$$

$$\bar{I} = \int_0^{T_0} p\,dt = \frac{p_0}{\beta}\left[1 - e^{-\beta T_0}\right]$$

Since $e^{-\beta T_0}$ is usually small compared to 1

$$\bar{I} \simeq \frac{p_0}{\beta}$$

Case 2.



$$\bar{I} = \int_0^{T_0} p_0 \sin \frac{\pi t}{T_0}\,dt = \frac{2 p_0 T_0}{\pi}$$

Case 1 is typical of an explosion pulse or a severe slamming pulse whereas case 2 would be a pulse which has a definite time of rise. According to this theory the damage will be dependent on $\bar{I}$. The degree of compressibility in the fluid for slamming will be felt in the value of p_0 and β.

To answer the question whether compressibility is important in slamming, approximate values of p_0 and β should be evaluated from a theory considering compressibility and a theory neglecting it. It is seen that damage from type 1 pulses could equal damage from type 2 pulses. The relative magnitudes of p_0, β, T_0 are the important factors.

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