

600566

D1-82-0342

19-P-80.50

**BOEING** SCIENTIFIC  
RESEARCH  
LABORATORIES

**Some Problems Involving Circular  
and Spherical Targets**

**DDC**  
**JUN 3 1964**  
**TISIA B**

**George Marsaglia**

**Mathematics Research**

**April 1964**

D1-82-0342

SOME PROBLEMS INVOLVING CIRCULAR  
AND SPHERICAL TARGETS

by  
George Marsaglia

Mathematical Note No. 343

Mathematics Research Laboratory  
BOEING SCIENTIFIC RESEARCH LABORATORIES

April 1964

12

### Summary

↘ This article is concerned with some problems which occur in certain tactical considerations: how should one place  $k$  circles (spheres) in the plane (3-space) so that their union has the greatest standard normal probability measure, that is, so as to maximize the probability that a random normal point will fall in one or more of the circles [spheres]. For  $k > 3$  the problem seems hopeless, (except for certain special situations); the case for  $k = 3$  is still unresolved and is being worked on by a number of investigators, and the case for  $k = 2$  is solved completely in this paper. The results for  $k = 2$  have some practical value when applied to actual problems arising in tactical considerations, and some theoretical value, as a method of attacking the problem for  $k \geq 3$ .

( )

$k \geq 3$ .

h

## Preface

This paper summarizes some results obtained in the summer of 1961. Most of the work is mine, although the problem arose, and several good suggestions came, from discussions among a group of us (Harold Ruben, Otto Hanš, Lajos Takacs, and M. D. MacLaren) working at the Boeing Scientific Research Laboratories during that summer. Since that time, the problem has come up in various forms; most recently a special case of the problem was discussed by D. C. Gilliland, "A Note on the Maximization of a Non-Central Chi-Square Probability", Ann. Math. Stat. 35, 441 (March 1964).

G. M.

## 1. Introduction

If you were shooting a rifle at two balloons 100 yards away, where would you aim in order to maximize the probability of hitting at least one? If you, and the rifle, were very accurate, you might aim at the center of one balloon, while if you were a lousy shot, it might be best to aim at the center of the pair. This problem is a special case of a class of problems we will consider in this note. To be precise, this is the basic problem: How should one place two circles (spheres) in the plane (3-space) so as to maximize the standard normal probability measure of the union? Both have radius  $r$ . We consider three cases:

Case I.    Both circles free to move.

Case II.   One circle fixed; one free to move.

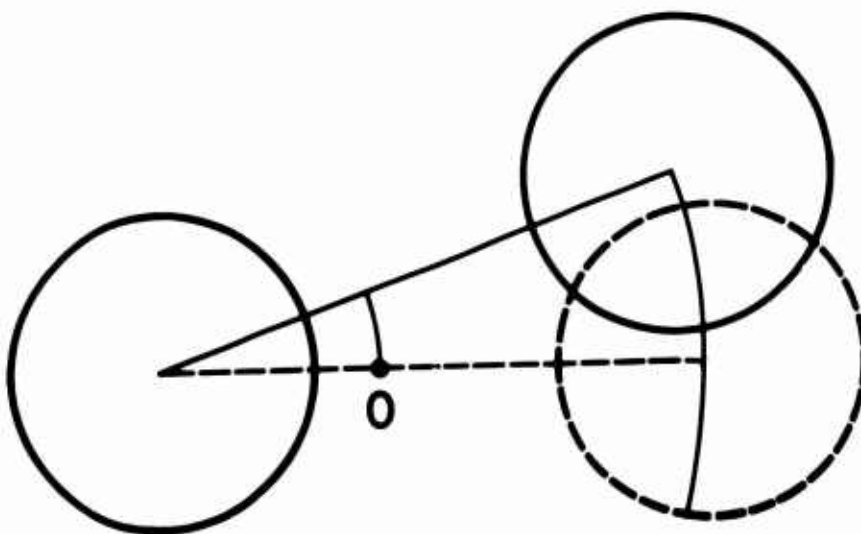
Case III.   Both circles free to move, but the distance between centers,  $2b$ , is fixed.

Such problems arise in tactical considerations; for example, Case I might be of interest to an anarchist who wishes to place two bombs at a reception where the location of the dignitary (target) has a circular normal distribution, while Case III may be viewed as the problem of how to aim at a pair of circular targets, as in the balloon problem, above. These problems have been described as cookie cutter problems - given a sheet of cookie dough with thickness proportional to the normal density, how should one punch out two pieces with a circular cookie cutter so as to get the greatest amount of dough?

The solution to all three of these cases is given in the summary in Section 5. Sections 2, 3, and 4 are devoted to finding solutions to each of the three cases.

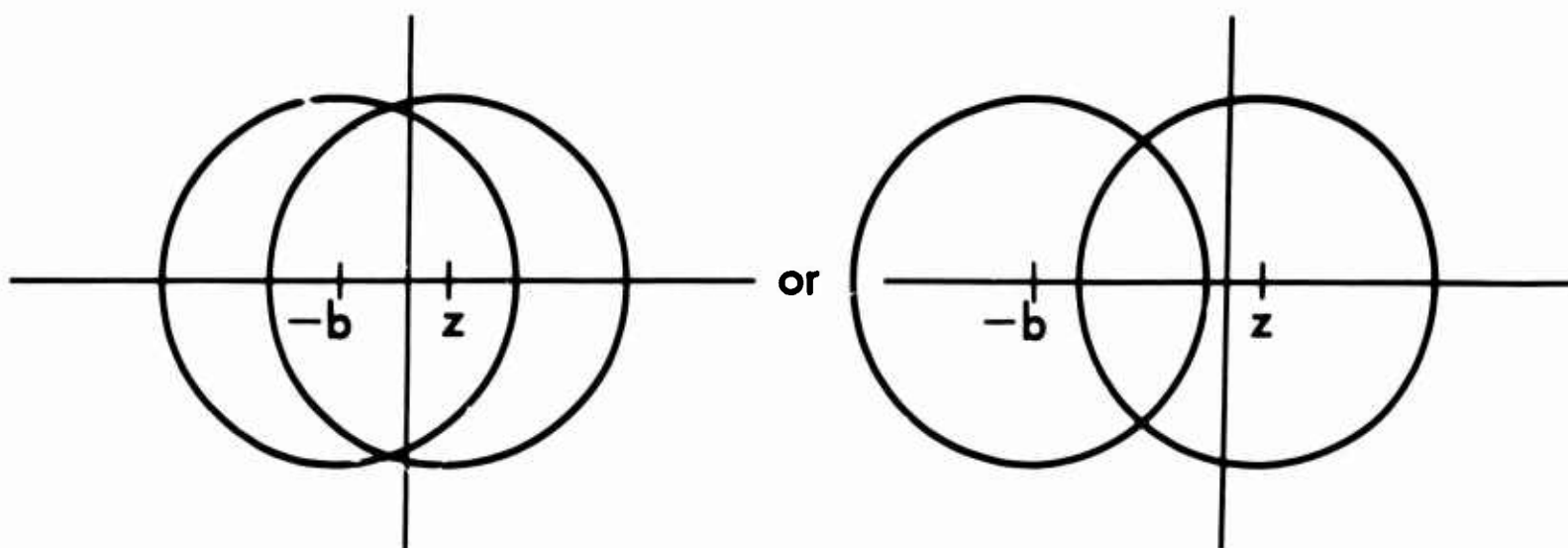
One may choose from several lines of attack for this problem - take advantage of the convexity of the normal density near the origin, for example, or perhaps use special properties of the non-central chi distributions, such as their representation as a mixture of chi-squares. It turns out, however, that a straightforward attack along elementary lines gives a unified approach to each of the three cases and provides fairly easy solutions. The solutions are general, and apply to the placement of  $n$ -spheres in  $n$ -space, although the chance of an application for  $n > 3$  seems remote.

We begin by noting that if two  $n$ -spheres are optimally placed, i.e., so that the standard normal measure of their union is a maximum, in any one of the three cases above, then the origin must lie on the line joining the centers. For example, in this diagram the position in bold line cannot be optimum, for if we rotate about the center of one circle until the origin,  $O$ , lies on the line joining the centers, we keep the measure of the left circle fixed and increase the measure of the right.



## 2. Case II. One Sphere Fixed

We consider Case II before Case I, as we shall use it to solve Case I. Suppose one  $n$ -sphere of radius  $r$  has center at  $(-b, 0, \dots, 0)$ . We want to place another  $n$ -sphere of radius  $r$  so as to maximize the standard normal measure of the two. Clearly if  $b > 2r$ , the best choice for the movable sphere is with center at the origin. Otherwise, if  $b < 2r$ , let the center of the movable sphere be at  $(z, 0, \dots, 0)$ , as pictured for  $n = 2$ :



Let  $\varphi$  be the normal density,  $\varphi(x) = (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}x^2}$ , and let  $F_n$  be the chi-square- $n$  distribution:

$$(1) \quad F_n(a) = 2^{-n/2} \Gamma(n/2)^{-1} \int_0^a e^{-x/2} x^{(n-2)/2} dx = k \int_0^a e^{-x/2} x^{(n-2)/2} dx.$$

Then the measure of the union of the two spheres, say  $M_n(z)$ , is

$$M_n(z) = \int_{-b-r}^{\frac{1}{2}(z-b)} F_{n-1}[r^2 - (x+b)^2] \varphi(x) dx + \int_{\frac{1}{2}(z-b)}^{z+r} F_{n-1}[r^2 - (x-z)^2] \varphi(x) dx.$$

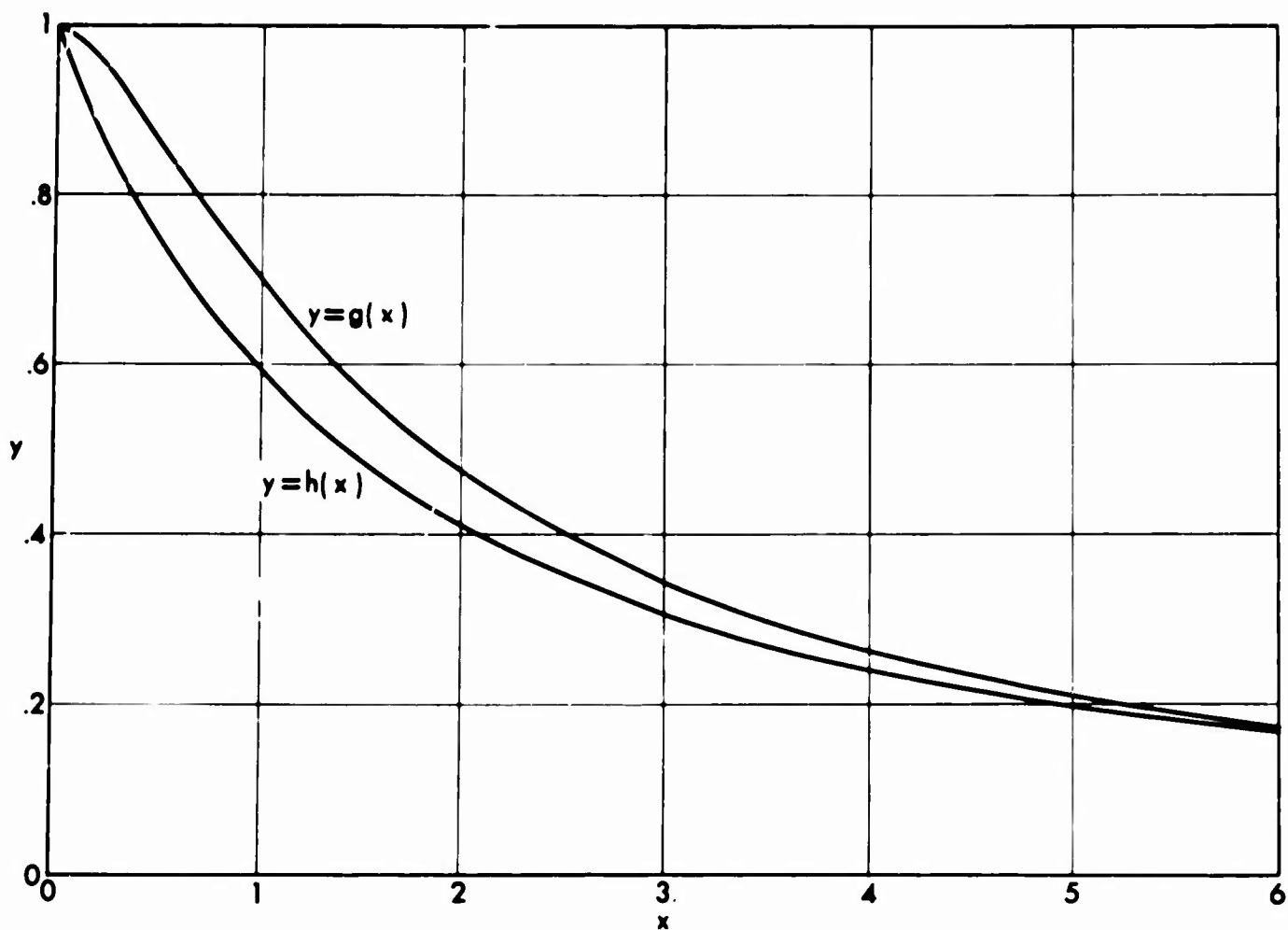


Figure 1. Functions  $g$  and  $h$  used to determine the placement of one circle or sphere so as to maximize the measure of its union with a fixed circle or sphere. (Case II).  $g$  and  $h$  are both asymptotic to  $1/x$ ; they are defined by

$$\int_{-1}^{g(x)} e^{xy} y(1-y^2)^{-1/2} dy = 0$$

$$\int_{-1}^{h(x)} y e^{xy} dy = 0 .$$



If we differentiate with respect to  $z$ , we get

$$M'_n(z) = 2 \int_{\frac{1}{2}(z-b)}^{\frac{z+r}{2}} F'_{n-1}[r^2 - (x-z)^2] (x-z) \phi(x) dx.$$

The condition  $M'_n(z) = 0$  becomes, after substituting the density of  $F_{n-1}$  from (1) and removing extraneous factors,

$$(2) \quad \int_{-1}^{(z+b)/2r} (1-y^2)^{(n-3)/2} y e^{rzy} dy = 0.$$

For the case  $n = 2$ , let  $g(x)$  be defined by the condition

$$\int_{-1}^{g(x)} e^{xy} y (1-y^2)^{-\frac{1}{2}} dy = 0.$$

Then the solution to (2) is the value of  $z$  for which  $g(rz) = (b+z)/2r$ . Thus the critical value of  $z$  may be found from the intersection of the curve  $y = g(x)$  with the line  $y = (rb+x)/2r^2$ . If  $(rb+x_0)/2r^2 = g(x_0)$  then  $z = x_0/r$  solves (2) and is the optimum  $z$ .

The function  $g(x)$  is drawn in Figure 1. As an example of the use of  $g(x)$ , when  $r = .5$  and  $b = .4$ , the line  $y = .4 + 2x$ , drawn in Figure 1, intersects the curve  $y = g(x)$  at  $x_0 = .27$ , so that if the fixed circle of radius .5 has center at  $(-.4, 0)$ , then the movable circle of radius .5 which maximized the standard normal measure of the two has center at  $(.54, 0)$ .

When  $n = 3$ , we need

$$(3) \quad \int_{-1}^{(z+b)/2r} y e^{zry} dy = 0.$$

We may integrate directly to get the condition

$$e^{(z^2+bz+2zr)/2} = \frac{2zr+2}{2-z^2-bz}.$$

However, if we let  $h(x)$  be defined by the condition

$$\int_{-1}^{h(x)} ye^{xy} dy = 0, \quad h(x) > 0,$$

then the solution to (3) may be found at the intersection of the line  $y = (rb+x)/2r^2$  with the curve  $y = h(x)$ . A graph of  $h$  is also in Figure 1.

Example. Let one sphere have center at  $(-1,0,0)$  and radius 2. We want to place another sphere with center at  $(z,0,0)$  so as to maximize the measure of the union. We require that

$$\frac{z+1}{4} = h(2z).$$

The intersection of the line  $y = (2+x)/8$  with  $y = g(x)$  is at  $x = 1.77$ , hence  $2z = 1.77$  and  $z = .885$  is the optimum value.

### 3. Case 1. Both Spheres Free to Move

Given two  $n$ -spheres of radius  $r$ , the problem is to place them in  $n$ -space so as to maximize the measure of their union. The argument for  $n = 2$  will apply to the general  $n$ . Let the centers of the optimally placed circles be at  $(-b, 0)$  and  $(a, 0)$ . We use the results from Case II. In that case we found that the best value of  $a$  for the fixed center at  $(-b, 0)$  required that  $g(ra) = (b+a)/2r$ . Since the left circle is also optimally placed with respect to the right, we must also have  $g(rb) = (b+a)/2r$ . Hence  $g(ra) = g(rb)$  and  $a = b$ , since  $g$  is monotone. We conclude that the optimal choice of centers is at  $(-a, 0)$  and  $(a, 0)$ , where  $g(ra) = a/r$ . If  $g(x_0) = x_0/r^2$ , then  $a = x_0/r$ .

For  $n = 3$ , the best choice of centers of two spheres is at  $(-a, 0, 0)$  and  $(a, 0, 0)$ , where if  $h(x_0) = x_0/r^2$  then  $a = x_0/r$ .

The functions  $g(x)$  and  $h(x)$  are both asymptotic to  $1/x$ . Hence if  $r$  is large, the best choice for two circles or spheres is when  $a = x_0/r$  and  $1/x_0 = x_0/r^2$ , that is,  $x_0 = r$  and  $a = 1$ .

Knowing that the best placement of two spheres is symmetric with respect to the origin, we can derive an expression for that common distance directly. If the centers of two overlapping spheres are at  $(-a, 0, \dots, 0)$  and  $(a, 0, \dots, 0)$  then the measure of the union is

$$2 \int_0^{a+r} F_{r-1} [r^2 - (x-a)^2] \varphi(x) dx.$$

Differentiating with respect to  $a$  and reducing leads to this condition

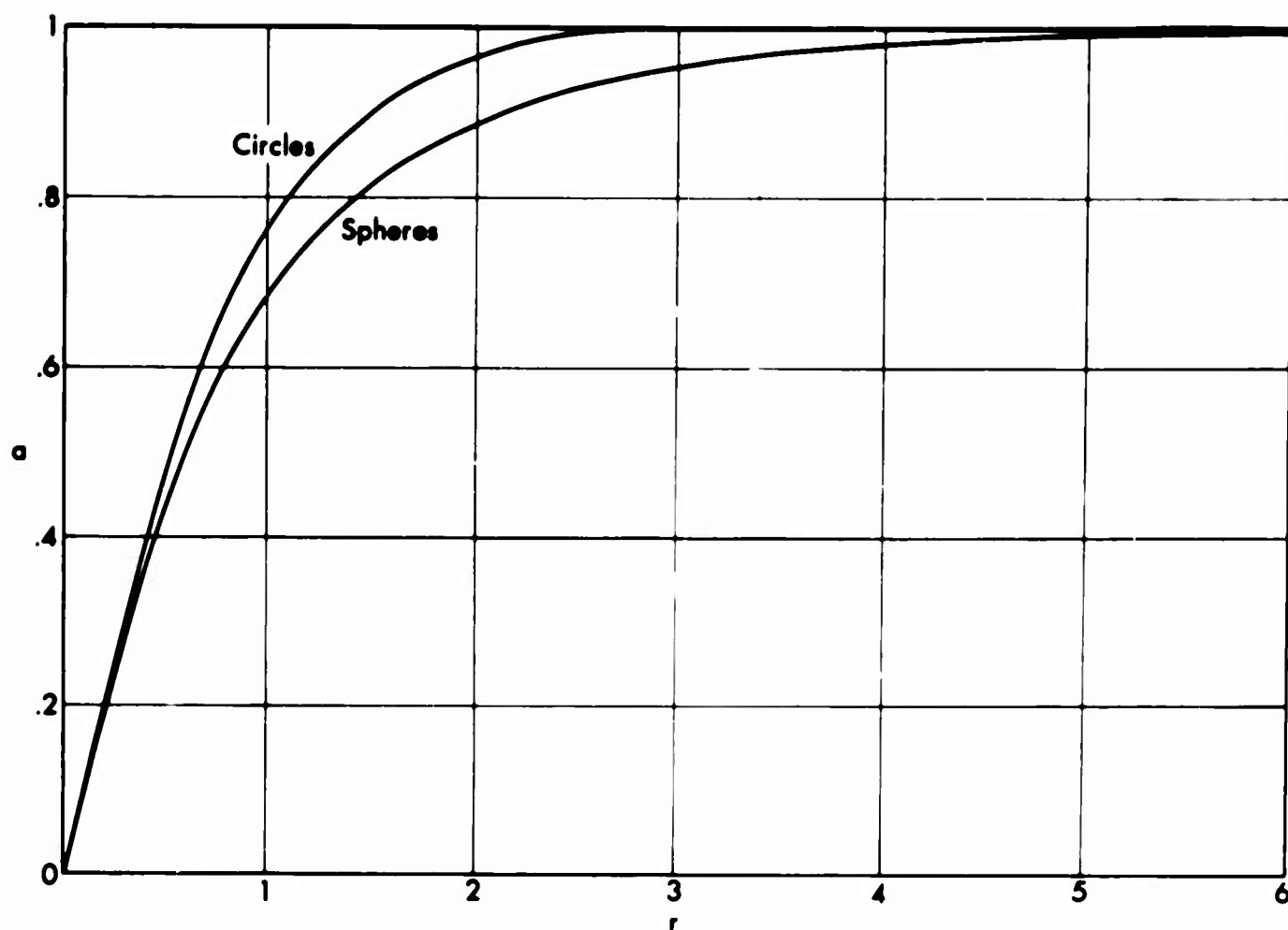


Figure 2. To maximize the normal measure of the union of two circles or spheres of radius  $r$ , place the centers symmetrically, centers distance  $a$  from the origin. These curves show  $a$  as a function of  $r$ . Relations between  $a$  and  $r$ :

$$\int_{-a/r}^1 (1-y^2)^{-1/2} y e^{-ary} dy = 0 \quad (\text{circles})$$

$$e^{-ar}(1+ar) = e^{a^2}(1-a^2) \quad (\text{spheres})$$

for an optimum value of  $a$ :

$$\int_{-a/r}^1 (1-y^2)^{(n-3)/2} y e^{-ary} dy = 0.$$

When  $n = 3$ , this leads to

$$e^{-ar}(1+ar) = e^{a^2}(1-a^2),$$

and when  $n = 2$ ,

$$\int_{-a/r}^1 (1-y^2)^{-1/2} y e^{-ary} dy = 0.$$

Graphs of solutions  $r, a$  of these two conditions are in Figure 2.

Thus Figure 2 shows the best placement of two circles or spheres as a function of the radius  $r$ .

#### 4. Case III. Distance between Centers Fixed

Let two  $n$ -spheres of radius  $r$  have centers at  $(z-b, 0, \dots, 0)$  and  $(z+b, 0, \dots, 0)$  so that the midpoint of the line joining their centers is at  $(z, 0, \dots, 0)$ . We assume first that  $b \leq r$ , so that the spheres overlap. We want to choose  $z$  so as to maximize  $M_n(z)$ , the standard normal measure of the union of the two spheres. We have

$$M_n(z) = \int_{z-b-r}^z F_{n-1}[r^2-(x-z+b)^2] \phi(x) dx + \int_z^{z+b+r} F_{n-1}[r^2-(x-z-b)^2] \phi(x) dx.$$

We let  $x = y$  in the first integral, and  $x = z + y$  in the second, to get

$$M_n(z) = \int_0^{b+r} F_{n-1}[r^2-(y-b)^2] [\phi(y-z) + \phi(y+z)] dy.$$

Differentiating, we have

$$(4) \quad M'_n(z) = \int_0^{b+r} F_{n-1}[r^2-(y-b)^2] [(y-z)\phi(y+z) - (y+z)\phi(y-z)] dy.$$

Since  $M'_n(0) = 0$ , we need  $M''(0)$  to see if  $M(0)$  is a local maximum or minimum. We have

$$(5) \quad M''_n(0) = 2 \int_0^{b+r} F_{n-1}[r^2-(y-b)^2] \phi(y) (y^2 - 1) dy.$$

If we integrate (4) and (5) by parts, we get useful variant forms:

$$M'_n(z) = 2 \int_0^{b+r} F'_{n-1}[r^2-(y-b)^2] (y-b) [\phi(y+z) - \phi(y-z)] dy$$

$$M''_n(0) = 4 \int_0^{b+r} F'_{n-1}[r^2-(y-b)^2] (b-y) y \phi(y) dy.$$

Using  $F'_n$  from (1), we have

$$M'_n(z) = k_1 \int_0^{b+r} e^{-\frac{1}{2}[r^2-(y-b)^2]} [r^2-(y-b)^2]^{(n-3)/2} (y-b) [\varphi(y+z) - \varphi(y-z)] dy$$

and

$$M''_n(0) = k_2 e^{-\frac{1}{2}(r^2-b^2)} \int_0^{b+r} e^{-by} (b-y)y [r^2-(y-b)^2]^{(n-3)/2} dy.$$

When the two spheres do not overlap ( $b > r$ ), the same formulas as those above apply, except that the lower limit of the integrals is  $b - r$  instead of zero:

$$M_n(z) = \int_{b-r}^{b+r} F_{n-1} [r^2-(y-b)^2] [\varphi(y-z) + \varphi(y+z)] dy$$

$$M'_n(z) = \int_{b-r}^{b+r} F_{n-1} [r^2-(y-b)^2] [(y-z)\varphi(y-z) - (y+z)\varphi(y+z)] dy$$

$$M'_n(z) = k_1 \int_{b-r}^{b+r} e^{-\frac{1}{2}[r^2-(y-b)^2]} [r^2-(y-b)^2]^{(n-3)/2} (y-b) [\varphi(y+z) - \varphi(y-z)] dy$$

$$M''_n(0) = 2 \int_{b-r}^{b+r} F_{n-1} [r^2-(y-b)^2] \varphi(y) (y^2-1) dy$$

$$M''_n(0) = k_2 e^{-\frac{1}{2}(r^2-b^2)} \int_{b-r}^{b+r} e^{-by} (b-y)y [r^2-(y-b)^2]^{(n-3)/2} dy.$$

Of particular interest are the cases  $n = 2, 3$ . When  $n = 3$ ,  $M''(0)$  becomes

$$M''_3(0) = k_2 e^{-\frac{1}{2}(r^2-b^2)} \int_0^{b+r} e^{-by} (b-y)y dy \quad \text{when } b \leq r$$

$$M''_3(0) = k_2 e^{-\frac{1}{2}(r^2-b^2)} \int_{b-r}^{b+r} e^{-by} (b-y)y dy \quad \text{when } b \geq r.$$

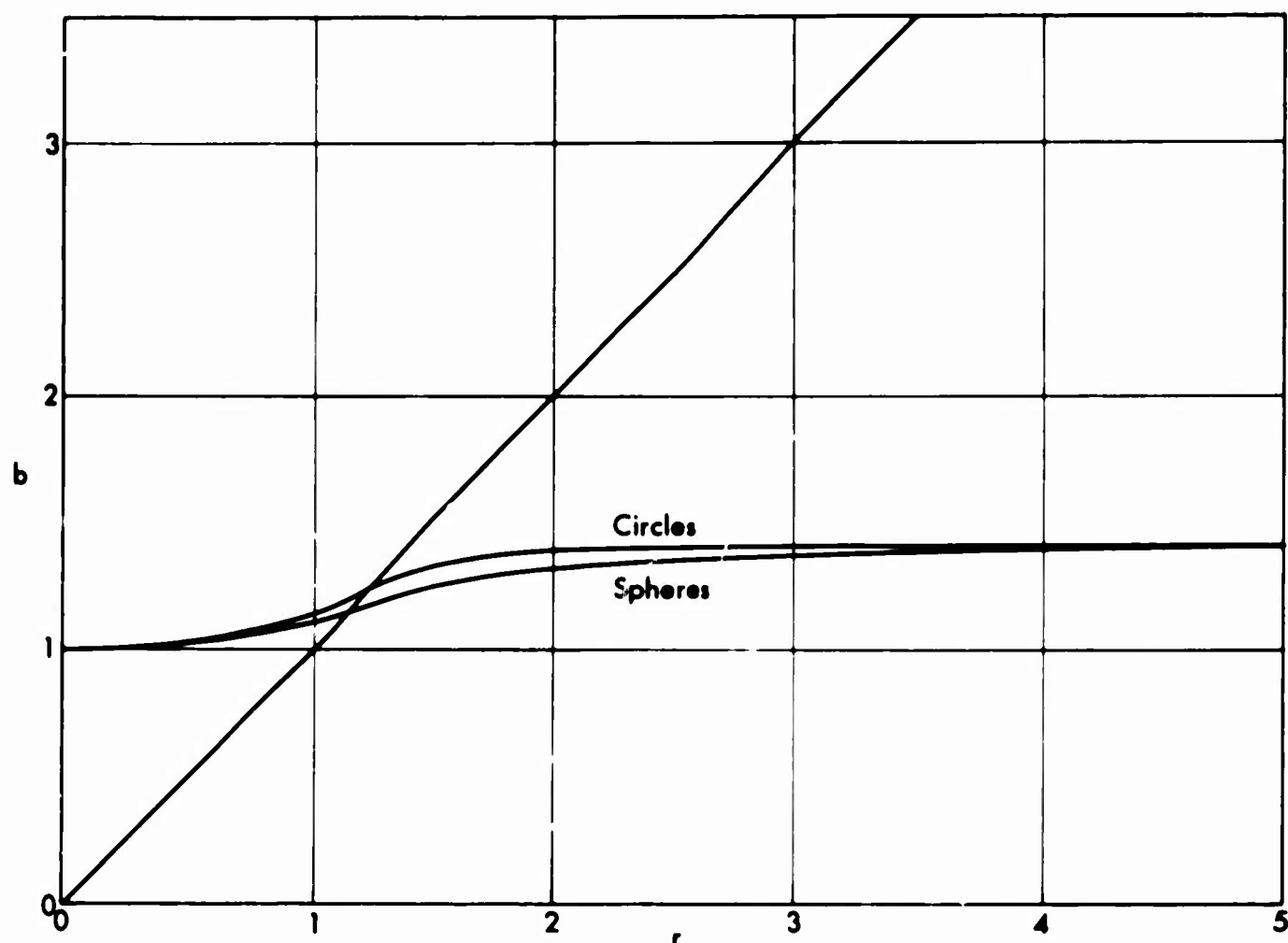


Figure 3. Curves for determining the optimum location of two circles or spheres with centers a fixed distance  $2b$  apart. If the point  $(r,b)$  lies below the appropriate curve, then place the centers at  $(-b,0,\dots,0)$  and at  $(b,0,\dots,0)$ . If  $(r,b)$  lies above the appropriate curve, place the centers at  $(z-b,0,\dots,0)$  and  $(z+b,0,\dots,0)$ , where  $z$  is the positive solution to the equation

$$\int_{-1}^{\min(1,b/r)} e^{-bry} (1-y^2)^{(n-3)/2} y \sinh[zy+b] dy = 0 .$$



Furthermore, the solutions to  $M'_3(z) = 0$  are those of

$$(6) \quad \int_0^{b+r} e^{-yb}(y-b)(e^{-yz}-e^{yz})dy = 0 \quad \text{when } b \leq r$$

and

$$(7) \quad \int_{b-r}^{b+r} e^{-yb}(y-b)(e^{-yz}-e^{yz})dy = 0 \quad \text{when } b \geq r$$

Figure 3 shows how the solutions to (6) and (7) divide the  $(r,b)$  plane into regions which either have  $z = 0$  for an optimum value or else require the solution of  $M'_n(z) = 0, z > 0$ . Figure 3 also gives corresponding results for  $n = 2$ .

## 5. Summary

In each of the three cases, an optimum placement of the spheres may be found with the centers on the  $X$ -axis.

Case I. To place two circles (spheres) of radius  $r$  in the plane (3-space) so as to maximize the standard normal probability measure of the union: place the centers at  $(-a, 0, \dots, 0)$  and  $(a, 0, \dots, 0)$  where  $a$  as a function of  $r$  is plotted in Figure 2. The relations between  $a$  and  $r$  are:

$$\int_{-a/r}^1 (1-y^2)^{-\frac{1}{2}} ye^{-ary} dy = 0 \quad (\text{circles})$$

$$e^{-ar}(1+ar) = e^{a^2} (1-a^2) \quad (\text{spheres})$$

$$\int_{-a/r}^1 (1-y^2)^{(n-3)/2} ye^{-ary} dy = 0 \quad (n\text{-spheres})$$

Case II. To place one sphere so as to maximize the standard normal measure of the union with another fixed sphere having center at  $(-b, 0, \dots, 0)$ : place the center at  $(z, 0, \dots, 0)$  where

$$\int_{-1}^{(z+b)/2r} (1-y^2)^{(n-3)/2} ye^{rzy} dy = 0.$$

A nomograph for finding  $z$  may be based on the function  $g_n(x)$ , defined by

$$\int_{-1}^{g_n(x)} e^{xy} y(1-y^2)^{(n-3)/2} dy = 0.$$

Then  $z = x_0/r$ , where  $x_0$  is at the intersection of the curve  $y = g_n(x)$  with the line  $y = (rb+x)/2r^2$ . The functions  $g_2$  and  $g_3$  are plotted as  $g$  and  $h$  in Figure 1.

Case III. To place two circles (spheres) which are a fixed distance  $2b$  apart so as to maximize the standard normal measure of their union: place the centers at  $(z-b, 0, \dots, 0)$  and  $(z+b, 0, \dots, 0)$  where  $z = 0$  if the point  $(r, b)$  lies below the curve in Figure 3, otherwise  $z$  may be found as the solution to

$$\int_{-1}^{\min(1, b/r)} e^{-bry} (1-y^2)^{(n-3)/2} y \sinh[z(ry+b)] dy = 0.$$