THE JOHNS HOPKINS UNIVERSITY
APPLIED PHYSICS LABORATORY
8621 Georgia Avenue, Silver Spring, Maryland

CM-1040

Operating under Contract NOw 62-0604-c with the Bureau of Naval Weapons Department of the Navy

Copy No. 93

58-p $1.50

IONOSPHERIC CONTRIBUTIONS TO THE DOPPLER SHIFT AT VHF FROM NEAR-EARTH SATELLITES

by W. H. Guier

July 1963

CM-1040
July 1963

Ionospheric Contributions to the Doppler Shift at VHF from Near-Earth Satellites

by W. H. Guier

THE JOHNS HOPKINS UNIVERSITY
APPLIED PHYSICS LABORATORY
SILVER SPRING, MARYLAND

The Johns Hopkins University
APPLIED PHYSICS LABORATORY
Silver Spring, Maryland

ABSTRACT

The ionospheric contributions to the non-relativistic Doppler shift at VHF from satellites above the ionosphere are considered to $O(1/f_s^3)$ where f_s is the satellite transmitter frequency and the vacuum Doppler shift is considered of ~~$O(f_s)$~~. It is shown that to $O(1/f_s^3)$, the phase of the electromagnetic radiation field from the satellite cannot be approximated by the usual Fermat integral of geometric optics. ~~Through a con~~sideration of the characteristics of an ideal doppler tracking receiver, boundary conditions are imposed on the solution to the wave equation containing the ionosphere electron contribution to the refractive index such that the solution to $O(1/f_s^3)$ in the phase is obtained. It is shown that to this order Fermat's principle can still be applied if the index of refraction is modified by terms containing gradients of the electron density. This generalized Fermat's principle is used to obtain the ionospheric contributions to the Doppler shift to $O(1/f_s^3)$ at VHF. Upper bounds for the various terms are estimated. It is shown that: 1) additional terms not given by the geometrical optics approximation are negligible so that geometrical optics theory is valid at VHF to $O(1/f_s^3)$, and 2) except for very disturbed conditions in the ionosphere, the use of the two frequency doppler data to eliminate the first order refraction contribution should yield negligible higher order refraction errors so long as the lower of the two frequencies is above 100 mc/s. ( )

The principal conclusions of this paper are reported in reference 10.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS AND TABLES

The Johns Hopkins University
APPLIED PHYSICS LABORATORY
Silver Spring, Maryland

IONOSPHERIC CONTRIBUTIONS TO THE DOPPLER SHIFT AT VHF FROM NEAR EARTH SATELLITES

INTRODUCTION

Considerable effort is being spent on developing methods for using accurate measurements of the radio Doppler shift for satellite tracking and geodesy.[1,2,3] The satellite frequencies, f_s, currently being employed are predominantly in the 50 to 500 mc/s region where the ionosphere contributes significant additions to the vacuum Doppler shift. Accurate doppler measurements have utilized the dispersive nature of the ionosphere to eliminate the first order ionospheric refraction contribution which is of the order of $1/f_s$. This is done by receiving two coherent frequencies transmitted from the satellite (usually a factor of two to six apart) and combining them experimentally to eliminate the first order ionospheric contribution on a data-point-by-data-point basis.[4] With the use of only two frequencies the re.ulting "vacuum" Doppler shift still contains all contributions that depend upon higher powers of $(1/f_s)$. In particular, the Faraday rotation contribution remains as well as contributions resulting from sharp gradients in the ionosphere electron density.

To this date, there is some experimental evidence that ionospheric contributions higher than $O(1/f_s)$ are significant for frequencies above 100 mc/s.[9] However, because of the low powe· levels transmitted from those satellites containing multiple, coherent, frequencies, the noise level in the experimental doppler data is sufficiently high that combining three or more satellite frequencies to accurately measure the higher order contributions have not been as definitive as one would like.

The principal objective of this paper is to consider theoretically the refracted Doppler shift to $O(1/f_s^3)$ in order to determine whether contributions higher than $O(1/f_s)$ should be significant.* The principal motivation for this study is to aid in the experimental justification that two coherent satellite frequencies are adequate to yield accurate Doppler shift data for tracking and geodesy.

The computation of the ionospheric contributions to $O(1/f_s^3)$ at VHF frequencies is made relatively easy because the transmitter frequencies are far above any significant resonance frequencies in the ionosphere. However, it is shown in Section II that to $O(1/f_s^3)$ the usual optics approximation (through using Fermat's principle) is not sufficient and consequently the more difficult task of finding the actual solution to Maxwell's equations to $O(1/f_s^3)$ in the phase is required.

Section I considers in some detail the expression for the vacuum Doppler shift when the ionosphere is assumed to be absent. In this section the characteristics of an idealized doppler tracking receiver are considered to a sufficient extent to derive special boundary conditions on the vacuum electromagnetic field created by the satellite transmitter. The resulting analysis yields an expression for the vacuum field at an orbitrary point in space which can itself be used as a boundary condition for the refracted electromagnetic field. In Section II, the solution to Maxwell's equations for an arbitrary ionospheric electron density is obtained to $O(1/f_s^3)$ in the phase for frequencies near the

*The vacuum Doppler shift is $O(f_s)$ so that contributions of $O(1/f_s^3)$ are of 4th order in $1/f_s$.

satellite transmitter frequency. The Fourier synthesis of the refracted frequency components is matched to the vacuum field expression in Section III. The two fields are matched in the region of the satellite, where it is assumed that the satellite is sufficiently above the ionosphere that the electron density is negligible. Section III also presents the detailed results of the expansion of the phase in powers of $1/f_s$ to $O(1/f_s^3)$, and the expression for the refracted Doppler shift that results. Section IV presents an estimate of the magnitude of the higher order terms in the expression for the Doppler shift where it is shown that except possibly for Faraday rotation and extremely sharp electron density gradients, the higher order terms are most likely negligible at VHF. Appendices I and II present some of the more tedious analysis.

I. The Vacuum Doppler Shift

Figure 1 indicates schematically an ideal doppler tracking receiver. It is assumed that the antenna is circularly polarized and tracks (in angle) the satellite to minimize fading due to ionospheric Faraday rotation. In Figure 1, the circularly polarized component of the satellite's transmitter field that matches the polarization of the antenna is labled $E_s(t)$. The output of the receiver has a phase represented by

$$\exp\left[-\,i\omega_{IF}t + i\,\frac{\omega_s}{c}\,\Phi_s(t)\right] \quad ,$$

which is fed to the tracking loop. The signal from the receiver is mixed with the signal from the VCO to produce a nearly constant angular frequency output at $(\omega_{IF} - \omega_F)$. This signal is passed through an IF-amplifier, $B_{IF}(\Delta\omega)$, centered at $(\omega_{IF} - \omega_F)$, whose bandwidth is $\Delta\omega$ and whose phase response is made as linear as possible. The phase of the signal after passing through the IF-amplifier is represented as

$$\exp\left[i\,\frac{\omega_s}{c}\,\Delta\Phi(t) - i(\omega_{IF} - \omega_F)t\right]$$

in Figure 1.

This signal is fed to a phase detector along with a reference signal whose angular frequency is $(\omega_{IF} - \omega_F)$ and the output of the phase detector is presented by

$$\frac{\omega_s}{c}\,\Delta\Phi(t) \quad .$$

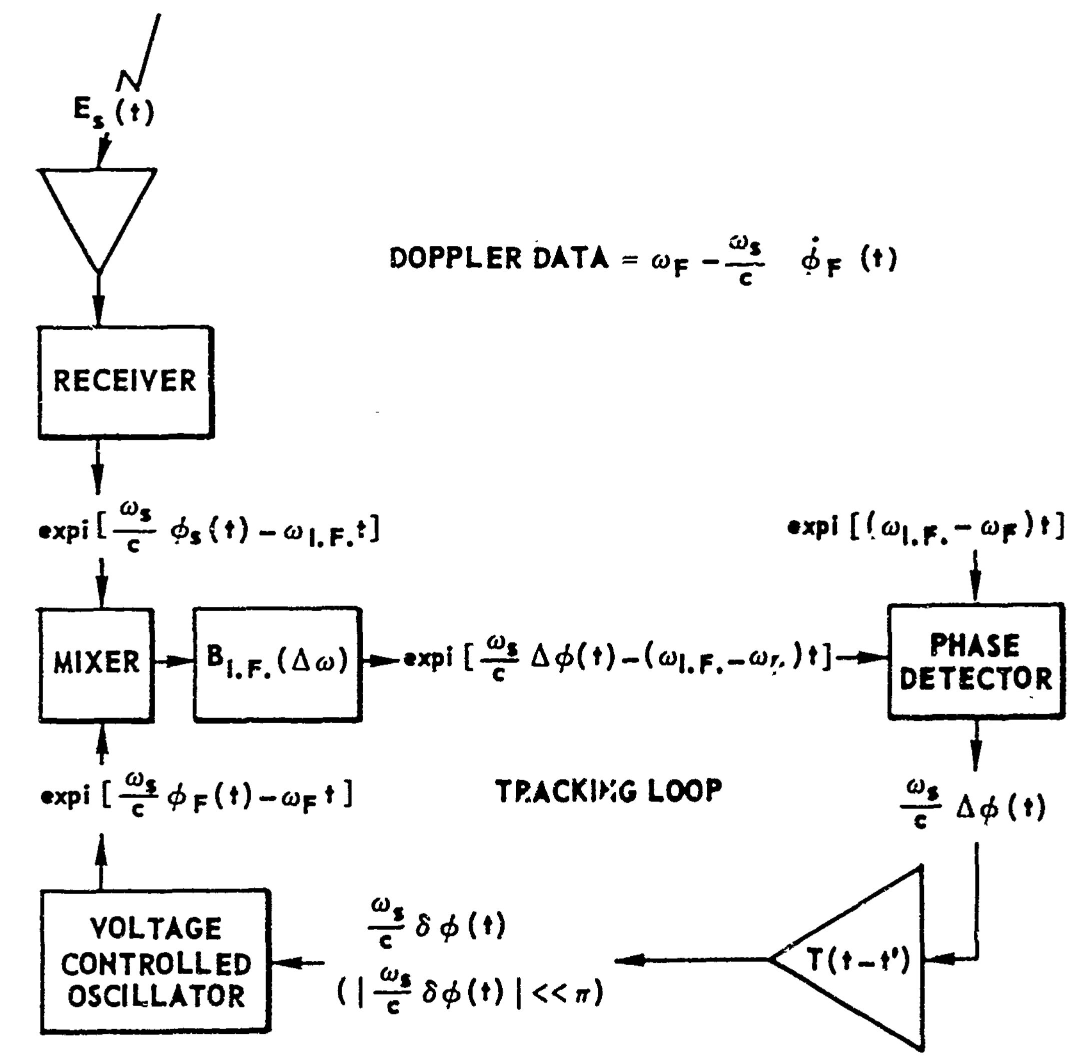

Fig. 1 IDEAL DOPPLER TRACKING RECEIVER

This phase difference is then fed through a servo amplifier with response $T(t - t')$ producing a phase error in Figure 1 by

$$\frac{\omega_s}{c} \delta\Phi(t).$$

This phase error alters the phase (time dependent frequency) of the VCO in such a way that the tracking loop produces as small a phase error $\frac{\omega_s}{c} \delta\Phi(t)$, as practicable In particular, the servo amplifier, $R(t - t')$, has enough prediction and long enough time constants that, for frequency changes typical of satellite Doppler shifts, the phase error remains very small once the loop has "locked on". The doppler data is then taken as the time derivative of the phase of the output signal from the VCO.

For present purposes, it is sufficient to neglect noise contributions to the phase and to consider the tracking loop as ideal, so that

$$\delta\Phi(t) = \Delta\Phi(t) = 0.$$

For this idealized case, the phase error, $\Delta\Phi(t)$, can be represented to a sufficient approximation by:

$$\Delta\Phi(t) = \Phi_s(t - t_D) - \Phi_F(t - t_D)$$

where t_D is the time delay in the tracking loop IF amplifier and is the order of the reciprocal of the IF bandwidth, $\Delta\omega$. Consequently, to a sufficient approximation:

$$\Delta\Phi(t + t_D) = \frac{\omega_s}{c} \Phi_F(t) - \frac{\omega_s}{c} \Phi_s(t) = 0 \tag{1A}$$

and the experimental Doppler shift is:

$$\Delta f(t) \equiv - \frac{f_s}{c} \quad \frac{d}{dt} \Phi_F(t) = - \frac{1}{2\pi} \frac{\omega_s}{c} \frac{d}{dt} \Phi_s(t) \quad . \qquad (1B)$$

Present day tracking loops have their IF amplifier bandwidth, $\Delta\omega$, between approximately 5 kc/s and 5 mc/s. Thus the time delay in the instrumentation prior to detection of the phase error is no greater than

$$t_D = O(\frac{1}{\Delta\omega}) \lesssim 10^{-4} \text{ sec.} \qquad (2)$$

Consequently, the doppler tracking receiver must 'sample' the field created by the satellite transmitter for times of the order of 100 microseconds around the current time, $\underline{t}$. In particular, components of the field radiated from the satellite during a time interval of about $2t_D$ centered around the time

$$t_o = t - t_D - \frac{R_s(t_o)}{v(t_o)} , \qquad (3)$$

where $R_s(t_o)$ is the slant range from station to satellite and $v(t_o)$ is the average speed of propagation from satellite to tracking station; and frequency components within about $1/t_D$ of the satellite frequency are the only contributions to the transmitter field that need to be considered. Since the satellite moves a negligible distance in time intervals of the order of t_D, and no significant ionosphere resonance occur near the satellite transmitter frequency, the calculation of the transverse radiation field is considerably simplified.

Consider now the field created by the satellite transmitter in the absence of the ionosphere and in a coordinate system at rest with respect to the tracking station. From the preceding discussion, only the appropriate circularly polarized component of the transverse radiation field need be considered. For the purposes of this paper, the non-relativistic expression suffices,* and

$$E_V(\vec{r},t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \, \frac{E_V(\omega,t_o)}{R(\vec{r},t_o)} \exp\left[-i\omega\left[t - \frac{R(\vec{r},t_o)}{c}\right]\right] \tag{4}$$

where

$E_V(\vec{r},t)$ = circularly polarized component of vacuum radiation field from satellite transmitter at the space point, $\vec{r}$, and at the current time, t, in a coordinate system at rest with respect to the tracking station,

$\vec{r}_s(t_o)$ = position of the satellite at the time, t_o ,

$R(\vec{r},t_o) = |\vec{r} - \vec{r}_s(t_o)|$ = slant range from the satellite to the space point ,$\vec{r}$,

and where

$$\left|t - t_o - t_D - \frac{R(\vec{r},t_o)}{c}\right|$$

is negligible.

*The relativistic field can be obtained by assuming a spherical outgoing wave in a reference system at rest with respect to the satellite, and then transforming to the coordinate system in which the station is at rest. In this transformation, it is sufficient to use as the relative velocity of the two coordinate systems:

$$\vec{v} = \frac{d}{dt_o} \vec{r}_s(t_o)$$

The additional terms of $O(v^2/c^2)$ do not contribute significantly to the ionospheric contributions. Consequently the additional complication of including relativistic corrections has been avoided.

To evaluate $E_V(\omega,t_o)$, consider the space point, $\vec{r}$, to be placed at the antenna of the tracking station, $\vec{r}_T$. From the preceding discussion, it is sufficient to choose $E_V(\omega,t_o)$ such that the field $E(\vec{r}_T, t - t_D)$ is very small for times far outside the region

$$t_o + \frac{R(t_o)}{c} - 2\,t_D \leqslant t - t_D \leqslant t_o + \frac{R(t_o)}{c}$$

and for angular frequencies far away from $\omega_s = 2\pi f_s$. For convenience, $E_V(\omega,t_o)$ is now chosen to be

$$E_V(\omega,t_o) = \exp\left[-\frac{t_D^{\,2}}{2}(\omega - \omega_s)^2 + i\omega t_o\right] \quad , \qquad (5A)$$

which yields

$$E(\vec{r}_T, t - t_D) = \frac{1}{R_s(t_o)} \; \frac{\exp\left[-\frac{1}{2t_D^{\,2}}\left[t - t_D - t_o - \frac{R_s(t_o)}{c}\right]\right]}{\sqrt{2\pi t_D^{\,2}}}$$

$$\exp\left[-\,i\omega_s\left[t - t_D - t_o - \frac{R_s(t_o)}{c}\right]\right], \qquad (5B)$$

where

$\vec{r}_T$ = position of antenna of tracking station,

$R_s(t_o)$ = slant range from tracking station to satellite.

The phase of the tracking filter is obtained by taking the expression for the phase in equation (5B) and substituting equation (3) with v = c into the argument of $R_s(t_o)$.

$$\frac{\omega_s}{c}\,\Phi_F(t - t_D) = \frac{\omega_s}{c}\,R_s\left(t - t_D - \frac{R_s(t_o)}{c}\right) .$$

Consequently, from equations (1), the non-relativistic expression for the vacuum Doppler shift becomes

$$\Delta f_V(t) = -\frac{1}{2\pi}\;\frac{\omega_s}{c}\;\frac{d}{dt}\,R_s\left(t - \frac{R_s(t)}{c}\right) + O\left(\frac{\dot{R}_s^{\,2}}{c^2}\right). \qquad (5C)$$

The results of this section, while not new in themselves, will be used in the following sections where the ionospheric contributions to the Doppler shift are evaluated to $O(1/\omega_s^3)$. In particular, the same idealized doppler tracking receiver will be assumed and the equation for the vacuum field, equation (4), together with equation (5A) for $E_V(\omega,t_o)$ at an arbitrary space point, will be used as a boundary condition for the refracted field when in the region of the satellite.

II. Refracted Field for Single Frequency Components at VHF

In this section the solution to Maxwell's equations is presented to $O(1/\omega^3)$ in the phase for a single frequency component, ω, near ω_s, which obeys the radiation condition. The principal result of this section is that the phase to $O(1/\omega^3)$ cannot be assumed to be given by Fermat's integral

$$\frac{\omega}{c} S(\vec{r}) = \frac{\omega}{c} \int_{\vec{r}_s(t_o)}^{\vec{r}} d\vec{r}\,' \cdot \hat{s}(\vec{r}\,')\; n(\vec{r}\,', t_o|\omega)$$

where the path is taken such that $S(\vec{r})$ is an extremum, but can be represented by

$$\frac{\omega}{c} \Phi_s(\vec{r}) = \frac{\omega}{c} \int_{\vec{r}_s(t_o)}^{\vec{r}} d\vec{r}\,' \cdot \hat{s}(\hat{r}\,')\; f(\vec{r}\,', t_o|\omega)$$

where $f(\vec{r}, t_o|\omega)$ is given by equation (10) of this section and the path is taken such that $\Phi_s(\vec{r})$ is an extremum. The appendices contribute to the proof of this.

Let

$\rho(\vec{r},t)$ = ionosphere electron density where it is assumed that it changes sufficiently slowly with time that its change during the transmission time from satellite to station is negligible,

$\vec{H}_E(\vec{r})$ = Earth's magnetic field which is assumed to have negligible time dependence,

e = charge of the electron in M.K.S. units,

m = Mass of the electron in M.K.S. units,

ϵ_o, μ_o = permittivity and permeability respectively of free space in M.K.S. units,

$E_s(\vec{r},t_o|\omega)$ = circularly polarized component of transverse field at the angular frequency, ω, emitted from the satellite at nearly the time t_o,

$\hat{s}(\vec{r})$ = unit vector in the direction of signal propagation,

$n(\vec{r},t_o|\omega)$ = equivalent index of refraction for the ionosphere at the frequency ω, time t_o , and space point $\vec{r}$.

The wave equation for each frequency component, $E_s(\vec{r},t_o|\omega)$, is

$$\vec{\nabla}^2 E_s(\vec{r},t_o|\omega) + \frac{\omega^2}{c^2} n^2(\vec{r},t_o|\omega)\, E_s(\vec{r},t_o|\omega) = 0 \quad , \tag{6A}$$

where

$$n^2(\vec{r},t_o|\omega) = 1 - \frac{(e^2/_{m\epsilon_o})\ \rho(\vec{r},t_o)}{\omega^2 \left[1 \pm (^{e\mu}o/_m)\ \dfrac{s(\vec{r})\ \cdot\ \vec{H}_E(\vec{r})}{\omega}\right]} \quad , \tag{6B}$$

and where the sign depends upon which circularly polarized component is considered. Let the solution for arbitrary $\vec{r}$ be represented by:

$$E_s(\vec{r},t_o|\omega) = \frac{E_o(\vec{r},t_o|\omega)}{R(\vec{r},t_o)} \quad \exp\left[i\ \frac{\omega}{c}\ \Phi_s(\vec{r},t_o|\omega)\right] \quad , \tag{7A}$$

where

$$R(\vec{r},t_o) = |\vec{r} - \vec{r}_s(t_o)| \quad .$$

In equation (7A) the functions $E_o(\vec{r},t_o|\omega)$ and $\Phi_s(\vec{r},t_o|\omega)$ are chosen to be real and:

$$\Phi_s(\vec{r},t_o|\omega) = \int_{\vec{r}_s(t_o)}^{\vec{r}} d\vec{r}\,' \cdot \hat{s}(\vec{r}\,')\, f(\vec{r}\,',t_o|\omega) \quad , \tag{7B}$$

where the path of the line integral is such that Φ_s is an extremum. So far, $f(\vec{r},t_o|\omega)$ is arbitrary.

Substituting equations (7) into equations (6), and then equating real and imaginary terms:

$$\vec{\nabla}^2 E_o - 2\hat{R} \cdot \nabla E_o + \frac{\omega^2}{c^2}(n^2 - \vec{\nabla}\Phi_s \cdot \vec{\nabla}\Phi_s)\, E_o = 0 \quad , \tag{8A}$$

$$2\vec{\nabla}\Phi_s \cdot \vec{\nabla}E_o - 2\frac{E_o}{R}\hat{R} \cdot \vec{\nabla}\Phi_s + E_o \vec{\nabla}^2 \Phi_s = 0 \quad , \tag{8B}$$

$$\vec{R}(\vec{r},t_o) = \vec{r} - \vec{r}_s(t_o) \quad , \quad \hat{R} = \frac{\vec{R}(\vec{r},t_o)}{R(\vec{r},t_o)} \quad . \tag{8C}$$

In Appendix I it is shown that

$$\vec{\nabla}\Phi(\vec{r},t_o|\omega) = \hat{s}(\vec{r})\, f(\vec{r},t_o|\omega) \quad . \tag{8D}$$

Thus, equations (8) can be written in the form:

$$\vec{\nabla}^2 E_o - 2\frac{\hat{R}\cdot\vec{\nabla}E_o}{R} + \frac{\omega^2}{c^2}(n^2 - f^2)E_o = 0 \quad , \tag{9A}$$

$$2f\hat{s}\cdot\vec{\nabla}E_o - 2\frac{E_o}{R}f\hat{R}\cdot\hat{s} + E_o f\vec{\nabla}\cdot\hat{s} + E_o\hat{s}\cdot\vec{\nabla}f = 0 \quad . \tag{9B}$$

In Appendix II, it is shown that

$$\hat{s}(r) = \frac{\hat{R}(\vec{r},t_o) + \vec{\xi}\,'(\vec{r},t_o|\omega)}{\sqrt{1 + \vec{\xi}\,'\cdot\vec{\xi}\,'}}$$

where

$$\vec{\xi}\,'\cdot\hat{R} = 0 \quad , \quad \vec{\xi}\,' = 0\left(\frac{1}{\omega^2}\right) \quad ,$$

and

$$n^2 - f^2 = 0\left(\frac{1}{\omega^4}\right) \quad .$$

Since

$$f = n + 0\left(\frac{1}{\omega^4}\right) = 1 + 0\left(\frac{1}{\omega^2}\right) \quad ,$$

then

$$|\vec{\nabla} f| = O(\frac{1}{\omega^2}) \quad .$$

Finally, since $E_o(\vec{r},t_o|\omega)$ should have a very slow dependence upon $\vec{R}$, assume that:

$$|\vec{\nabla} E_o| = O(\frac{1}{\omega^2}) \quad ,$$

$$|\vec{\nabla}^2 E_o| = O\left(\frac{|\vec{\nabla} E_o|}{R}\right) \quad .$$

Using the above relations, it can now be seen that:

$$\hat{s} = \hat{R} + \vec{\xi}\,' + O(\frac{1}{\omega^4}) \quad ,$$

$$\hat{s} \cdot \vec{\nabla} E_o = \hat{R} \cdot \vec{\nabla} E_o + O(\frac{1}{\omega^4}) \quad ,$$

$$\hat{R} \cdot \hat{s} = 1 + O(\frac{1}{\omega^4}) \quad ,$$

$$\vec{\nabla} \cdot \hat{s} = \frac{2}{R} + \vec{\nabla} \cdot \vec{\xi}\,' + O(\frac{1}{\omega^4}) \quad ,$$

$$\hat{s} \cdot \vec{\nabla} f = \hat{R} \cdot \vec{\nabla} f + O(\frac{1}{\omega^4}) \quad .$$

Consequently, equation (9B) reduces to:

$$2\hat{R} \cdot \vec{\nabla} E_o = - E_o [\hat{R} \cdot \vec{\nabla} f + \vec{\nabla} \cdot \vec{\xi}'] + O(\frac{1}{\omega^4}) \quad .$$

Substituting this equation into equation (9A) and neglecting terms of $O(\vec{\nabla}^2 E_o)$, the following differential equation results for $f(\vec{r},t_o|\omega)$.

$$f^2(\vec{r},t_o|\omega) = n^2(\vec{r},t_o|\omega) + \frac{c^2}{\omega^2} \frac{[\hat{R} \cdot \vec{\nabla} f + \vec{\nabla} \cdot \vec{\xi}']}{R} + O(\frac{1}{\omega^6}) \quad .$$

However, since,

$$\vec{\nabla} f = \vec{\nabla} n + O\left(\frac{1}{\omega^4}\right) \quad ,$$

the expression for $f(\vec{r},t_o|\omega)$ finally reduces to:

$$f(\vec{r},t_o|\omega) = n(\vec{r},t_o|\omega) + \frac{c^2}{2\omega^2} \frac{\hat{R} \cdot \vec{\nabla} n + \vec{\nabla} \cdot \vec{\xi}'}{R} + O(\frac{1}{\omega^6}) \quad . \qquad (10)$$

From equations (7) and (10), it can be seen that to $O(\frac{1}{\omega^4})$ the phase of the field is given through integrals of the Fermat type if the integrand contains the function, $f(\vec{r},t_o|\omega)$, instead of the index of refraction, $n(\vec{r},t_o|\omega)$. To $O(\frac{1}{\omega^4})$, the difference between $f(\vec{r},t_o|\omega)$ and the index of refraction is

$$\frac{c^2}{2\omega^2} \frac{\hat{R} \cdot \vec{\nabla} n + \vec{\nabla} \cdot \vec{\xi}'}{R} \quad .$$

The first term depends upon the gradient of the index of refraction along the geometric path. From Appendix II, the vector, $\vec{\xi}'(\vec{r},t_o|\omega)$, is the gradient of the deviation of the optical path from the geometric path taken in the direction of the geometric path. Therefore, the second term in the difference between f and n is also proportional to the spatial change of the index of refraction. Consequently, the 'corrected index of refraction' is proportional to the spatial change in the index of refraction and can become large if the electron density in the ionosphere is not a slowly varying function of position.*

Substituting equation (10) into equation (7B), evaluating the 'corrected optical path', and then performing the integral along this path,

$$\Phi_s(\vec{r},t_o|\omega) = \int_{\vec{r}_s(t_o)}^{\vec{r}} d\vec{r}' \cdot \hat{s}(\vec{r}')\, n(\vec{r}',t_o|\omega) + \frac{c^2}{2\omega^2} \int_{\vec{r}_s(t_o)}^{\vec{r}} d\vec{r}' \cdot \hat{s}(\vec{r}')\, \frac{R \cdot \vec{\nabla}' n + \vec{\nabla}' \cdot \vec{\xi}'}{R'} + O(\frac{1}{\omega^3})$$

However, it is shown in Appendix II that the optical path deviates from the geometrical path by an amount, $\vec{\xi}(\vec{r},t_o|\omega) = O(\frac{1}{\omega^2})$. Thus, to $O(\frac{1}{\omega^2})$ the geometrical path can be used in integrating the corrections to the index of

*It should be noted that equation (10) was derived assuming the satellite frequen is significantly above any ionospheric resonance frequency. This expression is not valid for frequencies near to or at any of the ionosphere resonance frequenc Equation (10) should not be used, for example, in the consideration of ionosonde data. See, for example, reference (6).

The first term depends upon the gradient of the index of refraction along the geometric path. From Appendix II, the vector, $\vec{\xi}'(\vec{r},t_o|\omega)$, is the gradient of the deviation of the optical path from the geometric path taken in the direction of the geometric path. Therefore, the second term in the difference between f and n is also proportional to the spatial change of the index of refraction. Consequently, the 'corrected index of refraction' is proportional to the spatial change in the index of refraction and can become large if the electron density in the ionosphere is not a slowly varying function of position.*

Substituting equation (10) into equation (7B), evaluating the 'corrected optical path', and then performing the integral along this path,

$$\Phi_s(\vec{r},t_o|\omega) = \int_{\vec{r}_s(t_o)}^{\vec{r}} d\vec{r}\,' \cdot \hat{s}(\vec{r}\,')\, n(\vec{r}\,',t_o|\omega) + \frac{c^2}{2\omega^2} \int_{\vec{r}_s(t_o)}^{\vec{r}} d\vec{r}\,' \cdot \hat{s}(\vec{r}\,')\, \frac{R \cdot \vec{\nabla}' n + \vec{\nabla}' \cdot \vec{\xi}'}{R'} + O(\frac{1}{\omega^6}).$$

However, it is shown in Appendix II that the optical path deviates from the geometrical path by an amount, $\vec{\xi}(\vec{r},t_o|\omega) = O(\frac{1}{\omega^2})$. Thus, to $O(\frac{1}{\omega^2})$ the geometrical path can be used in integrating the corrections to the index of

*It should be noted that equation (10) was derived assuming the satellite frequency is significantly above any ionospheric resonance frequency. This expression is not valid for frequencies near to or at any of the ionosphere resonance frequencies. Equation (10) should not be used, for example, in the consideration of ionosonde data. See, for example, reference (6).

refraction in equation (10), and the phase is given by:

$$\frac{\omega}{c}\,\Phi_s(\vec{r},t_o|\omega) = \frac{\omega}{c}\int_{\vec{r}_s(t_o)}^{\vec{r}} d\vec{r}\,' \cdot \hat{s}(\vec{r}\,')n(\vec{r}\,',t_o|\omega)$$

$$+ \frac{c}{2\omega}\int_{\vec{r}_s(t_o)}^{\vec{r}} dr' \cdot \hat{R}\;\frac{\hat{R}\cdot\vec{\nabla}'n(\vec{r}\,',t_o|\omega) + \vec{\nabla}'\cdot\vec{\xi}'(r',t_o|\omega)}{R(r',t_o)} + O\left(\frac{1}{\omega^5}\right), \tag{11}$$

where the usual optical path can now be used in the first integral in equation (11).

Equations (10) and (11) are the principal results of this section. If the usual form of Fermat's principle is used, the phase is given by

$$\frac{\omega}{c}\,S(\vec{r}) = \frac{\omega}{c}\int_{\vec{r}_s(t_o)}^{\vec{r}} dr' \cdot \hat{s}(\vec{r}\,')\;n(\vec{r}\,',t_o|\omega)$$

where the path, $\hat{s}(\vec{r})$, is taken such that this integral is an extremum. From equation (11), it can be seen that

$$\frac{\omega}{c}(\Phi_s - S) = \frac{c}{2\omega}\int_{\vec{r}_s(t_o)}^{\vec{r}} dr' \cdot \hat{R}(r', t_o)\, \frac{\hat{R} \cdot \vec{\nabla}' n + \vec{\nabla}' \cdot \vec{\xi}'}{R(v_i t_o)} + O\left(\frac{1}{\omega^4}\right) \quad ,$$

and the right hand side of this equation must be negligible before the usual optical approximation for the ionosphere can be used if results to $O(\frac{1}{\omega^3})$ are desired.

The Johns Hopkins University
APPLIED PHYSICS LABORATORY
Silver Spring, Maryland

III. Expression for the Refracted Doppler Shift

In this section the single frequency components are combined to yield the expression for the field. The evaluation of the resulting Fourier integral is performed in the following way.

1. Assume that the satellite is sufficiently above the ionosphere that in a region near the satellite the electron density is negligible,
2. match the expression for the refracted field to the vacuum field at space points close to the satellite,
3. assume that when the space point is placed at the tracking station antenna, no singularities are introduced into the amplitude of each frequency component for frequencies near the satellite frequency, and then
4. evaluate the integral over the frequency domain by the method of steepest descents.

The resulting expression for the phase is then expanded to $O(1/\omega_s{}^3)$ and the time derivative taken to yield the expression for the refracted Doppler shift to $O(1/\omega_s{}^4)$ of the vacuum Doppler shift itself. Appendix II contains a detailed calculation of the extremum path for the phase function, Φ_s , which is required for the final evaluation of the phase in powers of $1/\omega_s$.

From equation (7A), the expression for the field at an arbitrary space point is:

$$E_s(\vec{r},t_o|t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \; \frac{E_o(\vec{r};t_o|\omega)}{R(\vec{r},t_o)} \; \exp[-i\omega[t - t_o - \frac{1}{c}\,\Phi\,(\vec{r},t_o|\omega)]] \; . \tag{12}$$

Now assume that the satellite is sufficiently above the ionosphere that a region in the vicinity of the satellite can be found where the far field pattern of the radiation field is still essentially the vacuum field and the electron density is negligible. For space points $\vec{r}$, in this region the refraction index is unity and equation (12) reduces to:

$$E_s(\vec{r},t_o|t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega \, \frac{E_o(\vec{r},t_o|\omega)}{R(\vec{r},t_o)} \exp\left[-i\omega\left[t - t_o - \frac{R(\vec{r},t_o)}{c}\right]\right] \equiv E_v(\vec{r},t_o),$$

where

$$R(\vec{r},t_o) \ll R_s(t_o) = |\vec{r}_T - \vec{r}_s(t_o)| \quad .$$

From equations (4) and (5A), it can be seen that within this region,

$$E_o(\vec{r},t_o|\omega) = \exp\left[-\frac{t_D^2}{2}(\omega - \omega_s)^2\right] \quad ,$$

$$R(\vec{r},t_o) \ll R_s(t_o) \quad .$$

Let the field point now be placed at the antenna of the doppler tracking station, $\vec{r} = \vec{r}_T$, and assume that for frequencies in the complex plane near the (real) satellite frequency, ω_s , no singularities or branch points appear in $E_1(\vec{r}_T,t_o|\omega)$, where:

$$E_o(\vec{r}_T, t_o|\omega) \equiv E_1(\vec{r}_T, t_o|\omega) \quad \exp\left[- \frac{t_D^{\,2}}{2} (\omega - \omega_s)^2 \right] .$$

It is shown in Appendix II that no singularities or branch points appear in the phase for ω near ω_s , and therefore the integral in equation (12) for the radiation field can be evaluated by the method of steepest descents. Equation (12) then reduces to:

$$E_s(\vec{r}_T, t_o|t) = \frac{E_1(\vec{r}_T, t_o|\omega_s)}{R_s(t_o)} \cdot \frac{\exp\left[\frac{1}{2t_D^{\,2}} \left[t - t_o - \frac{1}{c} \frac{\partial}{\partial\omega_s} (\omega_s \Phi_s) \right]^2 \right]}{\sqrt{2\pi\, t_D^{\,2}}} .$$

$$\exp\left\{ i\omega_s [t - t_o - \frac{1}{c} \Phi_s(\vec{r}_T, t_o|\omega_s)] \right\} . \tag{13}$$

The time interval, t_D, is the order of 10^{-4} sec. and in this time the satellite moves a negligible amount. Thus, in the phase, the time, t_o, can be identified with the time, t, by the expression

$$t_o = t - \frac{1}{c} \frac{\partial}{\partial\omega_s} [\omega_s \Phi_s(\vec{r}_T, t|\omega_s)] \quad . \tag{14A}$$

The Doppler shift is proportional to the time derivative of the phase, and thus the refracted Doppler shift is represented by:

$$\Delta f(t) = \frac{1}{2\pi} \frac{\omega_s}{c} \frac{d}{dt} \Phi_s(r_T, t_o | \omega_s) \quad . \tag{14B}$$

It now remains to evaluate the function Φ_s in detail.

In Appendix II, it is shown that the optical path deviates from the geometrical path by an amount of $O(1/\omega_s^2)$. Thus, from equation (6B),

$$n(\vec{r}, t_o | \omega_s) = 1 - \frac{1}{\omega_s^2} \frac{e^2}{2m\epsilon_o} \rho(\vec{r}, t_o) \pm \frac{1}{\omega_s^3} \frac{e^3}{2m^2} \frac{\mu_o}{\epsilon_o} \rho(\vec{r}, t_o) \hat{R}_s(t_o) \cdot \vec{H}_E(\vec{r})$$

$$- \frac{1}{\omega_s^4} \frac{e^4}{8m^2\epsilon_o^2} \rho^2(\vec{r}, t_o) + O\left(\frac{1}{\omega_s^5}\right) \quad ,$$

$$\hat{R}_s(t_o) = \frac{\vec{r}_T - \vec{r}_s(t_o)}{|\vec{r}_T - \vec{r}_s(t_o)|} \quad .$$

Substituting this expression into equation (10), and defining:

$$a_2(\vec{r}.t_o) = - \frac{e^2}{2m\epsilon_o} \rho(\vec{r}, t_o) \quad , \tag{15A}$$

$$a_3(\vec{r}, t_o) = \pm \frac{e^3}{2m^2} \frac{\mu_o}{\epsilon_o} \rho(\vec{r}, t_o) \hat{R}_s(t_o) \cdot \vec{H}_E(\vec{r}) \quad , \tag{15B}$$

$$a_4(\vec{r},t_o) = -\frac{e^4}{8m^2\epsilon_o^2}\rho^2(\vec{r},t_o) + \frac{c^2e^2}{4m\epsilon_o}\frac{\hat{R}_s(t_o)\cdot\vec{\nabla}\rho(\vec{r},t_o)}{|\vec{r}-r_s(t_o)|} \tag{15C}$$

$$+\frac{c^2\omega_s^2}{2}\frac{\vec{\nabla}\cdot\vec{\xi}'(\vec{r},t_o|\omega_s)}{|\vec{r}-\vec{r}_s(t_o)|} \quad ,$$

the expression for $f(\vec{r},t_o|\omega_s)$ becomes:

$$f(\vec{r},t_o|\omega_s) = 1 + \sum_{k=2}^{4}\frac{a_k(\vec{r},t_o)}{\omega_s^k} + 0\left(\frac{1}{\omega_s^5}\right) \quad . \tag{16}$$

Also in Appendix II, the integral of this function over the optical path is given to $0(1/\omega_s^4)$ by:

$$\Phi_s(\vec{r}_T,t_o|\omega_s) = \int_{\vec{r}_s(t_o)}^{\vec{r}_T} d\vec{r}'\cdot\hat{R}_s(t_o)\,[f(\vec{r}',t_o|\omega_s) - \frac{1}{2}|\vec{\xi}'(\vec{r}',t_o|\omega_s)|^2] + 0\left(\frac{1}{\omega_s^5}\right) \quad , \tag{17A}$$

where $\hat{R}_s(t_o)\cdot\vec{\xi}'(\vec{r},t_o|\omega_s) = 0$, and each of the two components of $\vec{\xi} = (\xi_1,\xi_2)$ are given by:

$$\omega_s{}^2 \xi_k'(\vec{r},t_o|\omega_s = \int_{\vec{r}_s(t_o)}^{\vec{r}_T} d\vec{r}' \cdot \hat{R}_s(t_o) \left[\frac{\partial a_2(\vec{r}',t_o)}{\partial \xi_k} \right]_{\vec{\xi}=0}$$

$$- \frac{1}{R_s(t_o)} \int_{\vec{r}_s(t_o)}^{\vec{r}} dr' \cdot \hat{R}_s(t_o) \int_{\vec{r}_s(t_o)}^{\vec{r}'} d\vec{r}'' \cdot \hat{R}(t_o) \left[\frac{\partial a_2(\vec{r}'',t_o)}{\partial \xi_k} \right]_{\vec{\xi}=0} , \tag{17B}$$

$k = 1,2$.

The symbol, $\int_{\vec{r}_s(t_o)}^{\vec{r}} d\vec{r}' \cdot \hat{R}_s(t_o)$ denotes taking the integral along the geometric path. The derivatives

$$\left[\frac{\partial a_2(\vec{r}',t_o)}{\partial \xi_k} \right]_{\vec{\xi}=0} , \quad k = 1,2,$$

denote taking the gradient of $a_2(\vec{r},t_o)$ in two orthogonal directions, both orthogonal to the direction of the geometric path, and evaluated on the geometric path.

Finally, substituting equation (16) into equation (17A) and letting

$$\Phi_2(R_s,t_o) = \frac{1}{(2\pi)^2} \int_{\vec{r}_s(t_o)}^{\vec{r}_T} d\vec{r}\,' \cdot \hat{R}_s(t_o)\, a_2(\vec{r}\,',t_o) \quad , \tag{18A}$$

$$\Phi_3(R_s,t_o) = \frac{1}{(2\pi)^3} \int_{\vec{r}_s(t_o)}^{\vec{r}_T} d\vec{r}\,' \cdot \hat{R}(t_o)\, a_3(\vec{r}\,',t_o) \quad , \tag{18B}$$

$$\Phi_4(R_s,t_o) = \frac{1}{(2\pi)^4} \int_{\vec{r}_s(t_o)}^{\vec{r}_T} d\vec{r}\,' \cdot \hat{R}(t_o) \left[a_4(\vec{r}\,',t_o) - \frac{\omega_s^2}{2} \left| \vec{\xi}\,'(\vec{r}\,',t_o|\omega_s \right|^2 \right] \quad , \tag{18C}$$

it can be seen that the phase is given by:

$$\Phi_s(\vec{r}_T,t_o|\omega_s) = R_s(t_o) + \sum_{k=2}^{4} \left(\frac{1}{f_s^{\,k}}\right) \Phi_k(R_s,t_o) + 0\left(\frac{1}{f_s^{\,5}}\right) \tag{19}$$

where $f_s = \omega_s/2\pi$ has been used.

To order v/c (non-relativistic approximation), the time derivative of the phase (to give the Doppler shift) is given by taking the time derivative of the phase with respect to t_o and then substituting the value of t_o given by equation (14A). Letting

$$\Delta f_k(t_o) = - \frac{f_s}{c} \frac{d}{dt_o} \Phi_{k+1}(R_s, t_o) \quad , \tag{20A}$$

the expression for the refracted Doppler shift becomes

$$\Delta f(t) = \Delta f_v(t_o) + \sum_{k=1}^{3} \frac{\Delta f_k(t_o)}{f_s^{\,k}} + O\left(\frac{1}{f_s^{\,4}}\right) \quad , \tag{20B}$$

where, from equations (14A) and (19), to $O(^v/c)$:

$$t_o = t - \frac{R_s(t)}{c} + \sum_{k=2}^{4} \frac{(k-1)}{cf_s^{\,k}} \Phi_k(R_s, t) + O\left(\frac{1}{cf_s^{\,5}}\right) \quad . \tag{20C}$$

Equations (20), together with equations (15), (17B) and (18), provide the required expressions for the non-relativistic refracted Doppler shift. The expressions are valid to $O(^1/f_s^{\,3})$ or to $O(^1/f_s^{\,4})$ of the vacuum Doppler shift itself. Section IV contains a summary of these results and some numerical upper bounds to aid in estimating the significance of the various terms.

IV. Summary and Conclusions

The equations given in Section III for the refracted Doppler shift are written in a notation especially chosen to display those variables important in matching boundary conditions. This previous notation is somewhat awkward for purposes of discussion and the final equations of Section III will now be rewritten in a slightly different notation. Figure 2 schematically shows the geometry of the satellite motion during a small time increment, Δt , relative to the tracking station. For convenience, the origin of the new coordinate system is taken at the tracking station antenna. From Figure 2, it can be seen that the velocity vector, $\dot{\vec{\xi}}_s(t_o)$, lies in the plane defined by the position and velocity vectors of the satellite, $\vec{r}_s(t_o)$ and $\dot{\vec{r}}_s(t_o)$ respectively, and is normal to $\vec{r}_s(t_o)$. Finally, it can be seen that the new unit vector, $\hat{r}_s$, is just the negative of the unit vector, $\hat{R}_s(t_o)$, used in the previous sections

With this notation, for an arbitrary function, $g(\vec{r},t_o)$, and $\vec{r} = r\hat{r}_s$:

$$\int_{\vec{r}_s(t_o)}^{\vec{r}} d\vec{r}\,' \cdot \hat{R}_s(t_o)\, g(\vec{r}\,',t_o) = \int_{r}^{r_s(t_o)} dr'\, g(r'\hat{r}_s,t_o) \quad .$$

Also letting

$$\eta_\ell(\vec{r},t_o) = \omega_s^{\,2}\, \frac{2m\epsilon_o}{e^2}\, \xi_\ell'(\vec{r},t_o) \quad , \qquad \ell = 1,2,$$

the various expression for the refracted Doppler shift can be written in the following way.

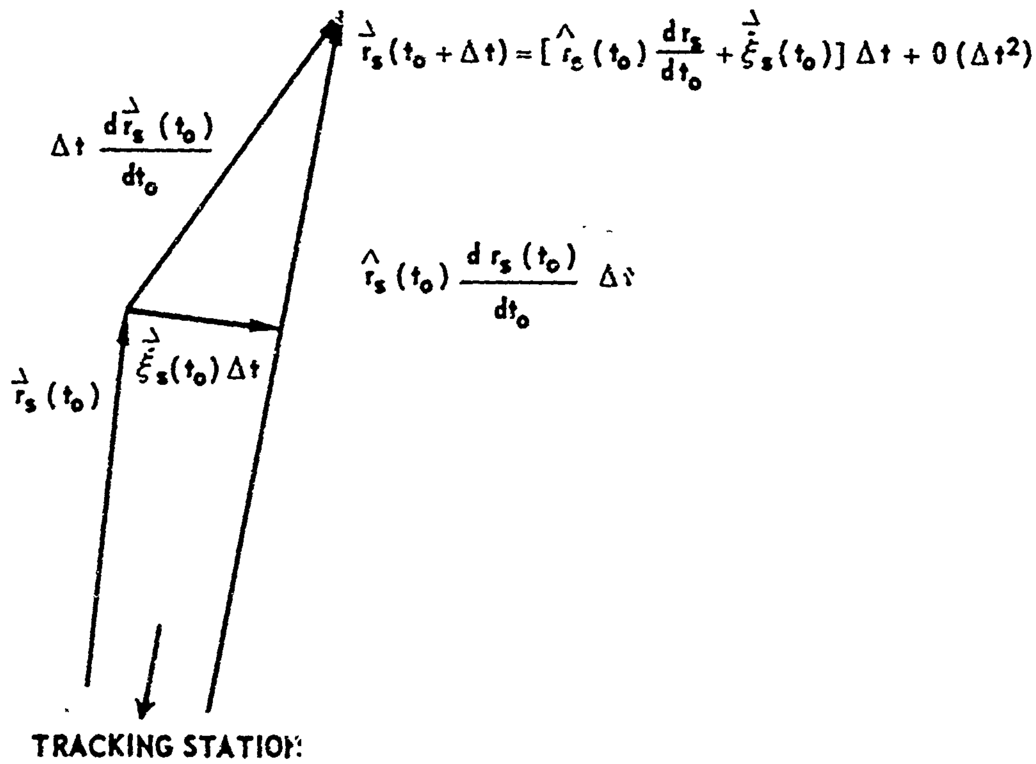

Fig. 2 GEOMETRY FOR SMALL CHANGE IN t_o

$$\Delta f(t_o) \doteq \Delta f_v(t_o) + \sum_{k=1}^{3} \frac{\Delta f_k(t_o)}{f_s^{\,k}} + 0\left(\frac{1}{f_s^{\,4}}\right) \quad ; \tag{21A}$$

where

$$\Delta f_v(t_o) = \text{vacuum Doppler shift,} \tag{21B}$$

$$\Delta f_k(t_o) = -\frac{f_s}{c}\,\frac{d}{dt_o}\,\Phi_{k+1}\,(r_s,t_o) \quad , \quad k = 1,2,3, \tag{21C}$$

$$t_o = t - \frac{r_s(t_o)}{v_p}^{\,*} \quad , \tag{22A}$$

$$v_p = C\left[1 - \frac{1}{f_s^{\,2}}\,\frac{\Phi_2(r_s,t_o)}{r_s(t_o)}\right] + 0\left(\frac{C}{f_s^{\,3}}\right) \quad , \tag{22B}$$

$$\Phi_2(r_s,t_o) = -\frac{1}{(2\pi)^2}\,\frac{e^2}{2m\epsilon_o}\int_0^{r_s(t_o)} dr'\rho(\hat{r}_s r',t_o) \quad , \tag{23A}$$

$$\Phi_3(r_s,t_o) = \mp \frac{1}{(2\pi)^3}\,\frac{e^3}{2m^2}\,\frac{\mu_o}{\epsilon_o}\int_0^{r_s(t_o)} dr'\rho(\hat{r}_s r',t_o)\;\hat{r}_s \cdot \vec{H}_E(\hat{r}_s r') \quad , \tag{23B}$$

*If the relativistic expression is desired, the relativistic vacuum doppler expression should be $\Delta f_v(t_o)$ and equation (22A) altered to be the corresponding relativistic expression. The refraction correction terms, $\Delta f_k(t_o)$, are sufficiently small that the non-relativistic expressions are still valid.

$$\Phi_4(r_s,t_o) = -\frac{1}{(2\pi)^4}\frac{e^4}{8m^2\epsilon_o^2}\int_o^{r_s(t_o)} dr'[\rho^2(\hat{r}_s r',t_o) - \eta_1^2(\vec{r}',t_o) - \eta_2^2(\vec{r}',t_o)]$$

$$-\frac{1}{(2\pi)^4}\frac{c^2e^2}{4m\epsilon_o}\int_o^{r_s(t_o)}\frac{dr'}{(r_s(t_o) - r')}\left[\frac{\partial}{\partial_{r'}}\rho(\hat{r}_s r',t_o) + \right.$$

$$\left. + \left|\frac{\partial}{\partial\xi_1}\eta_1(\hat{r}_s r' + \vec{\xi},t_o)\right|_{\vec{\xi}=0} + \left|\frac{\partial}{\partial\xi_2}\eta_2(\hat{r}_s r' + \vec{\xi},t_o)\right|_{\vec{\xi}=0}\right] \quad , \tag{23C}$$

$$\eta_\ell(\vec{r},t_o) = \int_r^{r_s(t_o)} dr' \quad \frac{\partial}{\partial\xi_\ell}\rho(\hat{r}_s r' + \vec{\xi},t_o) \quad {}_{\vec{\xi}=0}$$

$$-\frac{1}{r_s(t_o)}\int_r^{r_s(t_o)} dr' \int_{r'}^{r_s(t_o)} dr''\left[\frac{\partial}{\partial\xi_\ell}\rho(\hat{r}_s r'' + \vec{\xi},t_o)\right]_{\vec{\xi}=0} \quad . \tag{23D}$$

In these expressions, the following identity can be used.

$$\frac{d}{dt_o} \int_{r}^{r_s(t_o)} dr' \, h(\hat{r}r', t_o) = \frac{dr_s(t_o)}{dt_o} h(\vec{r}_s(t_o), t_o) + \int_{r}^{r_s(t_o)} dr' \left[\frac{\dot{\vec{\xi}}_s(t_o)}{r_s(t_o)} \cdot \vec{\nabla}' h(\hat{r}_s r', t_o) + \frac{\partial h}{\partial t_o} (\hat{r}_s r', t_o) \right], \tag{24}$$

where, by assumption number 1 of Section III, the first term is zero when the function $h(\vec{r}_s(t_o), t_o)$ is proportional to the electron density or its gradients evaluated at $\vec{r}_s(t_o)$. Except possibly in $\Delta f_1(t_o)$, the explicit time dependence of the electron density can be neglected, so that usually $(\frac{\partial h}{\partial t_o})$ can be neglected in equation (24).

Equations (21) through (23) are the non-relativistic expressions to $O(1/f_s^3)$ for the Doppler shift corresponding to a circularly polarized receiving antenna. Admittedly they are rather complicated. As expected, the lead term is the vacuum Doppler shift, evaluated at the satellite position corresponding to the retarded time, $t - r_s(t_o)/v_p$, where v_p is the average signal velocity to $O(1/f_s^2)$. Higher corrections to the signal velocity are negligible when considering the non-relativistic Doppler shift.

The first order refraction term, $\Delta f_1(t_o)/f_s$, is the term that is eliminated when two-frequency doppler data is used. From equation (23A) it can be seen that this first order term is proportional to the time rate of change of the total number of electrons in a tube of unit cross-section extending from the tracking station to the satellite. Figure 3 shows a typical set of experimental data taken from the satellite 1960 ETA 1. On rare occasions, $\Delta f_1(t_o) \times 10^{-8}$ has been observed to be as large as 10 (cps)^2.

The second order contribution $\Delta f_2(t_o)/f_s^2$, can be seen from equation (23B) to correspond to the same term in the ionospheric refractive index that gives rise to Faraday rotation since its sign depends upon whether the antenna is left or right circularly polarized. From equations (23A) and (23B), it can be seen that $\Delta f_2(t_o)/f_s^2$ has an upper bound which is approximately

$$\left| \frac{\Delta f_2}{f_s^2} \right| \lesssim \frac{e}{2\pi m} \frac{\mu_o H_E}{f_s} \left| \frac{\Delta f_1}{f_s} \right| . \tag{25}$$

Choosing an upper bound for $\Delta f_1 \times 10^{-8}$ of 10 (cps)^2,

$$\left| \Delta f_2 \times 10^{-16} \right| \lesssim 0.14 \text{ (cps)}^3 \quad ,$$

or at f_s = 100 mc/s, $\Delta f_2/f_s^2 \lesssim 0.14$ cps. Under normal ionosphere conditions, the magnitude of this term should be a factor of three or four smaller.

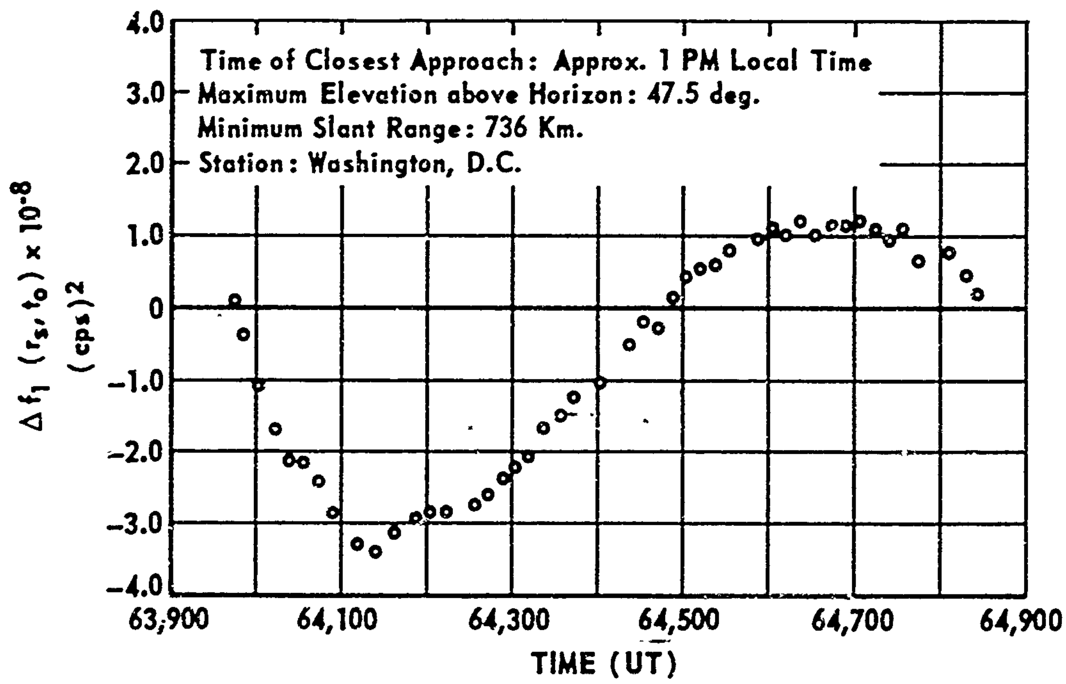

Fig. 3 FIRST ORDER REFRACTION DATA FROM SATELLITE 1960 ETA 1

From equation (23C) the third order contribution is composed of four distinct terms.

$$\Phi_{4,1}(r_s,t_o) = -\frac{1}{(2\pi)^4}\,\frac{e^4}{8m^2\epsilon_o{}^2}\int_0^{r_s(t_o)} dr'\rho^2(\hat{r}_s r',t_o) \quad , \tag{26A}$$

$$\Phi_{4,2}(r_s,t_o) = \frac{1}{(2\pi)^4}\,\frac{e^4}{8m^2\epsilon_o{}^2}\int_0^{r_s(t_o)} dr'\,[\eta_1{}^2(\vec{r}',t_o) + \eta_2{}^2(\vec{r}',t_o)] \quad , \tag{26B}$$

$$\Phi_{4,3}(r_s,t_o) = -\frac{1}{(2\pi)^4}\,\frac{c^2e^2}{4m\epsilon_o}\int_0^{r_s(t_o)} \frac{dr'}{(r_s(t_o) - r')}\,\frac{\partial}{\partial r'}\,\rho(\hat{r}_s r',t_o) \quad , \tag{26C}$$

$$\Phi_{4,4}(r_s,t_o) = -\frac{1}{(2\pi)^4}\,\frac{c^2e^2}{4m\epsilon_o}\int_0^{r_s(t_o)} \frac{dr'}{(r_s(t_o) - r')}\left[\frac{\partial\eta_1}{\partial\xi_1} + \frac{\partial\eta_2}{\partial\xi_2}\right]_{\vec{\xi}=0} \quad , \tag{26D}$$

where the $\eta_\ell(\vec{r},t_o)$ are given by equation (23D).

The first term arises from the fourth order term in the expansion of the refractive index in powers of $1/f_s$. The second term represents the contribution to $O(1/f_s^3)$ of the difference between the optical path and the geometric path. The last two terms of equations (26) represent the additional terms obtained from the solution of the wave equation that are not obtained when the geometrical optics approximation is used. Crude upper bounds to the magnitudes of their contributions to the refracted Doppler shift are now given.

From equation (24), neglecting the explicit time derivative,

$$\left|\frac{d}{dt_o} \Phi_{4,1}(r_s,t_o)\right| = \left|\frac{1}{(2\pi)^4} \frac{e^4}{4m^2\epsilon_o^2} \int_0^{r_s} dr' \, \rho(\hat{r}_s r',t_o) \frac{\dot{\vec{\xi}}_s(t_o)}{r_s(t_o)} \cdot \vec{\nabla}\rho(\hat{r}_s r',t_o)\right|$$

$$\lesssim \left|\frac{1}{(2\pi)^2} \frac{e^2 \rho \text{ max.}}{2m\epsilon_o} \cdot \frac{1}{(2\pi)^2} \frac{e^2}{2m\epsilon_o} \int_0^{r_s} dr' \frac{\dot{\vec{\xi}}_s}{r_s} \cdot \vec{\nabla}\rho\right|$$

$$\lesssim f_{max}^2 \left|\frac{d\Phi_2}{dt_o}\right| \quad , \quad f_{max}^2 = \frac{1}{(2\pi)^2} \frac{e^2 \rho \text{ max}}{2m\epsilon_o} \quad ,$$

where f_{max} is the maximum plasma resonance frequency in the ionosphere;

$$f_{max} \lesssim 15 \text{ mc/s.}$$

From equation (21C), using an upper bound for $\Delta f_1(t_o) \times 10^{-8}$ of 10 $(cps)^2$,

$$|\Delta f_{3,1}(t_o)| \lesssim f^2_{max} \mid \Delta f_1(t_o)| \quad , \quad f^2_{max} = \frac{1}{(2\pi)^2} \frac{e^2\rho\ max}{2m\epsilon_o} \quad , \tag{27}$$

or

$$|\Delta f_{3,1}(t_o)| \times 10^{-24} \lesssim .23\ (cps)^4 \quad .$$

From equations (24) and (26), it can be seen that the remainder of the terms depend upon various derivatives of the electron density. To obtain approximate upper bounds, it is assumed that over a distance of about one kilometer, the electron density can change by no more than its maximum value and that its second derivative can be no larger than ten times the maximum electron density.

Accordingly, assume:

$$|\vec{\nabla}\rho(\hat{r}r,t_o)| \lesssim \rho_{max}\ \delta(r - r_g) \quad , \tag{28A}$$

$$|\vec{\nabla}^2\rho(\hat{r}r,t_o)| \lesssim 10\ \rho_{max}\ \delta(r - r_g) \quad . \tag{28B}$$

From equations (24) and (28A),

$$|\eta_\ell| \lesssim \rho_{max} \left[\int_r^{r_s(t_o)} dr' \; \delta(r' - r_g) - \frac{1}{r_s(t_o)} \int_r^{r_s(t_o)} dr' \int_{r'}^{r_s(t_o)} dr'' \; \delta(r'' - r_g) \right]$$

$$\lesssim \rho_{max} \left[1 - S(r_g) \right] \left[1 - \frac{r_g - r}{r_s(t_o)} \right] \quad , \qquad \ell = 1,2,$$

where $S(r_g)$ is the unit step function at $r = r_g$. Thus,

$$\left| \int_0^{r_s} dr' \left[\eta_1^2 + \eta_2^2 \right] \right| \lesssim \rho_{max}^2 \, r_s(t) \cdot \frac{r_g}{r_s} \left[1 - \frac{r_g}{r_s} \right] \simeq \rho_{max}^2 \, \frac{r_s(t_o)}{4} \quad ,$$

so that

$$|\Phi_{4,2}| \lesssim \frac{f_{max}^2}{4} \, r_s(t_o) \qquad .$$

Substituting this expression into (21C), and using the fact that the vacuum Doppler shift at 100 mc/s is not greater than about 2500 cps,

$$|\Delta f_{3,2}(t_o)| \lesssim \frac{f_{max}^2}{4} \; |\Delta f_v(t_o)| \qquad ,$$

or: (29)

$$|\Delta f_{3,2}(t_o)| \times 10^{-24} \lesssim .08(\text{cps})^4 \qquad .$$

Employing similar analysis using equations (28), it can be shown that

$$|\Delta f_{3,3}(t_o)| \times 10^{-24} \lesssim 0[10^{-5}(\text{cps})^4] \qquad ,$$

$$|\Delta f_{3,4}(t_o)| \times 10^{-24} \lesssim 0[10^{-4}\,(\text{cps})^4] \qquad .$$

Therefore, unless the satellite is below VHF (near the plasma resonance frequencies), the two terms not obtainable from the geometrical optics approximation are negligible. The fact that these two additional terms are negligible affords a proof that for each circularly polarized component of the received signal from the satellite, the Doppler shift at VHF is given to $0(1/f_s^3)$ to sufficient accuracy by

$$\Delta f = \frac{d}{dt}\,\Phi(t) \qquad , \qquad \Phi = \int n ds \tag{30A}$$

where the integration path is the optical path which makes Φ an extremum. From equations (21), (22), and (23), it can be seen that to $0(1/f_s^2)$,

$$\Delta f = \frac{d}{dt}\int n dr \tag{30B}$$

where this integral is taken along the geometric path.

Table I summarizes the results pertaining to upper bounds on the various contributions to the refracted Doppler shift. Assuming that these bounds are realistic, the second and third order refraction contributions may be significantly large. It is believed that the upper bound for $\Delta f_2(t_o)$ is correct to within a factor of two, so that if the ionosphere is unusually dense and the propagation path is parallel to the earth's magnetic field, the Faraday contribution should not be ignored at the lower end of the VHF region. The estimated upper bound for $\Delta f_{3,1}(t_o)$ may be in error by a factor of five since it is dependent upon the square of the electron density.

The upper bound for $\Delta f_{3,2}(t_o)$ is very difficult to estimate. It is relatively easy to show that if the ionosphere electron density contains no sharp gradients, $\Delta f_{3,2}(t_o)$ is negligible. Consequently, only unusually large gradients can contribute significantly, and equations (28) were assumed to aid in establishing upper bounds to the contributions from regions of sharp density gradients. However, equations (28) assume that only a single region contributes at any one time. When the ionosphere is unusually disturbed, for example during a severe magnetic storm or in auroras, there could be many such regions contributing along the transmission path. If these should constructively contribute (have the same sign) the contributions from density gradients could be considerably higher. Such cases should be rare, however, since there is the probability that if there are many regions of large gradients they would contribute with random signs and partially cancel.

Table I contains a summary of the estimated maxima. Except for the vacuum term, typical values are probably a factor of three to ten smaller. If very accurate measurements of the vacuum Doppler shift are required, only time

TABLE I

ESTIMATED MAXIMUM CONTRIBUTIONS TO REFRACTED DOPPLER SHIFT

(Entries are in cps.)

f_s (mc/s)	Δf_v	$\frac{\Delta f_1}{f_s}$	$\frac{\Delta f_2}{f_s^2}$	$\frac{\Delta f_{3,1}}{f_s^3}$	$\frac{\Delta f_{3,2}}{f_s^3}$	$\frac{\Delta f_{3,3}}{f_s^3}$	$\frac{\Delta f_{3,4}}{f_s^3}$
50	1,200	20.0	.60	1.92	.63	-	-
100	2,500	10.0	.15	.23	.08	-	-
150	3,700	6.7	.07	.04	.02	-	-
200	5,000	5.0	.04	.03	.01	-	-
300	7,500	3.3	.02	.01	-	-	-
400	10,000	2.5	.01	-	-	-	-
500	12,500	2.0	-	-	-	-	-

periods when the ionosphere is normal should be used. For such carefully selected periods, contributions higher than first order should be negligible.

Table II presents estimated maxima of the second and third order contributions to the 'vacuum' Doppler shift when dual frequency data are combined to eliminate the first order refraction contribution. Letting the two transmitter frequencies be f_1 and f_2 and the corresponding experimental data be Δf_{E_1} and Δf_{E_2} , the combined Doppler shift becomes:

$$f_s \left[\frac{f_1 \Delta f_{E_1} - f_2 \Delta f_{E_2}}{f_1^2 - f_2^2} \right] = \Delta f_v + \frac{\Delta f_2}{f_s^2} \left[\frac{\frac{f_s}{f_1} - \frac{f_s}{f_2}}{\left(\frac{f_1}{f_s}\right)^2 - \left(\frac{f_2}{f_s}\right)^2} \right]$$

$$+ \frac{\Delta f_3}{f_s^3} \left[\frac{\left(\frac{f_s}{f_1}\right)^2 - \left(\frac{f_s}{f_2}\right)^2}{\left(\frac{f_1}{f_s}\right)^2 - \left(\frac{f_2}{f_s}\right)^2} \right] + 0\left(\frac{1}{f_s^4}\right), \qquad (31)$$

where f_s is a chosen reference frequency. Table II lists three typical combinations where the entries in the table have been converted from frequency shift to errors in the range rate in meters per second. From this table, it can be seen that if the lower of the two frequencies is not less than 100mc/s and the ionosphere is not extremely disturbed, the second and third order refraction contributions should be negligible.

TABLE I

ESTIMATED MAXIMUM CONTRIBU

DOPPLER SH

(Entries are i

f_s (mc/s)	Δf_v	$\frac{\Delta f_1}{f_s}$	$\frac{\Delta f_2}{f_s^2}$
50	1,200	20.0	.60
100	2,500	10.0	.15
150	3,700	6.7	.07
200	5,000	5.0	.04
300	7,500	3.3	.02
400	10,000	2.5	.01
500	12,500	2.0	-

E I

RIBUTIONS TO REFRACTED

R SHIFT

re in cps.)

$\frac{\Delta f_{3,1}}{f_s^3}$	$\frac{\Delta f_{3,2}}{f_s^3}$	$\frac{\Delta f_{3,3}}{f_s^3}$	$\frac{\Delta f_{3,4}}{f_s^3}$
1.92	.63	-	-
.23	.08	-	-
.04	.02	-	-
.03	.01	-	-
.01	-	-	-
-	-	-	-
-	-	-	-

periods when the ionosphere is normal should be used. For such carefully selected periods, contributions higher than first order should be negligible.

Table II presents estimated maxima of the second and third order contributions to the 'vacuum' Doppler shift when dual frequency data are combined to eliminate the first order refraction contribution. Letting the two transmitter frequencies be f_1 and f_2 and the corresponding experimental data be Δf_{E_1} and Δf_{E_2}, the combined Doppler shift becomes:

$$f_s\left[\frac{f_1\Delta f_{E_1} - f_2\Delta f_{E_2}}{f_1^2 - f_2^2}\right] = \Delta f_v + \frac{\Delta f_2}{f_s^2}\left[\frac{\frac{f_s}{f_1} - \frac{f_s}{f_2}}{\left(\frac{f_1}{f_s}\right)^2 - \left(\frac{f_2}{f_s}\right)^2}\right]$$

$$+ \frac{\Delta f_3}{f_s^3}\left[\frac{\left(\frac{f_s}{f_1}\right)^2 - \left(\frac{f_s}{f_2}\right)^2}{\left(\frac{f_1}{f_s}\right)^2 - \left(\frac{f_2}{f_s}\right)^2}\right] + 0\left(\frac{1}{f_s^4}\right), \qquad (31)$$

where f_s is a chosen reference frequency. Table II lists three typical combinations where the entries in the table have been converted from frequency shift to errors in the range rate in meters per second. From this table, it can be seen that if the lower of the two frequencies is not less than 100mc/s and the ionosphere is not extremely disturbed, the second and third order refraction contributions should be negligible.

BLANK PAGE

TABLE

ESTIMATED MAXIMUM CONTRIB

DOPPLER

(Entries are in

f_1 (mc/s)	f_2	2nd Order
50	300	.10
100	300	.05
150	400	.02

BLE II

RIBUTIONS TO DUAL FREQUENCY

LER DATA

in meters/sec.)

	3rd Order
	.26
	.01
	.01

References

1. Guier, W. H., "The Doppler Tracking of Project Transit Satellites", Proc. of IRE, 48 , No. 9, Sept. 1960.

2. Kershner, R. B., "Status Report on the Transit Navigation System", Proc. of IRE, 48 , No. 9, Sept. 1960.

3. Newton, R. R., "Potential Geodetic Applications of the Transit Satellite System", Proc. Am. Geophys. Union, Spring 1961, Washington, D. C.

4. Guier, W. H. and Weiffenbach, G. C., "A Satellite Doppler Navigation System", Proc. of IRE, 48 , No. 4, April 1960.

5. Stratton, J. A., "Electromagnetic Theory", McGraw-Hill Book Co., p. 327 ff, 1941.

6. Ratcliffe, J. A. "The Magneto-Ionic Theory and its Applications to the Ionosphere", Cambridge Univ. Press, 1959.

7. Landau, L. and Lifshitz, E., "The Classical Theory of Fields", pp. 136-144, Addison-Wesley, 1951.

8. Sommerfeld, A. and Runge, J., "Anwendung der Vektorrechnung auf die geometrischen Optik", Ann. Physik, 35, p. 277 ff, 1911.

9. Willman, J. F. and Doyle, J. F., "Ionospheric Refraction Errors Measured in the Doppler Shift of Radio Transmissions from Artificial Earth Satellites", APL Report CF-2991, 11 February 1963.

10. Guier, W. H., "Ionospheric Contributions to the Doppler Shift at VHF from Near-Earth Satellites", (L), Proc. IRE, Vol. 49, No. 11, pp. 1680-1681, (1961).

APPENDIX I. Proof of Equation (8D).

While well known, the author has not found in the current literature a detailed proof of the fact that if the line integral,

$$\Phi = \int_{\vec{r}_o}^{\vec{r}_1} d\vec{r} \cdot \hat{s}(r)\, f(\vec{r}) \quad , \tag{A1.1}$$

is taken over that path going from the point $\vec{r}_o$ to the point $\vec{r}_1$ which makes Φ an extremum, then

$$\vec{\nabla}_{r_1} \Phi = f(\vec{r}_1)\, \hat{s}(\vec{r}_1) \quad , \tag{A1.2}$$

where $\hat{s}(\vec{r}_1)$ is the unit vector tangent to the path of integration at the point $\vec{r}_1$*. Equation (A1.1) is given as equation (8D) in the main body of the paper. A proof is given in this appendix for the benefit of the reader.

Let a parameter, σ , vary from zero to one when following the path from $\vec{r}_o$ to $\vec{r}_1$, and let the path be specified by:

$$\vec{r} = \vec{r}(\sigma,\vec{r}_1) = x(\sigma,\vec{r}_1) \ , \ \ y(\sigma,\vec{r}_1) \ , \ z(\sigma,\vec{r}_1) \quad .$$

*This theorum forms the basis of the transition from electromagnetic theory to geometrical optics. See, for example, references (5), (7), and (8).

Writing

$$\vec{r}\,' = \frac{d\vec{r}}{d\sigma} \quad ,$$

the line integral can be written in the form

$$\Phi = \int_0^1 d\sigma \; L(\vec{r}\,',\vec{r},\sigma) \quad ,$$

$$L(\vec{r}\,',\vec{r},\sigma) = [x'^2 + y'^2 + z'^2]^{1/2} \; f(x,y,z) \quad . \tag{A1.3}$$

With this notation the path, $\vec{r}(\sigma,\vec{r}_1)$, is the extremum path when

$$\frac{d}{d\sigma} \frac{\partial L(\vec{r}\,',\vec{r},\sigma)}{\partial x'} - \frac{\partial L(\vec{r}\,',\vec{r},\sigma)}{\partial x} = 0 \quad ,$$

$$\left.\begin{aligned} x_1 &= x(1,\vec{r}_1) \quad , \\ x_0 &= x(0,\vec{r}_1) \quad , \end{aligned}\right\} \tag{A1.4}$$

with similar equations in $y(\sigma,\vec{r}_1)$ and $z(\sigma,\vec{r}_1)$. The fact that $\vec{r}(\sigma,\vec{r}_1)$ depends upon the end points has been explicitly noted so far as the upper end point is concerned.

Now consider the change of Φ with a change in the value of $x_1 = x_1(1,\vec{r}_1)$, where a new extremum path is chosen corresponding to the new value of x_1 .

$$\frac{\partial \Phi}{\partial x_1} = \int_0^1 d\sigma \left[\frac{\partial L}{\partial x'} \frac{\partial x'}{\partial x_1} + \frac{\partial L}{\partial x} \frac{\partial x}{\partial x_1} \right] + \int_0^1 d\sigma \left[\frac{\partial L}{\partial y'} \frac{\partial y'}{\partial x_1} + \frac{\partial L}{\partial y} \frac{\partial y}{\partial x_1} \right]$$

$$+ \int_0^1 d\sigma \left[\frac{\partial L}{\partial z'} \frac{\partial z'}{\partial x_1} + \frac{\partial L}{\partial z} \frac{\partial z}{\partial x_1} \right] .$$

Partially integrating and using equations (A1.4):

$$\frac{\partial \Phi}{\partial x_1} = \left[\frac{\partial x}{\partial x_1} \frac{\partial L}{\partial x'} \right]_{\sigma=0}^{\sigma=1} + \left[\frac{\partial y}{\partial x_1} \frac{\partial L}{\partial y} \right]_{\sigma=0}^{\sigma=1} + \left[\frac{\partial z}{\partial x_1} \frac{\partial L}{\partial z} \right]_{\sigma=0}^{\sigma=1} .$$

At the end points, with a change in only x_1 , $\left.\frac{\partial x}{\partial x_1}\right|_{\sigma=1} = 1$, with all other similar partials being identically zero. Thus,

$$\frac{\partial \Phi}{\partial x_1} = \left[\frac{\partial L}{\partial x'} \right]_{\sigma=1} = \frac{x_1'}{\sqrt{x_1'^2 + x_2'^2 + x_3'^2}} f(\vec{r}_1) \qquad \text{(A1.5)}$$

$$= \hat{s}_x(\vec{r}_1)\, f(\vec{r}_1) \quad ,$$

where $\hat{s}(\hat{r}_1)$ is the unit vector tangent to the extremum path at the point $\vec{r}_1$.

Partial derivatives with respect to y_1 and z_1 yield equations similar to (A1.5). Consequently, the three partial derivatives combine in vector notation to give:

$$\vec{\nabla}_{r_1} \bar{\phi} = \hat{s}(\vec{r}_1)\ f(\vec{r}_1) \quad .$$

APPENDIX II. Evaluation of the Phase Integral and Extremum Path.

In this appendix, the line integral,

$$\Phi = \int_{\vec{r}_s(t_o)}^{\vec{r}} d\vec{r}\,' \cdot \hat{s}(\vec{r}\,')\ f(\vec{r}\,') \tag{A2.1}$$

is evaluated to $0\left(\frac{1}{\omega^4}\right)$ together with the deviation , $\vec{\xi}$, of the extremum path from the geometrical path to $0\left(\frac{1}{\omega^2}\right)$. Let:

$\hat{R}$ = unit vector from $\vec{r}_s(t_o)$ to $\vec{r}$,

$\vec{\xi} = \xi_1, \xi_2$ = vector difference between the extremum path and geometrical path, $\vec{\xi} \cdot \hat{R} = 0$.

It is assumed that the function, $f(\vec{r})$, is of order unity and that

$$|\vec{\nabla} f(\vec{r})| = 0\left(\frac{1}{\omega^2}\right) .$$

It will be shown that

$$|\vec{\xi}| = 0\left(\frac{1}{\omega^2}\right) .$$

The vector integration variable in equation (A2.1) is given by:

$$\vec{r}\,' = r'\hat{R} + \vec{\xi}(\vec{r}\,') \quad , \quad 0 \leq r' \leq R = |\vec{r} - \vec{r}_s(t_o)| \quad ,$$

where, at the end points of the integration,

$$\vec{\xi}(0) = \vec{\xi}(R) = 0 \quad .$$

It can be seen that

$$d\vec{r}' = \hat{R}dr' + d\vec{\xi} = (\hat{R} + \vec{\xi}')dr'$$

where

$$\vec{\xi}' = \frac{d\vec{\xi}}{dr} \quad .$$

Therefore,

$$\hat{s}(r') = \frac{\hat{R} + \vec{\xi}'}{\sqrt{1 + |\vec{\xi}'|^2}} \quad , \tag{A2.2}$$

so that

$$dr' \cdot \hat{s}(r') = dr' \sqrt{1 + |\vec{\xi}'|^2} \quad ,$$

and equation (A2.1) can be written in the form.

$$\Phi = \int_0^R dr' \sqrt{1 + |\vec{\xi}'|^2} \quad f(r', \vec{\xi}(r')) \quad . \tag{A2.3}$$

By definition, $\bar{\phi}$ is an extremum if the extremum path, $\vec{\xi}(r')$, is such that

$$\frac{d}{dr}\left[\frac{\xi'_k}{\sqrt{1+|\vec{\xi}'|^2}}\, f(r,\vec{\xi})\right] = \frac{\partial}{\partial \xi_k} f(r,\vec{\xi}) \quad . \quad k = 1,2.$$

Since, $f = O(1)$ and $|\nabla f| = O\left(\frac{1}{\omega^2}\right)$, $|\vec{\xi}'| = O\left(\frac{1}{\omega^2}\right)$ and the above equation to $O(1/\omega^2)$ can be written in the form:

$$\frac{d}{dr}\left[\xi'_k(r)\, f(r,\vec{\xi})\right] = \frac{\partial}{\partial \xi_k} f(r,\vec{\xi}) + O\left(\frac{1}{\omega^4}\right) \quad , \quad k = 1,2.$$

Consequently, $\vec{\xi}$ should also be of $O(1/\omega^2)$.

Assuming both $\vec{\xi}$ and $\vec{\xi}'$ are small, and expanding equation (A2.4) to $O(1/\omega^2)$,

$$\frac{d}{dr}\left[\xi'_k(r)\, f(r,o)\right] = \frac{\partial}{\partial \xi_k} f(\vec{r},\vec{\xi})\Bigg|_{\vec{\xi}=0} + O\left(\frac{1}{\omega^4}\right) \quad , k = 1,2 . \qquad (A2.4)$$

Integrating this equation with respect to r and re-expanding to $O(1/\omega^2)$,

$$\xi'_k(r) = \int_o^r dr' \frac{\partial}{\partial \xi_k} f(r',\vec{\xi}(r'))\Bigg|_{\vec{\xi}=0} + \xi'_k(0) \quad , \quad k = 1,2. \qquad (A2.5)$$

Integrating this equation and using the fact that $\vec{\xi}$ at the end points of the integration must be zero:

$$0 = \int_{o}^{R} dr' \int_{o}^{r'} dr'' \frac{\partial}{\partial \xi_k} f(r'',\vec{\xi}) \Bigg|_{\vec{\xi}=0} + R\, \vec{\xi}_k(o)$$

or,

$$\xi_k'(o) = \frac{1}{R} \int_{o}^{R} dr' \int_{o}^{r'} dr'' \frac{\partial f}{\partial \xi_k} \Bigg|_{\vec{\xi}=0} .$$

Substituting this equation into equation (A2.5),

$$\xi_k'(r) = \int_{o}^{r} dr' \frac{\partial f}{\partial \xi_k}\Bigg|_{\vec{\xi}=0} - \frac{1}{R}\int_{o}^{R} dr' \int_{o}^{r'} dr'' \frac{\partial f}{\partial \xi_k}\Bigg|_{\vec{\xi}=0} , \qquad k = 1,2 . \tag{A2.6}$$

Finally, integrating this equation,

$$\xi_k(r) = \int_{o}^{r} dr' \int_{o}^{r'} dr'' \frac{\partial f}{\partial \xi_k}\Bigg|_{\vec{\xi}=0} - \int_{o}^{R} dr' \int_{o}^{r'} dr'' \frac{\partial f}{\partial \xi_k}\Bigg|_{\vec{\xi}=0} , \qquad k = 1,2, \tag{A2.7}$$

which is of $O(1/\omega^2)$, justifying the above assumption.

To evaluate the phase integral Φ, given by equation (A2.3), expand the integrand to $O(1/\omega^4)$. Then,

$$\Phi = \int_0^R dr' \left[f(r',o) + \frac{\vec{\xi}' \cdot \vec{\xi}'}{2} f(r',o) + \vec{\xi} \cdot \vec{\nabla} f(r',o)\right] + O\left(\frac{1}{\omega^6}\right)$$

Partially integrating,

$$\int_0^R dr' \, \vec{\xi}' \cdot \vec{\xi}' \, f(r',o) = - \int_0^R dr' \, \vec{\xi} \cdot \frac{d}{dr'}(\vec{\xi}' f(r',o) + \left[\vec{\xi} \cdot \vec{\xi}' f(r',o)\right]_0^R$$

Therefore, since $\vec{\xi} = 0$ at the end points,

$$\Phi = \int_0^R dr' \left[f(r',o) + \vec{\xi} \cdot \left(\vec{\nabla} f(\vec{r}',o) - \frac{1}{2} \frac{d}{dr'} (f\vec{\xi}')\right)\right] + O\left(\frac{1}{\omega^6}\right) .$$

Partially integrating once more after substituting from equations (A2.4), the final expression becomes:

$$\Phi = \int_0^R dr' \, f(r',o) \left[1 - \frac{1}{2}\left(\xi_1'^2 + \xi_2'^2\right)\right] + O\left(\frac{1}{\omega^6}\right), \tag{A2.8}$$

where the $\xi_k'(r')$ are given to sufficient order by equation (A2.6).

The Johns Hopkins University
APPLIED PHYSICS LABORATORY
Silver Spring, Maryland

Initial distribution of this document has been made in accordance with a list on file in the Technical Reports Group of The Johns Hopkins University, Applied Physics Laboratory.