THIS REPORT HAS BEEN DELIMITED
AND CLEARED FOR PUBLIC RELEASE
UNDER DOD DIRECTIVE 5200.20 AND
NO RESTRICTIONS ARE IMPOSED UPON
ITS USE AND DISCLOSURE.

DISTRIBUTION STATEMENT A

APPROVED FOR PUBLIC RELEASE;
DISTRIBUTION UNLIMITED.

THE EFFECT OF ASPECT RATIO ON THE LIFT

OF FLAT PLANING SURFACES,

Byrne Perry

Sep 52

24/6.

Hydrodynamics Laboratory
California Institute of Technology
Pasadena, California

Report No. E-24.5
September, 1952

J. P. O'Neill
Project Supervisor

174 250

ABSTRACT

Previous theories and empirical formulas
for the lift of flat planing surfaces are reviewed,
and the resemblance of the planing surface to the
airfoil noted. A simple expression, which con-
verges to the correct limits for exactly known
cases, is assumed for the ratio of planing to air-
foil lift, and the planing lift is then estimated by
using airfoil experimental data. The resulting
calculated values are in satisfactory agreement
with planing experiments.

- i -

SYMBOLS

For convenience, the symbols used in the following text are listed here. Note that the definition of the aspect ratio $Æ$ follows airfoil prac· tice instead of the definition often taken in planing work, which is the re- ciprocal of that given here.

$Æ$, aspect ratio: $Æ=b^2/S$ ($=b/\ell$ for rectangle)

b, plate width (span for airfoil)

C_a, lift coefficient of airfoil: $C_a = L_a / \frac{\rho}{2} V^2 S$

C_b, lift coefficient due to bottom pressure on airfoil

C_L, lift coefficient of planing surface: $C_L = L / \frac{\rho}{2} V^2 S$

C_N, normal force coefficient: $C_N = N / \frac{\rho}{2} V^2 S$

C_p, pressure coefficient: $C_p = (p - p_o) / \frac{\rho}{2} V^2$

L, lift force on planing surface

L_a, lift force on airfoil

L_b, lift force due to bottom pressure on airfoil

ℓ , wetted length, plate length

N, normal force on planing surface or airfoil

p, dynamic pressure

p_o, free stream pressure

S, area of planing surface or airfoil ($S=\ell b$ for rectangle)

V, free stream velocity

α, attack angle

α_i, downwash angle

β, auxiliary parameter. See Eqs. (22) and (23)

δ, spray thickness

μ, parameter defined by Eq. (27)

θ, parameter defined by Eq. (29)

ρ, fluid density

Introduction

The hydrodynamics of planing surfaces has been of interest for many years now because of the need for design information applicable to planing boats and seaplanes. While most of the data on planing refer to specific hull shapes on which the flow is rather complex, a number of systematic measurements have also been made on planing flat plates because, in addition to an occasional need for such data in design, it is easier to gain an understanding of the various effects when the geometry of the model is kept simple. To date a considerable amount of data has been collected in several countries on planing flat plates, although the range of aspect ratio and attack angle covered is by no means so complete as might be wished.

Meanwhile a theory of planing has been developed, chiefly by H. Wagner, which leads to a much better understanding of the hydrodynamic phenomena. Unfortunately the theory applies exactly only for very simple limiting configurations, none of which are ordinarily encountered in engineering practice or experiments to date so that no direct comparison of theory and experiment is possible. The difficulty in extending the theory to practical cases lies in the fact that the air-water interface introduces a nonlinear boundary condition which greatly complicates the calculations.

While thus limited in direct application, however, the theory has established an important result which sometimes permits indirect calculation of quantitative results, namely, the hydrodynamic resemblance of the planing problem to the airfoil problem. Hence, if quantitative results are available for the airfoil, as often happens, it may be possible to use them for estimating planing effects. In particular, W. Bollay has proposed that the aspect ratio effect on flat planing surfaces might be predicted from the extensive theory and data available for flat plate airfoils.

The purpose of this report is to review briefly the pertinent planing theory, which will in course bring out the resemblance to the airfoil, and then to estimate the lift of flat planing surfaces of any aspect ratio by following Bollay's suggestion. The effects of viscosity, gravity, and surface tension are for the most part neglected so that the resulting estimate applies especially for high speed. The calculation of the lift in this manner by no means represents an exact solution to the problem, but it is hoped that it will provide estimates accurate enough for engineering purposes and that,

moreover, it will allow a more direct correlation of theory and experiment than has heretofore been possible.

Wagner's Theory for the Planing Flat Plate

In order to calculate the flow of an ideal, weightless fluid about a two-dimensional (infinitely wide) flat planing surface, Wagner[1, 2, 3*] used Kirchoff's method to solve the flow configuration sketched in Fig. 1. The calculations give the velocity and pressure distributions on the plate for any attack angle α. By integration one can then find the normal force on the plate, which can then be resolved into lift and drag.

The lift force L per unit width b computed in this fashion is

$$\frac{L}{b} = \rho \frac{V^2}{2} \, \ell \, \mu \cdot 2\pi \sin \alpha \tag{1}$$

where ρ is the fluid density, V the velocity at infinity, and ℓ the so-called wetted length, defined as shown in Fig. 1. The quantity μ is a function of α only (s. Eq. 27, Appendix) and has been plotted in Fig. 1, which is the same as F. 18 of Ref. 2, with $\mu = 1/2 \times \cos \alpha$. For the lift coefficient defined in the usual way, $C_L = L/\frac{1}{2}\rho V^2 \, \ell \, b$ there results

$$C_L = \mu \cdot 2\pi \sin \alpha \quad . \tag{1a}$$

It should be noted that, in so far as Wagner's hydrodynamic model in Fig. 1 is correct, the result is exact and does not involve the assumption of small α so that Eq. (1a) should hold even for large attack angles. Wagner also calculated the effect of small amounts of longitudinal curvature, and Green[4] took into account a finite plate length or stream depth, but these modifications will not be considered here.

Wagner was able to extend his results to include finite width plates if the attack angles are vanishingly small. The corrections are of the same type as those used by Prandtl[5] for the airfoil and might be expected to hold for the same range of aspect ratio, that is, from $\mathcal{R} = \infty$ down to $\mathcal{R} = 3$, or possibly a little less. (For aspect ratios as low as one, however, the lifting line theory is inadequate for describing the flow.) For flat plates

---

*Numbers indicate references listed at end of report.

of sufficiently great width, then, Wagner calculates the lift to be (Ref. 2, Eq. 19)

$$C_L = \frac{\pi \sin a}{1 + \frac{2}{\mathcal{R}}} \quad (a \to 0, \quad \mathcal{R} > 3) .$$ (2)

Case of Zero Aspect Ratio

The only other case for which an exact solution is available is that of zero aspect ratio, which again becomes a two-dimensional problem since the flow is the same at all points along the length (Fig. 2). At any section A - B, the flow is of the so-called cavity type[6] with a velocity at infinity of (V sin a) as indicated in Fig. 2b. The force on the plate under these conditions is known to be

$$N = \frac{1}{2} \rho (V \sin a)^2 \frac{(2\pi)}{\pi + 4} b\ell ,$$ (3)

or in terms of a normal force coefficient

$$C_N = \frac{2\pi}{\pi + 4} \sin^2 a,$$ (4)

a result which was given by Bollay.[7] Since the lift is L = N cos a, the lift coefficient is

$$C_L = \frac{2\pi}{\pi + 4} \sin^2 a \cos a .$$ (5)

Eqs. (1), (2), and (5) then represent the available limiting solutions for the flat plate which are theoretically exact. Before the connection between these results and the airfoil theory is shown, it will be convenient to review some empirical formulas and a recent semi-empirical theory which approaches the calculation of the lift more directly.

Empirical Formulas

Approaching the problem from the experimental side, several investigators have proposed completely empirical formulas for the lift. The ETT formula,[8] which is an improvement on Sedov's[9] is

$$C_L = 1.03 \sqrt{\frac{b}{\ell}} \ a^{1.1} \quad (a \text{ in radians})$$ (6)

for gravity-free motion (Fig. 3). It is very similar to the formula proposed by Sottorf,[10] namely,

$$C_L = 0.85 \sqrt{\frac{b}{l}}\, \alpha \qquad (\alpha \text{ in radians}) \qquad (7)$$

which is also plotted in Fig. 3. Both these formulas are based on essentially the same data and give practically the same results. The actual variation of experimental parameters is indicated in the figure by plotting only the aspect ratios $0.2 = AR = 2$ for α up to 12°. It is worth noting that the form of both Eq. (6) and Eq. (7) is wrong for either very high or very low aspect ratio since in the former case they predict infinite lift and for the latter no lift at all. However, in the range indicated, either formula represents the experimental data reasonably well, and for the purposes of this report they will be used when comparing theory with experiment.

Theory of Korvin

As mentioned previously, Eqs. (1), (2), and (5) represent the theoretically exact solutions. However, the lift at finite attack angles for any aspect ratios except infinity and zero remains undetermined. In order to cover this range Korvin[11] recently proposed an equation for the lift at small attack angles which is composed of two terms, the first like Eq. (2) and the second like Eq. (5). The first term, however, was empirically corrected by a factor of 0.73 (= 0.04/0.055) in order to get the best fit with experimental data, so that the result reads (Ref. 11, Eq. 13)

$$C_L = 0.73 \frac{\pi \sin \alpha}{1 + \frac{2}{AR}} + \frac{2\pi}{\pi + 4} \sin^2 \alpha \qquad (8)$$

In Fig. 4 this equation is compared with the ETT empirical formula, both with and without the correction to the first term. As can be seen, the agreement is satisfactory when the correction is used.

The necessity for the correction factor may be explained by the fact that the first term of Eq. (8) is derived from Prandtl's lifting line theory, which is not accurate for $AR < 3$, and should be replaced by a term derived from lifting surface theory, such as Lawrence[12] has used for the airfoil. Whether this approach, if it could be used, would effect an improvement remains to be seen, but in any event, Eq. (8) must for the moment

be regarded as a semi-empirical result.

Connection with Airfoil Theory

Returning now to the theoretical problem of a planing surface of any aspect ratio, we consider Bollay's[7] approach to the calculation of lift, which is suggested by the similarity of the planing surface to the airfoil of the same geometry. In order to see this resemblance, it is only necessary to recall that the lift coefficient of a high aspect ratio flat plate airfoil[5] is

$$C_a = \frac{2\pi \sin \alpha}{1 + \frac{2}{\mathcal{R}}} \quad \begin{pmatrix} \mathcal{R} > 3 \\ \alpha \gtrsim 0 \end{pmatrix} \tag{9}$$

which is seen to be identical in form with Eq. (2), but with a multiplying factor of 2. On the other hand, the lift coefficient of a zero aspect ratio airfoil[13,14] is

$$C_a = 2 \sin^2 \alpha \cos \alpha \tag{10}$$

so that, dividing Eq. (5) by Eq. (10), we have

$$\frac{C_L}{C_a} = \frac{\pi}{\pi + 4} - 0.44 \quad \begin{pmatrix} \mathcal{R} = 0 \\ \alpha \gtrsim 0 \end{pmatrix} \tag{11a}$$

and, dividing Eq. (2) by Eq. (9),

$$\frac{C_L}{C_a} = 0.50 \quad \begin{pmatrix} \mathcal{R} > 3 \\ \alpha \rightarrow 0 \end{pmatrix} . \tag{11b}$$

Since the lift of airfoil and planing surface obeys analogous laws at each extreme of aspect ratio, and the percentage variation in the ratio C_L/C_a is not too large, Bollay suggests that the analogy probably holds for intermediate aspect ratios. Hence it is only necessary to find C_L/C_a as a function of aspect ratio and, since the variation is not too great, a fairly good estimate of C_L could be made even if the ratio of C_L/C_a were estimated only moderately well. Bollay did not give any specific values for C_L/C_a other than to recommend using Eq. (11a) for low $\mathcal{R}$ and Eq. (11b) for high $\mathcal{R}$. However, as will be shown presently, it is possible to make a better estimate for high $\mathcal{R}$ than that given by Eq. (11b). Also, it seems

safe to assume that the variation in C_L/C_a must be smooth with changes in $\mathcal{R}$. When these restrictions are taken into account it becomes a fairly simple matter to guess a reasonable value for C_L/C_a. It is the purpose of this report to carry out these calculations and to compare the resulting value of the lift with experiment; but first it will be convenient to summarize the airfoil theory and data from which C_a can be computed.

Airfoil Theory and Experiment

So far as the high aspect ratio ($\mathcal{R} \geq 3$) flat plate airfoil is concerned, the lift is well predicted by Eq. (9)[*] if the attack angles are not too high. For $\mathcal{R} < 3$, however, no completely satisfactory theory is available for large α[10], but both the theory of Bollay[13] and Weinig[14] predict the non-linear effect correctly. The theory of Weinig, while physically obscure, is in best agreement with the data[12] and has been plotted in Fig. 5 for comparison with Winter's[15] measurements on rectangular airfoils. Winter's data, as presented, has been corrected by Sambraus[16] for the effect of leading edge bevel. According to Weinig the lift is

$$C_a = 2\pi \frac{\tanh \dfrac{1}{\dfrac{\mathcal{R}}{2} + \dfrac{2}{\pi}\sin\alpha}}{1 + \tanh \dfrac{1}{\dfrac{\mathcal{R}}{2} + \dfrac{2}{\pi}\sin\alpha}} \left(\frac{\mathcal{R}}{2} + \frac{2}{\pi}\sin\alpha\right)\sin\alpha \ . \qquad (12)$$

Weinig uses $\tan\alpha$ instead of $\sin\alpha$ in Eq. (12) to get better fit with the data at high attack angles, but this makes no essential difference in the range of α considered here. As shown in Fig. 5, the lift predicted by the theory is too low for the range $0.134 = \mathcal{R} = 2$, but for $\mathcal{R} = 0.033$ the agreement is better. For such a low aspect ratio the lift value at $\mathcal{R} = 0$, as given by Eq. (10), is not too far off. It may be mentioned in passing that the planing experiments to date (Fig. 3) have not extended to nearly so low an aspect ratio as the airfoil measurements (Fig. 5), or to nearly so high an attack angle.

---

[*]Equation (9) holds exactly for elliptic lift distribution, but in practice the rectangular wing has a lift distribution sufficiently close to elliptic, and the same is probably true for planing surfaces. See Wagner's remark, Ref. 2, p. 16, footnote.

Further Discussion of Airfoil vs. Planing Surface

Equation (1) is written with the factor $2\pi \sin \alpha$ to facilitate comparing the planing lift with that on an infinitely thin flat plate airfoil, the lift on which is

$$L_a = \rho \frac{V^2}{2} \, l \, b \, (2\pi \sin \alpha) \tag{13}$$

Dividing Eq. (1) by Eq. (13) we obtain the ratio of the lift coefficients for airfoil and planing surface,

$$\frac{C_L}{C_a} = \mu \qquad (\mathcal{R} = \infty). \tag{14}$$

As the attack angle α goes to zero, $\mu \to 0.5$, which shows that the lift of the planing surface is one-half that of the airfoil for a vanishingly small attack angle. As the attack angle increases, however, the planing surface develops considerably less than half of the airfoil lift, the effect being of the order of 20% for $\alpha = 10^o$ (see Fig. 1). This state of affairs exists because the planing surface lift, as opposed to the airfoil, does not increase linearly with attack angle. The resulting loss of lift, which is predicted by Wagner's two-dimensional theory, Eq. (1), and has nothing to do with real fluid effects, has been overlooked by most subsequent investigators who assume that the planing surface always has half the airfoil lift; that is, they assume $\mu = 0.5$ for any attack angle. The approximation is poor, even for fairly small attack angles, as can be seen from Fig. 1. Wagner uses $\mu = 0.5$ in his equations for finite aspect ratio apparently because they hold exactly only for $\alpha \to 0$, in which case, of course, $\mu = 0.5$ is the correct value.

Note that while the total force on the airfoil is directed normal to the free stream and is identical with the lift, the total force on the planing surface is normal to the plate and is larger than the lift. Also, while the airfoil theory predicts the observed lift force only for small attack angles because of viscous separation on the upper surface, the planing theory should predict the force for much larger angles since very little danger of separation exists in the favorable pressure gradient. On the other hand, the planing surface flow is dependent on the formation of spray sheets which may be affected unpredictably by surface tension.

Use of Airfoil Pressure Distribution for Planing Surface

In order to gain some further insight into the usefulness of the airfoil theory for predicting planing parameters, we can examine the flow about the corresponding two-dimensional flat plates in more detail. In Fig. 6 the pressure distribution for the planing surface and airfoil bottom at $a = 10^o$ have been superposed with the scale arbitrarily chosen so that the trailing edges and stagnation points coincide. As shown, the pressure is very nearly the same over the greater part of the lengths, with a small discrepancy at the trailing edge. At the leading edge the differences are much greater because the flow on the airfoil has an infinite velocity rounding the thin front edge, while the velocity on the planing surface smoothly decreases back to the free stream magnitude. Wagner showed that the pressure distributions become more alike as the attack angle decreases, the differences becoming of second order as $a \rightarrow 0$. On the other hand even for attack angles as large as 30° the agreement remains fair but the differences at the leading and trailing edges become much more marked.[*]

Suppose now that with only the airfoil pressure distribution known we wish to estimate the lift on the planing surface. Of course, this problem is trivial in the two-dimensional case where, after all, the planing problem can be solved directly but the concept to be demonstrated may prove helpful for other situations where the planing flow has so far not permitted theoretical analysis. If we now look at the airfoil pressure distribution, Fig. 6, we know physically that the negative pressure near the leading edge will not occur on the planing surface since the planing flow velocity ahead of the stagnation point gradually increases back to free stream value, but never exceeds it. Hence, we might assume for a rough estimate, that the planing lift can be reckoned by integrating the airfoil bottom pressure from the trailing edge up to the point where the pressure becomes zero. When the calculation is carried out in this way (Appendix) the result shown in Fig. 7 is obtained. The agreement with the exact value for the lift is fairly good, within 6% at 10°, so that one gains some confidence in using this approach.

---

[*]The similarity of the pressure distribution for $a > 0$ was first pointed out by Weinig[2]. The values he presents in Fig. 24, Ref. 2, however, are not in complete agreement with those shown in Fig. 6. See the remarks in the appendix.

Extension of Wagner's High-Aspect-Ratio Theory for $a > 0$

We return now to the problem of finding the lift when the planing surface has a finite aspect ratio. Reviewing briefly our exact results, we see that we have the following equations:

$$C_L = \mu\, 2\pi \sin a \quad (\mathcal{R} = \infty, \quad a \geqq 0) \tag{1a}$$

$$C_L = \frac{\pi \sin a}{1 + \dfrac{2}{\mathcal{R}}} \quad (\mathcal{R} > 3, \quad a \to 0) \tag{2}$$

$$C_L = \frac{2\pi}{\pi + 4} \sin^2 a \cos a \quad (\mathcal{R} = 0, \quad a \geqq 0) . \tag{3}$$

We need, then, to find estimates for the remaining ranges of a and $\mathcal{R}$ by utilizing the airfoil theory.

Before utilizing Bollay's method of doing this, however, we can apply the airfoil theory more directly by assuming that Prandtl's lifting line analysis for the airfoil is valid for the planing surface. The factor in the denominator of Eq. (9) represents Prandtl's correction for the so-called downwash velocity and, if we assume the same downwash for airfoil and planing surface, even for attack angles $a > 0$, then the planing lift is easily calculated. In the case of the airfoil, the downwash velocity reduces the effective attack angle from a to $(a - a_i)$ where a_i is the downwash angle and is calculated from the three-dimensional potential theory as set up by Prandtl.[5] Then the flow at each section of the wing is assumed to be two dimensional and the lift is calculated from two-dimensional theory except that the geometrical attack angle a is replaced with the effective attack angle. It turns out that if the lift distribution is elliptic, the effective attack angle is the same everywhere along the span and is equal to

$$a - a_i = \frac{a}{1 + \dfrac{2}{\mathcal{R}}} \tag{15}$$

so that for a small we have the result of Eq. (9)

$$C_a = \frac{2\pi \sin a}{1 + \dfrac{2}{\mathcal{R}}} . \tag{9}$$

If we now assume that the downwash correction is the same for the planing surface, that is, that Eq. (15) applies to the planing surface as well, we write Eq. (1) in the form

$$C_L = \mu \; 2\pi \; \sin (\alpha - \alpha_i) \tag{16}$$

and substituting from Eq. (15) gives (α small)

$$C_L = \mu \; \frac{2\pi \; \sin \alpha}{1 + \dfrac{2}{\mathcal{R}}} \quad (\mathcal{R} > 3, \; \alpha \gtrless 0). \tag{17}$$

We know the assumption is valid for $\alpha \to 0$ because then Eq. (17) converges to Eq. (2). Moreover, as $\mathcal{R} \to \infty$, Eq. (17) converges to Eq. (1), which is reasonable. If the assumption is good, Eq. (17) should be usable up to 10° or so provided $\mathcal{R} > 3$, just as in the airfoil case. Although Eq. (17) is directly implied by Wagner's theory, he did not present it, possibly because it has not been derived rigorously. It appears worthwhile to attempt such a derivation by calculating the downwash due to a lifting line in a free surface, but no attempt will be made to do this here.

Unfortunately there are no reliable data with which to check Eq. (17) since, for $\mathcal{R} > 3$, the required small wetted length measurements are difficult to carry out. In Fig. 8 the theory has been plotted to see if at least the trends are correct, although the validity of the empirical formula is doubtful for $\mathcal{R}$ as high as 2, while the theory is not expected to apply very well below $\mathcal{R} = 3$. In any event, it can be seen that for $\mathcal{R} = 2$, the theory gives the right order of magnitude but the trend is noticeably wrong since the theory does not predict a sufficiently high lift for the higher attack angles. The discrepancy becomes more marked as $\mathcal{R}$ decreases, but this merely brings out the error one would be making in applying the Prandtl corrections for such small aspect ratios. Similar errors occur if Prandtl's theory is used for airfoils of small aspect ratio.[13] For $\mathcal{R} > 3$ the agreement between Eq. (17) and experiment should be better, but more measurements must be made before the exact range of validity will be known.

Estimate of Ratio of Planing to Airfoil Lift

We are now in a good position to estimate the ratio of the planing surface lift to that on an equivalent airfoil. Since several limiting cases are known and the variation of the ratio in between is fairly small, any reasonable

guess which converges to the proper limits should furnish a useful answer.

The ratio C_L/C_a of Eq. (11b) can now be replaced by a better estimate by utilizing the result derived just above. Dividing Eq.(17) by Eq. (9) we have

$$\frac{C_L}{C_a} = \mu \quad (\mathcal{R} > 3, \ a \gtrless 0) \tag{18}$$

which now replaces Eq. (11b). For the other limit we still have the result of Eq. (11a):

$$\frac{C_L}{C_a} = \frac{\pi}{\pi + 4} \quad (\mathcal{R} = 0, \ a \gtrless 0). \tag{11a}$$

A fairly simple expression which converges to these required limits is

$$\frac{C_L}{C_a} = \frac{\mu \frac{\mathcal{R}}{2} + \frac{2}{\pi+4} \sin a}{\frac{\mathcal{R}}{2} + \frac{2}{\pi} \sin a} \quad (\text{all } \mathcal{R}). \tag{19}$$

It is not proposed that Eq. (19) is the "correct" expression, but rather that it is correct in the limits represented by Eqs. (11a) and (18). Moreover, it varies monotonically in between and it seems reasonable that the lift ratio should vary smoothly between the extremes of $\mathcal{R} = 0$ and $\mathcal{R} = \infty$.

An experimental check on Eq. (19), not involving planing data, is given by the pressure measurements of Winter[15] on airfoils. As discussed above (Fig. 7), the ratio of C_L/C_a should be about the same as the ratio of the lift on the bottom side C_b of the equivalent airfoil to its total lift C_a. Winter's measurement for $\mathcal{R} = 1$ and $a = 10.5°$ (Sambraus' correction) gives[10]

$$\frac{C_b}{C_a} = 0.39$$

while Eq. (19) gives

$$\frac{C_L}{C_a} = 0.388,$$

which is very close indeed. This check, along with the limiting cases, gives one considerable assurance that Eq. (19) is a good assumption. The computed values for Eq. (19) are plotted in Fig. 9.

Calculation of Planing Lift from Airfoil Theory and Experiment

We may now proceed to calculate the lift on the planing surface by using Eq. (19) and either some suitable airfoil theory or airfoil experimental measurements. The former course would be preferable but, as discussed previously, the low aspect ratio airfoil theory is only in fair agreement with data and it seems unlikely that it would prove reliable for use in the planing surface analogy. Just the same, we can carry out the calculation to see in what way the resulting theory is deficient. From Eqs. (12) and (19) we have

$$C_L = \left(\frac{C_L}{C_a}\right) \cdot (C_a)_{theory} \tag{20}$$

or

$$C_L = \mu \, 2\pi \, \frac{\tanh \dfrac{1}{\dfrac{R}{2} + \dfrac{2}{\pi} \sin \alpha}}{1 + \tanh \dfrac{1}{\dfrac{R}{2} + \dfrac{2}{\pi} \sin \alpha}} \left(\frac{R}{2} + \frac{2 \sin \alpha}{\mu(\pi + 4)}\right) \sin \alpha \tag{20a}$$

which has been plotted in Fig. 10. The discrepancies between theory and experiment are of the same type and order of magnitude as for the airfoil (Fig. 5). A better agreement would be expected if a more accurate airfoil theory were available for use.

Turning now to Winter's airfoil data, Fig. 5, we calculate the lift on the planing surface from

$$C_L = \left(\frac{C_L}{C_a}\right) \cdot (C_a)_{experiment} \tag{21}$$

where (C_L/C_a) is given by Eq. (19) as before. The results with experimental values for comparison are presented in Fig. 11. As can be seen, the agreement is satisfactory.

It is interesting to note that, while the lift vs. attack angle curve is nonlinear for both $R = 0$ and $R = \infty$ (Fig. 11), it is nearly linear at some

intermediate aspect ratio near $\mathcal{R} = 1$. This may explain the success Sottorf[10] and Sedov[9] had in using a linear relationship. On the other hand, Sambraus'[16] efforts to use a nonlinear curve can be better appreciated, since for very small aspect ratio his approach would be justified. Hence both points of view are in part correct.

Conclusion

The estimate of the aspect ratio effect on flat planing surfaces has now been completed. For $\mathcal{R} > 3$, Eq. (17) can be used, while for $\mathcal{R} < 3$, the values in Fig. 11 should prove adequate, with Eq. (5) holding exactly for $\mathcal{R} = 0$. If more convenient, empirical formulas for the lift may be used, but it should be remembered that they cannot be expected to hold for any extreme range of parameters. It should be noted that the present estimate takes no account of viscosity, gravity, or surface tension, but in many applications these effects are negligible or can be accounted for in some separate calculation. It is hoped that the approach used here will again call attention to the usefulness of airfoil methods in predicting planing surface parameters.

APPENDIX

Pressure on Flat Plate Planing Surface

The pressure on the flat planing surface[1,3] located along the positive x-axis with trailing edge at $x = 0$ is given by solving simultaneously

$$C_p = 1 - \left(\frac{\cos \beta - \cos \alpha}{1 - \cos \beta \cos \alpha + \sin \alpha \sqrt{1 - \cos^2 \beta}} \right)^2 \qquad (22)$$

and

$$\frac{x \pi}{\delta} = \frac{1}{1 - \cos \alpha} \left[(1 + \cos \beta) \cos \alpha - (1 - \cos \alpha) \ln \left(\frac{1 - \cos \beta}{2} \right) \right.$$

$$\left. - \sqrt{1 - \cos^2 \beta} \, \sin \alpha - \beta \sin \alpha + \pi \sin \alpha \right] \qquad (23)$$

for the same values of the parameter β, where x is measured forward from the trailing edge and δ is the spray thickness. The "wetted length" ℓ is given by

$$\frac{\ell \pi}{\delta} = \left[\frac{1 + \cos \alpha}{1 - \cos \alpha} - \ln \left(\frac{1 - \cos \alpha}{2 \cos \alpha} \right) + \frac{\pi \sin \alpha}{1 - \cos \alpha} \right], \qquad (24)$$

and the stagnation point x_s, given by $C_p = 0$, is

$$\frac{x_s \pi}{\delta} = \frac{1}{1 - \cos \alpha} \left[(1 + \cos \alpha) \cos \alpha - (1 - \cos \alpha) \ln \left(\frac{1 - \cos \alpha}{2} \right) \right.$$

$$\left. - \sqrt{1 - \cos^2 \alpha} \, \sin \alpha - (\sin \alpha)(\alpha) + \pi \sin \alpha \right]. \qquad (25)$$

The normal force is found by integrating the pressure, so

$$C_N = \frac{N}{\rho V^2/2\ell} = \frac{2 \pi \sin \alpha}{1 + \cos \alpha - (1 - \cos \alpha) \ln \left(\frac{1 - \cos \alpha}{2 \cos \alpha} \right) + \pi \sin \alpha}, \qquad (26)$$

which must be multiplied by $\cos a$ to obtain the lift coefficient.[*] Finally, we have, defining μ such that $C_L = \mu \cdot 2\pi \sin a$,

$$\mu = \frac{\cos a}{1 + \cos a - (1 - \cos a)\ln\left(\frac{1 - \cos a}{2 \cos a}\right) + \pi \sin a} \tag{27}$$

which is plotted in Fig. 1.

Pressure Distributions on Flat Plate Airfoil

The velocity distribution on a flat plate airfoil of chord l located along the x-axis with leading edge at $x = -l/2$ and trailing edge at $x = +l/2$ in (Ref. 5, p. 38).

$$\frac{v_x}{V} = \frac{\sin a}{\sin \theta} - \sin a \cot \theta + \cos a \tag{28}$$

where V is the free stream velocity directed at an angle a to the plate and

$$\cos \theta = \frac{2x}{l} . \tag{29}$$

Since the pressure coefficient is defined by $C_p = (p - p_o)/\rho\frac{V^2}{2}$, we have from Bernoulli, $C_p = 1 - (v_x/V)^2$, or

$$C_p = \sin^2 a \left(1 - \tan^2 \frac{\theta}{2}\right) - 2 \cos a \sin a \tan \frac{\theta}{2} . \tag{30}$$

This relationship has been plotted in Fig. 6 for the angle $a = 10^o$. The scale has been adjusted so that the stagnation point is at one. This can be done by noting that the stagnation point is given by $v_x = 0$ or

$$x = -\frac{l}{2}(1 - 2 \sin^2 a) . \tag{31}$$

The values for the pressure on the airfoil given by Weinig[2] are not in agreement with those computed from Eq. (30) above. So far as can be learned from the brief explanation in Ref. 2, the results should be the same. It is not known why the discrepancy occurs, but it is clear that Weinig's curves

---

[*]Pierson and Leshnover[3] refer to Eq. (26)as the "mean normal lift", but the term "lift" is reserved here for the force perpendicular to the free stream, as is usual in aeronautical terminology.

cannot represent the pressure values desired because they all go to zero at the trailing edge of the airfoil, which is not the correct value for $a > 0$.

The point where the pressure is zero x_o is given by putting $v_x/V = -1$ in Eq. (28) so that

$$\theta = -\pi + a \tag{32}$$

corresponds to this point.

In order to find the lift due to the bottom positive pressure, L_b, we integrate the pressure from the trailing edge up to the point of zero pressure:

$$C_b = \frac{L_b}{\rho \frac{V^2}{2} l} = \int_{x=l/2}^{x=x_o} C_p \frac{dx}{l} \tag{33}$$

$$= \int_{\theta=-\pi+a}^{2\pi} \left[\sin^2 a (1 - \tan^2 \frac{\theta}{2}) - 2 \cos a \sin a \tan \frac{\theta}{2} \right] \left[\frac{d\theta}{-2 \sin \theta} \right]$$

so finally

$$C_b = \sin^2 a \left(1 + 2 \ln \sin \frac{a}{2} \right) + (\pi - a) \sin a \cos a \ . \tag{34}$$

Taking the ratio of C_b to the total airfoil lift $C_a = 2\pi \sin a$, we have

$$\frac{C_b}{C_a} = \frac{1}{2\pi} \sin a \left(1 + 2 \ln \sin \frac{a}{2} \right) + \frac{\pi - a}{2\pi} \cos a \ . \tag{35}$$

Equation (35) has been plotted in Fig. 7.

REFERENCES

1. Wagner, H., Uber Stoss- und Gleitvorgange an der Oberflache von Flussigkeiten, Z. angew. Math. u. Mech., Vol. 12, Aug. 1932, pp. 193-215.

2. Wagner, H., Planing of Watercraft, NACA T.M. No. 1139, Washington, D.C., 1948.

3. Pierson, J.D., and Leshnover, S., An Analysis of the Fluid Flow in the Spray Root and Wake Regions of Flat Planing Surfaces, Experimental Towing Tank Report No. 335, Stevens Inst. Tech., Hoboken, N.J., 1948. (Fairchild Pub. Fund preprint No. 166, Inst. Aero. Sc., New York, N.Y.)

4. Green, A.E., The Gliding of a Plate on a Stream of Finite Depth, Proc. Cambridge Phil. Soc., Vol. 31, 1935, pp. 589-603; also Vol. 32, 1936, pp. 67-85, 248-252; and Vol. 34, 1938, pp. 167-184.

5. Durand, W.F., (editor), Aeordynamic Theory, Julius Springer, Berlin, 1935, Vol. II.

6. Lamb, H., Hydrodynamics, 6th ed., Dover Publications, New York, N.Y., 1945, pp. 99-102.

7. Bollay, W., A Contribution to the Theory of Planing Surfaces, Proc. 5th Intl. Congress for App. Mech., John Wiley & Sons, New York, N.Y., 1939, pp. 474-477.

8. Korvin-Kroukowsky, B.V., et al., Wetted Area and Center of Pressure of Planing Surfaces, Experimental Towing Tank Report No. 360, Stevens Inst. Tech., Hoboken, N.J., 1949. (Fairchild Pub. Fund preprint No. 244, Inst. Aero. Sc., New York, N.Y.).

9. Sedov, Planing on a Water Surface, R.T.P. Translation No. 2506, reprinted by Durand Reprinting Committee, Pasadena, Calif.

10. Sottorf, W., Analysis of Experimental Investigations of the Planing Process on Water, NACA T.M. No. 1061, Washington, D.C., 1944.

11. Korvin-Kroukowsky, B.V., Lift of Planing Surfaces, J. Aeronaut. Sci., Vol. 17, 1950, pp. 597-599.

12. Lawrence, H.R., The Lift Distribution on Low Aspect Ratio Wings at Subsonic Speeds, J. Aeronaut. Sci., Vol. 18, 1951, pp. 683-695.

13. Bollay, W., A Nonlinear Wing Theory and its Application to Rectangular Wings of Small Span, Z. angew. Math. u. Mech., Vol. 19, 1939, pp. 21-35.

14. Weinig, F., Lift and Drag of Wings with Small Span ,
    NACA T.M. No 1151, Washington, D.C., 1947.

15. Winter, Flow Phenomena on Plates and Airfoils of Short
    Span, NACA T.M. No. 798, Washington, D.C., 1936.

16. Sambraus, A., Planing Surface Tests at Large Froude
    Number - Airfoil Comparison, NACA T.M. No. 848, Washington,
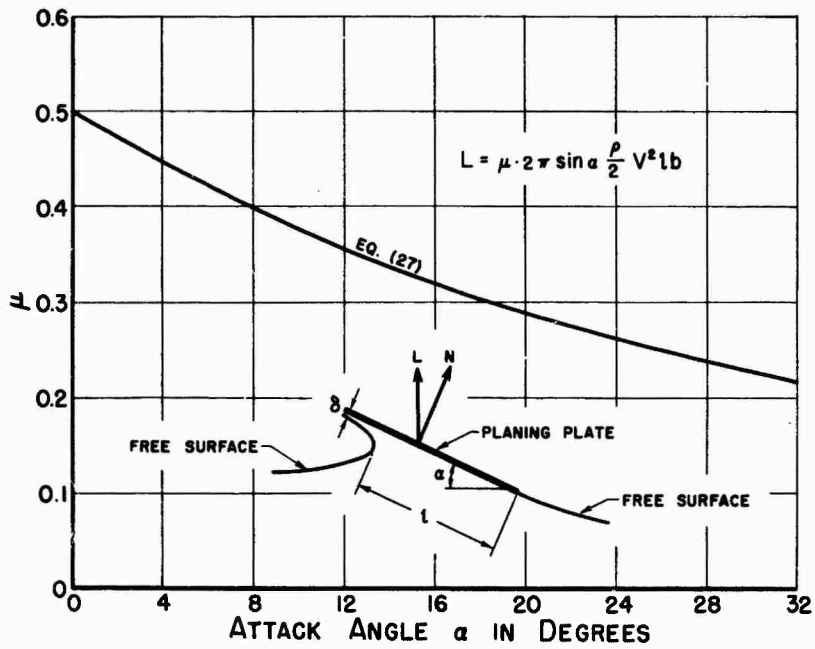   D.C., 1938.

$$L = \mu \cdot 2\pi \sin a \, \frac{\rho}{2} \, V^2 lb$$

EQ. (27)

Fig. 1 - The lift of a two-dimensional flat planing surface
according to the theory of Wagner
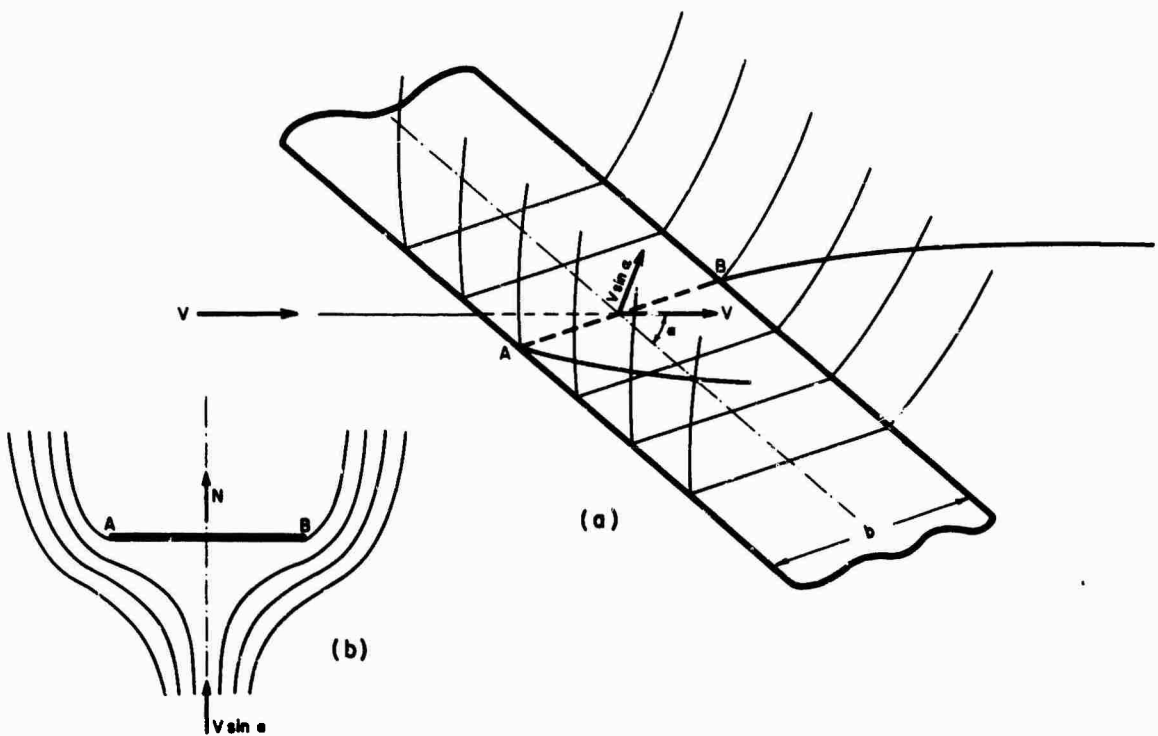
Fig. 2 - (a) The flow about an infinitely long flat planing surface
(after Bollay).   (b) Section A - B showing the flow
in a plane perpendicular to the plate.
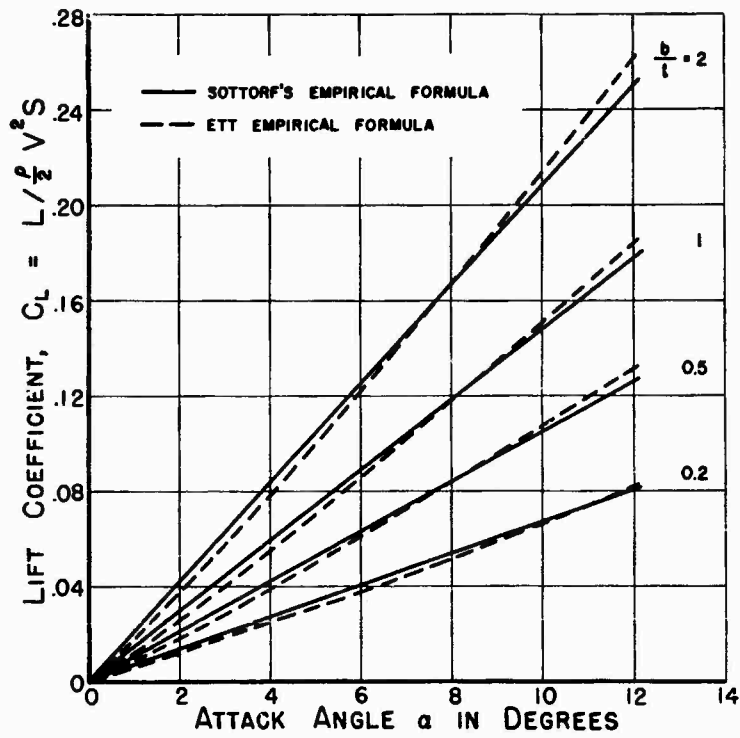
Fig. 3 - Empirical formulas for the lift of flat planing surfaces.
Gravity is neglected.
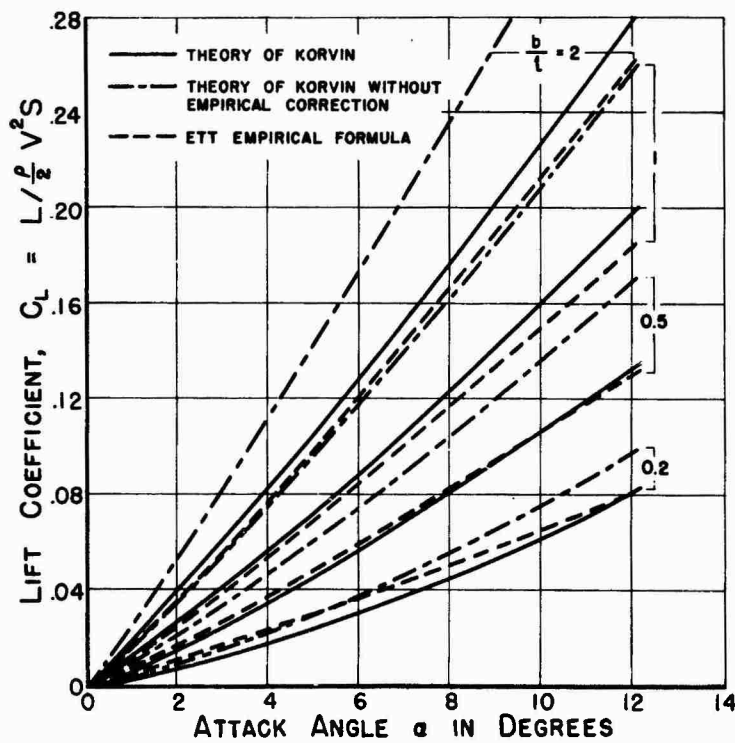
Fig. 4 - The lift of flat planing surfaces according to Korvin
compared with experiment
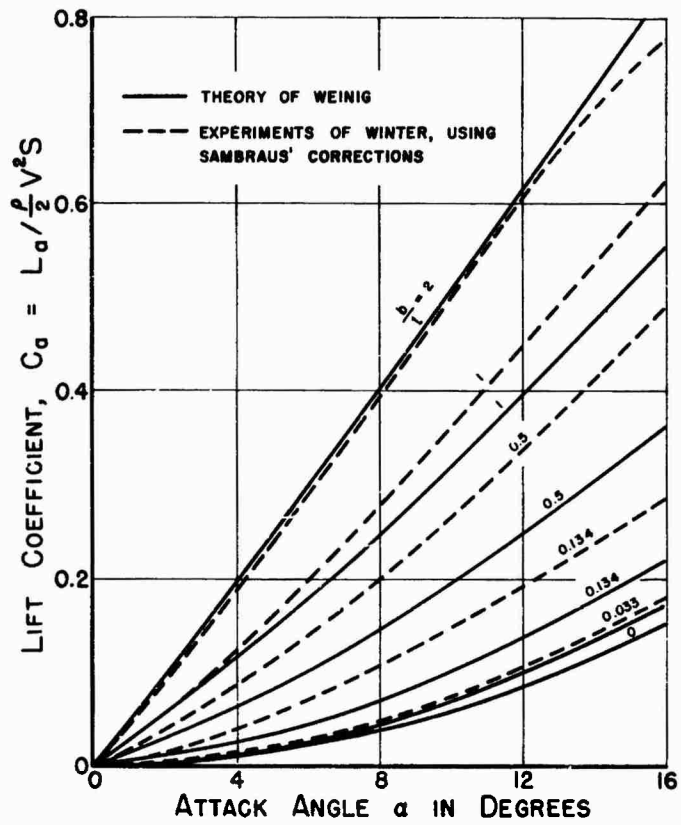
Fig. 5 - The lift of flat rectangular airfoils according to
theory and experiment
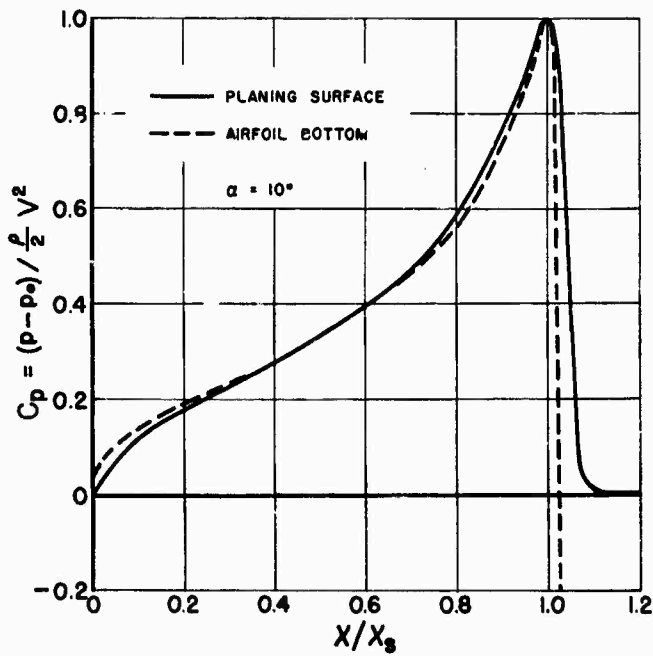
Fig. 6 - The calculated pressure distribution on two-dimentional
flat plate airfoil and planing surface at $\alpha = 10^{O}$
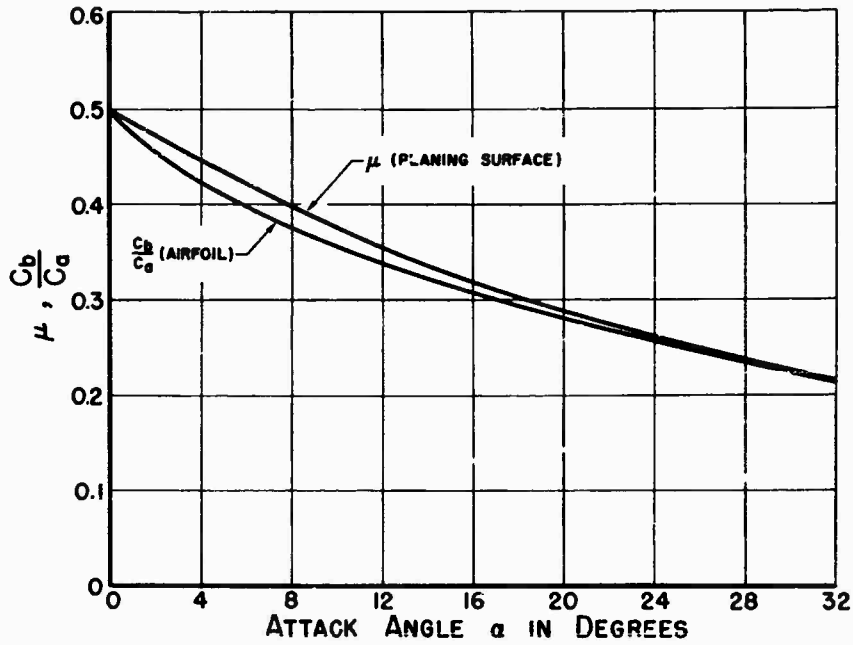
Fig. 7 - The lift due to the bottom pressure on a flat airfoil compared
with planing surface lift. See Appendix for calculations.
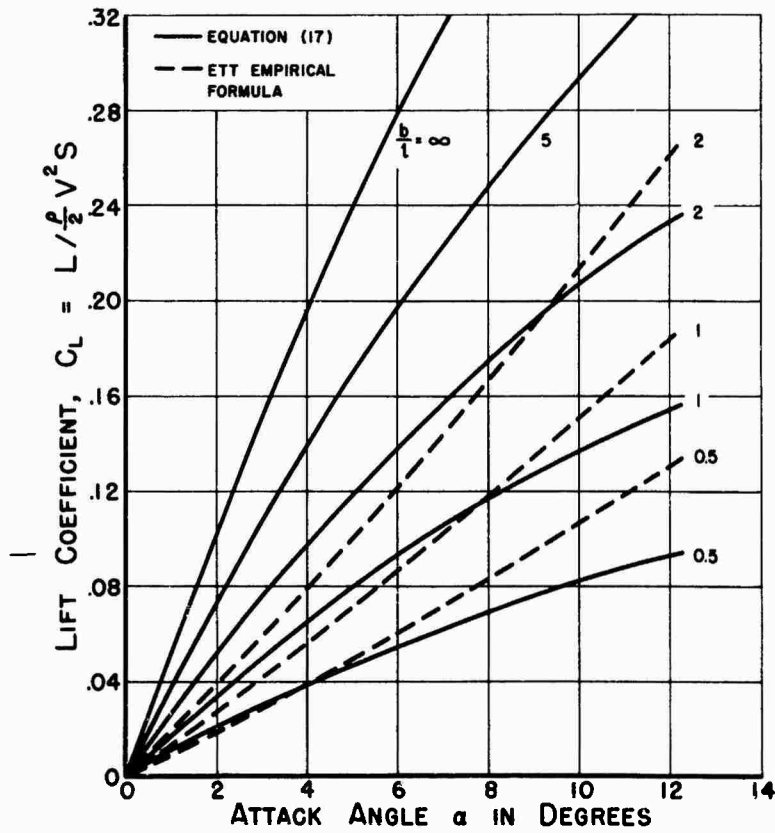
Fig. 8 - The planing lift as predicted by high aspect ratio theory,
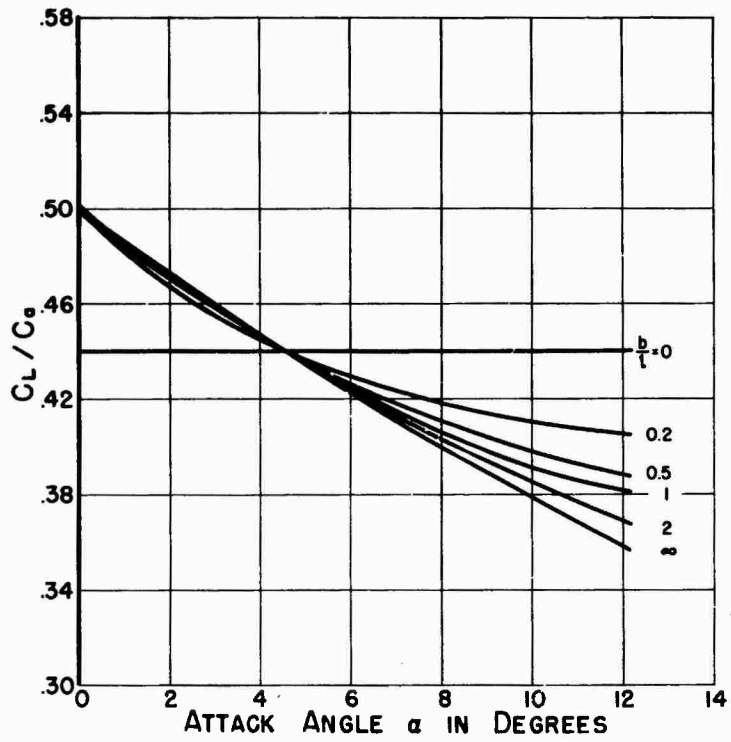Eq. (17), compared with experiment

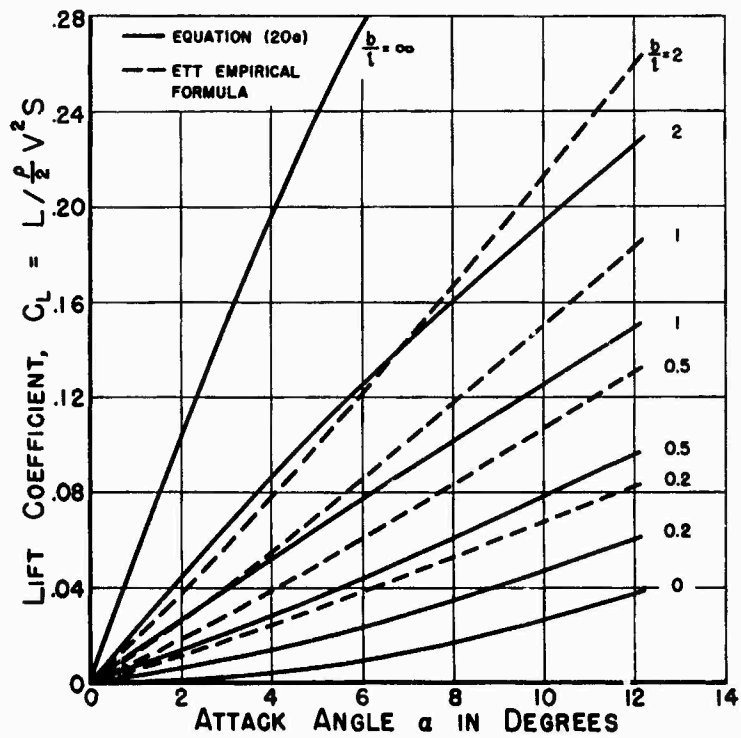Fig. 9 - The ratio of planing to airfoil lift as assumed by Eq. (19)

Fig. 10 - The lift of rectangular flat planing surfaces calculated from Weinig's airfoil theory and Eq. (19), compared with experiment
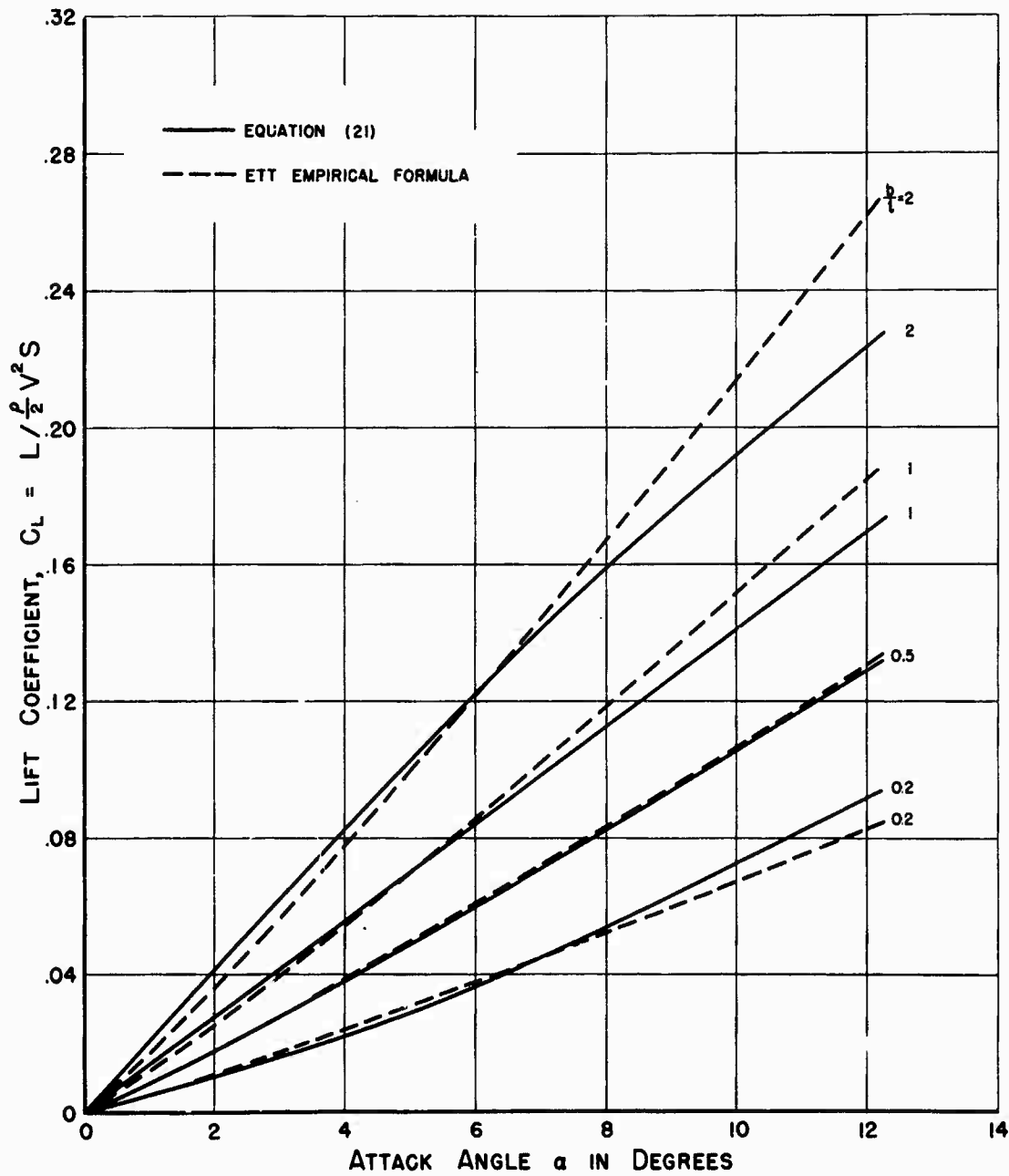
Fig. 11 - The lift of rectangular flat planing surfaces calculated from
Winter's airfoil data and Eq. (19), compared with experiment