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REPORT

AN ELEMENTARY TREATMENT OF THE MOTION OF A SPINNING PROJECTILE ABOUT ITS CENTER OF GRAVITY

(Revised Ballistic Research Laboratory Report No. 85)

BY

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AND

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REPORT NO. 459 11 April 1944

ORDNANCE RESEARCH CENTER
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Ballistic Research
Laboratory Report No. 459

Ordnance Research Center
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Kent/McShane/app
Aberdeen Proving Ground, Md.
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AN ELEMENTARY TREATMENT OF THE MOTION OF A SPINNING PROJECTILE ABOUT ITS CENTER OF GRAVITY

(Revision of Ballistic Research Laboratory Report No. 85)

Abstract

This report is intended to replace Ballistic Research Laboratory Report No. 85, by R.H. Kent. Its first twelve pages are taken without change from Report No. 85. In those pages the equations of motion are derived for a projectile traversing a rectilinear trajectory and acted on by drag, cross-wind force and overturning torque only. The method of determining the stability factor s is given. In the later sections the damping and the precessional yaw are discussed in the light of recent developments.

A spin is imparted to most artillery projectiles to prevent their tumbling. It is of interest in designing projectiles and the rifling of cannon to know what spin is required to produce a stable motion of the projectile about its centre of gravity. The object of this paper is to provide simply and clearly the theoretical basis for the estimation of the required spin and to describe analytically the motion of a yawing, spinning projectile. The treatment is, with a few exceptions, an amplification of that given in the British Report No. 422 A.A.E.S. M.I.D. which was written by Fowler, Gallop, Lock and Richmond.*

The procedure by which the equations of motion are deduced is as follows:

(a) Angular coordinates for the motion about the center of gravity are defined.

(b) An expression is obtained for the angular momentum about any line through the centre of gravity.

* See also their article in Phil. Trans. Roy. Soc. A. Vol. 222 pp. 295-387 (1920).

[redacted]

(c) An expression for the kinetic energy of spin is deduced.

(d) The system of forces acting on the projectile is described.

(e) The equations of motion are obtained by the application of the principle of the conservation of energy and of the principle that the vector rate of change of the angular momentum is equal to the vector moment of the resultant force.

Coordinate System

The projectile is assumed to be moving in a straight line. The position of the axis of the projectile with reference to its rectilinear trajectory is defined by two angles δ and φ (see figure 1). The angle of yaw, δ, is the angle between the axis of the projectile and the trajectory, while φ, called the angle of orientation, is the angle between the vertical plane including the trajectory and the plane of the yaw including the trajectory and the axis.

Angular Velocities or Spins

The angular velocity or spin of a rigid body is a vector, i.e., it has a direction and magnitude.* The direction of the spin is by definition the direction of the axis about which the body spins.

The spin $\frac{d\delta}{dt}$, where t is the time, is designated by $\dot{\delta}$. It is evident that $\dot{\delta}$ is the spin about the axis which is perpendicular to the plane of the yaw. The spin $\frac{d\varphi}{dt}$, designated by $\dot{\varphi}$, is the spin about the trajectory, OT. The component of the spin of the projectile about its axis of symmetry is designated by N.

We now define three perpendicular axes, X,Y,Z of (see Fig. 1) the projectile and ascertain the components of the vector spin about the three axes.

The X axis is the axis of symmetry of the projectile pointing toward the nose. The Y axis is perpendicular to X and to the trajectory OT and passes through the centre of gravity O. The Z axis is perpendicular to the X and Y axes and hence lies in the plane of the yaw. If φ and δ are less than $\pi/2$, to an observer looking along the trajectory, Y points to the left of, and Z above the trajectory.

* See Jeans' Theoretical Mechanics for a discussion of spin, moment of momentum and kinetic energy of spin. In this discussion, the word 'spin' is defined to be fully equivalent to 'angular velocity'.

We now designate the components of the vector spin about the three axes by ω_x, ω_y and ω_z respectively and evaluate ω_x, ω_y and ω_z in terms of δ, $\dot{\varphi}$ and N.

N is by definition the component of the vector spin about the axis of the projectile X, hence

$$\omega_x = N.$$

The spin ω_y is the spin about Y, but $\dot{\delta}$ is also the spin about Y; hence

$$\omega_y = \dot{\delta}.$$

The spin $\dot{\varphi}$ is the spin about OT. It has no component about Y since OT and Y are perpendicular. The component of $\dot{\varphi}$ about Z is

$$\dot{\varphi} \cos \text{ZOT} = \dot{\varphi} \cos (\pi/2 + \delta) = -\dot{\varphi} \sin \delta.$$

Neither $\dot{\delta}$ nor N has a component about Z, hence

$$\omega_z = -\dot{\varphi} \sin \delta.$$

The results for ω_x, ω_y and ω_z are tabulated below.

$$\begin{aligned} \omega_x &= N \\ \omega_y &= \dot{\delta} \\ \omega_z &= -\dot{\varphi} \sin \delta. \end{aligned} \qquad (1)$$

Angular Momentum

It has been mentioned that spin is a vector. It may be shown that angular momentum is also a vector. If the moments of inertia of a rigid body about the three axes, X, Y, Z are respectively

$$I_x, I_y, I_z,$$

and the X, Y, Z axes are principal* axes of inertia, the components of the vector angular momentum about these axes are as follows:

* A principal axis is one about which the moment of inertia is an extremum. The X, Y, Z axes of the projectile as defined on page 2 are principal axes.

Axis	Component of Angular Momentum
X	$I_x\omega_x$
Y	$I_y\omega_y$
Z	$I_z\omega_z$

In the case of the projectile the moments I_y and I_z are equal and each may be represented by B while I_x is represented by A. For the present however, we shall continue to use the symbols I_x, I_y, I_z.

Suppose a line through O makes angles, ξ, η and ζ with the axes X, Y, and Z; then it follows from the vector properties of the angular momentum that its component along the given line is

$$I_x\omega_x \cos\xi + I_y\omega_y \cos\eta + I_z\omega_z \cos\zeta \,. \tag{2}$$

Kinetic Energy of Spin

If the projectile has a spin ω_x about X while ω_y and ω_z are zero, its kinetic energy of rotation is $\frac{I_x\omega_x^2}{2}$. If ω_x and ω_z are zero, then its kinetic energy is $\frac{I_y\omega_y^2}{2}$, while if $\omega_x = \omega_y = 0$, its kinetic energy is $\frac{I_z\omega_z^2}{2}$. If the projectile has at a given instant spins ω_x, ω_y, and ω_z about the three axes, and if X, Y, and Z are principal axes of inertia (as they are in the present problem) then the kinetic energy of spin is

$$\frac{1}{2}\left[I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2\right] \,. \tag{3}$$

Force System of the Projectile*

Experiment has shown that aside from the force of gravity, W, the force system acting on the projectile may be approximately represented by a single air force, R (see figure 2) acting in the plane of the yaw and intersecting the axis of the projectile at P, called the center of pressure. The component of R in a direction opposite to the direction of motion is called the drag and is designated by D while the component of R perpendicular to the direction of motion is called the cross wind force and is designated by L.

Experiment indicates that for small angles of yaw D is approximately constant while L varies as sin δ. We accordingly represent L by $\lambda \sin \delta$, where λ is taken to be constant. The forces D and L tend to overturn the projectile. To obtain the overturning moment, we first add the components of D and L perpendicular to the axis and then multiply by the arm (see figure (2)). The sum of the two components is

$$D \sin \delta + \lambda \sin \delta \cos \delta,$$

while the arm is $\overline{OP}$.

The moment, designated by M, about O is therefore

$$\overline{OP}(D \sin \delta + \lambda \sin \delta \cos \delta).$$

If cos δ is taken as unity and $\overline{OP}(D + \lambda)$ is represented by μ, called the moment factor, we have

$$M = \mu \sin \delta.$$

This is the overturning moment acting on the projectile. The factor μ is taken to be constant, which implies that $\overline{OP}$ is constant.

The Equations of Motion

As mentioned in the preceding, one of the principles used in obtaining the equations of motion is that the rate of change of the vector angular momentum is equal to the vector moment of the applied force. For this principle to be valid however, the rate of change of angular momentum must be obtained with reference to a suitable reference frame. A reference frame stationary

* See the chapter on Exterior Ballistics of the forthcoming Hayes' Ordnance and Gunnery for a more nearly complete description.

with respect to the earth satisfies the requirements closely enough. We use this principle to determine the variation of N, the axial spin.

Consider a line, Q fixed in the reference frame. If this line makes angles ξ, η, and ζ with the three axes and these are principal axes of inertia then the component of the angular momentum along Q is given by (2) as

$$I_x\omega_x \cos \xi + I_y\omega_y \cos \eta + I_z\omega_z \cos \zeta \, . \qquad (2)$$

The moment about Q is given by the derivative of (2) with respect to t,

$$\begin{aligned} &I_x (\cos \xi)\, \dot\omega_x + I_y (\cos \eta)\, \dot\omega_y + I_z (\cos \zeta)\, \dot\omega_z \\ &-I_x\omega_x (\sin \xi)\, \dot\xi - I_y\omega_y (\sin \eta)\, \dot\eta - I_z\omega_z (\sin \zeta)\, \dot\zeta \end{aligned} \qquad (2a)$$

Let Q coincide with the X axis at the time t, then as may be seen from Fig. 6

$$\xi = 0^{\circ}, \quad \eta = \zeta = \pi/2; \quad \dot\eta = \omega_z, \quad \dot\zeta = -\omega_y.$$

The moment about Q given by (2a) is zero at time t, since both W and R intersect the X axis which coincides with Q at time t. Hence at time t,

$$I_x\dot\omega_x + \omega_y\omega_z (I_z - I_y) = 0.$$

But, since the projectile is symmetrical

$$I_z = I_y, \text{ and } I_x\dot\omega_x = 0.$$

Since the instant chosen is any instant it follows that $\dot\omega_x$ is always zero and hence $\omega_x = N =$ constant.*

We now derive an expression for the angular momentum about the trajectory OT. According to equation (2) this is

$$I_x\omega_x \cos \xi + I_y\omega_y \cos \eta + I_z\omega_z \cos \zeta \, ,$$

if ξ, η, and ζ are the angles OT makes with X, Y and Z respectively.

*This proof that N is constant is due to Dr. Charters. Since this report was written it has been discovered that there is a spin destroying couple which causes an appreciable loss of spin for the longer times of flight. See Ballistic Research Laboratory Report No. 154.

It may be seen from figure (3) that

$$\xi = \delta$$

$$\eta = \pi/2$$

$$\zeta = \pi/2 + \delta.$$

If these values are used and if A, B and B are substituted for I_x, I_y and I_z respectively and the values of ω_x, ω_y and ω_z are obtained from (1), the angular momentum about OT is found to be

$$AN \cos \delta + B \sin^2\delta\, \dot{\varphi}.$$

Since the force R is in the plane of the yaw it intersects OT and has no moment about it, neither has W, hence

$$\frac{d(AN \cos \delta + B \dot{\varphi} \sin^2\delta)}{dt} = 0$$

and $$AN \cos \delta + B \sin^2\delta\, \dot{\varphi} = F, \qquad (4)$$ ✓

where F is a constant.

As the yaw increases, the moment, $\mu \sin \delta$, does work on the projectile and therefore increases its kinetic energy. If K is the kinetic energy at any time, t, K_o is the initial kinetic energy and W is the work done, we have

$$K = K_o + W. \qquad (5)$$

The work done by the couple while δ increases from δ_o to δ is

$$\mu \sin \delta\, d\delta = \mu(\cos \delta_o - \cos \delta) = W. \qquad (6)$$

From (3)

$$K = \frac{1}{2}\left[I_x\omega_x^2 + I_y\omega_y^2 + I_z\omega_z^2\right].$$

If the I's are expressed in terms of A and B and the ω's in terms of δ, φ and N (see (1)) it is found that

$$K = \frac{1}{2}\left[AN^2 + B(\dot{\delta}^2 + \sin^2\delta\, \dot{\varphi}^2)\right].$$

Hence from (5) and (6)

$$B(\dot{\delta}^2 + \sin^2\delta\, \dot{\varphi}^2) = 2K_o + 2\mu(\cos \delta_o - \cos \delta) - AN^2. \qquad (7)$$ ✓

[redacted]

Now $\cos \delta \approx 1 - \frac{\delta^2}{2}$ and $\cos \delta_0 \approx 1 - \frac{\delta_0^2}{2}$. If these substitutions are made and the constant E is substituted for the constant sum,

$$2K_0 + \frac{2\mu\delta_0^2}{2} - AN^2, \text{ (7) becomes}$$

$$B(\dot{\delta}^2 + \sin^2\delta\, \dot{\varphi}^2) = \mu\delta^2 + E,$$

or, if $\sin \delta$ is replaced by δ,

$$B(\dot{\delta}^2 + \delta^2\dot{\varphi}^2) = \mu\delta^2 + E. \tag{8}$$

If the initial values are designated by subscript zeros, it follows that

$$E = B\,(\dot{\delta}_0^2 + \delta_0^2\dot{\varphi}_0^2) - \mu\delta_0^2\,. \tag{8a}$$

If $\sin \delta$ is replaced by δ in (4), $\cos \delta$ by $1 - \frac{\delta^2}{2}$ and the equation is divided by B it becomes.

$$\delta^2\,\dot{\varphi} = \frac{AN}{2B}\,\delta^2 - \frac{AN}{B} + \frac{F}{B}\,.$$

If the initial value of δ, δ_0 is designated β, it follows that

$$\delta^2\dot{\varphi} = \beta^2\dot{\varphi}_0 + \frac{AN}{2B}\,(\delta^2 - \beta^2) = \beta^2\dot{\varphi}_0 + \frac{\Omega}{2}\,(\delta^2 - \beta^2), \tag{9}$$

if $\Omega = \frac{AN}{B}$. If h is substituted for $\dot{\varphi}_0 - \Omega/2$, (9) may be written as

$$\dot{\varphi} = \frac{h\,\beta^2}{\delta^2} + \frac{\Omega}{2}\,. \tag{10}$$

If $\dot{\varphi}$ from (10) is substituted in (8) the latter becomes upon multiplying by δ^2 and dividing by B,

$$\delta^2\dot{\delta}^2 + (h\,\beta^2 + \tfrac{1}{2}\,\Omega\,\delta^2)^2 = \frac{E}{B}\,\delta^2 + \frac{\mu\delta^4}{B}\,. \tag{11}$$

The substitution

$$\delta^2 = x,\ 2\,\delta\,\dot{\delta} = \dot{x}$$

is now made and equation (11) becomes

[redacted]

$$\frac{1}{4}\dot{x}^2 + \frac{1}{4}(\Omega^2 - 4\mu/B)\, x^2 + a_1 x + a_2 = 0 \tag{12}$$

where

$$a_1 = h\beta^2\Omega - \frac{E}{B} = -\beta^2\left[-\dot{\varphi}_0\Omega + \Omega'^2/2 + \dot{\varphi}_0^2 + \frac{\dot{\delta}_0^2}{\beta^2} - \frac{\mu}{B}\right] =$$

$$-\beta^2\left[(\dot{\varphi}_0 - \Omega/2)^2 + \frac{\dot{\delta}_0^2}{\beta^2} + \frac{\Omega^2}{4}\left(1 - \frac{4B\mu}{A^2N^2}\right)\right] \tag{13}$$

and the value of the constant a_2 is of no interest because it will be eliminated in the process of forming the equation of motion. We now make the substitution

$$s = \frac{A^2N^2}{4B\mu} = \frac{B\Omega^2}{4\mu}\ .$$

Equation (12) becomes, on differentiation and division by $\frac{\dot{x}}{2}$,

$$\ddot{x} + \Omega^2(1 - 1/s)x + 2a_1 = 0.$$

Making the substitution,

$$x = y - \frac{2a_1}{\Omega^2(1 - 1/s)}\ ,$$

we obtain

$$\ddot{y} + \Omega^2(1 - 1/s)y = 0,$$

or if

$$p^2 = \Omega^2(1 - 1/s), \tag{14}$$

$$\ddot{y} + p^2 y = 0. \tag{15}$$

The solution is of the form

$$y = C_1 e^{pit} + C_2 e^{-pit}$$

where

$$i = \sqrt{-1}.$$

If p^2 is negative, p is imaginary and pi is real. Hence, unless $C_2 = 0$, the value of y will increase indefinitely.

Because of this fact the motion about $y = 0$ is said to be unstable when p^2 is negative.

It is evident from (14) that

$$p^2 \text{ is negative if } s < 1.$$

If $s > 1$, then p^2 is positive, and the solution of (15) is known to be

$$y = a \cos (p t + \varkappa) \tag{16}$$

where a and $\varkappa$ are constants depending upon the initial conditions.

It is thus shown that when $s = \frac{A^2N}{4B\mu} > 1$, y oscillates about $y = 0$ or δ^2 about $\frac{-2a_1}{\Omega^2(1-1/s)}$. Never does y depart more than 'a' from $y = 0$. The motion is stable about $y = 0$.

In view of that fact that the motion is stable only when $s > 1$, s is called the stability factor.

From the previous substitutions and (16) it may be shown that

$$\delta^2 = b + a \cos (pt + \varkappa), \tag{17}$$

where

$$b = \frac{-2a_1}{\Omega^2(1-1/s)} \pm \frac{2\beta^2 \left[(\phi_0 - \Omega/2)^2 + \frac{\dot{\delta}_0^2}{\beta^2} + \frac{\Omega^2}{4} (1-1/s) \right]}{\Omega^2 (1 - 1/s)}. \tag{18}$$

The origin of time is taken as the instant when the projectile leaves the muzzle. It is assumed that the minimum value of δ occurs at this time, $t = 0$. Therefore $\varkappa = 0$ or π, $\dot{\delta}_0 = 0$, and β is the minimum yaw.

We take $\varkappa = \pi$, and then equation (17) becomes

$$\delta^2 = b - a \cos pt. \tag{19}$$

If α is the maximum value of δ and β is the minimum, we have

$$b - a = \beta^2$$

whence $$a = b - \beta^2 ;$$

and $$b + a = \alpha^2, \text{ or}$$

$$\alpha^2 = b + a = 2b - \beta^2. \tag{20}$$

Experiment indicates that β is approximately equal to ε, the yaw of the projectile in the gun at the muzzle. Furthermore, $\dot\varphi_0$ is the spin of the axis of the projectile about the axis of the bore and hence $N = \dot\varphi_0 \cos \varepsilon$. If $\cos \varepsilon$ is replaced by unity, we have

$$\dot\varphi_0 = N/(1+\gamma)$$

If this value of $\dot\varphi_0$ is substituted in (18), $\dot\delta_0$ is taken as zero and α is computed by (20) it is found that

$$\alpha = \left(\frac{2B}{A} - 1\right)\frac{\varepsilon}{\sqrt{1 - 1/s}} \; * \tag{21}$$

By expressing a and b in (19) in terms of α and β, it may be changed to

$$\delta^2 = \frac{1}{2}(\alpha^2 + \beta^2) - \frac{1}{2}(\alpha^2 - \beta^2)\cos pt,$$

which may be transformed into

$$\delta^2 = \alpha^2 \sin^2\left(\frac{pt}{2}\right) + \beta^2 \cos^2\left(\frac{pt}{2}\right). \tag{22}$$

If β is negligible,

$$\delta = \alpha \sin\left(\frac{pt}{2}\right). \tag{23}$$

Equation (22) or (23) provides a means of determining s. If T is the observed time interval from zero yaw to zero yaw, or minimum to minimum then

$$\pi = \frac{pT}{2}.$$

From this,

$$1 - 1/s = (2\pi/\Omega T)^2 \tag{24}$$

* This expression for α was first given in a paper by Kent and Hitchcock on the "Effect of Cross Wind on Yaw".

from which s may be determined from the observed value of T, $\Omega = \frac{AN}{B}$ being assumed known. A and B may be measured and it can be shown that

$$N = \frac{2\pi v_o}{nd} , \tag{25}$$

where v_o is the muzzle velocity, n is the number of calibers for one turn of the rifling and d is the caliber. Of course v_o and d must be measured in consistent units.

The motion in φ may be obtained by substituting δ^2 from (22) in (10) and integrating. The result is

$$\varphi = \varphi_o + \frac{1}{2} \Omega t \pm \frac{2 \beta h}{p \alpha} \tan^{-1}\left(\frac{\alpha}{\beta} \tan \frac{pt}{2}\right)$$

or

$$\varphi = \varphi_o + \frac{1}{2} \Omega t \pm \tan^{-1} \left(\frac{\alpha}{\beta} \tan \frac{pt}{2}\right) . \tag{26}$$

The choice of the sign in (26) depends upon the initial conditions. If the + sign holds, the motion is called 'stepped up': if the - sign holds, 'stepped down'.

The Damping of the Yaw

Equation (22) was derived on the following assumptions:

(a) The trajectory is rectilinear.

(b) The air force system has no elements except D and L, both of which lie in the plane of the yaw.

(c) μ and hence also s are constants.

(d) The yaw is small.

These assumptions are only approximately correct. Both the cross wind force, L, and the weight, W, tend to cause the trajectory to curve. There is an air couple opposing the angular motion of the axis of the shell, which has a component about the trajectory. Because of the diminishing air speed of the projectile μ is not a constant; it decreases, while s increases.

While for well designed artillery projectiles, the yaw near the gun is small, this is not always the case.

The Epicyclic Motion

We outline briefly without proof the modifications of equation (22) required to take account of the failure of assumptions (a), (b), and (c).

Assumption (b) is incorrect to the extent that in addition to the overturning moment there are three couples capable of producing an appreciable effect on the motion. One of these is the "yawing moment due to yawing". If the angular velocity of the axis of the projectile is denoted by ω, there is a couple which tends to reduce the angular velocity of the axis. This is represented by $K_H \rho d^4 v\omega$ or by $H\omega$, where $H = K_H \rho d^4 v$ and K_H is a dimensionless coefficient depending on the Mach number $v \div$ speed of sound. This couple may be regarded as the moment of a force, the "pitching force" which can be designated by $K_S \rho d^3 v\omega$ and tends to move the center of gravity in the same direction as the nose is turning.

Since the projectile is both spinning and yawing, it is acted upon by the same type of force, the "Magnus force", that causes a golf ball to slice or hook. The magnitude of this force is directed by $K_K \rho d^3 vN \sin\delta$ where K_K is another dimensionless coefficient. The moment of this force about the center of gravity is the "Magnus moment", and is denoted by $K_J \rho d^4 vN \sin\delta$. According to established custom, based on a guess made about 1920 by Fowler, this is regarded as acting in front of the center of gravity. However, experiments indicate that the Magnus force usually acts behind the center of gravity causing the slight inconvenience that K_J is usually negative.

The last couple which we shall consider is the spin-decelerating couple, which acts to reduce the spin of the projectile about the axis. This is denoted by $K_A \rho d^4 Nv$, where K_A is another dimensionless coefficient, ordinarily much smaller than K_H, K_J, etc. This couple and the Magnus moment act together in damping the yaw; their coefficients will not be found except in the combination $K_J - \frac{1}{2} K_A$.

It is evident that a couple will affect the yaw, for the angular momentum vector points nearly along the axis of the shell (there is a small cross-component due to the spin of the axis) and the impressed couple is equal to the rate of change of angular momentum. It is less evident at first glance that a force perpendicular to the trajectory will affect the yaw. The explanation is that even if the axis of the shell kept constant direction, the cross-force would curve the trajectory. This would affect the yaw, which is the angle between shell's axis and the line tangent to the trajectory. The complete study of the motion is rather tedious, and will not be attempted here. The result is that the axis of the shell undergoes a damped epicyclic motion; the motion described in previous pages is an undamped epicyclic motion. We now describe an epicyclic motion in some detail. Imagine a plane set perpendicular to the trajectory and one unit of length ahead of the center of gravity. The axis of the shell will not intersect this plane at the same point as the trajectory, unless the yaw happens to be zero. The intersection of the axis of the shell with the plane describes a curve which can be readily visualized thus. Imagine an arm of length k_1 rotating about the origin at a rate of n_1 degrees per foot of travel of the shell. At the end of this arm we attach another arm, of length k_2, rotating at a rate of n_2 degrees per foot of travel of the shell. The end of this second arm describes an epicyclic curve. The arms k_1 and k_2 are not constant, but change exponentially with distance travelled. Moreover, n_1 and n_2 are not quite constant, but change slowly along the trajectory. Thus the turn of the first epicyclic curve is not exactly $n_1 z$ degrees in z feet, but is better represented as $\int n_1 dz$, where n_1 is a slowly changing quantity.

If we denote the length of arc of trajectory from muzzle by ℓ, and the initial values of k_1 and k_2 by K_1 and K_2, then the right and downward components of yaw, in radians, are (if the small effects of gravity are ignored) given by

$$\text{Right component of yaw} = k_1 \cos \varphi_1 + k_2 \cos \varphi_2, \tag{27}$$

$$\text{Downward component of yaw} = k_1 \sin \varphi_1 + k_2 \sin \varphi_2,$$

where

$$k_1 = K_1 \sqrt{\frac{N(0)\sigma(0)}{N(\ell)\sigma(\ell)}} \exp\left\{-\frac{1}{2}\int_0^{\ell} \rho d^2\left(\frac{K_L}{m} + \frac{d^2K_H}{B} + \frac{1}{\sigma}\left[-\frac{K_L}{m} + \frac{d^2K_H}{B} - \frac{d^2}{A}(K_A - 2K_J)\right]\right) d\ell\right\}$$

$$k_1 = K_1 \sqrt{\frac{N(0)\sigma(0)}{N(\ell)\sigma(\ell)}} \exp\left\{-\frac{1}{2}\int_0^{\ell} \rho d^2\left(\frac{K_L}{m} + \frac{d^2K_H}{B} - \frac{1}{\sigma}\left[-\frac{K_L}{m} + \frac{d^2K_H}{B} - \frac{d^2}{A}(K_A - 2K_J)\right]\right) d\ell\right\}$$

$$\varphi_1 = \varphi_1(0) + \tfrac{1}{2}\int_0^{\ell} \frac{AN}{Bv}(1 + \sigma)\, d\ell \text{ radians}$$

$$= \varphi_1(0) + \tfrac{1}{2}\int_0^{t} \frac{AN}{B}(1 + \sigma)\, dt \text{ radians,} \tag{28}$$

$$\varphi_2 = \varphi_2(0) + \tfrac{1}{2}\int_0^{\ell} \frac{AN}{Bv}(1 - \sigma)\, d\ell \text{ radians}$$

$$= \varphi_2(0) + \tfrac{1}{2}\int_0^{t} \frac{AN}{B}(1 - \sigma)\, dt \text{ radians.}$$

$$\sigma = \sqrt{1 - 1/s}\,,$$

$$s = \frac{A^2N^2}{4BK_M\rho d^3 v^2}$$

The theory leading to these formulas is applicable with only minor modifications to the spinning rocket in which, during burning, the spin is in a constant ratio to the velocity The effect of propulsion can be treated as though it were a large negative drag. However, experiment indicates that at least some of the aerodynamic coefficients are considerably different during burning from the values which they have after burning. So the possibility of the mathematical extension of the formulas to the burning period does not mean that the numbers in the formulas are also unchanged.

Stability

From the epicyclic representation it is clear that a maximum of yaw occurs when the two arms have the same direction in which case the yaw is

$$\alpha = k_1 + k_2. \tag{29}$$

A minimum occurs when the two arms have opposite directions, at which time the yaw is

$$\beta = |k_1 - k_2|. \tag{30}$$

In order that the shell may have stable flight the maxima of yaw must diminish as ℓ increases. That is, $k_1 + k_2$ must diminish. This can only occur if each diminishes separately, for if either increases its rate of increase also increases, being exponential, and will eventually nullify any decrease of the other. For k_1 to decrease the integral whose exponential occurs in the formula for k_1 must have a positive integrand ~~and~~, since it is preceded by a minus sign. Therefore it must be true that

$$\frac{K_L}{m} + \frac{d^2K_H}{B} + \frac{1}{\sigma}\left[-\frac{K_L}{m} + \frac{d^2K_H}{B} - \frac{d^2}{A}(K_A - 2K_J)\right] > 0.$$

Likewise, for k_2 to diminish it must be true that

$$\frac{K_L}{m} + \frac{d^2K_H}{B} - \frac{1}{\sigma}\left[-\frac{K_L}{m} + \frac{d^2K_H}{B} - \frac{d^2}{A}(K_A - 2K_J)\right] > 0.$$

As a result, the product of the left members must be positive:

$$\left(\frac{K_L}{m} + \frac{d^2K_H}{B}\right)^2 - \frac{1}{\sigma^2}\left[-\frac{K_L}{m} + \frac{d^2K_H}{B} - \frac{d^2}{A}(K_A - 2K_J)\right]^2 > 0.$$

Replacing σ by its definition and performing an algebraic simplification leads to the necessary condition for stability

$$\frac{1}{s} < \frac{\left[2\frac{d^2K_H}{B} - \frac{d^2}{A}(K_A - 2K_J)\right]\left[2\frac{K_L}{m} + \frac{d^2}{A}(K_A - 2K_J)\right]}{\left(\frac{K_L}{m} + \frac{d^2K_H}{B}\right)^2}. \quad (31)$$

The right member can never exceed 1, so this implies the familiar stability condition $1/s < 1$. The converse is not true; the stability condition just derived can be decidedly more stringent than $1/s < 1$.

$$\cancel{s > \frac{(h+\kappa)^2}{4(h+\gamma)(\kappa-\gamma)}}$$

$$\frac{1}{s} < \frac{4(h+\gamma)(\kappa-\gamma)}{(h+\kappa)^2}$$

Drift

So far, the trajectory has been treated as a straight line, except that the effects of the aerodynamic force perpendicular to the trajectory were considered in computing the yaw. We now consider the curvature of the trajectory caused by gravity. If the trajectory is in a fixed XY* plane, and no aerodynamic force except the drag is considered, and the angle between the horizontal and the tangent to the trajectory is designated by θ, then

$$\ddot{x} = \frac{d^2x}{dt^2} = -\frac{D\cos\theta}{m},$$

$$\ddot{y} = \frac{d^2y}{dt^2} = -\frac{D\sin\theta}{m} - g.$$

Now $\tan\theta = \dot{y}/\dot{x}$; hence by differentiation

$$\sec^2\theta\,\frac{d\theta}{dt} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^2}$$

$$= \frac{\frac{-D\sin\theta\cdot\dot{x}}{m} - g\dot{x} + \frac{D\cos\theta\cdot\dot{y}}{m}}{\dot{x}^2}.$$

But

$$\sin\theta = \dot{y}/\sqrt{\dot{x}^2+\dot{y}^2},\quad \cos\theta = \dot{x}/\sqrt{\dot{x}^2+\dot{y}^2},\quad \sec^2\theta = \frac{\dot{x}^2+\dot{y}^2}{\dot{x}^2},$$

and

$$v = \sqrt{\dot{x}^2+\dot{y}^2},$$

so

$$v^2\dot{\theta} = -g\dot{x},$$

$$\dot{\theta} = -g\cos\theta/v. \qquad (32)$$

This is the rate of turning of the trajectory about a horizontal axis perpendicular to the trajectory.

* These axes are different from the XY axes of the projectile.

It will now be shown that there is a mathematically possible motion of the shell in which the axis makes an angle ξ to the right of the trajectory and an angle η above the trajectory, both ξ and η being small and slowly changing. This means that the shell "trails" properly, continuing to point nearly along the tangent to the trajectory.

At some particular time t_o we choose the position of the center of gravity as origin and construct a rectangular axis system (x_1, x_2, x_3); the first is tangent to the trajectory, the third is horizontal and to the right, and the second is perpendicular to the other two and is upwardly directed. If the angles ξ and η are small, at all times t, the direction of the axis has components nearly equal to $(\cos(\theta - \theta_o), \sin(\theta - \theta_o), 0)$, where θ_o is the inclination of the trajectory at time t_o. Hence the angular momentum is nearly

$(AN \cos(\theta - \theta_o), AN \sin(\theta - \theta_o), 0)$, and the rate of change of angular momentum is

$$(-AN \sin(\theta - \theta_o) . \dot{\theta}, AN \cos(\dot{\theta} - \theta_o) \dot{\theta}, 0),$$

which at time t_o is

$$(0, -\frac{ANg \cos\theta_o}{v}, 0). \tag{33}$$

If the nose of the shell is directed at an angle ξ (radians) to the right of the trajectory, there results an overturning moment $K_M \rho d^3 v^2 \sin\xi$ tending to increase ξ. It is permissible to replace $\sin\xi$ by ξ, since ξ is small. The couple tends to cause a clockwise (negative) rotation about the x_2 - axis, so it is represented by the vector $(0, -K_M \rho d^3 v^2 \xi, 0)$. Likewise, an upward angle η produces a moment $K_M \rho d^3 v^2 \eta$, tending to cause counterclockwise (positive) rotation about the x_3 - axis. This moment is represented by the vector

$$(0, 0, K_M \rho d^3 v^2 \eta).$$

If the nose of the projectile is to the right of the trajectory by an amount ξ, the Magnus torque has magnitude $K_J \rho d^4 N v \xi$. For right-hand rifling, the shell presents its rising side forward, and so the Magnus force is upward.

Assuming as required in the definition of K_J that this force acts in front of the center of gravity, this tends to produce counterclockwise rotation about the x_3 axis, so its components are

$$(0,\ 0,\ K_J \rho d^4 N v \xi\ ).$$

Likewise an upward tilt of angle η produces a Magnus torque with components

$$(0,\ K_J \rho d^4 N v \eta\ ,\ 0).$$

The combined torques, due to both overturning and Magnus moments caused by a yaw of ξ to the right and η upward, must equal the rate of change ($\mathfrak{Z}\upsilon$) of angular momentum. This yields the two equations

$$\begin{aligned} -K_M \rho d^3 v^2\ \xi + K_J \rho d^4 N v \eta &= -\frac{AN\mathfrak{g}\cos\theta_0}{v}, \\ K_J \rho d^4 N v\ \xi + K_M \rho d^3 v^2 \eta &= 0\ . \end{aligned} \tag{34}$$

These equations have solutions

$$\begin{aligned} \xi &= \frac{AN\mathfrak{g}\cos\theta_0}{\rho d^3 v^3} \cdot \frac{K_M}{K_M^2 + K_J^2\,(dN/v)^2}, \\ \eta &= \frac{-\,ANg\cos\theta_0}{\rho d^3 v^3} \cdot \frac{K_J\,(dN/v)}{K_M^2 + K_J^2\,(dN/v)^2}\ . \end{aligned} \tag{35}$$

The quantity $K_J\ (dN/v)$ is usually considerably smaller than K_M, and its square is negligible compared with K_M^2. Also θ_0 is the angle at time t_0; which is arbitrary, so we may as well drop the subscript zero. This yields

$$\begin{aligned} \xi &= \frac{AN\mathfrak{g}\cos\theta}{\rho d^3 v^3 K_M}, \\ \eta &= -\frac{ANg\cos\theta}{\rho d^3 v^3 K_M} \cdot \frac{K_J}{K_M} \cdot \frac{Nd}{v}. \end{aligned} \tag{36}$$

If the projectile points to the right of the trajectory by the amount ξ (in radians) and above the trajectory by the amount η, the rate of turning of the projectile about the x_3 (or Z) axis will equal the rate of turning of the trajectory, and the projectile will continue to make these angles ξ, η with the trajectory. This shows that a possible motion of the projectile when gravity causes the trajectory to turn is one in which the projectile points to the right and above the trajectory by the amounts ξ, η just given. A more nearly complete theory shows that the general motion consists very approximately of the epicyclic motion as previously given, but with the center of motion at (ξ, η) instead of at the tangent (0,0) to the trajectory, provided always that the yaw never exceeds a few degrees, say 8° or 10°. This theory has received a rather satisfactory experimental confirmation.

As a consequence of the yaw (ξ, η) the shell is acted on by the cross-wind force, of magnitude

$$K_L \rho d^2 v^2 \sin \delta, \quad \delta = \sqrt{\xi^2 + \eta^2}$$

being the yaw, and also by the Magnus force

$$K_K \rho d^3 vN \sin \delta .$$

The cross-wind force acts in the plane of the yaw, while the Magnus force acts perpendicularly to it, its direction being upward and to the left. The resultant force has component

$$K_L \rho d^2 v^2 \xi - K_K \rho d^3 vN \eta = \rho d^2 v^2 [K_L \xi - K_K (Nd/v) \eta]$$

to the right and

$$K_L \rho d^2 v^2 \eta + K_K \rho d^3 vN \xi = \rho d^2 v^2 [K_L \eta + K_K (Nd/v) \xi]$$

upward and perpendicular to the trajectory. Although K_K has not been very accurately evaluated for any shell, experiments indicate that it cannot greatly exceed K_L, while Nd/v is decidedly less than 1 and η is smaller than ξ. Hence the force to the right is positive, and in fact differs little from

$$K_L \rho d^2 v^2 \xi = \frac{K_L}{K_M} \frac{ANg \cos \theta}{dv} . \tag{37}$$

It is this steadily acting force which causes the right drift of projectiles fired from guns with right-handed twists.

Except for large projectiles with small muzzle velocity, the yaw due to gravity will be of small magnitude, much smaller ordinarily than the epicyclic yaw near the muzzle due to launching conditions.

Determination of Aerodynamic Coefficients from Yaw Card or Spark Photograph Data

If a projectile is fired through a sequence of cards, from the shape of the hole it is possible to deduce the amount and orientation of the yaw at each card; and if desired, the right and downward components can be found from this. Another method of recording yaw, very satisfactory in accuracy, is to take horizontal and vertical photographs of the projectile at each of several stations. From these data it is possible in either of two ways to deduce the aerodynamic coefficients of the shell.

A first possibility is to locate the maxima and minima of yaw and to measure their magnitudes.

If h_1 and h_2 are defined by the equations

$$h_1 = \frac{1}{2}\int_0^{\alpha} \rho d^2 \left(\frac{K_L}{m} + \frac{d^2 K_H}{B}\right) d\ell, \tag{38}$$

$$h_2 = \frac{1}{2}\int_0^{\alpha} \rho \frac{d^2}{\sigma}\left(-\frac{K_L}{m} + \frac{d^2 K_H}{B} - \frac{d^2}{A}\left[K_A - 2K_J\right]\right) d\ell,$$

the equations for the maximum and minimum yaw take the form

$$\alpha = \sqrt{\frac{N(0)\sigma(0)}{N(\ell)\sigma(\ell)}}\left\{K_1 \exp(-h_1 - h_2) + K_2 \exp(-h_1 + h_2)\right\},$$

$$\beta = \pm\sqrt{\frac{N(0)\sigma(0)}{N(\ell)\sigma(\ell)}}\left\{K_1 \exp(-h_1 - h_2) - K_2 \exp(-h_1 + h_2)\right\}, \tag{39}$$

where in the last equation the + sign is to be used if

$K_1 \exp(-h_1 - h_2)$

exceeds

$K_2 \exp(-h_1 + h_2)$ and the - sign otherwise.

But

$$K_1 \exp(-h_1 - h_2) + K_2 \exp(-h_1 + h_2)$$

$$= \exp(-h_1)\left\{K_1 e^{-h_2} + K_2 e^{+h_2}\right\}$$

$$= e^{-h_1}\left\{e^{\log K_1 - h_2} + e^{\log K_2 + h_2}\right\}$$

$$= e^{-h_1 + \log\sqrt{K_1K_2}}\left\{e^{\log\sqrt{K_1K_2} - h_2} + e^{-\log\sqrt{K_1/K_2} + h_2}\right\}$$

$$= 2\sqrt{K_1K_2}\, e^{-h_1} \cosh\left(\log\sqrt{K_1/K_2} - h_2\right),$$

so

$$\alpha = 2\sqrt{\frac{K_1K_2N(0)\sigma(0)}{N(\ell)\sigma(\ell)}}\ e^{-h_1}\cosh\left(\log\sqrt{K_1/K_2} - h_2\right). \qquad (40)$$

In the same way

$$\beta = \pm\sqrt{\frac{K_1K_2N(0)\sigma(0)}{N(\ell)\sigma(\ell)}}\ e^{-h_1}\sinh\left(\log\sqrt{K_1/K_2} - h_2\right), \qquad (41)$$

the choice of sign being as explained after (~~40~~ 39). Let letters with subscript 1 refer to measurements and values corresponding to distance ℓ_1 from the muzzle and those with subscript 2 to measurements and values at distance ℓ_2 from the muzzle. Assume N constant; this is very nearly true over any short arc. Write J for $2\sqrt{K_1K_2}$ and j for $\log\sqrt{K_1K_2}$. Then

$$\alpha_1^2 = J^2\frac{\sigma(0)}{\sigma(\ell)_1}\ e^{-h_{11}}\cosh^2(j - h_{21}),$$

$$\beta_1^2 = J^2\frac{\sigma(0)}{\sigma(\ell_1)}\ e^{-2h_{11}}\sinh^2(j-h_{21}).$$

Since $\cosh^2 x - \sinh^2 x = 1$, it follows that

$$\alpha_1^2 - \beta_1^2 = J^2\frac{\sigma(0)}{\sigma(\ell_1)}\ e^{-2h_{11}}.$$

Similarly

$$\alpha_2^2 - \beta_2^2 = J^2\frac{\sigma(0)}{\sigma(\ell_2)}\ e^{-2h_{12}}.$$

From the last two equations

$$-2h_{11} = \log(\alpha_1^2 - \beta_1^2)\sigma(\ell_1) - \log J^2\sigma(0),$$

$$-2h_{12} = \log(\alpha_2^2 - \beta_2^2)\sigma(\ell_2) - \log J^2\sigma(0).$$

Subtracting,

$$-2h_{11} + 2h_{12} = \log\frac{(\alpha_1^2 - \beta_1^2)\sigma(\ell_1)}{(\alpha_2^2 - \beta_2^2)\sigma(\ell_2)}.$$

If ρ, K_L and K_H, are constants, this takes the form

$$\rho d^2\left(\frac{K_L}{m}+\frac{d^2K_H}{B}\right)(\ell_2-\ell_1)=\log\frac{(\alpha_1^2-\beta_1^2)\sigma(\ell_1)}{(\alpha_2^2-\beta_2^2)\sigma(\ell_2)},$$

or

$$\frac{K_L}{m}+\frac{K_Hd^2}{B}=\frac{1}{\rho d^2(\ell_2-\ell_1)}\log\frac{(\alpha_1^2-\beta_1^2)\,\sigma(\ell_1)}{(\alpha_2^2-\beta_2^2)\,\sigma(\ell_2)}, \qquad (42)$$

the logarithms being natural logarithms.

From (40) and (41)

$$\tanh(\log\sqrt{K_1/K_2}-h_2)=\pm\beta/\alpha,$$

the choice of sign being as explained after (39). If β/α is small, as is ordinarily the case in firing from fixed mounts, this can be adequately replaced by the approximation

$$\log\sqrt{K_1/K_2}-h_2=\pm\beta/\alpha,$$

since $\tanh z \doteq z$ for z near zero. Applying this at distances ℓ_1 and ℓ_2 from the muzzle and subtracting yields

$$h_{22}-h_{21}=\left(\frac{\pm\beta_1}{\alpha_1}\right)-\left(\frac{\pm\beta_2}{\alpha_2}\right)$$

If K_H, etc., are constants, by (39)

$$h_{22}-h_{21}=\frac{\rho d^2}{2}\left(-\frac{K_L}{m}+\frac{K_Hd^2}{B}-\frac{d^2}{A}[K_A-2K_J]\right)\int_{\ell_1}^{\ell_2}\frac{d\ell}{\sigma}.$$

Hence

$$-\frac{K_L}{m}+\frac{K_Hd^2}{B}-\frac{d^2}{A}[K_A-2K_J]=\frac{2\left(\frac{\pm\beta_1}{\alpha_1}-\frac{\pm\beta_2}{\alpha_2}\right)}{\rho d^2\int_{\ell_1}^{\ell_2}\frac{d\ell}{\sigma}}. \qquad (43)$$

By (28), the fast-turning epicyclic arm gains on the slow one at the rate

$$\frac{d}{dt}(\varphi_1-\varphi_2)=\frac{AN}{B}\sigma \text{ radians/sec},$$

or

$$\frac{d}{ds}(\varphi_1-\varphi_2)=\frac{AN}{Bv}\sigma \text{ radians/foot}.$$

Since minima occur whenever φ_1 and φ_2 differ by 180°, or π radians, the distance between minima is the distance required for $\varphi_1 - \varphi_2$ to increase through 2π radians. Thus if Δ is the distance between minima

$$2\pi = \int_{\ell}^{\ell+\Delta} \frac{d}{ds}(\varphi_1 - \varphi_2) \cdot ds$$

$$= \int_{\ell}^{\ell+\Delta} \frac{AN}{BV} \varphi ds .$$

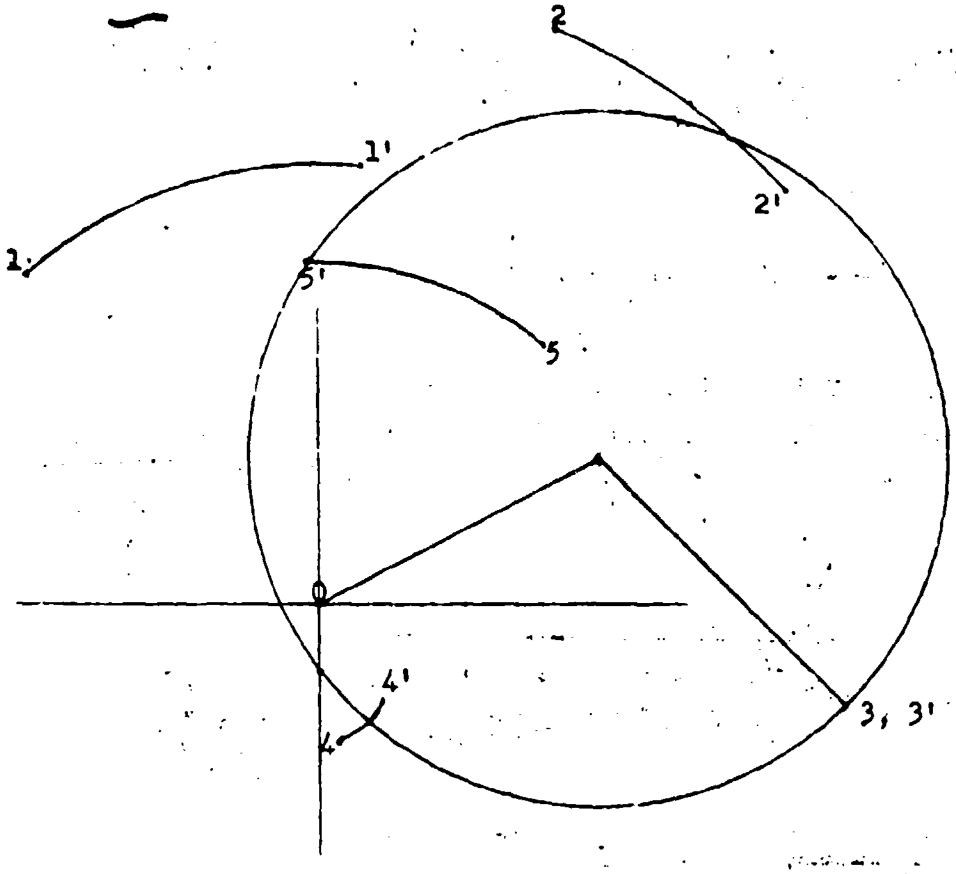
If v and φ are supposed constant between two consecutive minima Δ feet apart, this gives

$$\varphi = \frac{2\pi Bv}{AN\Delta} .$$

Thus by the last two of equations (28) it is possible to determine K_M.

The drift of the projectile is seen from (37) to be the product of K_L/K_M by a readily computable quantity, provided that K_L/K_M is assumed constant. If this quantity is computed, and the drift is measured by means of special firings, K_L/K_M can be found. Since K_M is known, K_L is also known. Now (42) determines K_H, and (43) determines $[K_A - 2K_J]$.

A different technique of reduction is in current use in connection with spark-photograph data. The projectile is photographed as it passes a group of stations, say five, equally spaced. By measurement of the photographs the upward and right components of the yaw are determined and plotted. If we knew the rate of rotation of the slow epicyclic arm, and imagined the paper turned at this rate, the motion of the axis registered on the paper would be only that due to the fast epicyclic arm. Hence the points 1, 2, 3, 4, 5 would be equally spaced around the circumference of some circle. Accordingly, we make several guesses at the slow rate (which in practice turns out to be rather easy) and plot the positions 1', 2', 3', 4', 5' which the axis would have had at stations 1, 2, 3, 4, 5 had the paper been rotating with the small arm, and had coincided with its actual position at some station, say 3. For our best guess



the points 1', etc. will be almost equally spaced about some circle. The center of this circle will represent the end of the slow arm at station 3, while the line from center to 3' represents the fast arm at station 3. These may be in error by several degrees.

The process is now repeated, using the photographs taken at another group of stations at some distance, say 150 feet, from the first. The positions of the arms are thus found for one of these stations. Having these positions of the epicyclic arms at two widely separated points, their rates of rotation are determined with considerable accuracy. If desired these well-determined rates may be used instead of the previous guessed values to give a revised graphical construction at each group.

5. Now that the epicyclic rates $d\varphi_1/d\ell$ and $d\varphi_2/d\ell$ are known, equations (28) yield σ, s and K_M. The length of the epicyclic

arms k_1 and k_2 are known at two values of ℓ, so by equations (28) we can find the quantities

$$
\begin{aligned}
&K_L/m + K_H d^2/B, \\
&-K_L/m + K_H d^2/B - (d^2/A)\,(K_A - 2K_J).
\end{aligned}
\tag{44}
$$

As before, K_L/K_M may be determined by drift measurements. It is also possible to deduce K_L from the swerve of the center of gravity of the projectile caused by the cross-wind force. This known, K_H and $(K_A - 2K_J)$ are determined by the known values of the quantities (44).

Dependence of Yaw on Travel

If in (27) and (28) we change variable from ℓ to $\lambda = \ell/d$, which is the distance traveled expressed in calibers, we find (for example)

$$
k_1 = K_1 \sqrt{\frac{N(0)\sigma(0)}{N(\lambda)\sigma(\lambda)}}\, \exp\left\{-\frac{1}{2}\int_0^{\lambda} \frac{\rho d^3}{m}\left(K_L + \frac{md^2}{B} K_H + \frac{1}{\sigma}\left[-K_L + \frac{md^2}{B} K_H - \frac{md^2}{A}(K_A - 2K_J)\right]\right) d\lambda\right\}
$$

Given two projectiles having the same shape and composed of material of equal density similarly distributed, the masses will be proportional to d^3 and the moments of inertia will therefore be proportional to md^2 or to d^5. Hence for the two projectiles the expressions d^3/m, md^2/B and md^2/A will have equal values. At any given velocity, then, the two projectiles will give the same value to k_1/K_1 at equal values of λ. Moreover, even at different velocities this will be nearly true, since the velocity enters only through the coefficients K_L, etc., which are not sensitive to changes in velocity (or rather Mach number).

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BIBLIOGRAPHY

R. H. Fowler, E. G. Gallop, C. N. H. Lock and H. W. Richmond, "Aerodynamics of a Spinning Shell", Proceedings of the Royal Society of London, Series A, Vol. 221, (1920) pp. 295-387.

R. H. Kent, "An elementary treatment of the motion of a spinning projectile about its center of gravity", Ballistic Research Laboratory Report No. 85, (1937).

K. L. Nielsen and J. L. Synge, "On the motion of a spinning shell"; reproduced by the Ballistic Research Laboratory by permission of the National Research Council of Canada, 1942.

J. L. Kelley and E. J. McShane, "On the motion of a projectile with small or slowly changing yaw," Ballistic Research Laboratory Report No. 446 (1944).

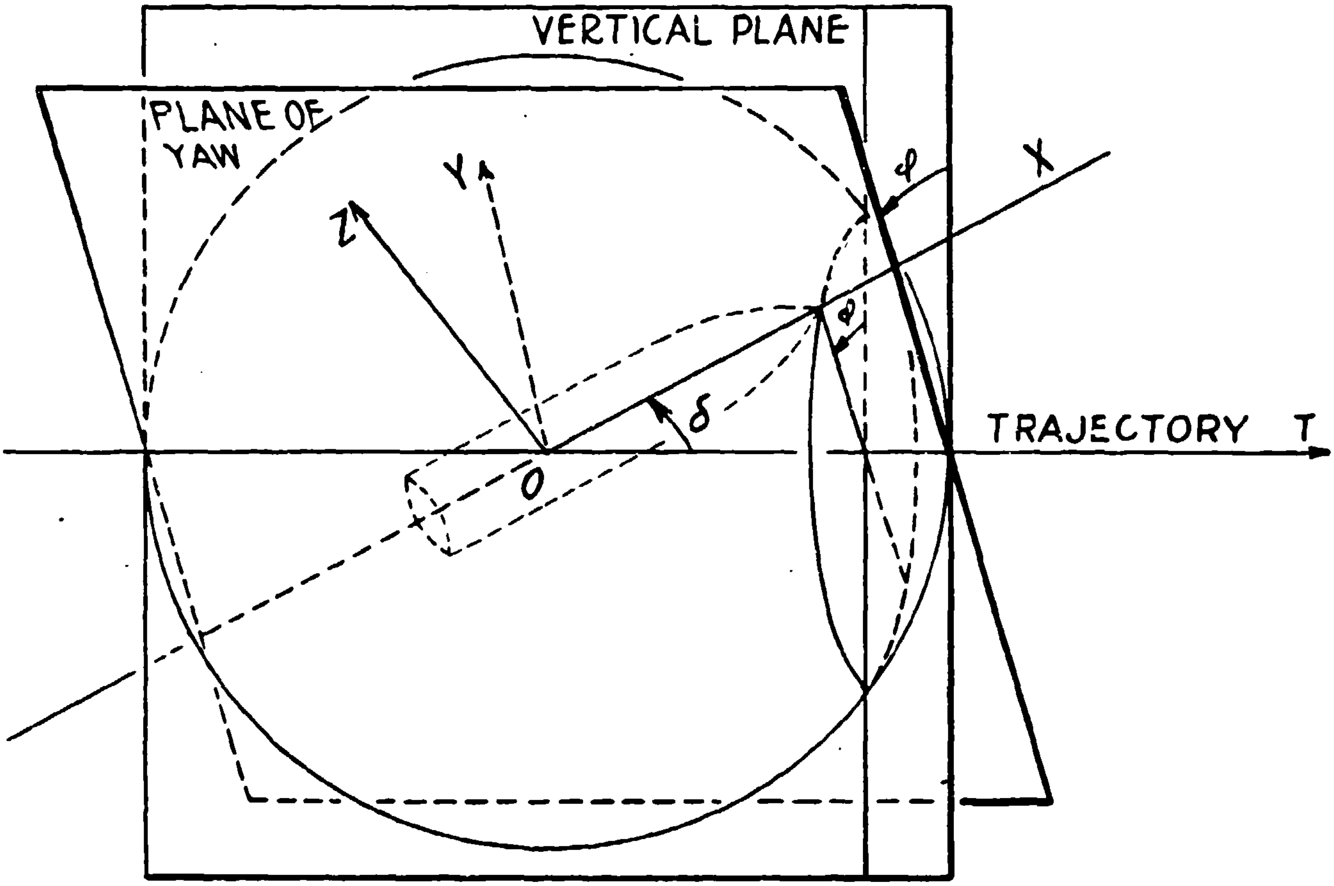

FIG. 1

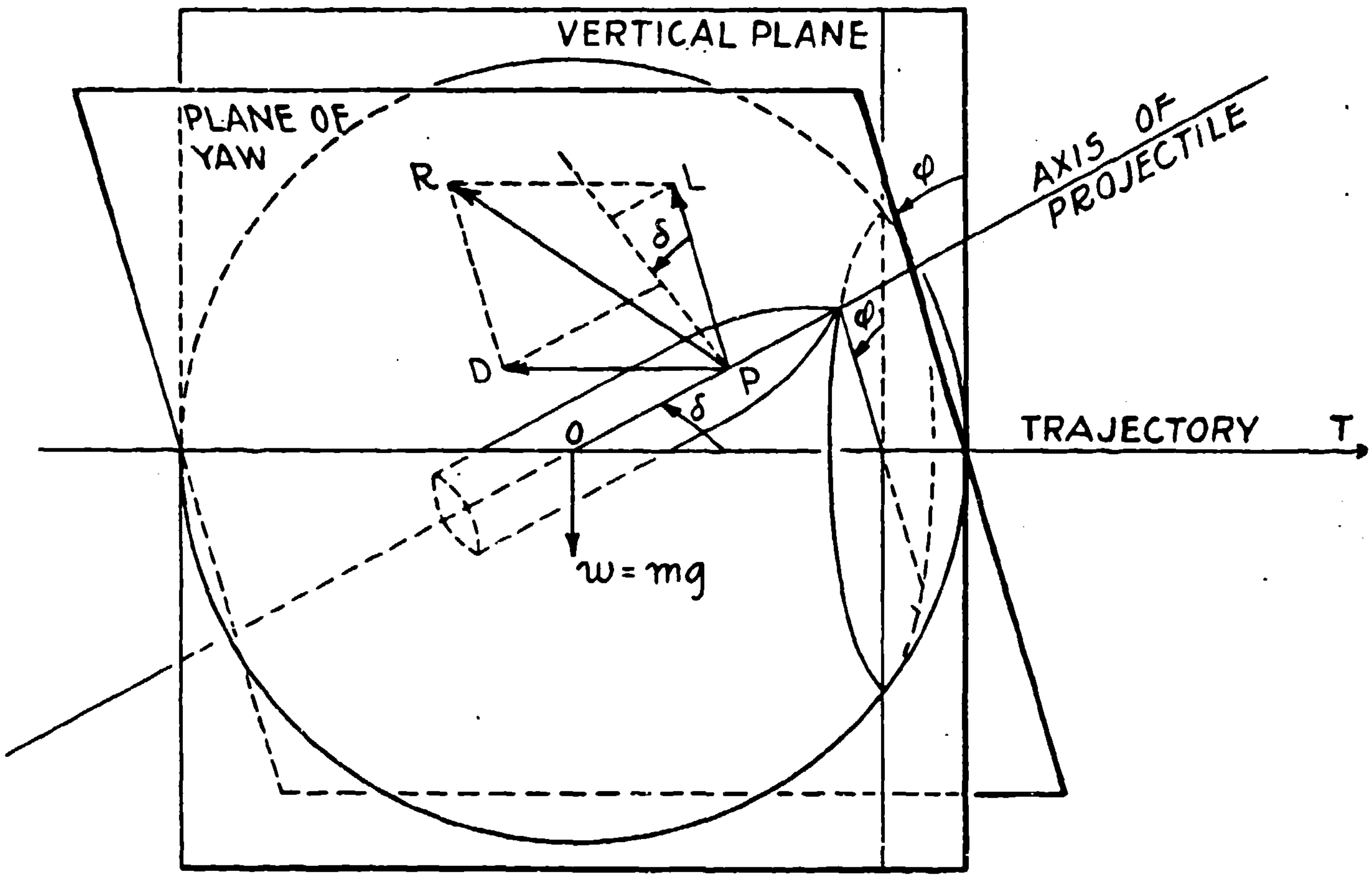
VERTICAL PLANE
PLANE OF YAW
AXIS OF PROJECTILE
R
L
δ
φ
D
P
O
δ
TRAJECTORY
T
w = mg

FIG. 2

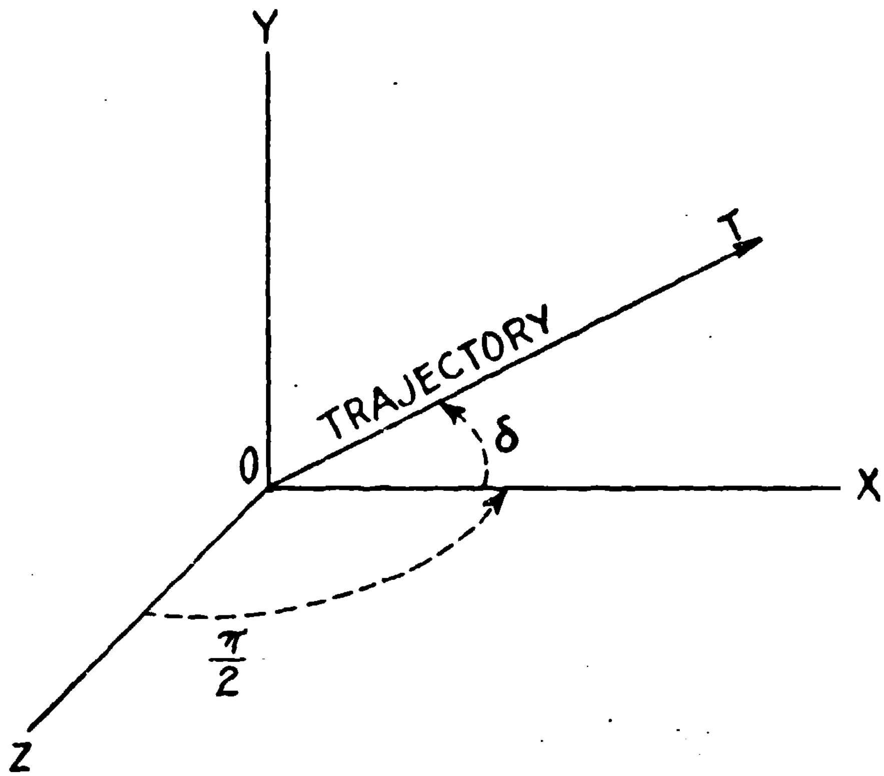

FIG. 3

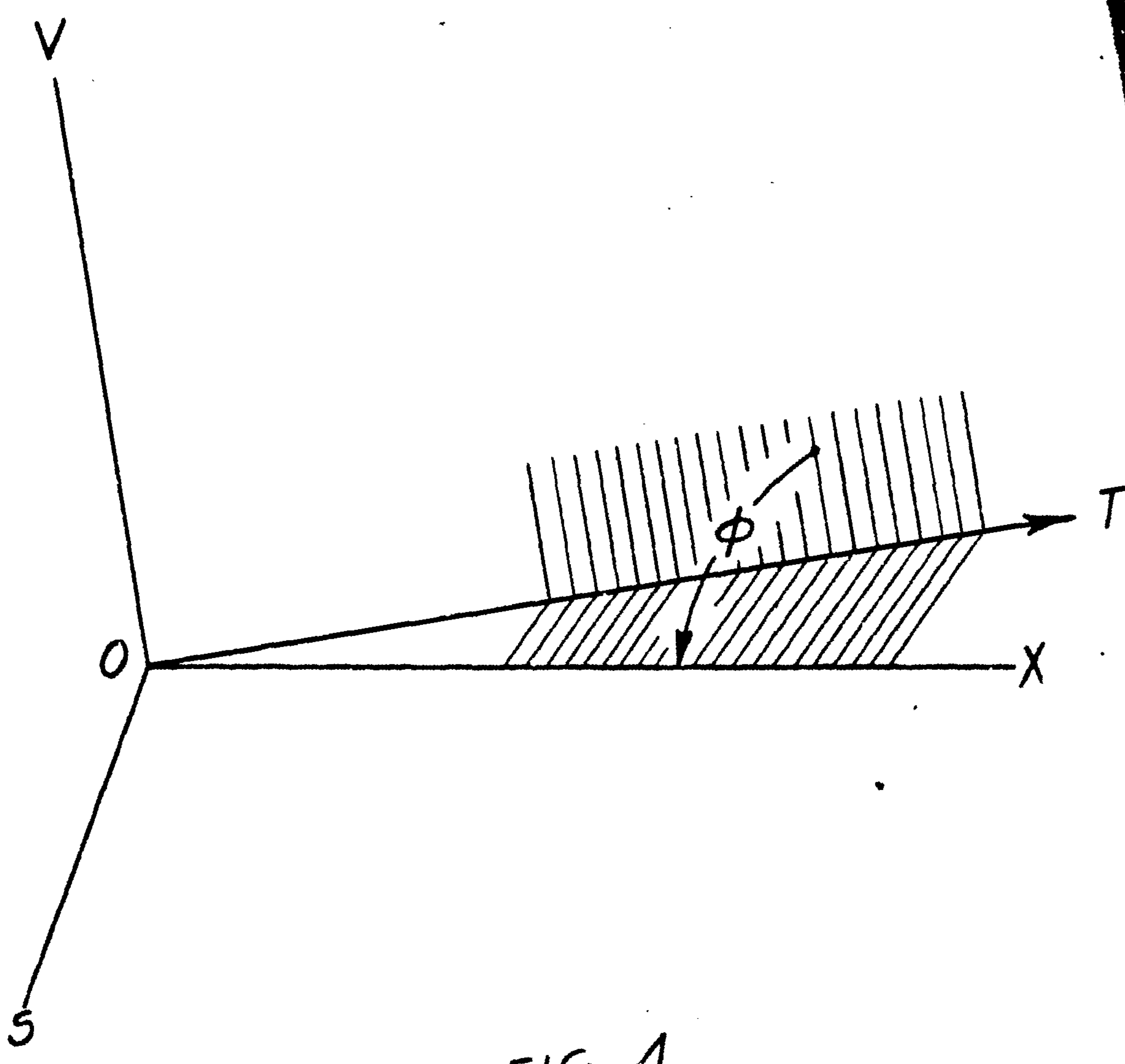

FIG. 4

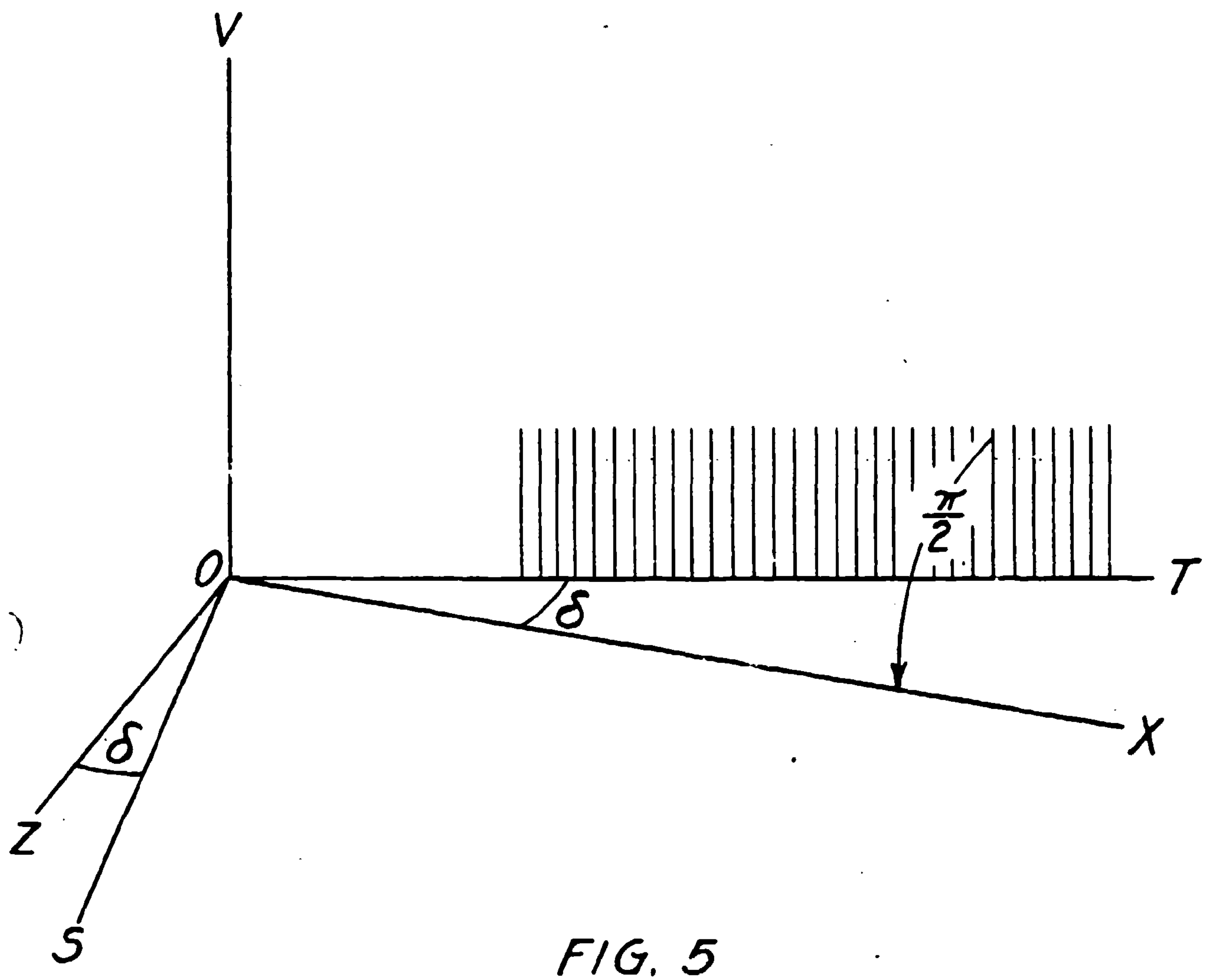

FIG. 5

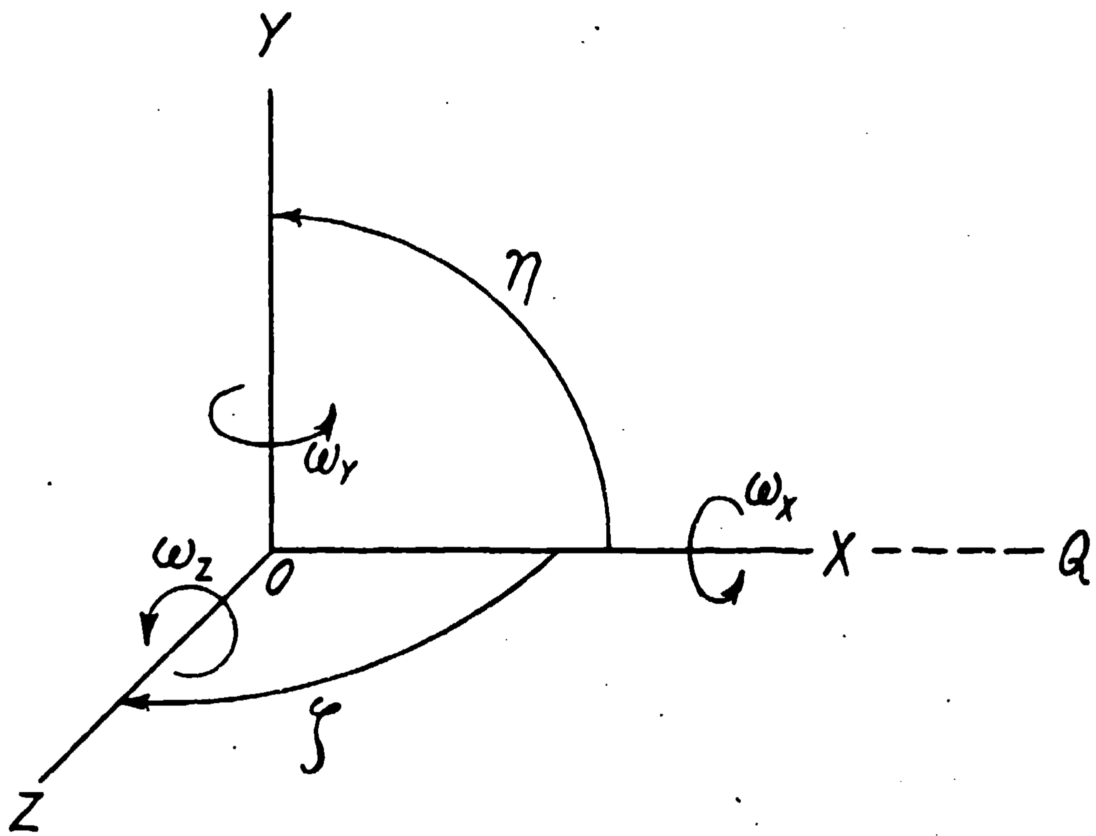

FIG. 6