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FLOW NOISE TRANSMITTED THROUGH DOMES OR ACOUSTICALLY MODIFIED BY NON-RIGID BOUNDARIES

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Submitted to:

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Office of Naval Research
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ABSTRACT

Flow induced noise generated by pressure fluctuations in a turbulent boundary layer is studied here theoretically with regard to its acoustic transmission through a dome to a transducer element or array. Particular attention is given to a uni-layer dome in the form of a finite slab, described as a fluid but serving as a prototype for a realizable solid slab. A similar dome with a thin outer cover having nonvanishing impedance is also considered. Likewise, attention is given to acoustic modification of the flow noise on a flush element due to mobility of the surrounding baffle.

To demonstrate clearly what properties of the boundary-layer pressure fluctuations are essential, high- and low wavenumber ranges are distinguished in the wavenumber spectrum of pressure at given frequency, the former range corresponding to possible generation by a convected eddy field. By various approximations and simplification of boundary conditions, the contributions of these respective ranges to the frequency spectrum of average pressure on a dome-shielded element are related to their respective contributions in the reference case of an element flush-mounted in a rigid plate. The high-wavenumber component in the flush case varies with element radius R_o as R_o^{-3}, whereas the low-wavenumber component may vary more as R_o^{-2}. If the latter component predominates in the spectrum at given frequency for a large flush element, it predominates still more for a shielded element. The high-wavenumber component for a shielded element contains a part that is independent of lateral dome size (face area) when this is large, but this part is highly attenuated by the dome if $L \gg U_\infty/\omega$, where L is dome thickness, ω angular frequency, and U_∞ asymptotic flow velocity. The other part of this high-wavenumber component is reduced somewhat as though the pressure were averaged over the face area of the dome section. On assumption that the wavenumber spectrum changes

little in the pertinent interval (in accord with area dependence as R_o^{-2} for a flush element), the low-wavenumber component is reduced rather as though averaged over an area of radius given roughly by the smaller of one-third the sound wave length or three times the dome thickness whenever this area is larger than the actual area of the element. With reference to the noise-to-signal ratio for an array of elements, though a thin unilayer dome can thus be very effective against the high-wavenumber component of flow noise, it can reduce the low-wavenumber component by no more than the array factor, or by not as much if $L \lesssim D/5$, where D is the element spacing.

In the case of a dome with cover having impedance, the effect of flexural resonances of the dome-fluid system is studied.

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PART 0
INTRODUCTION AND SUMMARY

0.1 Purpose and Context

The present report is concerned with the modification of flow noise by acoustic means. The main instance treated is that of transmission of flow noise to a transducer, or array of transducers, shielded from the flow by a dome. A secondary instance is that of a transducer flush-mounted in a non-rigid plate bounding the flow, where the pressure on the transducer is modified by the acoustic field excited by plate vibration.

Two types of dome are considered. The first is a fluid dome or sheath, i.e., a slab of fluid separated from the outside flow by an impedanceless membrane. The fluid sheath is viewed as a conceptual prototype for an elastic sheath covering an array of flush-mounted tran ducers. The second type of dome is a covered fluid dome, i.e. a fluid dome separated from the flow by a membrane or plate having appreciable impedance.

In each case we are concerned with the frequency spectrum of average pressure on the transducer generated by pressure fluctuations in the turbulent boundary layer. This spectrum is to be compared with that for a transducer flush-mounted in a rigid plate bounding the flow (the referent).

We especially wish to demonstrate what properties of the boundary-layer pressure fluctuations are essential to describe the flow noise transmitted to a dome-shielded transducer. In this connection we distinguish low- and high-wavenumber components in the excitation spectrum and show that the corresponding contributions to the dome-shielded spectrum are related to the excitation components and the dome parameters in distinct characteristic ways.

More explicitly, in some contrast to previous work in this area, the following points are emphasized. First, the transmission of flow-induced noise by domes and modification of flow noise by plate vibration depend on the statistical properties of the boundary-layer pressure fluctuations mainly via certain gross features of the wavenumber spectrum at the frequency in question. They depend especially on the spectral density of the exciting pressure in the region $k \lesssim (\omega^2/c^2 + L^{-2})^{1/2}$, where L is the dome depth, and at the wave numbers of any fairly unattenuated dome resonances, relative to the spectral density in the range $k \gtrsim \omega/U_\infty$, where the density is greatest. Second, information on these relative magnitudes is implied by the observed area dependence of pressure spectra for flush-mounted transducers, but the observations leave appreciable uncertainty, and a supporting theoretical account of the region of low wave numbers does not exist. Assumption of any explicit form for the space-time correlation function or wavenumber spectrum of boundary-layer pressure fluctuations in models for the effects in question has no reliable basis and is unnecessary. Unless the pertinent gross features are settled, it is not possible to conclude what flow-noise reduction by domes is theoretically possible.

A great deal of this report pertains to contributions from the high-wavenumber component of the excitation pressure. The treatment of the nonresonant contribution from low wave numbers depends less on the explicit dome model, and this contribution is probably the more important one. The amount of attention devoted to the former contribution resulted from a theoretical prejudice that the other, low-wavenumber portion should be very small. In any case, the actual form of the wavenumber spectrum of boundary-layer pressure fluctuations is uncertain, so that a treatment of the contribution from high wavenumbers is not impertinent.

Considerable discussion of the properties of the boundary-layer pressure fluctuations is given in the following.

The model dome employed here is closely similar to the one introduced in Ref. 2. That reference is concerned with much the same questions as this report. In the classified literature, we note Ref. 13.* Similar models have been used to treat the problem of transmission and radiation by plates representing a fuselage and excited to vibration by pressure fluctuations of the turbulent boundary layer. This application differs from the sonar application mainly in that the bounding fluids are air which, being much less dense than water, yields an acoustic impedance that may be treated as small in certain respects in the coupled plate-fluid system (see also Ref. 2).** The physical quantities ordinarily computed or measured differ. Nevertheless, elements of the approach and some results of the present work are applicable in this related area.

Among the pertinent references, including some experimental work, Refs. 5-12 are mentioned as representative.

0.2 Summary Discussion with Conclusions

In this section we summarize remarks, results, and conclusions of this report. In the first subsection some of the salient points are stated. The principal relevant sections of the report are noted in the heading for each configuration.

0.2.1 Salient Points in Summary

Fluid dome (no cover or cover with negligible impedance, depth L, lateral radius a). (Sec 0.2.3, 2.6.1)

1. Measured average-pressure spectra $Q_o(\omega)$ on flush elements of radius R_o suggest an area dependence $Q_o \propto R_o^{-2}$ a range $\pi U_\infty/\omega \ll R_o < R_m$ for some R_m. In a range where such dependence holds, boundary-layer pressure components with wave numbers $k \lesssim 2\pi R_o^{-1}$ contribute the major part of $Q_o(\omega)$ and also the major part of the average-pressure spectrum

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* Also Ref. 16, related to the present report.

** Likewise, for example, the regime where conditions (0-24) or (0-32) below hold is outside that of interest.

$Q(\omega,-L)$ on a dome-shielded element. This part of $Q(\omega,-L)$, say $Q_-(\omega,-L)$, and hence the total spectrum $Q(\omega,-L)$ itself, has a magnitude roughly equal to the average-pressure spectrum for a flush element of radius R_e defined by

$$R_e^{-2} = (1/4)(\omega^2/c^2 + 1/2L^2), \qquad (0\text{-}1)$$

(or possibly a lower spectrum if $R_e \gtrsim R_m$), where the radius r_o of the shielded element satisfies $r_o \lesssim R_e$. If $r_o \gtrsim R_e$, $Q_-(\omega,-L)$ is instead roughly the same as the average-pressure spectrum $Q_o(\omega)$ for a flush element of the same radius, r_o. The correlation area of the pressure field at the shielded element is πR_e^2.

1a. If $Q(\omega,-L)$ is in fact dominated by a part Q_- having these properties, the spectrum of flow-noise pressure averaged over the active area of a sheathed array of elements, and hence also the noise-to-signal ratio for a fixed incident signal, are reduced relative to the value for a flush array of the same active area at least by the array factor [$\equiv$(active area)/(total area) (<1)], provided the sheath thickness satisfies

$$L \gtrsim (1/5)D(1-\pi D^2/\lambda^2)^{-1/2}, \qquad (0\text{-}1.1)$$

where $\lambda(=2\pi c/\omega)$ is the sound wave length at the frequency in question and D is the center spacing of array elements. The noise-to-signal ratio is not appreciably reduced at all unless the less stringent condition

$$L \gtrsim 0.36 r_o(1-\pi^2 r_o^2/\lambda^2)^{-1/2} \qquad (0\text{-}1.2)$$

is met.

2. The bulk of the excitation spectrum presumably nevertheless lies in the convective range of high wave numbers, $k \gtrsim \omega/U_\infty$. This range contributes to the spectrum $Q(\omega,-L)$ a residual part Q_+^∞ that remains when the lateral dome size $\underline{a}$ becomes infinite but is very small for $L >> U_\infty/\omega$. Also, however, it contributes to Q a part Q_+^p that varies as a^{-3} for large $\underline{a}$ and is of the order of magnitude of the convective (high-wavenumber) contribution to the

average-pressure spectrum on a flush element of radius equal to that of the dome, a. This pressure field is coherent over a shielded element of radius $R_o \lesssim c/\omega$. Its contribution to the spectrum of flow-noise pressure averaged over the active area of a sheathed array of elements is small compared to its contribution for a flush array with equal active area at least when, roughly, the lateral size a of each adjacent section of the sheath is rather larger than the wave length, $2\pi c/\omega$, and is large compared to the size of individual elements in both arrays.

Covered Dome (dome with cover of appreciable acoustic impedance). (Section 0.2.4, 2.6.2)

3. The dome cover introduces resonant contributions to the spectrum $Q(\omega,-L)$ and alters the non-resonant contributions. For a laterally large dome flexible enough that the free flexural wavenumber in the loaded dome cover, $k_r(\omega)$, satisfies

$$k_r(\omega) \gtrsim (\omega^2/c^2 + L^{-2})^{1/2}, \qquad (0\text{-}2)$$

the low-wavenumber part Q_-^p of the non-resonant contribution differs from Q_- for the fluid dome by terms of relative order h/L, for h/L << 1, where h is the depth of fluid having the same effective mass per unit area as the dome cover (effective fluid-equivalent thickness).(Good acoustic signal transmission requires h/L << 1).

4. Two types of resonant contributions may be distinguished, one due to wavenumbers near resonance ($k \simeq k_r$) and the other to convective wavenumbers ($k \gtrsim \omega/U_\infty$), where it is assumed that $k_r \lesssim \omega/U_\infty$ (i.e., the regime is not one of hydrodynamic coincidence). Both types derive from nearly resonant response modes $k_n \simeq k_r$). In consequence, if the structural damping is small, the frequency spectrum of average pressure $Q(\omega,-L)$ may contain a series of rapidly declining widely spaced resonance peaks. As the damping is made larger, the peaks become lower and the half-widths borader, having only the background contributions Q_-^p, Q_+^p, and Q_+^∞. For a

large dome, Q_+^p is nearly the same as for a fluid dome, and Q_+^∞ is similar but smaller than for a fluid dome (see point 2; regarding Q_-^p, see point 3). In a given frequency range, the more completely condition (0-2) is satisfied, i.e., the more mass-dominated is the dome-cover impedance, the more attenuated are the resonant waves in the dome interior.

<u>Acoustic modification for flush element in nonrigid boundary.</u> (Section 0.2.5, 2.6.3)

5. In the acoustic modification to the average-pressure spectrum on a flush element due to boundary vibration, the same contributions may be distinguished as in the case of an element within a covered dome, except that the one analogous to Q_+^∞ is presumably negligible. From the evidence of point 1, low wavenumbers are expected to make the major contribution. For example, in the special regime where $\omega/c \lesssim R_o^{-1} \lesssim k_r(\omega)$, a nonresonant part of this contribution can be distinguished and is of the order of $-1.7(R_o/h')Q_o(\omega)$ for $h'/R_o \gg 1$, where h' is the effective fluid-equivalent plate thickness. (Good acoustic signal preservation requires $h'/R_o \gg 1$; in the opposite limit of negligible plate impedance, with $\omega R_o/c \ll 1$, the element, as well as the boundary, is a pressure-release surface.)

6. The convective nonresonant contribution to the acoustic modification is similar to Q_+^p in the case of a dome (point 2), i.e., for large plate size it is of the order of the convective contribution to the direct (nonacoustic) average-pressure spectrum on a flush element of radius equal to that of the whole plate section.

7. The resonance contributions are similar to those for the covered dome, but there is no attenuation of the resonance peaks analogous to that which occurs in a dome interior when (0-2) is satisfied.

<u>Extended Summary</u>

These points are elaborated in the following more extended discussion and summary. Some of the results are independent of the dome model employed; others are not, though stated in more general terms; before acceptance of the results as applicable to an actual configuration, the degree of correspondence with the present model must be considered.

0.2.2 Excitation Pressure Due to the Turbulent Boundary Layer

We regard this pressure field as driving the nonrigid boundary without being affected by the resulting vibration. Consideration of its properties is needed to compute the desired acoustic pressure spectrum transmitted to a dome-housed element, or exerted on a flush element, and also to compare these spectra with the spectrum of average pressure on a similar flush element in a rigid boundary. The frequency-wavenumber spectrum $P_o(\bar{k},\omega)$ $[\bar{k} = (k_1,k_3)]$ of exciting pressure on a plane boundary presumably increases rapidly with streamwise wave number k_1 just beyond $k_1 \simeq \omega/U_\infty$, U_∞ being the asymptotic flow velocity, and reaches a maximum somewhat above this wave number. Wavenumber components with $k_1 > \omega/U_\infty$ (and hence $k \equiv |\bar{k}| > \omega/U_\infty$) are termed convective, in the sense that they can be generated by time-independent frozen-in eddies convected downstream at speeds $\leq U_\infty$. Components with low wave number, $k << \omega/U_\infty$, are presumably of much smaller amplitude, but important, as indicated below.

We define a spectral density $I_o(k,\omega)$ integrated over wavenumber direction θ:

$$I_o(k,\omega) = \int d\theta \; P_o(\bar{k},\omega) . \tag{0-3}$$

Let $Q_o(\omega)$ denote the frequency spectrum of pressure averaged over a circular area πR_o^2 and write

$$Q_o(\omega) = Q_{o-}(\omega) + Q_{o+}(\omega) , \tag{0-4}$$

where Q_{o-} derives from wave numbers $k < \omega/U_\infty$ and Q_{o+} from $k > \omega/U_\infty$. In a reasonable approximation, we obtain in the case $\omega R_o/U_\infty >> \pi$ (large area)

$$Q_{o-}(\omega) \simeq \int_0^{mR_o^{-1}} dkk[2J_1(kR_o)/kR_o]^2 I_o(k,\omega) , \tag{0-5}$$

$$Q_{o+}(\omega) \simeq (4/\pi)R_o^{-3} \int_{\omega/U_\infty}^{\infty} dk\, k^{-2} I_o(k,\omega), \tag{0-6}$$

where m is regarded as a fixed number large compared to unity.*

The area dependence of Q_{o+} is thus as R_o^{-3}. As for Q_{o-}, assume that $I_o(k,\omega)$ varies moderately and smoothly in the interval $0 < k < mR_o^{-1}$. In this event

$$Q_{o-} \simeq 2R_o^{-2} I_o(\sim R_o^{-1}, \omega), \tag{0-7}$$

where $I_o(\sim R_o^{-1},\omega)$ denotes an appropriately weighted average over the interval but is assumed nearly independent of R_o. If R_o is sufficiently large, (0-7) indeed holds; in fact

$$Q_{o-} \to 2R_o^{-2} I_o(0,\omega), \tag{0-8}$$

since (except by neglect of compressibility, viscosity, and inhomogeneity) we have $I_o(0,\omega) \neq 0$. For the range of R_o of usual interest, however, the average I_o of (0-7) is not necessarily identifiable with I_o of (0-8).

When $\omega R_o/c \lesssim m$, with c the sound velocity, the validity of (0-7) seems open to doubt.** It appears possible, in fact, that $I_o(k,\omega)$ has a peak where $k \simeq \omega/c$, corresponding to the wavenumber of sound; if so, a "radiation" contribution to Q_{o-} could be distinguished that would be independent of R_o so long as $\omega R_o/c \lesssim 1$.

Based on (0-6) and acceptance of (0-7), but with allowance for the tenuous possibility of the radiation contribution, we would have an area dependence for the total Q_o of the form

$$Q_o(\omega) = A(\omega)R_o^{-2} + B(\omega) + C(\omega)R_o^{-3}. \tag{0-9}$$

---

* The lower limit in (0-6) (and in the definition of Q_{o+}) may be taken $< \omega/U_\infty$, provided it remains larger than mR_o^{-1} for the range of R_o.

** Throughout, we limit consideration the regime of low Mach number, $U_\infty \ll c$.

Observed spectra for different transducer size, and measured narrow-band spatial correlations, place experimental constraints on A, B, and C. Observation has indicated that Q_o varies more nearly as R_o^{-2} than as R_o^{-3} (see Ref. 3). Hence the first term, in the range of $\omega R_o/U_\infty$ in question, is apparently larger than the third, i.e., $Q_{o-} \gtrsim Q_{o+}$.* Some observational justification for the approximation (0-7) is thus also provided.

0.2.3 Flow-Induced Noise Transmitted by a Fluid Dome

We consider in this and the following section the frequency spectrum $Q(\omega,y)$ of average pressure on a circular area πR_o^2 at depth $-y(>0)$ in the dome interior at or near the inner surface $(y = -L)$ and laterally near its center (see Fig. 0-1). The dome will be nominally regarded as circular with lateral radius $a \gg R_o$. In the specific model assumed in the text, the dome cover is supported or constrained only along its periphery. Accordingly, the wavenumber spacing of modes of dome-fluid structure is $\sim\pi/a$. Inner and outer fluids are assumed to have equal density ρ and sound velocity c.

We may divide $Q(\omega,y)$, just as $Q_o(\omega)$ in (0-4), into contributions $Q_-(\omega,y)$ and $Q_+(\omega,y)$ from $k < k_+$ and $k > k_+$, respectively, with $k_+ \simeq \omega/U_\infty$.

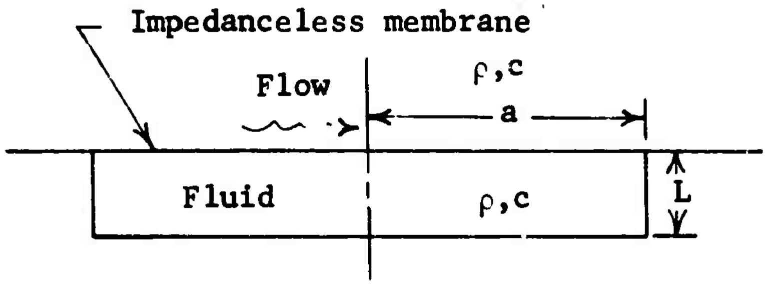

FIGURE 0-1. FLUID DOME

*The possibility that the observed dependence is more nearly as $B + CR_o^{-3}$ than as AR_o^{-2} may still bear examining. Plausible estimates of $Q_{o+}(\omega)$ can be made from crude convection models, but $Q_{o-}(\omega)$ is more inaccessible theoretically.

In the present section the interior fluid is regarded as separated from the flow by a dome cover with no stiffness or mass.

0.2.3.1 <u>Contribution from Excitation at Low Wave Numbers</u>

To obtain a rough estimate of $Q_-(\omega,y)$, we assume that $I_o(k,\omega)$ does not change substantially in most of the range $0 < k < (\omega^2/c^2 + L^{-2})^{1/2}$, an assumption similar to that at (0-7).* For each excitation wave number k, Q_- derives mainly from response modes of the dome having wave numbers nearly equal to k. Assuming a rigid inner surface and

$$\omega a/c \gg \pi, \tag{0-9.1}$$

$$(\omega L/c)(\omega a/c)^{-1/2} \ll 1, \tag{0-10}$$

we find the contribution Q_- at this surface to be

$$Q_-(\omega,-L) \simeq 2R_e^{-2} I_o(\sim R_e^{-1},\omega), \qquad (R_o \lesssim R_e) \tag{0-11}$$

where R_e is an effective averaging radius defined by

$$R_e^{-2} = (1/4)[(\omega/c)^2 + (1/2)L^{-2}] \doteq (1/4)\left[(2\pi/\lambda)^2 + 1/2L^2\right] \tag{0-12}$$

and $I_o(\sim R_e^{-1},\omega)$ again denotes a suitable average but is assumed nearly independent of R_e. By comparison of (0-7) and (0-11), we have as the relation between spectra for dome-shielded and flush-mounted elements of radius R_o

$$Q_-(\omega,-L) \sim H(R_o/R_e)^2 Q_{o-}(\omega) \qquad (R_o \lesssim R_e) \tag{0-13}$$

where $H = I_o(\sim R_e^{-1})/I_o(\sim R_o^{-1})$. We presume that $H \sim 1$; it may be smaller, however, for $R_e \gg R_o$, where R_o refers to the largest

*The possibility that $I_o(k,\omega)$ in the major part of this range is much larger than $I_o(0,\omega)$ need not be excluded.

radius for which $Q_o(\omega)$ is observed to vary as R_o^{-2}. If $R_e \lesssim R_o$, we find in place of (0-13)

$$Q_-(\omega,-L) \simeq Q_o(\omega). \tag{0-14}$$

Thus for a dome-shielded element the average-pressure spectrum due to the low-wavenumber contribution is roughly the same as for a flush-mounted element of radius $R_e = R_e(\lambda,L)$ given by (0-12). By use of a dome then, one can reduce this part of the flow-noise spectrum substantially below that for a flush element of given radius R_o only for frequencies corresponding to sound wave lengths rather larger than $4R_o$ and then only by a dome (sheath) of thickness rather larger than $R_o/2$.

According to (0-13) and (0-12), at frequencies low enough that $\lambda >> 9L$ we have $Q_-(\omega,-L)$ varying with ω, U_∞, and other flow parameters exactly as Q_{o-} for a large flush element. At frequencies high enough that $\lambda << 9L$, on the other hand, Q_- contains an additional factor ω^2. Accordingly, in the latter range $Q_-(\omega,L)$ does not scale with the boundary-layer parameters in the same way as Q_{o-}.

If there exists a peak in $I_o(k,\omega)$ at $k \simeq \omega/c$, as discussed preceding (0-9), there will be an additional contribution to $Q_-(\omega,-L)$ that will be just the same, in the case of a large dome ($\omega a/c >> \pi$), as the corresponding contribution to $Q_o(\omega)$ for a flush element of the same size, i.e., that will not be reduced by the dome for any λ and L.

If an infinitely deep interior medium is assumed instead of a rigid surface at $y = -L$, expressions (0-11) and (0-13) for $Q_-(\omega,-L)$ are reduced by a factor $\sim 1/4$.

0.2.3.2 <u>Spectra Averaged Over an Array of Elements</u>

We refer now to spectra of noise pressure averaged over the entire active area of an array of elements, confining consideration initially to the conbributions from low wave numbers. These average spectra are of practical interest.

Consider first an array of N flush elements of radius R_o, where $\omega R_o/U_\infty \gg \pi$ so that $Q_{o-} \propto R_o^{-2}$. Let the center spacing of elements be D_o, and represent the active and the total area of the array respectively by $A_o[=N(\pi R_o^2)]$ and $A_T[=ND_o^2]$. The spectrum of pressure averaged over the active area, say $Q_{o-}^{A_o}$, is given by

$$Q_{o-}^{A_o}(\omega) \simeq Q_{o-}(\omega)/N \simeq 2\pi(A_o/A_T)^{-1}A_T^{-1}I_o(\sim R_o^{-1}), \qquad (0\text{-}15)$$

where the second form corresponds to (0-7). To the extent that $I_o(\sim R_o^{-1},\omega)$ is independent of R_o, $Q_{o-}^{A_o}$ thus does not depend specifically on the elemental area πR_o^2 (in the assumed range $R_o \gg \pi U_\infty/\omega$) but varies inversely as the total active area A_o of the array.

Now consider an array of N' shielded elements of radius r_o and center spacing D yielding the same total area $A_T = N'D^2$.* The correlation area, or area scale, for the pressure field on the elements (at depth L) is πR_e^2. In two opposite limiting conditions the spectrum of pressure averaged over the total active area $A[=N'(\pi r_o^2)]$, say Q_-^A, is given by

1. Loosely packed array, $D^2 \gg \pi R_e^2$:

$$Q_-^A(\omega,-L) \simeq Q_-(\omega,-L)/N'$$

$$\simeq \begin{cases} 2R_e^{-2}I_o(\sim R_e^{-1})(D^2/A_T) \simeq H(A_o/A)(r_o^2/R_e^2)Q_{o-}^{A_o} & \text{if } R_e \gtrsim r_o, \\ 2r_o^{-2}I_o(\sim r_o^{-1})(D^2/A_T) \simeq (A_o/A)Q_{o-}^{A_o} & \text{if } R_e \lesssim r_o \end{cases} \qquad (0\text{-}16)$$

*We still assume a rigid inner surface (including the surface of the elements).

2. Tightly packed array, $D^2 << \pi R_e^2$:

$$Q_-^A(\omega,-L) \simeq 2\pi A_T^{-1} I_o(\sim A_T^{-1/2}) \simeq H_T(A_o/A_T) Q_{o-}^{A_o}, \qquad (0\text{-}17)$$

where $H_T = I_o(\sim A_T^{-1/2})/I_o(\sim R_o^{-1})$.

Suppose that the total area of the array, whether flush or sheathed, is fixed independently of flow-noise considerations, and likewise the minimum element radius and spacing in the shielded array. If incident signal pressure is independent of active area, as is roughly true even for an active array, the noise-to-signal ratio is proportional to the spectrum of noise pressure averaged over the active area. So far as determined by low excitation wave numbers, the signal-to-noise ratio for a flush array is then maximized by maximizing the array factor A_o/A_T. For a shielded array, with assumption that $H_T \sim H \sim 1$, the signal-to-noise ratio is increased relative to that for flush array by a factor A_T/A_o at frequencies below ω ($\doteq 2\pi c/\lambda$), provided that the spacing $D \lesssim \lambda/\pi^{1/2}$ and that the sheath thickness satisfies Eq. (0-1.1).* At higher frequencies, or with a thinner sheath or wider spacing, such that (0-1.1) is not satisfied, the factor of increase is smaller; it becomes roughly equal to the constant value A/A_o at frequencies such that $\lambda \lesssim r_o/\pi$, or at frequencies and sheath thicknesses such that condition (0-1.2) is violated.

*If $H_T << 1$, the factor of increase is greater.

0.2.3.3 Contribution from Excitation at High Wave Numbers

Two possibly significant contributions to $Q_+(\omega,y)$ may be distinguished according to the wavenumber range of the response modes of the dome. For given excitation wave number k, these correspond to response wave numbers k_n that are nearly (1) equal to k or (2) low enough that the corresponding waves in the interior fluid are propagated into the interior or at least are not attenuated to a negligible value at the interior surface [i.e., $k_n \lesssim (\omega^2/c^2+L^{-2})^{1/2}$]. These contributions to $Q_+(\omega,y)$ will be denoted here respectively by $Q_+^\infty(\omega,y)$ and $Q_+^P(\omega,y)$ and termed the direct convective and the propagating overlap contributions. (The contribution from low wave numbers considered in the preceding section may similarly be termed the direct non-convective contribution.) Since we are dealing here with a fluid dome, we do not yet have to consider resonant contributions.*

The direct convective part, $Q_+^\infty(\omega,y)$, for each contributing excitation wave number k, is affected by area averaging over the interior element in the same way as the excitation pressure, and differs from it by the absolute square of the interior acoustic response coefficient, $\Gamma(k,\omega,y)$. In order of magnitude, in the case of a rigid inner surface, Q_+^∞ is related to the high-wave-number part of the average-pressure spectrum Q_{0+}[see Eq. (0-6)] for a flush element of equal area by

$$Q_+^\infty(\omega,y) \sim \exp(-2\omega L/\eta U_\infty)\, ch^2[\omega(y+L)/\eta U_\infty] Q_{0+}(\omega), \qquad (0\text{-}18)$$

where the factor $\eta(<1)$, which defines an effective velocity ηU_∞, increases slowly with decreasing ω and increasing L but is regarded as being close to unity even in the higher range of frequencies. In any case the factor multiplying Q_{0+} is extremely small

*Resonances of the interior acoustic field for certain relations between the depth L of the dome and the sound wave length do not occur, or do not result in amplification of the driving pressure by a factor greater than unity, on account of damping by the radiation field produced in the outside medium.

for $|y| \gg U_\infty/\omega$, and this depth U_∞/ω is rather small except at very low frequency.

The propagating overlap part, $Q_+^p(\omega,y)$, though readily computed numerically in the model employed in the text, is not expressible in a transparent closed form except in restricted parameter domains. In a certain limit based on the largeness of $\omega a/c(=2\pi\lambda/a)$, a simple form applies; more specifically the form applies where conditions (0-9.1,) (0-10) and the condition

$$(\omega L/c)^{-1}(\omega a/c)^{-1/2} \ll 1 \tag{0-19}$$

are simultaneously satisfied. In order of magnitude in the case of a rigid inner surface and with $\omega R_o/c \lesssim 1$, Q_+^p in this regime is estimated, again in terms of Q_{o+} for an equal area, as

$$Q_+^p(\omega,y) \simeq (\pi/4)(\omega L/c)^2 f(\omega a/c)(R_o/a)^3 Q_{o+}(\omega), \tag{0-20}$$

where f(z) denotes a certain positive oscillatory function of the argument z having period π, a peak value 3.4 and a value less than half as large when averaged over z.* In the case of infinite depth in place of a rigid inner surface, L^2 in (0-20) is replaced by $(1/4)y^2$ (Q_{o+} still refers to flush mounting in a rigid surface).

Conditions (0-10) and (0-19) will not be satisfied in much of the parameter domain of interest.** Still, estimate (0-20) is perhaps not misleading.

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* The forms of (0-20) and of (0-27), (0-33), and (0-34) below depend on the lateral boundary condition assumed in the explicit model. The forms quoted correspond to the larger of the two limiting types of expression given in the text and therefore roughly apply also, with reduced coefficients, to mixed conditions, which may most closely simulate actual configurations.

** The paradoxical increase of Q_+^p with dome depth L according to (0-20) occurs only in the limited range of L(if any) defined by conditions (0-10) and (0-19).

In the limit where it holds, the contributing modes are those with lateral wave numbers near that of sound in the inner and outer fluids, ω/c, and hence with nearly vanishing wave numbers normal to the outer surface of the dome, corresponding to interior waves propagating parallel to this surface. Q_+^p is independent of R_o in the regime considered (in (0-20), $Q_{o+} \propto R_o^{-3}$). Crudely, in view of the factor $(R_o/a)^3$ multiplying Q_{o+} in (0-20), Q_+^p is similar to the direct pressure Q_{o+} averaged not over the area πR_o^2 but instead over the entire area of the dome (πa^2).*

On account of the factor $\omega^2 f(\omega a/c)$, Q_+^p as given by (0-20) does not scale with the boundary-layer parameters in the same way as Q_{o+}. In the case of Q_+^∞, estimated by (0-18), the difference in scaling relative to Q_{o+} is inferred to be less pronounced.

0.2.3.4 Comparison of Contributions; Possible Reduction by a Fluid Dome

The component Q_+^∞ estimated at (0-18) represents a contribution to the interior average-pressure spectrum that still remains in the limit of a laterally infinite dome ($a \to \infty$), as does the direct component Q_- due to low excitation wave numbers estimated in the preceding section. The overlap component Q_+^p, on the other hand, vanishes in this limit.

Suppose we accept the estimates (0-13), (0-18), and (0-20) of the various contributions to the interior average-pressure spectrum as roughly correct in their common domain of application. By choosing $L \gg U_\infty/\omega$, which is still consistent with a rather thin sheath, we can make Q_+^∞ small compared to Q_{o+}, or even compared to Q_+^p for any realizable lateral size a. Further, if we accept the observational indication stated after (0-9), we have $Q_{o-} \gtrsim Q_{o+}$. From (0-13) and (0-20), again considering $\omega L/c \sim 1$ (or at least not large) and recalling that $\omega a/c \gg 1$ for applicability of (0-20), we then infer that

*The factor $\omega L/c$ in (0-20) is regarded as of order of magnitude unity because of the assumed conditions (0-10) and (0-19).

$$Q_-(\omega,-L) \gg Q_+^p(\omega,-L)$$

and likewise $Q_-(\omega,-L) \gg Q_+(\omega,-L)[= Q_+^p + Q_+^\infty]$. Hence, under the present assumptions, the area dependence of the average-pressure spectrum on a flush element indicated by experiment, namely as R_o^{-2} (for $\omega R_o/U_\infty \gg \pi$), implies the predominance of the low-wave-number contribution to the average-pressure spectrum on an element shielded by a fluid dome or sheath. The earlier discussion of Eqs. (0-13) and (0-14) then applies also approximately to the total spectrum for an interior element.

Thus, with regard to flush and shielded arrays, according to the preceding discussion and Section 0.1.2.2 above, a fluid sheath (or comparable elastic-solid sheath) of thickness L permits significant noise reduction over a substantial frequency range only if $L \gtrsim r_o$, where r_o is the radius of the shielded elements used.* If instead the convective contribution from high excitation wave numbers predominated, significant noise reduction relative to a flush array could be achieved provided roughly only that $L \gg U_\infty/\omega$ and that the lateral size a of the sheath (or each section thereof) is rather larger than a wave length ($2\pi c/\omega$) and is large compared to the size of individual elements.

Present conclusions based on a treatment of a fluid sheath are believed to apply roughly to an elastic-solid sheath as well, provided that the transverse sound velocity c_t is of the order assumed with regard to c, notably that $c_t \gg U_\infty$.

0.2.4 Flow-Induced Noise Transmitted by a Covered Dome

In distinction from the preceding section we consider now a dome having a cover with significant impedance. We assume that, except at frequencies below the range of concern, the free flexural waves in the dome cover (coupled to the internal and

*If there should prove to be a spike in the wave number spectrum of excitation at $k \simeq \omega/c$, this contribution will not be reduced by a sheath independently of L. It would be somewhat reduced for both flush and shielded arrays by tight packing to yield coherent cancellation (provided the effective size satisfies $A_T^{1/2} \gg c/\omega$).

external fluids) have wave velocities greater than the asymptotic flow velocity (U_∞), so that the waves of excitation pressure in resonance with free waves are nonconvective ($k_1 < \omega/U_\infty$) and their amplitudes therefore small, i.e., the regime of "hydrodynamic coincidence" is avoided.

In the present instance, unlike that of the fluid dome, resonances of the dome-fluid structure occur, such that at a given frequency one or more modes may have acoustic response coefficients of magnitude greatly exceeding unity unless the structure is highly damped. [The resonant wave number, $k_r(\omega)$, at any frequency exceeds both the sound wavenumber in the fluid, ω/c, and the resonant wavenumber of the isolated plate, $\hat{k}_o(\omega)$.] In this case the previously defined low-wavenumber contribution $Q_-(\omega,y)$ to the interior average-pressure spectrum $Q(\omega,y)$, deriving, at each excitation wave number k, mainly from modes with $k_n \simeq k$, may have to be understood to include contributions from k's (and k_n's) near the resonant wave number, even though contributions from k's somewhat higher or lower are negligible. Likewise, the high-wavenumber contribution $Q_+(\omega,y)$, which formerly was composed of a direct convective part Q_+^∞, deriving mainly from modes with $k_n \simeq k$, and a propagating overlap part, deriving mainly from modes with $k_n \lesssim (\omega^2/c^2 + L^{-2})^{1/2}$, may have to be understood to include a further part deriving from modes near resonance, i.e, with $k_n \simeq k_r$, which will be called the resonance overlap contribution and denoted by $Q_+^r(\omega,y)$.

A given lateral mode ν of the structure resonates at some frequency, ω_ν, and a corresponding half-width $\delta\omega_\nu$ of the response may be defined. Likewise, for any given frequency ω, a resonant wavenumber $k_r(\omega)$ and a wavenumber half-width δk may be defined. If the structural damping is small enough, we have

$$\delta ka \lesssim \pi; \qquad \text{(0-21)}$$

in this case, at a given frequency at most a single mode can be near resonance, but the acoustic response at resonance can be large. The parameter δka increases with mode number. If the dome cover is highly damped and the mode number large, on the other hand, we have $\delta ka \gg \pi$, so that at a given frequency many modes lie almost equally near resonance; the response for each, however, is only moderately greater than for modes away from resonance, and furthermore their effects tend to cancel so far as their excitation by high wavenumbers is concerned.

The resonant modes are more rapidly attenuated in the dome interior if the resonant wave length $2\pi/k_r(\omega)$ is small compared to both the sound wave length and, say, six times the depth L, roughly in accordance with a factor $\exp[-2(k_r^2-\omega^2/c^2)^{1/2}L]$ [see Eq. (0-23b) below].

0.2.4.1 Contribution from Excitation at Low Wave Numbers

We refer first to the contribution from excitation at low wavenumbers ($<\omega/U_\infty$) removed from the resonant wavenumber, analogous to Q_- for the fluid dome.

It is not possible to design the dome so that noise due to excitation wavenumbers in most of the interval $k < \omega/c$ is suppressed without likewise suppressing an incident acoustic signal at least from certain angles of incidence; that is, the dome-cover impedance in this interval must be low for adequate signal transmission.

In general, even under the assumptions of Sec. 0.1.2.1 it is not possible to estimate $Q_-(\omega,y)$ for a covered dome by a simple closed expression analogous to (0-11). If (0-19) is satisfied, however, the resonant part of Q_-, say Q_-^r, can be clearly distinguished and is given in the neighborhood of a resonant frequency ω_ν, corresponding to modal wave number k_ν, roughly by the single-mode form

$$Q_-^r(\omega,y) \simeq (\pi/a)k_\nu I_o(k_\nu,\omega)\frac{b_\nu^2}{(\omega/\omega_\nu-1)^2+\epsilon_\nu^2}$$

$$\sim \frac{\pi b_\nu^2}{(\omega/\omega_\nu-1)^2+\epsilon_\nu^2}(R_o/a)k_\nu R_o Q_{o-}(\omega) \qquad (0\text{-}22)$$

for $k_\nu R_o \lesssim 1$, where the second form follows from (0-7) on assumption that $I_o(k_\nu,\omega) \sim I_o(\sim R_o^{-1},\omega)$. If $k_\nu R_o \gtrsim 1$, (0-22) requires an additional area-averaging factor $[2J_1(k_\nu R_o)/k_\nu R_o]^2$. The resonance parameters in (0-22) are given roughly, for a rigid inner surface, by

$$\omega_\nu \sim \omega_{o\nu}[1+2/K_{2\nu}h]^{-1/2}, \quad \text{(0-23a)}$$

$$b_\nu \sim (k_\nu h)^{-1}(1+2/k_\nu h)^{-1}\exp(-K_{2\nu}L)\,\mathrm{ch}[K_{2\nu}(y+L)], \quad \text{(0-23b)}$$

$$\epsilon_\nu \equiv \delta\omega_\nu/\omega_\nu, \quad \text{(0-23c)}$$

with $K_{2\nu}=(k_\nu^2-\omega_\nu^2/c^2)^{1/2}$, where $\omega_{o\nu}$ is the resonant frequency of the isolated dome cover for wavenumber k_ν, and $h = \sigma/\rho$, with σ the dome cover mass per unit area. The fractional frequency half-width ϵ_ν is of the order of a dimensionless damping coefficient for the motion of the dome structure. In the model explicitly pursued in the text this coefficient consists of a sum of terms for hysteretic and viscous damping of the dome cover, but more generally includes also damping at the joinings of the dome with the peripheral supporting structure (which commonly is the more important contribution).

Since at fixed frequency $k_r(\omega)$ increases with decreasing dome stiffness, so does the k_ν pertinent to a given frequency range and hence also the exponential-attenuation coefficient $K_{2\nu}$ in Eq. (0-23b) for b_ν. Thus b_ν and the amplitude of the resonance maximum decreases as the stiffness decreases, on assumption that the rate of increase of $I_o(k_\nu,\omega)$ is not precipitous so long as $k_\nu \lesssim \omega/U_\infty$.

In actuality, resonance peaks of the character of (0-22) or (0-27) below seem not to be ordinarily distinguishable in the frequency spectra of noise for dome-shielded hydrophones.

For a dome of sufficiently low stiffness that $\hat{k}_o(\omega) \gtrsim (\omega^2/c^2+L^{-2})^{1/2}$, the resonant modes with $k_n \simeq k_r(\omega)(>\hat{k}_o)$

are well removed and distinguishable from the other contributing range $k_n \lesssim (\omega^2/c^2+L^{-2})^{1/2}$ of propagating or weakly attenuated modes, whether or not condition (0-21) is satisfied. Furthermore, the dome-cover impedance is mass-dominated throughout the latter interval. In this instance in the regime where conditions (0-9.1), (0-10) and the additional condition

$$(|z_\omega|/\rho c)(\omega a/c)^{-1/2} \ll 1 \tag{0-24}$$

are simultaneously satisfied, where z_ω represents the acoustic impedance of the dome cover at a wavenumber ω/c, we can compute a simple estimate for the nonresonant contribution, say Q_-^p, to Q_-. If the ratio h/L of the fluid-equivalent dome-cover thickness ($h=\sigma/\rho$) to the dome depth is small compared to unity, as required for good signal transmission, $Q_-^p(\omega,-L)$ differs from the corresponding expression (0-11) for $Q_-(\omega,-L)$ in the case of a fluid dome by terms of relative order h/L.

0.2.4.2 Contribution From Excitation at High Wave Number

The direct convective contribution, as in the case of the fluid dome, is estimated as

$$Q_+^\infty(\omega,y) \sim \left|\Gamma(\omega/\eta U_\infty,\omega,y)\right|^2 Q_{0+}(\omega) \tag{0-25}$$

(with a slightly larger η), but now the acoustic response coefficient Γ takes account of the dome-cover impedance. If the cover is dynamically equivalent to a thin plate or membrane, for a rigid inner surface and with $\omega/U_\infty \gg k_r(\omega)$, we have

$$\left|\Gamma(\omega/\eta U_\infty,\omega,y)\right|^2 \sim \exp(-2\omega L/\eta U_\infty)\,\mathrm{ch}^2[\omega(y+L)/\eta U_\infty] \tag{0-26}$$

$$\times\; 4(\omega h/\eta U_\infty)^{-2}[\omega/\eta U_\infty \hat{k}_o(\omega)]^{-2n} Q_{0+}(\omega)$$

[cf. (0-18)], where n=4 for a plate and n=2 for a membrane. In the regime considered, $|\Gamma|^2$ as given by (0-26) is even smaller than for the fluid dome.

The propagating overlap contribution, $Q_+^p(\omega,y)$, as for the fluid dome, can be computed numerically for the model of the text, but not generally expressed in a transparent form. Again, however, in the regime where conditions (0-9.1), (0-10), (0-19), and (0-24) are satisfied, the simple estimate (0-20) applies.

As for the resonance overlap contribution, if condition (0-21) for a single-mode approximation holds, for $\omega/U_\infty >> k_\nu$ and $k_\nu R_o \lesssim 1$ we have, similarly to (0-22),

$$Q_+^r(\omega,y) \sim \frac{(\pi/2)b_\nu^2}{(\omega/\omega_\nu-1)^2+\epsilon_\nu^2} \; k_\nu a(R_o/a)^3 Q_{o+}(\omega) \tag{0-27}$$

Eqs. (0-22) and (0-27) relate Q_-^r and Q_+^r to quantities Q_{o-} and Q_{o+} pertinent to an imagined large flush element of radius R_o; Q_-^r and Q_+^r, for $k_\nu R_o \lesssim 1$, are themselves independent of the radius R_o of the interior element. Eqs. (0-22) and (0-27) yield as the order of magnitude of the ratio of direct-resonance to resonance-overlap contributions

$$Q_-^r/Q_+^r \sim 2(a/R_o)(Q_{o-}/Q_{o+}). \tag{0-28}$$

On acceptance of the observational evidence that $Q_{o-} \gtrsim Q_{o+}$, we infer $Q_-^r >> Q_+^r$.

If the condition $\delta ka >> \pi$ opposite to (0-21) holds, then in contrast to (0-27) a number of modes lie within the resonance peak. If certain additional conditions are satisfied (as they are for sufficiently large a and nonvanishing damping), contributions from adjacent modes near resonance, as elsewhere (except those with $k_n \simeq \omega/c$), nearly cancel one another. Q_+^r then vanishes relative to Q_+^p of (0-20)[or, pertinent to still larger a, relative to Q_+^∞ of (0-24)].

0.2.5 Acoustic Modification of Flow Noise on Flush Elements Due to a Non-Rigid Boundary

We proceed to the question of the acoustic modification of the spectrum of average pressure on a rigid circular plug (transducer) of radius R_o flush with, but cut out from, a non-rigid plate or membrane bounding a turbulent flow. The opposite side of the plate is supposed to be effectively vacuous. The effect of the vibrating plate on the excitation pressure is neglected.

The acoustic increment in the average-pressure spectrum $Q_o(\omega)$, which includes an interference between the direct and acoustic pressures, is denoted by $\delta\hat{Q}$. This is regarded, as usual, as composed of a sum of contributions from low wavenumbers, $\delta\hat{Q}_-$, and high, $\delta\hat{Q}_+$.

0.2.5.1 Contribution From Excitation at Low Wave Numbers

No simple approximation can be written for the sum representing this contribution in the general case. If (0-21) is satisfied, however, as in the case of the covered dome the resonant part of $\delta\hat{Q}_-$, say $\delta\hat{Q}_-^r$, can be distinguished and is given in the neighborhood of a resonant frequency $\omega_{1\nu}$, for $k_\nu R_o \lesssim 1$, roughly by the single-mode form

$$\delta\hat{Q}_-^r(\omega) \simeq (\pi/a)k_\nu I_o(k_\nu,\omega)\frac{b_{1\nu}^2}{(\omega/\omega_{1\nu}-1)^2+\epsilon_{1\nu}^2}(1-8k_\nu R_o/3\pi)^2$$

$$\sim \frac{\pi b_{1\nu}^2(1-8k_\nu R_o/3\pi)^2}{(\omega/\omega_{1\nu}-1)^2+\epsilon_{1\nu}^2}(R_o/a)k_\nu R_o Q_{o-}(\omega) \qquad (0\text{-}29)$$

[cf. Eq. (0-22)], where the resonance parameters are given roughly by

$$\omega_{1\nu} \sim \omega_{o\nu}[1 + 1/K_{2\nu}h]^{-1/2}, \qquad (0\text{-}30a)$$

$$b_{1\nu} \sim (2k_\nu h)^{-1}(1+1/k_\nu h)^{-1}, \qquad (0\text{-}30b)$$

with the fractional frequency half-width $\epsilon_{1\nu}$ again of the order of a damping coefficient for the motion.

Unlike b_ν of (0-23b), $b_{1\nu}$ contains no attenuating factor governed by $K_{2\nu}$ and is therefore less affected by the plate stiffness.

For a plate of sufficiently low stiffness that $\hat{k}_o(\omega)R_o >> 1$, whence also $k_r(\omega)R_o >> 1$, which represents a regime excluding that of (0-29), the resonant modes are well removed and distinguishable from the other contributing range $k_n \lesssim mR_o^{-1}$ of modes that are not suppressed by area averaging. The plate impedance is then mass-dominated in the latter interval.

In this instance, in the regime where conditions (0-9.1) and (0-24) hold, we can compute a simple crude estimate for the nonresonant contribution, say $\delta\hat{Q}_-^p$, to $\delta\hat{Q}_-$. Let $\epsilon = (1-q_o^2)h/R_o$, where q_o^2 represents the ratio of the stiffness contribution to the mass contribution to plate impedance, evaluated at a wavenumber $k_n \sim R_o^{-1}$, i.e., $(1-q_o^2)h$ is an effective fluid-equivalent plate thickness for the pertinent interval; since $\hat{k}_oR_o >> 1$ by assumption, we have $q_o^2 << 1$.* In the limit of zero plate impedance ($\epsilon \to 0$), for $\omega R_o/c \lesssim 1$, we necessarily have $\delta\hat{Q}_-^p \simeq -Q_o$, i.e., if the boundary is a pressure-release surface and the radiation impedance of the blocked and rigid plug is negligible, then the plug face is likewise nearly a pressure-release surface. The noise may not thus be cancelled without likewise cancelling an incident acoustic signal. In order that a signal not be substantially reduced by boundary vibration, then, it is required in the frequency range of interest that $\epsilon >> 1$.** For $\epsilon >> 1$ and $\omega R_o/c \lesssim 1$, we obtain the crude estimate

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* The equivalent thickness h introduced in connection with the contribution Q_-^p for the covered dome may similarly be construed as reduced by a factor $1-q_\omega^2$, where q_ω^2 pertains to the stiffness at a wavenumber $k_n \sim \omega/c$.

** If resonating devices, effective in a narrow frequency range, are employed for this purpose, they may be regarded as implying an equivalent plate thickness having a resonance-type frequency dependence.

$$\delta\hat{Q}^{p}_{-}(\omega) \simeq -\epsilon^{-1}(1.7-1.5\epsilon^{-1}\ln\epsilon)Q_{o-}(\omega). \quad (0\text{-}31)$$

More generally, in the regime in question, for arbitrary ϵ but with $\omega R_o/c \lesssim 1$, $\delta\hat{Q}^{p}_{-}/Q_{o-}$ depends substantially only on the parameter ϵ. The simultaneous conditions $\hat{k}_o R_o >> 1$ and $\omega\lambda_o/c \lesssim 1$ assumed above define a very restricted parameter regime, however.

0.2.5.2 Contribution from Excitation at High Wave Numbers

The propagating overlap contribution $\delta\hat{Q}^{p}_{+}$ to $\delta\hat{Q}_{+}$ can be simply expressed in a special regime analogous to that where (0-20) applies. The conditions for this type of approximation in this instance are (0-9.1), (0-24), and in addition

$$\left|z_{\omega}/\rho c + \xi\right|^{-1} (\omega a/c)^{1/2} << 1, \quad (0\text{-}32)$$

where $\xi = z_R/\rho c(\pi R_o^2)$ with z_R the radiation impedance of a piston of radius R_o at frequency ω in the given nonrigid boundary. The result is given roughly in terms of Q_{o+} for an equal area by

$$\delta\hat{Q}^{p}(\omega) \simeq (\pi/4)\left|z_{\omega}/\rho c + \xi\right|^{2} f(\omega a/c) (R_o/a)^3 Q_{o+}(\omega), \quad (0\text{-}33)$$

where it supposed as in (0-20) that $\omega R_o/c \lesssim 1$. Hence again, with reference to high excitation wave numbers, the non-resonance acoustic average-pressure spectrum on the element is similar to the direct pressure spectrum averaged over the entire plate area (πa^2) demarked by the bounding structural members. The regime defined by conditions (0-4) and (0-32) is rather limited.

The resonance overlap contribution $\delta\hat{Q}^{r}_{+}$ to $\delta\hat{Q}_{+}$, if condition (0-21) for a single-mode approximation holds, is given, for $\omega/U_\infty >> k_\nu$ and $k_\nu R_o \lesssim 1$, similarly to (0-27) by

$$\delta\hat{Q}^{r}_{+}(\omega) \sim \frac{(\pi/2)b_{1\nu}^2(1-8k_\nu R_o/3\pi)^2}{(\omega/\omega_{1\nu}-1)^2+\epsilon_{1\nu}^2} k_\nu a(R_o/a)^3 Q_{o+}(\omega) \quad (0\text{-}34)$$

The acoustic direct convective contribution to average pressure on the rigid plug, corresponding to modes of the surrounding plate with $k_n \simeq k \gtrsim \omega/U_\infty$ is reasonably neglected.

PART 1
DOMES OF INFINITE EXTENT

Part 1 is concerned with the simple limiting case where the dome and its surface in contact with the exciting flow may be regarded as of infinite extent. (The conditions for this to be so will appear in Part 2.) Properties of the exciting boundary-layer pressure field and of the response of the coupled plate-fluid system, pertinent also in Part 2 on finite dimes, are discussed here. The former topic is considered further in Section 2.5. Except for Section 1.1, Part 1 may be omitted and used merely as an appendix to Part 2.

1.1 Infinite-Dome Model

This dome model is constructed as follows. A turbulent boundary layer is generated by the flow of a semi-infinite fluid along an infinite elastic plate or membrane. (See Figure 1-1.) The boundary layer is regarded as having a finite thickness, however, such as would occur in a finite flow of interest. The plate is planar in its undeformed state and is assumed to be thin, in the usual sense, for all excitations encountered. On the opposite side of the plate is contained a second fluid at rest in the shape of an infinite slab bounded on its other face by an infinite surface parallel to the plate. For the acoustic field at this inner surface, a fixed impedance condition is assumed to be given for each frequency and wavenumber component parallel to the surface.

The boundary layer pressure driving the plate is regarded as the same as would exist if the boundary were rigid, i.e., the effect of the boundary motion on the turbulence is not considered.* Consideration will be restricted to the regime of low Mach numbers, $U_\infty << c^{\pm}$.

*The driving pressure should, nevertheless, be regarded as the entire fluctuating pressure existing in the fluid (with rigid boundary), including the effect of compressibility; at wave numbers $k \lesssim \omega/c$ the pressure components likely differ substantially from those for a similar incompressible fluid.

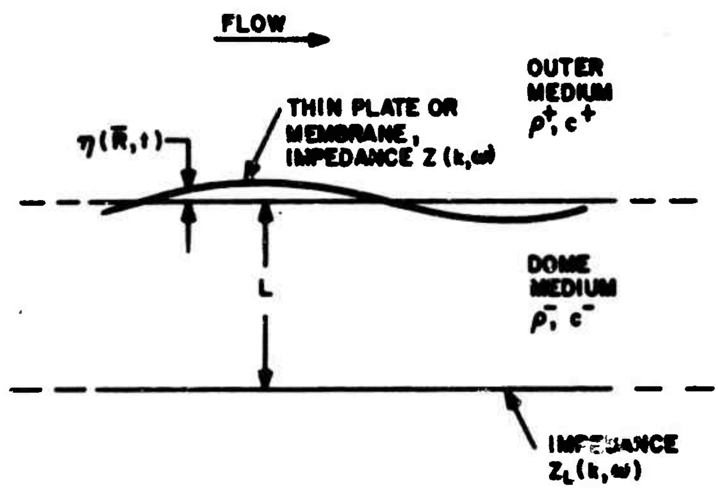

FIGURE 1-1. MODEL DOME OF INFINITE LATERAL EXTENT

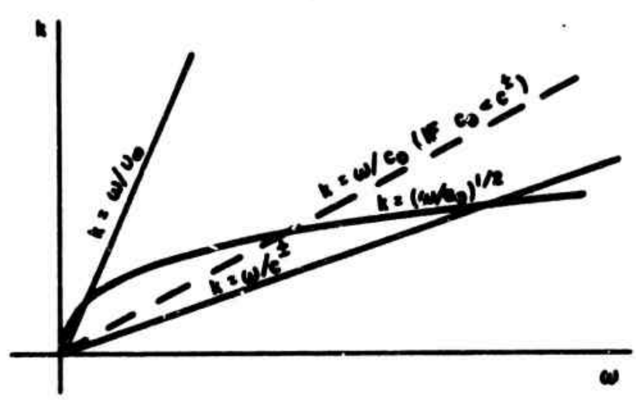

FIGURE 1-2. RELATION OF PERTINENT WAVE NUMBERS OF THE DOME-FLOW SYSTEM

This problem is regarded as a simplified and partial model for the transmission of flow noise to a dome-housed transducer via mechanical excitation of dome vibration by the fluctuating pressure on its surface and via the sound field within the dome generated by this vibration. The plate and inner fluid represent the dome, and the inner, parallel surface represents both the portion of the hull enclosed by the dome and the surface of the transducer itself, unless the latter, instead of being contiguous with the hull, is considered to project in front of it without disturbing significantly the acoustic field.

Many observations will be drawn that do not depend on assumption of a completely explicit form for the impedance z of the dome cover.* In some part, however, we treat specifically a thin plate or membrane subject to hysteretic and viscous damping, i.e., having impedance

$$z(k,\omega) = \sigma\omega \, [i(q^2-1) + q^2\zeta + \beta/\omega], \tag{1-1}$$

$$q = \begin{cases} a_o k^2/\omega \text{ (plate)} \\ c_o k/\omega, \text{ (membrane)} \end{cases} \tag{1-2}$$

where k denotes the wave number of an exciting pressure wave of angular frequency ω along the surface of the plate,

$$a_o{}^2 = (E/\rho_o)h_o{}^2/12(1-\sigma_o{}^2) = c_t{}^2 h_o{}^2/6(1-\sigma_o), \tag{1-3}$$

h_o is the plate thickness, ρ_o the mass density of the plate material, E Young's modulus, σ_o Poisson's ratio, c_t the infinite-medium shear-wave velocity, β a viscous damping coefficient (with units of frequency), ζ a hysteretic damping coefficient (dimensionless), σ the mass per unit area ($=\rho_o h_o$ in the case of the plate), and, with reference to the membrane,

$$c_o = (T/\sigma)^{1/2} \tag{1-4}$$

*In Part 2, some consideration is given to the fluid dome, which has z = 0.

with T the tension (force/length).* The quantity q of Eq. (2) may also be written

$$q = \begin{cases} (k/k_o)^2 & \text{(plate)} \\ k/k_o & \text{(membrane)} \end{cases} \tag{1-5}$$

with

$$k_o = \begin{cases} (\omega/a_o)^{1/2} & \text{(plate)} \\ \omega/c_o\,. & \text{(membrane)} \end{cases} \tag{1-6}$$

k_o is the resonant wave number (neglecting damping) of the plate or membrane in isolation, i.e., the wave number of free waves at frequency ω. The relation among wave numbers is illustrated in Fig.1-2.

These parameters fix the effective values of quantities that have significance also in more realistic and complex situations. For example, a hull or dome cover will not ordinarily be thin for much of the exciting spectrum of boundary-layer pressure fluctuation (i.e., we will not have $\omega h_o/U_\infty \lesssim 1$, where U_∞ denotes ship speed). Nevertheless, the dynamic behavior will not on this account differ essentially from that determined by our model.

1.2 Response of the Infinite Dome to an Exciting Wave

We denote the impedance at the inner surface (y=-L) for wave number k and frequency ω by $z_L = z_L(k,\omega)$. The density and speed for the outer fluid will be denoted by ρ^+, c^+ and for the inner by ρ^-, c^-. Suppose an exciting pressure wave

$$p(\bar{R},t) = p_o e^{i\bar{k}\cdot\bar{R}-i\omega t}[\bar{k} = (k_1,k_3),\ \bar{R} = (x,z)] \tag{1-7}$$

of definite frequency and wave number, e.g., a spectral component of boundary-layer noise pressure, is applied to the outer surface of the plate. The y components k_2^+, k_2^- of the wave vectors associated with the acoustic fields in the outer and inner fluids, respectively, are then given by

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*c_o is the free wave velocity in the membrane in the absence of damping.

$$k_2^{\pm} = \left[\left(k_\omega^{\pm}\right)^2 - k^2\right]^{1/2}, \quad k_\omega^{\pm} = \omega/c^{\pm} \tag{1-7.1}$$

(where the sign is chosen on assumption that $\omega > 0$). For use when $k^2 > (k_\omega^{\pm})^2$, corresponding to exponential dependence on y, we define

$$K_2^{\pm} = \left[k^2 - \left(k_\omega^{\pm}\right)^2\right]^{1/2}. \tag{1-7.2}$$

The resulting pressure in the interior fluid in this laterally infinite case has the form

$$p^-(\bar{r},t) = p_o^- f(y) e^{i\bar{k}\cdot\bar{R} - i\omega t}, \tag{1-8}$$

where f(0) may be normalized to unity.

In Appendix 1 [see Eqs. (13)] the ratio

$$\Gamma(k,\omega,y) = p^-(\bar{r},t)/p(\bar{R},t) = p_o^- f(y)/p_o \tag{1-8.1}$$

of the acoustic pressure (8) to the driving pressure (7) is obtained for arbitrary dome-cover impedance $z(k,\omega)$. The result may be written*

$$\Gamma(k,\omega,y) = \begin{cases} \dfrac{\cos[k_2^-(y+L)+\alpha]}{\cos\theta - i\sin\theta[(k_2^-\rho^+/k_2^+\rho^-) + (zk_2^-/\rho^-\omega)]} & (k < k_\omega^{\pm}) \quad (1\text{-}9) \\[2ex] \dfrac{\mathrm{ch}[K_2^-(y+L)+\hat{\alpha}]}{\mathrm{ch}\hat{\theta} + \mathrm{sh}\hat{\theta}[(K_2^-\rho^+/K_2^+\rho^-) + i(zK_2^-/\rho^-\omega)]} & (k > k_\omega^{\pm}) \quad (1\text{-}10) \end{cases}$$

where

$$\theta = k_2^- L + \alpha, \quad \hat{\theta} = K_2^- L + \hat{\alpha} (= -i\theta),$$
$$\tan\alpha = i\rho^-\omega/k_2^- z_L, \quad \mathrm{th}\,\hat{\alpha} = -i\rho^-\omega/K_2^- z_L \quad (\hat{\alpha} = -i\alpha). \tag{1-11}$$

(If the inner surface is rigid, we note,

$$|z_L| \to \infty, \quad \alpha = \hat{\alpha} = 0.)$$

With z given by (1), Eqs. (9) - (10) become

*For the case where the inner surface approaches pressure release ($z_L = 0$), a form more appropriate than (10) is given in Appendix 1, and likewise where the inner medium is infinite.

$$\Gamma(k,\omega,y) = \begin{cases} \dfrac{\cos[k_2^-(y+L)+\alpha]}{[\cos\theta + k_2^-h^-\sin\theta(q^2-1)]-ik_2^-h^-\sin\theta[(k_2^+h^+)^{-1}+\zeta q^2+\beta/\omega]} & (k < k_\omega^\pm) \quad (1\text{-}12) \\ \dfrac{ch[K_2^-(y+L)+\hat{\alpha}]}{ch\hat{\theta} + sh\hat{\theta}[(K_2^-h^-/K_2^+h^+)-K_2^-h^-(q^2-1)]+iK_2^-h^-sh\hat{\theta}(\zeta q^2+\beta/\omega)} & (k > k_\omega^\pm) \quad (1\text{-}13) \end{cases}$$

where $h^\pm$ is the effective thickness of the plate or membrane referred to the density of the outer (+) or inner (-) fluid:

$$h^\pm = \sigma/\rho^\pm. \tag{1-14}$$

(We note that $h^-/h^+ = \rho^+/\rho^-$; also, in the case of the plate, $h^\pm = (\rho_o/\rho^\pm)h_o$.)

In the factor q^2-1 appearing in the denominators of (12) and (13), the q^2 corresponds to cover stiffness and the -1 to cover mass. Assuming a pure imaginary z_L (purely reactive inner surface) and $\hat{\alpha} > -K_2^-L$, we see from (13) that both fluids act to increase the mass loading on the plate. The only imaginary (resistive) part of the denominator in this case, which will be inversely related to the maximum of Γ at a resonance, is contributed by the plate damping effects. We see from (12), on the other hand, that when $k < \omega/c^\pm$ (or more generally when $k < \omega/c^+$) there is an added resistive contribution due to the radiation impedance associated with an outgoing sound wave in the outer fluid.

We look apart momentarily from the dome problem to the question of modification of flow-noise pressure on a bounding flexible plate or membrane with no interior fluid, due to the radiation impedance associated with the acoustic field produced in the outer fluid by plate vibration. The ratio of acoustic to exciting pressure, $\Gamma_1(k,\omega)$, for this simpler instance is given for arbitrary $z(k,\omega)$ [Appendix 1, Eq. (15)] by

$$\Gamma_1(k,\omega) = \begin{cases} -[1+(zk_2/\rho\omega)]^{-1} & (k < k_\omega) \\ -[1+i(zK_2/\rho\omega)]^{-1} & (k > k_\omega) \end{cases} \tag{1-15}$$

the superscript + being dropped without ambiguity in this case. With z given by (1), this becomes

$$\Gamma_1(k,\omega) = \begin{cases} \dfrac{-1}{1 + k_2h(\zeta q^2+\beta/\omega) + ik_2h(q^2-1)} & (1\text{-}16) \\[2ex] \dfrac{-1}{1 - K_2h(q^2-1) + iK_2h(\zeta q^2+\beta/\omega)} & (1\text{-}17) \end{cases}$$

1.3 Spectra of Excitation and Acoustic Pressure Averaged over a Circular Area in Terms of the Wavenumber Spectrum of Excitation

The actual exciting pressure due to flow noise consists of a spectral distribution in wave number and frequency of components of the form (7) assumed above.

Let the ratio of some response pressure to the driving pressure for a particular component be denoted generally by $\Gamma(\bar{k},\omega)$. In particular, $\Gamma(\bar{k},\omega,y)$ $[=\Gamma(k,\omega,y)]$ denotes this ratio where the response in question is the pressure at depth -y within the dome, which was given at Eqs. (9)-(10), and $\Gamma_1(k,\omega)$ of Eq. (15) refers similarly to the acoustic pressure just outside the plate without interior fluid.

Now the spectral density of such a response pressure, say $P(\bar{k},\omega)$, is related to that of the driving pressure, say $P_o(\bar{k},\omega)$, by

$$P(\bar{k},\omega) = M(\bar{k},\omega)P_o(\bar{k},\omega); \tag{1-18}$$

$M(\bar{k},\omega)$ is related to the corresponding response ratio $\Gamma(\bar{k},\omega)$, provided

$$\Gamma(\bar{k},\omega) = \Gamma^*(-\bar{k},-\omega), \tag{1-19}$$

by

$$M(\bar{k},\omega) = |\Gamma(\bar{k},\omega)|^2. \tag{1-20}$$

Eq. (19) is in fact satisfied by the Γ and Γ_1 given in (12)-(17).*

With reference to either a dome-housed or flush-mounted transducer of finite size, it is appropriate to consider the spectral density in frequency of the average pressure over an area. This spectral density, say $Q(\omega)$, is given in terms of the spectral density in wave number and frequency of the point pressure, $P(\bar{k},\omega)$, where k_1 and k_3 $[\bar{k} = (k_1,k_3)]$ are conjugate to the two spatial coordinates in the plane of integration, by

$$Q(\omega) = A_o^{-2}\int_A d^2\bar{R}_1\int_A d^2\bar{R}_2\int d^2\bar{k}\ e^{i\bar{k}\cdot(\bar{R}_2-\bar{R}_1)}P(\bar{k},\omega), \qquad (1\text{-}21)$$

in which the first two integrations are taken over the finite area A_o in question and the third is taken over the entire $\bar{k}$ plane. Suppose this area is circular, of radius R_o. Then

$$\int_{A_o} d^2\bar{R}\ e^{i\bar{k}\cdot\bar{R}} = 2\pi R_o k^{-1}J_1(kR_o), \qquad (1\text{-}22)$$

and hence

$$Q(\omega) = \int_0^\infty dkk[2J_1(kR_o)/kR_o]^2\ I(k,\omega) \qquad (1\text{-}23)$$

where, by (18) and (20),

$$I(k,\omega) = \int_0^{2\pi} d\theta P(\bar{k},\omega) = \int_0^{2\pi} d\theta\left|\Gamma(\bar{k},\omega)\right|^2 P_o(\bar{k},\omega) \qquad (1\text{-}24)$$

with $\theta = \tan^{-1}(k_3/k_1)$. In the limit of $R_o \to 0$ (i.e., $k_m R_o \to 0$, where $P(\bar{k},\omega)$ is negligible at $k > k_m$) Eq. (23) becomes, as it must,

*To verify this, one notes that, if the sign of the frequency is reversed ($\omega \to -\omega$), then the sign of the hysteretic damping coefficient ζ must also be reversed in order that the plate strain will still lag the stress just as before, i.e., ζ is an odd function of frequency. Likewise, the sign of k_2^+ must be reversed in order that the acoustic wave in fluid + will be outgoing rather than incoming; however, if $(k_2^+)^2 < 0$ as in the case of Eq. (13) or (17), the sign of K_2^+ remains unchanged when $\omega \to -\omega$, since K_2^+ will then still correspond to attenuation with distance away from the plate.

$$Q(\omega) \to P(\omega), \tag{1-25}$$

where $P(\omega)$ is the spectral density in frequency of the point pressure:

$$P(\omega) = \int d^2\bar{k} P(\bar{k},\omega). \tag{1-26}$$

As R_o increases, the area averaging reduces $Q(\omega)$ below the limiting value $P(\omega)$.

If the spectral density of averaging pressure $Q(y,\omega)$ on an area of the inner surface ($y = -L$), computed from $P(\bar{k},\omega,y)$ $\left[= |\Gamma(\bar{k},\omega,y)|^2 P_o(\bar{k},\omega)\right]$ via (23) and (24), is regarded as representing the spectral density of force on a dome-housed transducer of this area according to the present model, the present assumption of impedance $z_L(k,\omega)$ over the entire inner surface implies the assumption that the result is not seriously affected by extending the condition over the transducer face as well as the area surrounding it. Similarly if $Q_1(\omega)$, the spectrum of average pressure on an area of the flow-bounding side of a plate, is regarded as representing the corresponding spectral density for a flush-mounted transducer (with the inner fluid omitted as at Eqs (15)-(17)), the use of the plate equation with impedance (1) unmodified for the separate transducer insert implies the assumption that the result is not seriously affected on this account. The assumption in the former (dome) case appears reasonable. In the latter it probably is invalid near a resonance if R_o is not small compared to the resonant wave length; a more appropriate approximation will be introduced later (Section 2.6.3).

If a response ratio $\Gamma(\bar{k},\omega)$ reduces to a function $\Gamma(k,\omega)$, i.e., is invariant to direction in the plane of the plate, as is true for all responses in the present model, Eqs. (23)-(24) may be written

$$Q(\omega) = \int_0^\infty dk \left[2J_1(kR_o)/kR_o\right]^2 M(k,\omega) \; I_o(k,\omega), \tag{1-27}$$

where

$$I_o(k,\omega) = \int_0^{2\pi} d\theta P_o(\overline{k},\omega) . \tag{1-28}$$

In (27) the driving pressure is characterized by the factor $I_o(k,\omega)$, the system response by $M(k,\omega)$, and the area-averaging process by $[2J_1(kR_o)/kR_o]^2$. Likewise, Eq. (26) may be written

$$P(\omega) = \int_0^{\infty} dkkM(k,\omega) I_o(k,\omega) . \tag{1-29}$$

1.4 Approximate Forms for Response Spectra Based On the Character Of the Excitation Spectrum and of the Response Function

We consider spectra of force and pressure at the inner surface, designated by the subscript L. First we discuss briefly and qualitatively the spectrum of driving pressure, $I_o(k,\omega)$, or $P_o(\overline{k},\omega)$.

In an approximation common to a fairly general class of calculational models of turbulent boundary-layer flow, some convection hypothesis is assumed such that the spectral density $P_o(\overline{k},\omega)$, expressed as an integral over distance (y) from the bounding wall, derives a contribution only from that distance, if any, such that the frequency ω and streamwise wave number k_1 are related by

$$k_1 = \omega/u(y) ,$$

where u(y) is the mean flow velocity at the distance y. In such an approximation $P_o(\overline{k},\omega)$ vanishes unless

$$k_1 \geq \omega/U_\infty . \tag{1-30}$$

Clearly any model that assumes pure convection of velocity fluctuations or eddies at a local mean flow velocity and therefore yields a pressure spectrum $P_o(\overline{k},\omega)$ that vanishes for $k < \omega/U_\infty$ is in this respect only approximate. In actuality, there must be a component, however rapidly decreasing, that extends on down to $k = 0$. Its magnitude and dependence are not yet known and will not be further considered here except for the following remarks, (see also Section 2.5).

Suppose there exists a wave number $k_m = k_m(\omega)$ such that $P_o(\overline{k},\omega)$ is negligible for $k < k_m$; e.g., if the convection hypothesis of (30) were accepted, one could take $k_m = \omega/U_\infty$. Then for R_o sufficiently large that $k_m R_o >> 1$, Eq. (23) applied to the excitation pressure ($I=I_o$) reduces approximately to

$$Q_o(\omega) \simeq 4(2/\pi) \int_{k_m}^{\infty} dkk(kR_o)^{-3}\cos^2(kR_o-3\pi/4)\, I_o(k,\omega)$$

$$\simeq (4/\pi \int_{k_m}^{\infty} dkk^{-2} I_o(k,\omega) \tag{1-31}$$

where the asymptotic form of $J_1^{\,2}$ is used in the first step and replaced in the second by its local average over a wave number interval Δk large enough that $\Delta k >> R_o^{-1}$ but small enough that the change in $I_o(k,\omega)$ may be neglected. Under the present assumption, the area scale of the pressure field, being proportional to $P_o(\overline{k},\omega)$, vanishes.* Correspondingly, Eq. (31) shows that $Q_o(\omega)$ decreases not as R_o^{-2} but as R_o^{-3}. As noted earlier, however, experimental evidence on the area dependence of average-pressure spectra due to a turbulent boundary layer indicates that this dependence, where $\omega R_o/U_\infty >> 1$, is more nearly as R_o^{-2} (e.g., see Ref. 3). The implication of such dependence is considered in Section 2.5.

It is a common approximation in treating boundary-layer pressure fluctuations to set the boundary condition that the normal derivative $\partial p/\partial y$ vanishes at the wall (Ref.1). To the extent of the validity of this approximation, it can be shown that $P_o(0,\omega)=0$ without use of assumptions related to convection (Refs. 1,18).** Again, however, this result will not hold exactly for the actual flow. In any event, the actual $I_o(k,\omega)$ [see (28)] will have a

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* A vanishing area scale, does not refer to vanishing correlation between points with nonvanishing spatial separation, but to a vanishing area integral of the correlation function.

** The vanishing of $P_o(0,\omega)$ is removed also by compressibility (Ref. 14) and by inhomogeneity of the flow in the boundary plane.

maximum at $k = \omega/\eta U_\infty$, say, where η is somewhat less than unity and depends on ω. On the low-wavenumber side of the broad peak, I_o will decline precipitately near $k = \omega/U_\infty$, and on the high side will gradually become very small between $k = \omega/6v_*$, the convective value of k_1 at the edge of the viscous sublayer, and $\omega/6v_* + v_*/6\nu$, where $6\nu/v_*$ is roughly the thickness of the viscous sublayer and v_* is the usual friction velocity, which is of the order of the rms fluctuating velocity (See Figure 1-3).

It is assumed here that the spectrum of turbulent pressure remains substantially unaffected by the resulting vibration of the bounding wall. If the damping of this wall is sufficiently small, however, this approximation must fail; in particular, if there is no damping, at those values of k, ω for which the spectral density of wall displacement per unit pressure becomes infinite on account of a resonance, the actual spectral density of turbulent pressure as altered by the interaction must vanish.

With regard to the response function $|\Gamma(k,\omega,-L)|^2 \equiv M_L(k,\omega)$ of the present model, mentioned earlier and considered in detail below, it possesses a resonant spike centered at some $k = k_r(\omega)$ and with some half-width δk.* The height and sharpness of this resonance decrease with increasing damping and decreasing plate thickness or membrane tension. In the limit, the plate or membrane loses its dynamic properties and becomes merely an impervious separator of inner and outer fluids. In this case the attenuation of pressure by the dome fluid is not compromised by resonances that may produce high response. The resonant wave number, as this limit is approached at fixed ω, increases without limit. With actual plates, this limit could not be usefully approached, and it would be preferable to choose plate parameters merely to place the resonant wave number k_r for the ω of concern well on the lower side of the peak in the excitation spectrum $I_o(k,\omega)$, i.e., at some $k_r \gg \omega/U_\infty$. On the other hand, one would prefer to have $k_r \gg \omega/c^-$, since otherwise the resonant wave is not rapidly attenuated within the inner fluid.**

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* These considerations apply also when $(0 >)\ y \neq -L$, with a suitable lower bound on $|y|$.

** If $k_r < \omega/c^-$, the sound wave is actually propagated in the inner fluid. (If $k_r < \omega/c^+$, we recall, there is an added contribution to plate damping due to a radiated sound wave in the outer medium.)

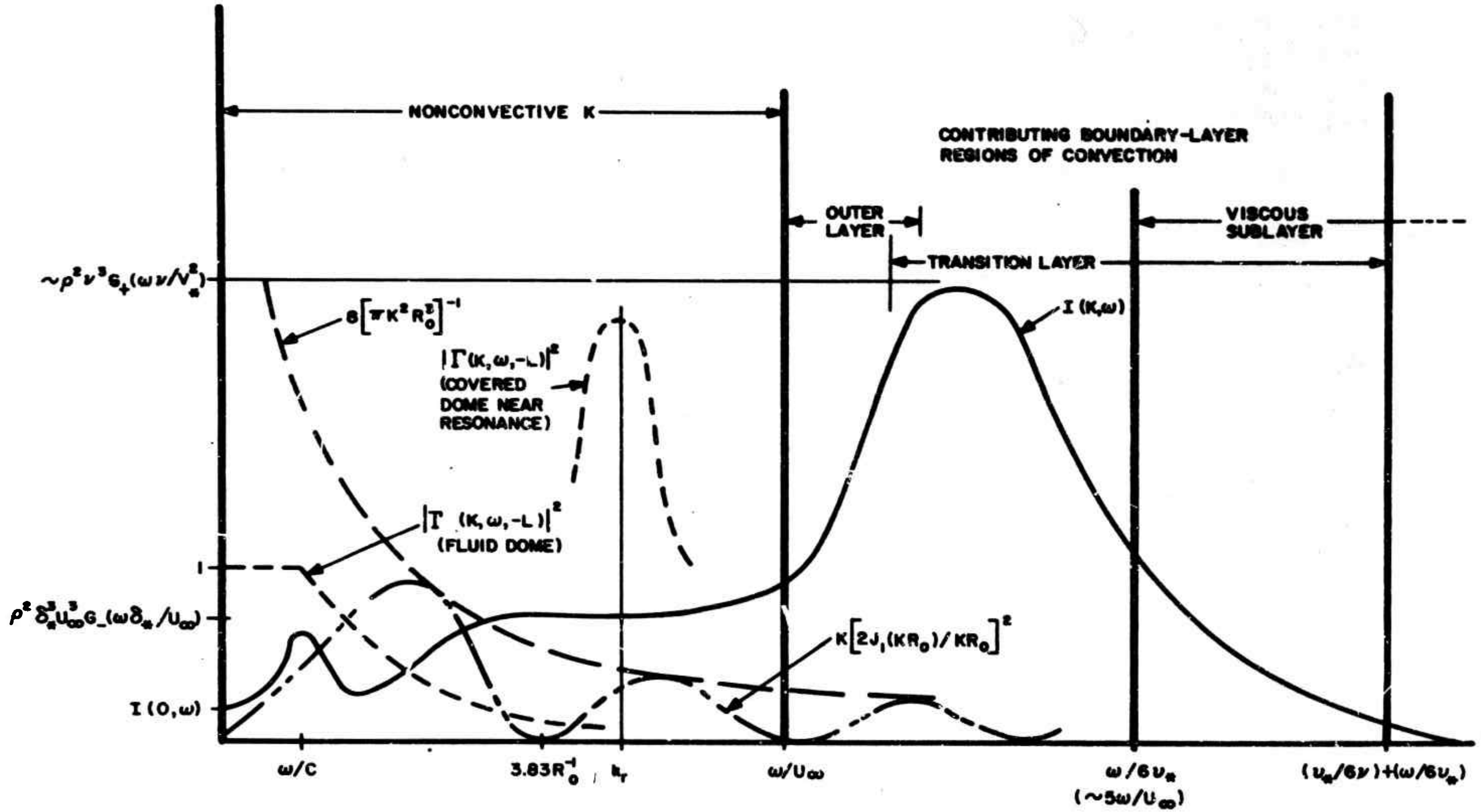

Figure 1-3. Factors in the wavenumber spectrum of average pressure acoustically transmitted through dome (high frequency). $I(k,\omega)$: spectrum of excitation pressure due to turbulent boundary layer; $k[2J_1(kR_o)/kR_o]^2$: averaging-weighting factor for circular area πR_o^2; $|\Gamma(k,\omega,-L)|^2$: acoustic-response function for depth L.

In these considerations for real domes, however, the finiteness of the dome is important, as seen in Part 2 to follow. Specifically, the finite size effectively diffuses the wave number spectrum $I_o(k,\omega)$ of noise excitation by coupling it for each k to a modal response function, say $|\Gamma_n(\omega)|^2$, which corresponds to a wave number $k_n \neq k$ (Ref.6). It thereby shifts the effective mean wave number (i.e., the mean wave number of the excitation weighted by the response function) to higher values, where the exciting spectrum is greater. Further consideration is deferred to Part 2.

If resonance occurs at a $k_r < \omega/U_\infty$, i.e., below the range of large and rapidly varying $I_o(k,\omega)$, the frequency spectrum of average pressure at the inner surface, $Q_L(\omega)$, given by (27), may be approximated roughly, assuming a pronounced resonance in $M_L(k,\omega)$, as

$$Q_L(\omega) \simeq Q_L^r(\omega) + Q_L'(\omega), \qquad (1\text{-}32)$$

with

$$Q_L^r(\omega) \simeq I_o(k_r,\omega)\int dkk[2J_1(kR_o)/kR_o]^2 M_L(k,\omega) \qquad (1\text{-}33a)$$

and $Q_L'(\omega)$ the same as the original integral (27) for $Q_L(\omega)$ but with the interval of k near resonance omitted. The resonant contribution (33a) can be further simplified if the resonance width $2\delta k$ and the approximate width πR_o^{-1} of a loop of $J_1^2(kR_o)$ are highly disparate. If $k_rR_o >> 1$, we have

$$Q_L^r(\omega) \simeq \begin{cases} (4/\pi)R_o^{-3}k_r^{-2}I_o(k_r,\omega)\int_0^\infty dkM_L(k,\omega) & \text{if } \delta kR_o >> 1 \\ 4R_o^{-2}k_r^{-1}J_1^2(k_rR_o)\,I_o(k_r,\omega)\int_0^\infty dkM_L(k,\omega) & \text{if } \delta kR_o << 1, \end{cases} \qquad (1\text{-}33b)$$

where an approximation similar to that in (31) is used in the first instance. If $k_rR_o << 1$, we may write instead

$$Q_L^r(\omega) \simeq k_rI_o(k_r,\omega)\int_0^\infty dkM_L(k,\omega). \qquad (1\text{-}33c)$$

The high- and low-wavenumber contributions to the non-resonant part Q_L' of (32) can be estimated in order of magnitude in terms of the high- and low-wavenumber parts of $Q_o(\omega)$. We defer consideration of the low-wavenumber contribution to Part 2, where a laterally finite dome is treated. As for the high-wavenumber part, say Q_{L+}, defined as due to wavenumbers $k \geq \omega/U_\infty$, let the wave number at the peak of $kI_o(k,\omega)$, or of $k^{-2}I_o(k,\omega)$ if an average over the factor $[2J_1(kR_o)/kR_o]^2$ is appropriate at the R_o in question, be denoted by $\omega/\eta U_\infty$. Then, evaluating the factor M_L in the integrand at this peak and extending the integration over all k, we obtain a simple product with the corresponding part, say Q_{0+} due to wave numbers $k \geq \omega/U_\infty$ in the average-pressure spectrum $Q_0(\omega)$ on a flush element of the same size:

$$Q_{L+}(\omega) \simeq M_L(\omega/\eta U_\infty,\omega)\,Q_{0+}(\omega) \qquad (1\text{-}34)$$

In approximations (33b) or (33c) for the resonance contribution, pertinent to the response function M_L we need only

$$\int_0^\infty dk M_L(k,\omega),$$

which will be given below. With reference to the excitation spectrum, for the resonance contributions we need $I_o(k_r,\omega)$, which, since $k_r < \omega/U$, we are presently at a loss to estimate reliably.

We note that it would not do, if one thinks the resonance contribution may be significant, to assume for the space-time auto-correlation function of the turbulent pressure some arbitrary, grossly plausible form, as for example

$$f(\zeta_1-\eta U_\infty\tau,\zeta_3)e^{-\alpha|\tau|}, \qquad (1\text{-}35)$$

where (ζ_1,ζ_3) represents the vector separation of the correlated points on the wall, τ the time difference, and ηU_∞ an effective convection velocity. For such an assumption yields a definite dependence of $I_0(k,\omega)$ on k in the region of importance, i.e., at $k = k_r \ll \omega/U_\infty$ in the tail of the spectrum, which has no rational basis and probably gives too large a value (see also Appendix 5).

1.5 The Response Function and Its Resonance Properties

Near resonance the form (13) is appropriate for $\Gamma(k,\omega,y)$, or more generally Eq. (10) or Appendix 1, Eq. (13c). From (1), appropriate when the impedance at the inner surface is not too low, writing the dome-cover impedance as

$$z(k,\omega) = R + iX,$$

where by (1)

$$R = \sigma\omega(q^2\zeta+\beta/\omega), \quad X = \sigma\omega(q^2-1), \tag{1-36}$$

we have

$$\Gamma(k,\omega,y) = \frac{ch[K_2^-(y+L)+\hat{\alpha}]}{ch\hat{\theta} + sh\hat{\theta}[(K_2^-\rho^+/K_2^+/\rho^-)-XK_2^-/\rho^-\omega] + ish\hat{\theta}(RK_2^-/\rho^-\omega)}\ . \tag{1-37}$$

If the dome damping is moderate and the depth $|y|$ not too great, in the neighborhood of resonance $\Gamma(k,\omega,y)$ can be approximated by the standard resonance form

$$\Gamma(y,\omega,y) \simeq \frac{b}{k-k_r-i\delta k} \tag{1-38}$$

where $b(\omega)$ and the half width $\delta k(\omega)$ may be regarded as roughly independent of k. Then M, given by

$$M(k,\omega,y) = \left|\Gamma(k,\omega,y)\right|^2 \simeq \frac{b^2}{(k-k_r)^2+(\delta k)^2}\ ,$$

attains a maximum roughly where $k=k_r(\omega)$. If the real part of the denominator of (37) is denoted for a given ω by D(k) and evaluation of functions at $k=k_r$ is specified by a subscript r and differentiation with respect to k by a prime, Eqs. (37) and (38) yield

$$b = \frac{ch[K_{2r}^-(y+L)+\hat{\alpha}]}{-D'(k_r)}, \quad \delta k = \frac{sh\hat{\theta}_r(R_r K_{2r}^-/\rho^- \omega)}{-D'(k_r)},$$

and k_r is given by the equation $D(k_r)=0$.

We now assume $\rho^+ = \rho^-$, $c^+ = c^-$. Eqs. (36)-(38) then yield for k_r the equation

$$q_r^{\;2} = 1 + 2/K_{2r}h(1-e^{-2\hat{\theta}_r}). \qquad (1\text{-}38.1)$$

If we suppose that at resonance we shall have $\hat{\theta}_r \equiv K_{2n}L + \hat{\alpha} \gtrsim 1$, (38.1) reduces approximately to

$$q_r^{\;2} \propto 1 + 2/K_{2r}h \qquad (1\text{-}39)$$

and (37) to

$$\Gamma(k,\omega,y) \propto \frac{2e^{-\hat{\theta}}ch[K_2(y+1)+\hat{\alpha}]}{2-(XK_2/\rho\omega)+i(RK_2/\rho\omega)}. \qquad (1\text{-}40)$$

This approximation together with (38), when the dome cover is a plate, yields also

$$b \simeq 2k_o^{\;4}K_{2r}(k_r^{\;3}h)^{-1}(5K_{2r} + k_\omega^{\;2})^{-1}e^{-\hat{\theta}_r}ch[K_{2r}(y+L)+\hat{\alpha}],$$

$$\delta k \propto k_o^{\;4}K_{2r}^{\;2}k_r^{\;-3}(5K_r^{\;2}+k_\omega^{\;2})^{-1}(q_r^{\;2}\zeta+\beta/\omega), \qquad (1\text{-}41)$$

where $k_\omega \equiv \omega/c$. Eq. (39) is seen to imply that $k_r > k_o$, i.e., the presence of the fluids bounding the plate decreases the resonant wave length, as expected for higher mass loading.

Up to moderately high frequency, we may assume $k_r^2 \gg k_\omega^2$ in order to obtain from (41) the more transparent forms

$$b \simeq (2/5)(h+2k_r^{-1})^{-1} e^{-(K_{2r}L+\hat{\alpha})} ch[K_{2r}(y+L)+\hat{\alpha}],$$

$$\delta k \simeq (1/5)k_r[\zeta+(\beta/\omega)(1+2/k_r h)^{-1}], \qquad (1\text{-}42)$$

in which the exact form $K_{2r}=(k_r^2-k_\omega^2)^{1/2}$ is retained only in the exponential functions. In the approximation of (42) it is further found that the integral appearing in Eq. (33b) and (33c) is given by

$$\int_0^\infty dkM(k,\omega,-L) \simeq (\pi/25)(1+e^{-2\hat{\alpha}})^2(h+2k_r^{-1})^{-2}$$

$$X \exp(-2K_{2r}L)(\delta k)^{-1}. \qquad (1\text{-}43)$$

In the case where the inner surface has low impedance and Eq. (37) is replaced by the corresponding equation based on Eq. (13c) of Appendix 1, in the approximation where $K_{2r}L + \hat{\gamma} \gtrsim 1$, analogous to the approximation of Eq. (40), k_r and δk are given by the same equations as before, and b differs from that given by Eqs. (41) or (42) only in having $\hat{\theta}_r[=K_{2r}L+\hat{\alpha}]$ replaced by $K_{2r}L+\hat{\gamma}$ and

$$ch[K_{2r}(y+L)+\hat{\alpha}] \text{ by } sh[K_{2r}(y+L)+\hat{\gamma}]. \qquad (1\text{-}44)$$

To approximate k_r, suppose first that the dome parameters and frequency have values such that

$$K_{2r}h \ll 1 \text{ and } k_r^2 \gg k_\omega^2. \qquad (1\text{-}45)$$

then (39) becomes

$$q_r^2 \simeq 2/k_r h.$$

In the case of the plate, by definitions (5), the desired resonant wave number is thus given by

$$k_r \simeq k_o(2/k_o h)^{1/5}. \qquad (1\text{-}46)$$

The first of assumptions (45) is thus equivalent to

$$k_r h \simeq 2^{1/5}(k_o h)^{4/5} << 1 \text{ or } k_o h << 1; \tag{1-47}$$

by Eqs (6) and (3) we have

$$k_o h = (\rho_o/\rho)(\omega h_o/c_t)^{1/2}[6(1-\sigma_o)]^{1/4}. \tag{1-48}$$

As a numerical example to indicate the magnitudes of the quantities in (45) and (47), consider a steel plate of thickness h_o = 0.25 in. with water on each side. We find

$a_o = 8.5\ m^2/sec$, $k_o = 0.86(\omega/2\pi)^{1/2} m^{-1}$, $k_o h = 0.042(\omega/2\pi)^{1/2}$,

$k_\omega = 0.0041(\omega/2\pi)m^{-1}$, with $\omega/2\pi$ in cps. At 3.5 kc, for example,* we have

$$k_o = 50.9\ m^{-1},\ k_\omega = 14.6\ m^{-1},\ k_o h = 2.5,$$

so that condition (47) no longer holds at frequencies as high as this.

In the case of the membrane, again assuming (45), Eq. (39) yields

$$k_r \simeq k_o(2/k_o h)^{1/3}. \tag{1-49}$$

A perturbation procedure may be used to obtain the maximum of $M(k,\omega,y)$ more exactly than given by (39). The results are of too little interest to be quoted here. If L is too large or ζ or $\beta h/\omega$ is insufficiently small, however, specifically if $k_r L \gtrsim 5/2\zeta$ when $2\zeta/k_r h \gtrsim \beta/\omega$ or if $k_r L \gtrsim 5/[k_r h(\beta/\omega)]$ when $2\zeta/k_r h \lesssim \beta/\omega$, then M_L has no resonance maximum at all; the resonant contributions are obliterated by exponential attenuation within the fluid, and contribution from low wave numbers is no longer correctly described by Eqs. (33a)-(33c); in this situation, however, these contributions tend to be unimportant.

*The relation (47) nevertheless holds in much of the limited regime where the thin-plate treatment is strictly valid (kh_o <<1) and the earlier assumption $k > k_r$ holds, since $k_r > k_o$, whence $k_o h = k_o h_o(\rho_o/\rho) << \rho_o/\rho$.

If the mass density of the plate (or membrane) is not sufficiently low or its stiffness sufficiently high, the condition (47), in view of (48), is not satisfied. From (39), in the opposite limit, where

$$K_{2r}h \gg 1, \tag{1-50}$$

we have simply

$$k_r \simeq k_o \left[1 + 2\Big/\left(k_o^2 - k_\omega^2\right)^{1/2} h\right]^{1/n}, \tag{1-51}$$

(plate n=4, membrane n=2) or simply $k_r \simeq k_o$, i.e., the resonant k is nearly that for an isolated plate. In view of the small fractional power involved in Eq. (46) for the opposite limit, it is clear that, except at very low frequency, k_r is larger than k_o by only a modest factor. For arbitrary $k_o h$, k_r is given implicitly from (39) by

$$k_r = k_o(1+2/K_{2r})^{1/n}, \tag{1-52}$$

where n = 4 for a plate and n = 2 for a membrane.

It is pertinent to dome design for control of resonant noise to consider the effect on the resonant wavenumber k_r of variation of the dome parameters. As the relative stiffness parameter a_o of the plate is increased at fixed density ρ_o, either by increasing thickness h_o or shear velocity c_t (see Eq. (3)), or as the tension in the membrane is increased at fixed area density σ (see Eq. (4)), for a given ω we have $k_o \to 0$ and $k_r \to k_\omega$. From (38.1) (assuming $\hat{\alpha} \simeq 0$), for $k_o/k_\omega \ll 1$ we find, in fact,

$$k_r \simeq k_\omega[1 + (k_\omega^2 hL)^{-1}(k_o/k_\omega)^n]^{1/2}$$

if $4(L/h)(k_o/k_\omega)^n \ll 1$, and

$$k_r \simeq k_\omega[1 + 4(k_\omega h)^{-2}(k_o/k_\omega)^{2n}]^{1/2}$$

if $4(L/h)(k_o/k_\omega)^n \gtrsim 1$; in the extreme limit of $k_o/k_\omega \to 0$, the former becomes

$$k_r \simeq k_\omega \, [1 + (2k_\omega^2 hL)^{-1}(k_o/k_\omega)^n], \qquad (1\text{-}53)$$

with $n = 4$ for a plate and $n = 2$ for a membrane. On the other hand, for a plate, if the thickness $h_o \to 0$ at fixed density and elastic properties, we have from (3) and (6) that $k_o \propto h_o^{-1/2} \to \infty$ and, from (46),

$$k_r \propto h_o^{-3/5} \to \infty.$$

In this limit where $h_o \to 0$, the plate impedance vanishes. Thus, as the limit of a fluid dome is approached, the resonant wave number increases without limit. For a membrane, if the mass per unit area $\sigma \to 0$, from (49) we have

$$k_r \to (2\rho\omega^2/T)^{1/3}.$$

If T remains fixed in this limit, we have $k_o = \omega(\sigma/T)^{1/2} \to 0$ and $k_r \to \text{const} \neq 0$, but if $T \to 0$ also, corresponding to a limit of vanishing membrane impedance, we again have $k_r \to \infty$ as a fluid dome is approached.

In addition to the resonance considered above, at which the denominator of (13) vanishes except for the imaginary, intrinsic dome-cover damping term and which occurs at a $k = k_r$ such that $k_r > k_o$ and $k_r > k_\omega$, another relative maximum in Γ may occur at a $k = k_r'$, say, such that $k_r' < k_o$ and $k_r' < k_\omega$, at which the denominator of (12) vanishes except for the imaginary part. In this case, however, the imaginary part, in addition to intrinsic dome damping term, contains a term $-\sin\theta$ corresponding to damping by radiation of energy into the outer medium. The value k_r' is given from (12) by the equation

$$q_r'^2 = 1 - \cos\theta_r'/k_{2r}'h \qquad (1\text{-}53.1)$$

(cf. (38.1)) where q_r' denotes q evaluated for $k=k_r'$ and likewise for k_{2r}' and θ_r'. Eq (53.1) can be satisfied only if

$$k_{2r}' h \tan\theta_r' > 1. \qquad (1\text{-}53.2)$$

The corresponding maximum of $|\Gamma|$, even for negligible dome damping ($\zeta=\beta=0$), satisfies approximately*

$$\left|\Gamma_r'\right| \leq 1/\left|\sin\theta_r'\right|.$$

Thus this value corresponds to a large amplification of the driving pressure only if $\theta_r' \equiv (k_\omega^{\,2}-k_r'^{\,2})^{1/2}L + \alpha \ll 1$. Assuming $\alpha \simeq 0$, this condition together with (53.2) would imply

$$\omega h/c \gg 1, \text{ i.e., } \omega\sigma/\rho c \gg 1. \qquad (1\text{-}53.3)$$

Condition (53.3) is not satisfied for reasonable cover mass at frequencies of concern, since in the present context ρ refers to water (cf. air). Accordingly, we do not further consider this type of resonance.

It will be pertinent in Part 2 to consider also the resonant behavior of $\Gamma(k,y,\omega)$ as a function of frequency ω for a given exciting wave number k. Analogously to (38), then, we write for ω in the neighborhood of resonance

$$\Gamma(k,\omega,y) \simeq \frac{B}{\omega-\omega_r+i\delta\omega}, \qquad (1\text{-}54)$$

*The maximum, of course, does not occur exactly where the real part of the denominator vanishes. One may usefully plot the denominator of Γ in the complex plane with k as a parameter.

where B(k) and the half-width $\delta\omega(k)$ are regarded as roughly independent of ω*. The resonant frequency ω_r is again given by Eq. (39) where a subscript r now denotes evaluation at $\omega=\omega_r$ with the given k. We let $\omega_o(k)$ denote the resonant frequency for the isolated plate or membrane at wave number k:

$$\omega_o = \begin{cases} a_o k^2 & \text{(plate)} \\ c_o k & \text{(membrane)}. \end{cases} \qquad (1\text{-}55)$$

Then from (39), similarly to (52), since now $q_r^{\ 2} = (\omega_o/\omega_r)^2$, ω_r is given implicitly by

$$\omega_r = \omega_o(1+2/K_{2r}h)^{-1/2} \qquad (1\text{-}56)$$

Corresponding to the approximation (51) when $K_{2r}h >> 1$, we have

$$\omega_r \simeq \omega_o \left[1+2 \Big/ \left\{k^2-(\omega_o/c)^2\right\}^{1/2} h\right]^{-1/2}. \qquad (1\text{-}57)$$

On assumption that the k of concern is such that $k >> k_\omega$, simple expressions may be written for B and $\delta\omega$ of (54) analogously to (42):

$$B \simeq \omega_r(kh)^{-1}(1+2/kh)^{-1}e^{-(K_{2r}L+\hat{\alpha})}\,ch[K_{2r}(y+L)+\hat{\alpha}],$$
$$\delta\omega \simeq (1/2)\omega_r\,[\zeta+(\beta/\omega_r)(1+2/kh)^{-1}]. \qquad (1\text{-}58)$$

In the case where the inner surface has low impedance, and again $K_{2r}L+\hat{\alpha} \gtrsim 1$, the quantities ω_r and $\delta\omega$ are still as given above and B differs by the replacement stated at (44).

---

*If the damping and hence also $\delta\omega$ are sufficiently small, the magnitude of the half-width $\delta\omega$ becomes of less interest than a frequency width defined as that fraction of ω_r over which Γ exceeds its neighboring nonresonant level by a factor large enough that the resonance contribution is significant.

At wave numbers high above resonance such that $k \gg k_r$ ($> k_\omega^-$) and $kL \gg 1$, the response function for a high or low-impedance inner surface, by Eq. (5) and (13) or Appendix 1, Eq. (13c), respectively, becomes approximately

$$\Gamma(k,y) \simeq \begin{cases} 2e^{-kL}\mathrm{ch}[k(y+L)+\hat{\alpha}](k_o/k)^n(kh^-)^{-1} \\ 2e^{-kL}\mathrm{sh}[k(y+L)+\hat{\gamma}](k_o/k)^n(kh^-)^{-1} \end{cases} \tag{1-59}$$

where $n = 4$ for a plate and $n = 2$ for a membrane and, we recall, $h^- = \sigma/\rho^-$.

In the nonresonant contribution Q_{L+}, Eq. (34), the requisite response function M_L evaluated at the wave number of peak excitation $\omega/\eta U_\infty \equiv k_u$ is given, assuming $k_r \ll k_u$, by the former of Eqs. (59) as

$$M_L(k_u,\omega) \simeq (1-e^{-2\hat{\alpha}})^2 \exp(-2k_u L)(k_u h^-)^{-2} (k_o/k_u)^{2n} \tag{1-60}$$

(plate $n = 4$, membrane $n = 2$). For typical parameters, the value of this response function is extremely small, but in the case of a strictly finite dome its value will no longer determine the response to excitation even at the high wave numbers ($\gtrsim \omega/U_\infty$).

In the present infinite-dome case, the efficacy of increasing the damping parameters ζ and β to reduce the response evidently depends, in view of Eq. (32)-(34) and (44)-(52), on whether the "tail" value of the excitation spectrum, $I_o(k_r,\omega)$, is large enough that the resonant contribution predominates in the transmitted pressure.

For reference in Part 2, we record the response coefficient Γ for a few special cases. For a fluid dome we have

$$z(k,\omega) = 0 \tag{1-61}$$

In this case $\Gamma(k,\omega,y)$ displays no resonance behavior at any k. In the important special case of a fluid dome with equivalent interior and exterior fludis, i.e., with $\rho^+ = \rho^- \equiv \rho$ and $c^+ = c^- \equiv c$, we have from (9) and (10)

$$\Gamma(k,\omega,y) = \begin{cases} e^{i(k_2L+\alpha)} \cos[k_2(y+L)+\alpha] & (k < k_\omega) \\ e^{-(K_2L+\hat{\alpha})} \mathrm{ch}[K_2(y+L)+\hat{\alpha}) & (k > k_\omega) \end{cases} \qquad (1\text{-}62)$$

or in case of a low-impedance inner surface, by Appendix 1, Eqs. (13c) and (13d),

$$\Gamma(k,\omega,y) = \begin{cases} -ie^{i(k_2L+\gamma)} \sin[k_2(y+L)+\gamma] & (k < k_\omega) \\ e^{-(K_2L+\hat{\gamma})} \mathrm{sh}[K_2(y+L)+\hat{\gamma}] & (k > k_\omega) \end{cases} \qquad (1\text{-}63)$$

in which γ or $\hat{\gamma}$ now characterizes z_L, or in case of an infinite inner fluid, by Appendix 1, Eqs. (13e) and (13f),

$$\Gamma(k,\omega,y) = \begin{cases} (1/2)e^{-k_2y} & (k < k_\omega) \\ (1/2)e^{K_2y} \; . & (k > k_\omega) \end{cases} \qquad (1\text{-}63.1)$$

1.6 Average Pressure on a Non-Rigid Flow-Bounding Surface

The work of Section 1.5 for a dome-shielded area may be carried through similarly with respect to an area in the outer surface, on which the average pressure due to flow noise is altered by the acoustic field produced outside by the flexible plate or membrane.

In this case the net pressure on the plate is the sum of the flow-noise pressure and the acoustic pressure just outside the plate, whence the spectral density is given by

$$P_1(\bar{k},\omega) = M_1(k,\omega)P_0(\bar{k},\omega),$$

$$M_1(k,\omega) = \left|1+\Gamma_1(k,\omega)\right|^2,$$

with Γ_1 given by Eqs. (16)-(17). If the resonance contribution to the spectrum of total average pressure is significant at all, we must have $|\Gamma_1|^2 \gg 1$ at resonance. In such a case the average-pressure spectrum $Q_1(\omega)$ may be approximated analogously to $Q_L(\omega)$ for the inner surface of a dome as at Eqs. (32)-(33) by replacing $M_L(k,\omega)$ in (33a)-(33c) by $|\Gamma_1(k,\omega)|^2$ and replacing $M_L(\omega/\eta U_\infty,\omega)$ in (34) by $|1+\Gamma_1(\omega/\eta U_\infty,\omega)|^2$.

Analogously to (37), we may write Γ_1 by (15) as

$$\Gamma_1(k,\omega) = \frac{-1}{1-XK_2/\rho\omega+iRK_2/\rho\omega} \qquad (1\text{-}64)$$

In the neighborhood of a resonance, we may write analogously to (38)

$$\Gamma_1(k,\omega) \simeq \frac{b_1}{k-k_{r1}-i\delta k_1} \qquad (1\text{-}65)$$

and analogously to (54)

$$\Gamma_1(k,\omega) \simeq \frac{B_1}{\omega-\omega_{r1}+i\delta\omega_1} \, . \qquad (1\text{-}66)$$

Comparison of (64) with (40) then shows that the quantities k_{r1}, δk_1, b_1, ω_{r1}, $\delta\omega_1$, B_1 can be obtained from the expressions previously given for their counterparts k_r, δk, b, ω_r, $\delta\omega$, B by replacing ρ wherever it appears by $\rho/2$, and hence also h by 2h, and also replacing k_r by k_{r1} [e.g., in expressions (42)], ω_r by ω_{r1}, and setting $L=\hat{a}=0$. In particular, the relation for k_{r1} analogous to (39) is

$$q_{r1}^2 \simeq 1 + 1/K_{2r1}h \qquad (1\text{-}67)$$

where $q_{r1} = q(k=k_{r1})$ and $K_{2r1} = K_2(k=k_{r1})$. Similarly, analogously to (43)

$$\int_0^\infty dk\,|\Gamma_1(k,\omega)|^2 \simeq (\pi/25)\left(h+k_{r1}^{-1}\right)^{-2}(\delta k_1)^{-1} \, . \qquad (1\text{-}68)$$

As, was true for the pressure interior to a dome, a resonance maximum of $|\Gamma_1|$ may occur at a $k=k'_{1r}$ such that $k'_{1r} < k_\omega$, for which the dome damping is supplemented by damping due to radiation into the outer medium [Eq. (16)]. It occurs, according to (16), roughly at $k'_{r1} \simeq k_o$ and hence only if $k_o < k_\omega$. The value of $|\Gamma_1|$ at this maximum sharply exceeds neighboring nonresonant values if $(k_\omega{}^2-k_o{}^2)^{1/2}h \gg 1$, a condition that implies (53.3); even if this condition is satisfied, however, still $|\Gamma_1| < 1$ at the maximum. We do not further consider this type of resonance.*

From Eq. (16), at wave numbers high above resonance ($k \gg k_{r1}$) the response function becomes, analogously to (59),

$$\Gamma_1(k,\omega) \simeq (kh)^{-1}(k_o/k)^n \qquad (1\text{-}69)$$

(plate n=4, membrane n=2).

If the wave number of resonance lies well below that of the peak excitation, i.e., $k_{r1} \ll k_u(\equiv\omega/\eta U_\infty)$, the non-resonant modification due to Γ_1 in the factor $|1+\Gamma_1(\omega/\eta U_\infty,\omega)|^2$ is small since in this case, by (69)

$$\Gamma_1(k_u,\omega) \simeq (k_u h)^{-1}(k_o/k_u)^n \ll 1$$

[cf. (60)].

1.7 Focusing Effect

If the dome surface is not planar, as assumed in the present model, but has pronounced curvature, a focusing effect can occur that will increase the pressure or force on an interior area over the value for the planar case. For example, if the dome surface over which the exciting flow passes is circular-cylindrical and the inner surface shielded by the interior fluid is also cylindrical and concentric, whether the flow passes parallel or normal to the elements of the cylinder, the acoustic pressure at the inner surface is amplified by the square root of the ratio of the radii of the outer and inner cylinders.

*In the context of radiation into air by a vibrating plate, such resonances require and receive consideration.

PART 2
DOMES OF FINITE EXTENT

2.1 Introduction

In Part 1 a simplified model of a sonar dome with transducer (and of a flexible plate with flush-mounted transducer) was studied with regard to the pressure fluctuations on the transducer that originate from the turbulent boundary-layer flow and are transmitted by excitation of dome vibration and generation of sound in the interior (or exterior) fluid. The dome (or plate) in part 1 was considered to be of infinite lateral extent.

In the present part, the same basic simplified model is assumed, but the dome and the interior fluid are taken to be finite, being bounded by a cylindrical wall on which a fixed-impedance condition is assumed to apply (see Figure 2-1). Much of the pertinent development is contained in Appendix 2, and results therefrom will be taken without discussion.

In the case of the infinite dome, suppose the dome parameters (plate thickness, etc.) are so chosen that, for a fixed operating frequency ω, the acoustic resonance of the plate fluid system lies at a lateral (x,z) wave number k_r substantially below the peak of the wave number spectrum of lateral pressure fluctuations due to the turbulent boundary layer, i.e., $k_r < \omega/U_\infty$. Then, as pointed out in Part 1, unless the damping at resonance is very small, the pressure fluctuations transmitted through the dome and interior fluid to the transducer are highly attenuated relative to the exciting fluctuations on the outer surface.* In the case of the finite dome, on the other hand, the excitation spectrum at wave numbers near its peak is coupled (by "functional overlap") to the response function at much lower modal wave numbers lying near resonance or corresponding to propagating waves. Hence the transmitted pressure can be much larger than for an infinite dome, and can depend differently on the properties and parameters of the dome and the turbulent flow.

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*It may be said that the wave number spectra of the excitation and of the response are highly mismatched.

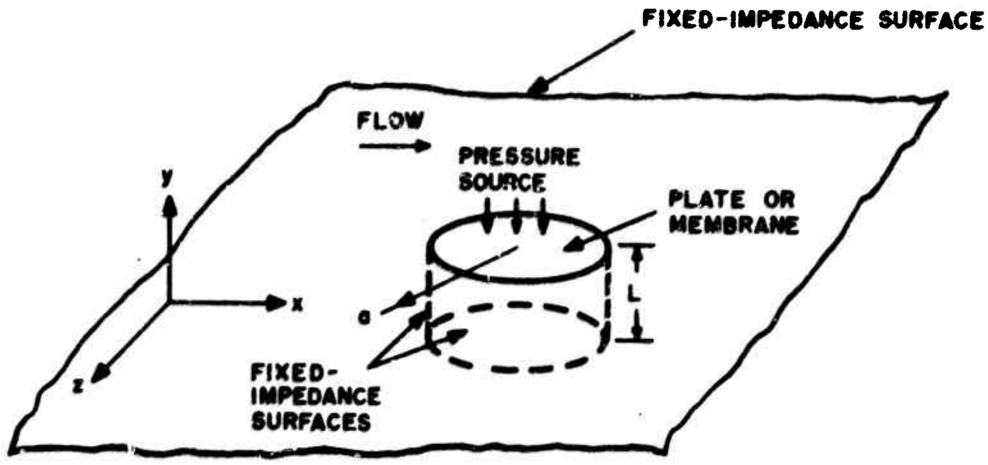

FIGURE 2-1. FINITE DOME-MODEL GEOMETRY.

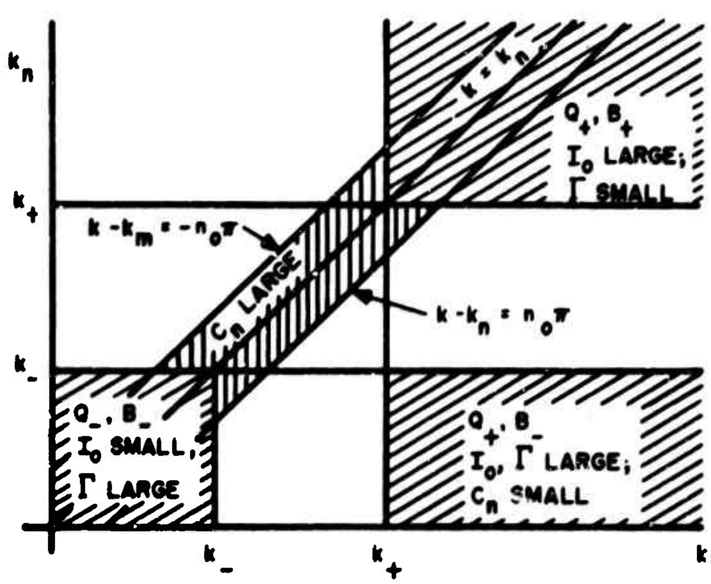

FIGURE 2-2. PLANE OF EXCITATION WAVE NUMBERS k AND MODAL WAVE NUMBERS k_n IDENTIFYING REGIONS TO BE RETAINED OR NEGLECTED.

2.2 General Relations Between Response and Excitation-Pressure Spectra

Let A denote the cross-section of the cylindrical dome (not necessarily circular) and $\bar{R}$ the two-dimensional position vector of a point in the cross section. Let $\left\{X_n(\bar{R})\right\}$ denote a complete set of orthonormal eigenfunctions of the two-dimensional Helmholz equation that satisfy the boundary condition prescribed on the boundary of A (i.e., on the lateral wall of the dome):

$$(\nabla^2+k_n{}^2)X_n(\bar{R}) = 0. \qquad (2\text{-}1)$$

Consider a random fluctuating pressure on the outer (y=0) end of the cylinder (i.e., on the outer surface of the dome) and assume it stationary in time and space with a spectral density in wave number and frequency $P_o(\bar{k},\omega)$ $[\bar{k}=(k_1,k_3)]$.

For any particular realization of the random pressure field let the spatial dependence of the exciting pressure at a given frequency be expanded in normal modes as

$$p_o(\bar{R}) = \sum_n b_n X_n(\bar{R}). \qquad (2\text{-}2)$$

Any response quantity of interest, say the pressure at a depth -y within the dome, has a similar expansion:

$$p(\bar{R},y) = \sum_n \beta_n g_n(y) X_n(\bar{R}), \qquad (2\text{-}3)$$

in which $g_n(y)$ is defined to satisfy the one-dimensional Helmholtz equation

$$\left[d^2/dy^2 + (k_\omega{}^2-k_n{}^2)\right] g_n(y)=0 \qquad (2\text{-}4)$$

and the boundary condition at y = - L, and is arbitrarily normalized to $g_n(0) = 1$; Eqs. (4)* and (1) ensure that $p(\bar{R},y)$ as given by (3) satisfies, as required, the three-dimensional Helmholtz equation

$$(\nabla^2+k_\omega{}^2)p(\bar{R},y)=0. \qquad (2\text{-}5)$$

(The ω-dependence of such quantities as $g_n(y)$ is here being suppressed.) The coefficients β_n in (3) are determined by the excitation coefficients b_n of (2) and the boundary condition for the

*An equation referred to by a number without hyphenated prefix is in the same part (1 or 2) as the reference; e.g., (4) here denotes Eq. (2-4), not (1-4).

acoustic field on the baffle surrounding the outer dome surface and bounding with it the exterior fluid. The relation is a linear one which we may write

$$\beta_n = \sum_s \gamma_{ns} b_s . \tag{2-6}$$

The coefficients γ_{ns} will be considered subsequently for a simplified form of the dynamic-acoustic model.

The cross-correlation of pressure at two points $\bar{R}_1$ and $\bar{R}_2$ at the same depth (-y) and time, per unit frequency range, is found by straightforward use of Fourier transforms and the above equations to be given by

$$\begin{aligned} P(\omega,\bar{R}_1,\bar{R}_2,y) &= \sum\sum_{nn'} X_n^*(\bar{R}_1) X_{n'}(\bar{R}_2) g_n^*(y) g_{n'}(y) \\ &\times \sum\sum_{ss'} \gamma_{ns}^* \gamma_{n's'} \int d^2\bar{k}\, a_s^*(\bar{k})\, a_{s'}(\bar{k})\, P_o(\bar{k},\omega) , \end{aligned} \tag{2-7}$$

where

$$a_s(\bar{k}) = \int_A d^2\bar{R} e^{i\bar{k}\cdot\bar{R}} X_s^*(\bar{R}) . \tag{2-8}$$

Hence, the frequency spectrum of the average pressure on any plane area A_o at depth -y is given by

$$\begin{aligned} Q(\omega,y) &= A_o^{-2} \int_{A_o} d^2\bar{R}_1 \int_{A_o} d^2\bar{R}_2 P(\omega,\bar{R}_1,\bar{R}_2,y) \\ &= A_o^{-2} \int d^2\bar{k}\; P_o(\bar{k},\omega) \left| \sum_n \left\{ \int_{A_o} d^2\bar{R} X_n(\bar{R}) \right\} g_n(y) \sum_s \gamma_{ns} a_s(\bar{k}) \right|^2 \\ &\equiv \int d^2\bar{k}\; Q(\bar{k},\omega,y) . \end{aligned} \tag{2-9}$$

Both $g_n(y)$ and γ_{ns} here depend on ω. According to Eq. (9) the response at lateral wave number k_n to a driving pressure at k_s, reflected in $\gamma_{ns} g_n(y)$, is coupled to the excitation spectrum at wave number k, given by $P_o(\bar{k},\omega)$, by an overlap coefficient, $a_s(\bar{k})$, between wave numbers k and k_s.

2.2.1 Elements with Fluid Boots

We digress here from domes to note other results of interest. If the averaging area is coextensive with the cylinder cross section (A_o=A) and the wall is rigid, then in Eq. (9), by orthogonality of modes,

$$\int_{A_o} d^2\bar{R} X_n(\bar{R}) = 0 \qquad (A_o=A,\ n\neq 0)$$

for all modes except the piston mode $X_o(\bar{R})=A^{-1/2}$. In this instance the average pressure at any y, formed by integrating Eq. (3) over the cross section [cf. Eq. (9)], is due solely to this piston mode. Likewise, the ratio of the average pressure (or frequency spectrum thereof) at any depth (y < 0) to that, say, just inside the outer surface at y=0 is independent of the source or wavenumber distribution of the exciting pressure, since, by use of Eq. (3), this ratio is simply $g_o(y)$.

Hence, if a flush-mounted transducer incorporates a co-extensive boot over its face that is equivalent to a fluid and has rigidly constrained sides, and if the transducer properly measures the force (in a given frequency band) integrated over its active face, the signal-to-noise ratio is expected to be independent of the thickness of the boot.* This result does not strictly apply to the limiting case of no boot, however, for the following reason. The noted lack of dependence on excitation wave number is displayed by the depth dependence of the averaged interior pressure . The latter, just inside the outer face ($y=0_-$), is equal to the sum of the averaged exciting pressure and exterior radiation pressure. For fixed excitation pressure, i.e., fixed force on a bootless flush element, this radiation pressure due to the outer acoustic field excited by surface vibration of the fluid boot is wavenumber-dependent. Hence

*This independence of wave number noted for the area-averaged interior pressure can be shown to apply similarly under certain boundary conditions when the cylinder is an elastic solid instead of a fluid. Possible boundary conditions include those where the sides are rigid but slippery and so also is the inner face.

the ratio of interior pressure to excitation pressure is also wavenumber dependent. But this ratio is just the ratio of the response of a booted element to the response of a bootless element. Since noise and signal have greatly different wavenumber spectra, the signal-to-noise ratios for bootless and booted elements may thus differ somewhat.

If the fluid cylinder has sides that are not rigid but instead approach pressure release, the interior pressure averaged over the cross-section is no longer due to a single acoustic mode, and its depth dependence becomes wavenumber dependent. In particular, the average pressure at greater depths is relatively smaller for higher wave numbers, since average-pressure-carrying modes excited by the higher wave numbers are more weighted toward higher modes that damp out more rapidly with depth. Hence in this instance the signal-to-noise ratio for a booted element is expected to increase with boot thickness.*

*The decreasing impedance of such an element is unwelcome in signal transmission, however.

2.2.2 Circular Cylindrical Dome

We now consider specifically a circular cylindrical dome of radius a. The eigenfunctions $X_n(\vec{R})$ are then identified as

$$X_{mn}(\vec{R}) = (2\pi)^{-1/2} e^{im\theta} N_{mn} J_m(k_{mn}R)$$

in two-dimensional polar coordinates, where the eigenvalues k_{mn} are to be fixed by the boundary condition at the wall and may be written

$$k_{mn} = x_{mn}/a$$

with x_{mn} independent of a. The normalizing coefficient N_{mn} is given by

$$\begin{aligned} N_{mn}^2 &= 2a^{-2}\left[J_m^2(x_{mn}) - J_{m-1}(x_{mn})J_{m+1}(x_{mn})\right]^{-1} \\ &= 2a^{-2}\left[J_m^2(x_{mn})\left\{1-(m/x_{mn})^2\right\} + J_m'^2(x_{mn})\right]^{-1}. \end{aligned} \tag{2-10}$$

From (8) one finds

$$a_{mn}(\vec{k}) = e^{-im\beta} c_{mn}(k), \quad (k^2 = k_1^2 + k_3^2) \tag{2-10.1}$$

where

$$c_{mn}(k) = (2\pi)^{1/2} i^m N_{mn} a^2 \frac{kaJ_m(x_{mn})J_m'(ka) - x_{mn}J_m(ka)J_m'(x_{mn})}{x_{mn}^2 - (ka)^2} \tag{2-11}$$

and $\beta = \tan^{-1}(k_3/k_1)$.

We wish to consider the spectrum of average pressure on a finite area A_0, representing a transducer in the dome interior, and in particular at the inner surface (y = -L). It will be consonant with the rudimentary model under consideration to assume that the area A_0 is circular of radius R_0 and concentric with the dome cylinder (hence $R_0 < a$). In this case the spectral density of average pressure (9) reduces to

$$Q(\omega,y) = 2(\pi R_0^2)^{-1} \int_0^\infty dkkI_0(k,\omega) \left| \sum_n k_n^{-1} J_n(k_nR_0) N_n g_n(y) \sum_{n'} \gamma_{nn'} c_{n'} \right|^2, \tag{2-12}$$

where I_o is the angle integral of the driving spectrum $P_o(\mathbf{k},\omega)$ [Eq. (1-28)], and the azimuthal modal number m is understood to be zero and is dropped as an index here and hereafter whenever only m=0 enters (e.g., $c_n \equiv c_{on}$, etc.). The unspecified boundary condition on the surface of the baffle is assumed not to depend on azimuth angle, so that, for arbitrary azimuthal modal numbers m,m', we have $\gamma_{mn;m'n'}=0$ if $m\neq m'$ or, say, $\gamma_{mn;m'n'}=\gamma^m_{nn'}\delta_{mm'}$.

2.2.3 Spectrum of Acoustically Modified Pressure Outside Boundary

Of interest also is the spectrum of average pressure on an area A_o in the outer surface of the dome, which is due to the exciting turbulent pressure modified by the acoustic pressure in the outer fluid at the boundary (y=0). As discussed in Part 1, when the inner fluid is considered absent, this configuration partially simulates that of a flush-mounted transducer in a finite flexible plate. Expression (9) again applies with the response coefficient $g_n(y)\gamma_{ns}$ becoming now rather

$$g_{1n}\gamma_{ns} + \delta_{ns},$$

where g_{1n} is the function analogous to $g_n(y)$ but pertaining to the outer medium at y=0 and the Kronecker delta takes account of the direct driving pressure due to turbulence. In the case of the circular cylinder with the area A_o taken to be circular and concentric, Eq. (12) similarly applies with $g_n(y)\gamma_{nn'}$ become $g_{1n}\gamma_{nn'}+\delta_{nn'}$, where g_{1n} now represents g_{1mn} and $\gamma_{nn'}$ again represents $\gamma^m_{nn'}$ for m=0.

The sum over n of the unity term in Eq. (9) or (12) (i.e. δ_{nn}) with $g_n(y)\gamma_{ns}$ replaced by $g_{1n}\gamma_{ns}+\delta_{ns}$ can be performed explicitly, and indeed we know that if $g_{1n}=0$ we must obtain the force spectrum due to the driving pressure alone. The sum is effected in the general case of Eq. (9) by use of the following closure property of the eigenfunctions $X_n(\mathbf{R})$:

$$\sum_n X^*_n(\mathbf{R}_1)X_n(\mathbf{R}_2) = \delta(\mathbf{R}_1-\mathbf{R}_2).$$

One immediately obtains for the spectrum of average pressure on the outer surface

$$Q_1(\omega)=A_o^{-2}\int d^2k\ P_o(k,\omega)\left|\int_{A_o} d^2\bar{R}\left[e^{i\bar{k}\cdot\bar{R}}+\sum_n g_{1n}\sum_s \gamma_{ns}a_s(k)\right]\right|^2 \tag{2-13}$$

or in the special case of Eq. (12)

$$Q_1(\omega)=2(\pi R_o^2)^{-1}\int_0^\infty dkkI_o(k,\omega)\left|(2\pi)^{1/2}k^{-1}J_1(kR_o)\right.$$
$$\left.+\ \Sigma_n k_n^{-1}J_1(k_nR_o)N_n g_{1n}\Sigma_{n'}\gamma_{nn'}c_{n'}\right|^2 . \tag{2-14}$$

2.3 Simplification of Boundary Conditions

The dynamics of the model enter solely via the response coeffients, e.g. $g_n(\omega,y)\gamma_{nn'}(\omega)$ in (12). The boundary conditions for the outer fluid have been specified only on the dome surface, and those for the periphery of the dome cover have not been specified at all. In both cases approximate boundary conditions can be introduced that drastically simplify the solution to the acoustic-vibration problem.

With regard first to the cover, consider a complete set of orthornormal modal functions $\psi_n(\bar{R})$ that satisfy prescribed boundary conditions at its periphery. In general this set of functions does not coincide with the set of modal functions $X_m(\bar{R})$ introduced earlier to describe the acoustic field in the inner fluid for a prescribed boundary condition at its wall. For each wall boundary condition, however, there exists a corresponding plate boundary condition, forming with it a conjugate pair, such that the two sets of functions do coincide. The boundary conditions on pressure and velocity at the interface of the plate and the interior fluid can then be treated trivially just as in the case of infinite lateral extent.

The same simplification is achieved at the interface of cover and outer fluid, and the whole problem rendered trivially tractable by introducing certain artificial, but not disruptive, boundary conditions on the outer fluid. Specifically, it is assumed that the effect of the outer radiated sound field on the quantities of interest is substantially the same as if the outer fluid were not laterally infinite, but were bounded by the (longitudinally infinite) cylindrical surface formed by extending outward the lateral wall bounding the inner fluid. The same boundary condition then applies on this extended surface as on the lateral wall itself (see Appendix 2, Figure A2-1). This artificial condition is probably without serious consequence at least if the dome diameter is no less than a few wave lengths of sound in the outer fluid and provided the quantities computed pertain to positions removed by a similar distance from the lateral walls (not, for example, the force integrated over an entire cross-section).

As for the conjugate pairs of boundary conditions, the two extreme conditions of a rigid, immovable lateral wall and of a free, pressure-release wall will be mentioned. The condition on the dome cover conjugate to the conditons of a rigid wall is that of perfectly free motion (normal to its plane) at the boundary. If the cover is a plate rather than a membrane, we must include the further condition of clamping to fix the slope at zero relative to the equilibrium plane (i.e., the condition of a sliding clamp). Inversely, the condition on the dome cover conjugate to the condition of a free wall is that of a fixed boundary, with free hinging at this boundary if the cover is a plate.*

Granted these assumptions, a response pressure in mode n is excited only by an excitation pressure in the same mode, i.e., the matrix $\{\gamma_{ns}\}$ of Eqs. (9) or (12) is diagonal, and furthermore the pertinent diagonal response coefficient, e.g., $g_n(\omega,y)\gamma_{nn}(\omega)$, is identical with the corresponding coefficient,

*Inclusion of viscous and hysteretic damping of the plate (or membrane) motion does not affect the stated conjugateness. Conjugate boundary conditions also exist that provide for damping at the periphery of the dome cover and a related damping at the lateral wall (see also Appendix 4).

$\Gamma(k,\omega,y)$, for the laterally infinite system of Part 1 evaluated at the modal wave number $k=k_n$, i.e., $g_n(\omega,y)\gamma_{nn}(\omega)=\Gamma(k_n,\omega,y)$ and similarly $g_{1n}(\omega)\gamma_{nn}(\omega)=\Gamma_1(k_n,\omega)$.*,+,++. The response coefficients Γ and Γ_1 have been studied in Part 1. Eqs. (12) and (14) may now be rewritten as

$$Q(\omega,y) = 2(\pi R_o^2)^{-1} \int_0^\infty dk k I_o(k,\omega)\left| \sum_n k_n^{-1} J_1(k_n R_o) C_n(ka)\Gamma(k_n,\omega,y)\right|^2 \quad (2\text{-}15)$$

$$Q_1(\omega)=2(\pi R_o^2)^{-1} \int_0^\infty dk k I_o(k,\omega) \Big|(2\pi)^{1/2} k^{-1} J_1(kR_o) + \sum_n k_n^{-1} J_1(k_n R_o) C_n(ka)\Gamma_1(k_n,\omega)\Big|^2, \quad (2\text{-}16)$$

where the product $N_n c_n(k)$ is denoted by $C_n(ka)$.

2.4 Approximate Forms for Average-Pressure Spectra

Average-pressure spectra can be computed numerically from the basic Eqs. (15) and (16) for arbitrary values of the parameters. Consistently with the rudimentary model, however, some orientation and insight can be provided by considering various simplified approximate forms, based for the most part on various assumptions of large dome radius $\underline{a}$. The pertinent mathematical work is largely done in Appendix 2.

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* Expressions for the power radiated by the dome cover in this model are given in Appendix 3.

\+ In general, the damping coefficients β and ζ could depend on mode number n.

++ We expect in a later report to treat numerically the problem where the actual coupling to the acoustic field in the infinite outer half-space is retained, with assumption of a rigid baffle surrounding the outer dome surface, in place of the present artificial extension of the lateral boundary condition. From the exact formulation, one can see that the off-diagonal coefficients $\gamma_{nn'}$ in Eq.(12) do vanish relative to the diagonal ones in the limit of a dome diameter large relative to the sound wave length. It may be expected that the extended-wall approximation is poorer if the impedance of the dome cover is low and the dome is shallow.

For convenience we rewrite Eqs. (15) and (16) as

$$Q(y) = 2(\pi R_o^2)^{-1} \int_o^\infty dkkI_o(k) \left|B(k,y)\right|^2, \qquad (2\text{-}17a)$$

$$Q_1 = 2(\pi R_o^2)^{-1} \int_o^\infty dkkI_o(k) \left|(2\pi)^{1/2}k^{-1}J_1(kR_o)+B_1(k)\right|^2, \qquad (2\text{-}17b)$$

where

$$B(k,y) = \sum_{n=0}^{\infty} (-)^n B_n(k,y), \quad B_1(k) = \sum_{n=0}^{\infty} (-)^n B_{1n}(k), \qquad (2\text{-}18)$$

$$B_n(k,y) = k_n^{-1} J_n(k_n R_o) C_n(ka) \Gamma(k_n,y), \qquad (2\text{-}19a)$$

and $B_{1n}(k)$ is similarly related to $\Gamma_1(k_n)$.

2.4.1 <u>Basic Expressions Appropriate to Rigid and Free Walls</u>

Before proceeding, we give B_n and B_{1n} explicitly for particular lateral boundary conditions corresponding to rigid or free (pressure-release) dome walls by introducing Eqs. (10) and (11) for $C_n(ka)[\equiv c_n N_n]$ with the eigenvalues $k_n \equiv x_n/a$ determined by

$$J_1(x_n)=0 \qquad \text{(rigid wall)} \qquad (2\text{-}20)$$

$$J_o(x_n)=0 \qquad \text{(free wall)} \qquad (2\text{-}21)$$

We then have

$$B_n(k,y) = \begin{cases} 2\pi a^{-1/2} k J_1(ka) T_n & \text{(rigid)} \\ 2\pi a^{-1/2} J_o(ka) S_n, & \text{(free)} \end{cases} \qquad (2\text{-}22)$$

where

$$T_n(k,y) = k_n^{-1} J_1(k_n R_o) e_n^{1/2} k_n^{1/2} (k^2-k_n^2)^{-1} \Gamma(k_n,y),$$
$$S_n(k,y) = -k_n^{-1} J_1(k_n R_o) e_n^{1/2} k_n^{3/2} (k^2-k_n^2)^{-1} \Gamma(k_n,y); \qquad (2\text{-}23)$$

$B_{1n}(k)$ is similarly related to $T_{1n}(k)$, $S_{1n}(k)$ and T_{1n}, S_{1n} to $\Gamma_1(k_n)$. Here e_n is the quantity introduced in the respective cases after Eqs. (47) and (67) of Appendix 2 and defined as

$$e_n = 2[\pi x_n J_i^{\,2}(x_n)]^{-1} \qquad (2\text{-}23.1)$$

where i=0 in the rigid case and i=1 in the free case and the sets of x_n differ for the two cases in accord with (20) and (21); in both cases $e_n \to 1$ as $n \to \infty$.

2.4.2 Factors Governing Contributions From Various Modal and Excitation Wave-Numbers

As a preliminary to introduction of simplifying approximations, it is useful to discuss more generally the pertinent factors in Eq. (19a) for B_n and B_{1n} and in Eqs. (17) for Q and Q_1. Generally, with regard to the sum over n and the integral over k in Eqs. (17)-(18), the size of the contribution to Q(y) or Q_1 from a particular pair k_n and k (actually a triplet k_n, $k_{n'}$, k, where a product of terms n, n' arises from the squaring of B or B_1) depends mainly on the size of the following factors: (1) the excitation spectrum $I_o(k)$, which, as previously noted, declines in a rapid but uncertain fashion below $k = \omega/U_\infty$; (2) $|C_n(ka)|$, which mainly declines with increasing $|k-k_n|$; (3) $|\Gamma(k,y)|$ (or $|\Gamma_1(k)|$), which may have a more or less sharp resonance peak at $k_n \simeq k_r$ and behaves otherwise as noted below. The alternation of signs of the terms in Σ_n is also highly pertinent to determining the contribution from a given interval in k_n as discussed in Appendix 2.

Regarding $\Gamma(k_n,y)$, pertinent to pressure in the interior fluid, we recall that it declines approximately exponentially with argument $(k_n^{\,2}-k_\omega^{\,2})^{1/2}|y|$ for $k_n > k_\omega$ [see Eq. (1-10)] except for a possible increase near resonance (k_r) if such occurs. Regarding $\Gamma_1(k_n)$, pertinent to pressure on the outer surface, we do not have an exponential factor as in $\Gamma(k_n,y)$ that cuts off contributions as k_n increases. Nevertheless, there is a fairly rapid decrease with k_n above resonance ($k_n > k_r$) due to the factor q^2-1 in $z(k,\omega)$ [Eq. (1-1)], which occurs in the denominator of Γ_1 [Eq. (1-15)]; specifically, at k_n roughly high enough that $q^2(k_n) \gg 1$ and $k_n \gtrsim k_\omega$ we have $|\Gamma_1|$ decreasing roughly as k_n^{-5} in the case of a plate or k_n^{-3} in the case of a membrane [Eq. (1-69)]. A similar factor declining for $k_n > k_r$ occurs in $\Gamma(k_n,y)$ in addition to the factor declining exponentially for $k_n > k_\omega$, and this factor, rather than the exponential factor, may determine the effective cutoff on k_n if L is sufficiently small. Even if the exponential factor virtually cuts off $\Gamma(k_n,y)$ at a $k_n < k_r$, contributions in a limited range of higher k_n near k_r may nevertheless be appreciable if the resonance is sharp enough.

2.4.3 Progressive Approximations

We thus may distinguish regions of k_n and k that will contribute much more to Q(y) and Q_1 than others and may attune our approximation and neglect accordingly.

We introduce first an approximation based on assumption that contributions to Q(y) (or Q_1) from wave number pairs (k,k_n) are negligible if both $k < k_+$ and $k_n > k_-$, because of the smallness of $I_o(k)$ and $\Gamma(k_n,y)$ [or $\Gamma_1(k_n)$] under these respective conditions, and on further assumption that

$$(k_+ - k_-)a/\pi \gg 1. \tag{2-24}$$

(Rough explicit conditions for this and subsequent approximations will be given later for the several cases.) We also neglect contributions from k not satisfying $ka/\pi \gg 1$, independently of k_n. In the region $k > k_+$, $k_n > k_-$, we also neglect the subregion $k > k_+$, $k_- < k_n < k_+$, and in the region $k < k_+$, $k_n < k_-$ we neglect the subregion $k_- < k < k_+$, $k_n < k_-$, since in these subregions (24) implies $(k-k_n)\, a/\pi \gg 1$, so that the factor $C_n(ka)$ in Eq. (19a), as well as one or the other of the pertinent factors $I_o(k)$, $\Gamma(k_n,y)$, is small there (see Figure 2-2). Setting

$$B \simeq B_- + B_+,\ B_1 \simeq B_{1-} + B_{1+},\ B_{10} = (2\pi)^{1/2}k^{-1}J_1(kR_o),$$
$$Q \simeq Q_- + Q_+,\ Q_1 \simeq Q_{1-} + Q_{1+}, \tag{2-25}$$

where B_- (and B_{1-}) derive from $k_n < k_-$, B_+ from $k_n > k_+$, Q_- from $k < k_-$, and Q_+ from $k > k_+$, we then obtain from Eqs. (17)-(18)

$$Q_- = 2(\pi R_o^2)^{-1}\int_0^{k_-} dk k I_o(k)|B_-|^2$$

$$Q_+ = 2(\pi R_o^2)^{-1}\int_{k_+}^{\infty} dk k I_o(k)|B_- + B_+|^2 \qquad (2\text{-}26)$$

$$Q_{1-} = 2(\pi R_o^2)^{-1}\int_0^{k_-} dk k I_o(k)|B_{10} + B_{1-}|^2$$

$$Q_{1+} = 2(\pi R_o^2)^{-1}\int_{k_+}^{\infty} dk k I_o(k)|B_{10} + B_{1-} + B_{1+}|^2 . \qquad (2\text{-}27)$$

The significant region of k_n with regard to B_+ and B_{1+} is that where $k_n \approx k$ so that $C_n(ka)$ is relatively large (e.g., see Appendix 2). In the present approximation this region is separated from the region $k_n < k_-$, and Γ, Γ_1 vary smoothly with k over intervals $\gg \pi/a$. Therefore we may further restrict the partial sums B_+, B_{1+} to this region, with the small contribution from the neighboring regions included approximately by use of Eq. (17) of Appendix 2 with residual terms omitted; thus

$$B_+ \equiv \sum_{n=n_+}^{\infty} (-)^n B_n \approx \sum_{n=s-n_o+1}^{s+n_o} (-)^n B_n - (1/2)B(s-n_o+1/2) + (1/2)B(s+n_o+1/2), \qquad (2\text{-}28)$$

and similarly for B_{1+}, where $n_\pm$ and s are defined by

$$k_{n_\pm} \leq k_\pm < k_{n_\pm+1}, k_s \leq k < k_{s+1}, \qquad (2\text{-}29)$$

$n_o \gg 1$ (but n_o not so large that $s-n_o < n_-$), and B(n) for non-integral n is given by B_n with use of the asymptotic expressions for large n,

$$k_n \to (n+1/4)\pi/a \quad \text{(rigid)}$$

$$k_n \to (n+3/4)\pi/a \quad \text{(free)}. \qquad (2\text{-}30)$$

A further approximation is based on negligible variation over the interval $s-n_o+1 \leq n \leq s+n_o$ in Eq. (28) of all functions of n in B_n or B_{1n}, as given by (22) and (23) or more generally (19a), except the factor $(k-k_n)^{-1}$. In this approximation, [see Appendix 2, Eqs. (84) and (50)] we have

$$B_+(k,y) \cong B_\infty(k,y) \equiv (2\pi)^{1/2}k^{-1}J_1(kR_o)\Gamma(k,y), \qquad (2\text{-}31)$$

and similarly for B_{1+} with Γ replaced by Γ_1. This approximation to the partial sum B_+ is just the result for the total sum B in the case of a laterally infinite dome ($a \to \infty$, k, L, and R_o fixed), treated in Part 1 (see also Ref. 15).

The factor $J_1(ka)$ (rigid case) or $J_o(ka)$ (free case) contained by B_n in Eq. (22) has disappeared in forming the approximation (31) to the partial sum B_+ on account of the related behavior of the factors $(k^2-k_n^{\;2})^{-1}$ for the various n in the vicinity of $k_n \cong k$.* The factor J_o (or J_1) in the partial sum B_- in Q_+ of Eq. (26), however, produces uncompensated oscillatory behavior, since $|k-k_n| \gg \pi/a$ throughout the pertinent region of (k,k_n). Under previous assumptions, and assuming reasonably that $I_o(k)$ varies negligibly over an interval $\sim a^{-1}$ in k, i.e.,

$$(dI_o/dk)\pi/a \ll I_o \qquad (2\text{-}32)$$

for all pertinent k, we may replace the factor $J_o^{\;2}(ka)$ [or $J_1^{\;2}(ka)$] in the squared term $|B_-|^2$ in Q_+ by its local average value $1/\pi ka$. Likewise, since $J_o(ka)$ oscillates in k about a vanishing mean, we may, under the same circumstances, neglect the cross term $2\mathrm{Re}(B_-B_+^*)$ in the integral for Q_+. We shall denote B_- as modified by replacing $J_o(ka)$ [or $J_1(ka)$] in (22) by $(\pi ka)^{-1/2}$ as $\hat{B}_-$.

Turning to Q_-, by Eq. (26) we have

$$Q_- = 2(\pi R_o^{\;2})^{-1}\int_o^{k_-} dk k I_o(k)\left|\sum_{n=0}^{n_-}(-)^n B_n\right|^2 \qquad (2\text{-}33)$$

*See Appendix 2, Eq. (83.1).

and similarly for Q_{1-}. We insert B_n explicitly from (22) and (23), employ the asymptotic forms for $J_o(ka)$ and $J_1(ka)$ (since $ka \gg 1$ for most of the range of integration), and interchange the double summation $(\Sigma_n \Sigma_{n'})$ with the integration in (33). Assuming that $I_o(k)$ varies smoothly in the range $0 < k < k_-$ and, in particular, that (32) holds there, we may approximate $I_o(k)$ by $I_o\left((1/2)(k_n+k_{n'})\right)$ for fixed n,n', and similarly for powers of k, and extend the k integration from $-\infty$ to $+\infty$. Since

$$\int_{-\infty}^{\infty} dz \, \frac{\sin^2 z}{(z-n\pi)(z-n'\pi)} = \pi\delta_{nn'},$$

we thereby obtain for both rigid and free cases

$$Q_-(y) \simeq (\pi/a) \sum_{n=0}^{n_-} k_n [2J_1(k_n R_o)/k_n R_o]^2 \, I_o(k_n) |\Gamma(k_n, y)|^2 . \tag{2-34}$$

In the case of Q_{1-}, Eq. (27) produces three types of terms due to the square of $B_{10}+B_{1-}$. Proceeding as with Q_-, we obtain a result that may be written

$$Q_{1-} \simeq (\pi/a) \sum_{n=0}^{n_-} k_n [2J_1(k_n R_o)/k_n R_o]^2 \, I_0(k_n) |1+\Gamma_1(k_n)|^2 \tag{2-35}$$

for both rigid and free cases, in which the direct (nonacoustic) force spectrum due to that part of the excitation spectrum with wave numbers $k < k_-$ has been approximated for uniformity by

$$\int_0^{k_-} dkk \left[2J_1(kR_o)/kR_o\right]^2 I_o(k) \simeq (\pi/a) \sum_{n=0}^{n_-} k_n [2J_1(k_n R_o)/k_n R_o]^2 I_o(k_n) . \tag{2-36}$$

As $a \to \infty$ at fixed $k_-(\simeq n_-\pi/a)$, we note, expression (35) becomes properly independent of a.

To summarize the results based on the rather modest approximations introduced up to this point, we have

$$Q \simeq Q_- + Q_+^{\infty} + Q_+^{-} \tag{2-37}$$

$$Q_-(y) \simeq (\pi/a) \sum_{n=0}^{n_-} k_n [2J_1(k_nR_o)/k_nR_o]^2 I_o(k_n) |\Gamma(k_n,y)|^2, \tag{2-38a}$$

$$Q_+^\infty(y) = \int_{k_+}^{\infty} dkk[2J_1(kR_o)/kR_o]^2 I_o(k)|\Gamma(k,y)|^2 \tag{2-38b}$$

$$Q_+^-(y) = 8R_o^{-2}a^{-2} \int_{k_+}^{\infty} dkkI_o(k) \times \begin{cases} k\left|\sum_{n=0}^{n-}(-)^n T_n\right|^2 & \text{(rigid)} \\ k^{-1}\left|\sum_{n=0}^{n-}(-)^n S_n\right|^2 & \text{(free)} \end{cases} \tag{2-38c}$$

with T_n, S_n given by (23);

$$Q_1 \simeq Q_{1-} + Q_{1+}^\infty + Q_{1+}^-, \tag{2-39}$$

$$Q_{1+}^\infty = \int_{k_+}^{\infty} dkk[2J_1(kR_o)/kR_o]^2 I_o(k)|1+\Gamma_1(k)|^2, \tag{2-40}$$

with Q_{1-} given by (35) and Q_{1+}^- by $Q_+^-(y)$ in (38c) with T_n, S_n replaced by T_{1n}, S_{1n} according to the remark following (23).

In a further approximation, employed below, if a is sufficiently large that many moded n have real wave numbers ($0 \leq k_n \leq k_\omega$), where equivalent inner and outer fluids are now assumed so that $k_\omega^+ = k_\omega^- \equiv k_\omega$, then, as discussed in Appendix 2, the modes that contribute significantly to Q_+^- or Q_{1+}^- from the sum over $0 \lesssim n \leq n_-$ separate into groups centered about wave numbers $k_n \simeq k_\omega$, $k_n \simeq k_r$ (resonance). This circumstance is represented by writing the sums in (38c) schematically as

$$\sum_{n=0}^{n-} \simeq \Sigma_\omega + \Sigma_r, \tag{2-41}$$

where the two terms denote partial sums (again supplemented in accord with Eq. (17) of Appendix 2) over $2n_2$ and $2n_r$ terms centered at $k_n \simeq k_\omega$ and $k_n \simeq k_r$, respectively:

$$\Sigma_\omega = \sum_{n=j-n_2+1}^{j+n_2} (-)^n T_n - (1/2)T(j-n_2+1/2) + (1/2)T(j+n_2+1/2), \tag{2-42a}$$

$$\Sigma_r = \sum_{n=r-n_r+1}^{r+n_r} (-)^n T_n - (1/2)T(r-n_r+1/2) + (1/2)T(r+n_r+1/2), \tag{2-42b}$$

except that the sum for Σ_r is understood to be truncated appropriately if it overlaps Σ_ω, and Σ_r is to be omitted for the fluid dome (for which no resonance peak occurs in $\Gamma(k,y)$). In Eqs. (42), we take $n_2 >> 1$ and n_r large enough to cover the resonance peak of width $2\delta k$ centered near n=r, i.e., $n_r \sim \delta ka/\pi$; j is defined by

$$k_j \leq k_\omega < k_{j+1}. \tag{2-42.1}$$

(Eqs. (42) apply also to the other cases related to (38c) with T_n replaced by S_n, T_{1n}, or S_{1n}.)

If the mode spacing ($\simeq \pi/a$) is small enough to satisfy certain further conditions, the variation of all factors of T_n in Σ_ω may be neglected relative to the factor $\Gamma(k_n,y)$ whose derivative becomes infinite at the value $k_n = k_\omega$, and the limiting results discussed in Appendix 2 then apply. These conditions are satisfied for only a restricted range of realistic values of the model parameters, but the results become transparent and easy to discuss and are probably not misleading in order of magnitude. Explicitly, we then have

$$\Sigma_\omega \simeq (2\pi)^{1/2} a^{-1/2} [J_1(k_\omega R_o)/k_\omega] k_\omega^s (k^2 - k_\omega^2)^{-1} (\partial \hat{g}/\partial k_2)_o \\ \times (-)^{j+1} [M(b) - iM(1-b)] \tag{2-43}$$

with s = 1 for rigid walls and s = 2 for free walls, where $\hat{g}(k,k_2,y)$ represents $\Gamma(k,\omega)$ in a notation separating out k_2 as an explicit argument and the subscript 0 on $\partial\hat{g}/\partial k_2$ signifies evaluation at $k_2=0$ (and hence at $k=k_\omega$); M(z) is a function defined (for $0 \leq z < 1$) in Appendix 2, Eq. (26), and shown in Fig. 2-3; argument b is the periodic (linear-sawtoothed)* function of $k_\omega a$ with approximate period π defined by

*Consistently with the nature of the approximation (43), we may replace x_j in (43.1) by its asymptotic form for large j. So doing restricts the range of b to $0 \leq b < 1$, which is the only range in which M(b) is usefully defined. The function $b=b(k_\omega a)$, we note, differs accordingly to the particular boundary condition.

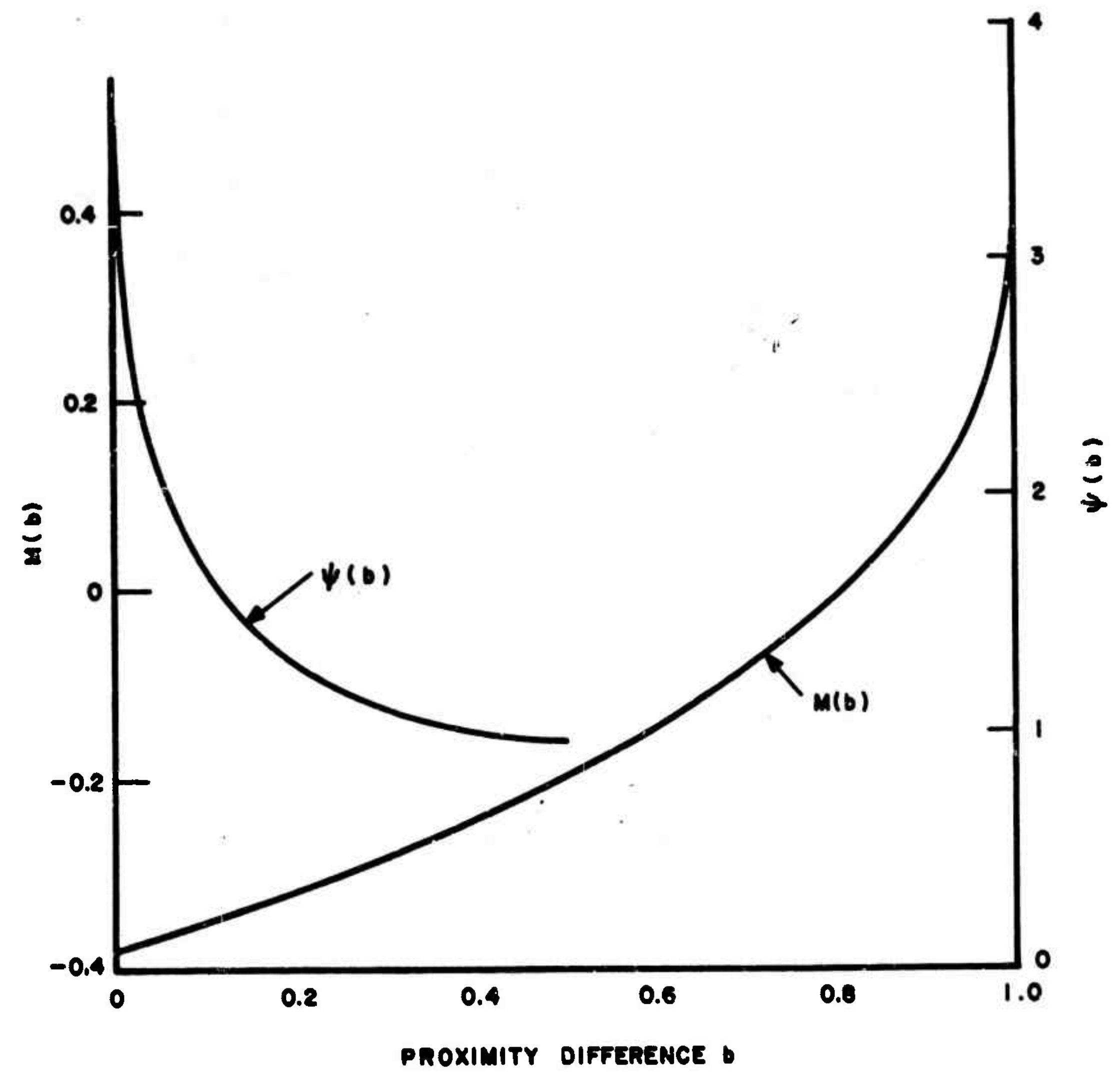

FIGURE 2-3. FUNCTION M(b) AND ψ(b) RELATED TO PRESSURE WITHIN DOME DUE TO HIGH EXCITATION WAVE NUMBERS AT VERY LARGE $\omega a/c$. [See Appendix 1, Eq. (26), and Eq. (2-91.1).] $\psi(1-b) = \psi(b)$.

$$b = k_\omega a/\pi - x_j/\pi \quad (x_j = k_j a) \tag{2-43.1}$$

which describes the position of k_ω between the modal wave numbers k_j, k_{j+1}. Eq. (1-9) yields for use in (43)

$$(\partial g/\partial k_2)_o = e^{i2\alpha}\left\{-ye^{-i\alpha}\sin\alpha + i[L+(z_\omega/\rho\omega)\sin\alpha \cos\alpha]\right\} \tag{2-44}$$

with $z_\omega = z(k_\omega)$, $\alpha = \alpha(k_\omega)$. In the case of a rigid inner surface ($\alpha=0$), (44) reduces to

$$(\partial g/\partial k_2)_o = iL \qquad \text{(rigid inner surface)} \tag{2-45}$$

independently of z_ω and y; in the case of a pressure-release inner surface it reduces to

$$(\partial g/\partial k_2)_o = -i(L+y); \text{ (free inner surface)} \tag{2-46}$$

in the case of an infinite inner medium, Appendix 1, Eq. (13e), similarly yields

$$(\partial g/\partial k_2)_o = -(1/4)(i2y+z_\omega/\rho\omega). \tag{2-46.1}$$

In a similar approximation Eqs. (43) apply also to the pressure outside a plate or membrane [i.e., to Q^-_{1+}], as well as to the pressure within a dome [i.e., to $Q^-_+(y)$], with $g(k,k_2,y)$ replaced by $g_1(k,k_2) \equiv \Gamma_1(k)$. In this case the required derivative is given by

$$(\partial g_1/\partial k_2)_o = z_\omega/\rho\varphi, \tag{2-47}$$

where again $z_\omega = z(k_\omega)$.

If the excitation spectrum $I_o(k,\omega)$ due to the turbulent boundary layer declines sufficiently rapidly with decreasing k for $k < \omega/U_\infty$ that the contributions to the sum for $Q_-(y)$ (or Q_{1-}) in (38a) are negligible except possibly for those modes n near resonance ($r-n_r+1 \leq n \leq r+n_r$),* then in the approximation of (43) the only sums left to be performed explicitly in order to form the average-pressure spectra are the resonance sums in (38a) and (41).

*The experimental result that the average-pressure spectrum due to a turbulent boundary layer varies as R_o^{-2}, corresponding to a non-vanishing area scale, and not as R_o^{-3}, we recall, casts doubt on this inference.

2.4.4 Conditions for Approximations

Conditions for previous approximations will be given here more explicitly for the distinct cases of a fluid dome, a covered dome, and a simple plate. Where a dome is in question, we restrict consideration to force on an area of the inner surface ($y = -L$).

2.4.4.1 Fluid Dome

On account of the attenuating factor $\exp[-(k_n^2 - k_\omega^2)^{1/2} L]$ in $\Gamma(k_n, -L)$ in Eq. (1-62), the condition for the initial approximation that led to Eqs. (25)-(28) may be expressed roughly as the condition that $I_o(k)$ be negligible for $k < k_+$ with reference to the contribution to $Q(y)$ in Eq. (18a), except possibly for n such that $(k-k_n)a \lesssim n_o\pi$, where

$$k_+ = k_- + N_2(\pi/a)$$
$$k_- = [(N_1/L)^2 + (k_\omega^-)^2]^{1/2} \qquad (2\text{-}48)$$

with N_1, N_2 taken such that $N_1 \gtrsim 1$, $N_2 >> 1$ [recall (24)]. If $I_o(k)$ is negligible for $k \lesssim \omega/U_\infty$ (apart from the possible exception for $(k-k_n)a < n_o\pi$) and if $U_\infty << c_-$ and $\omega a/U_\infty >> 1$, as in cases of concern, this condition requires simply that

$$L >> U_\infty/\omega, \qquad (2\text{-}49)$$

which at reasonably high frequency is a very weak condition.

Conditions for approximation (31) are

$$R_o/a << 1,$$
$$\frac{k_+}{[k_+^2 - (k_\omega^-)]^{1/2}} \frac{\pi L}{a} << 1. \qquad (2\text{-}50)$$

The condition for (41) is

$$k_\omega a >> \pi. \qquad (2\text{-}51)$$

Further conditions for (43), on assumption that $k_+ \gtrsim 2k_\omega$, are

$$k_\omega L(k_\omega a)^{-1/2} \ll 1,$$

$$\begin{cases} (k_\omega L)^{-1}(k_\omega a)^{-1/2} \ll 1 & \text{if } k_\omega R_o \lesssim 1 \\ (R_o/L)(k_\omega a)^{-1/2} \ll 1 & \text{if } k_\omega R_o \gtrsim 1. \end{cases} \qquad (2\text{-}52)$$

2.4.4.2 Covered Dome

Two alternative sufficient conditions for Eqs. (25)-(28) apply in this case. First, the condition given above for the fluid dome suffices if k_- is redefined to be the larger of the expression given in (48) and $k_r+\delta k$, where δk is still the resonance half-width, thus insuring also that $I_o(k)$ is negligible in the neighborhood of resonance. In view of Eq.(1-59), an alternative condition is the previous one with k_- redefined as satisfying

$$(k_o/k_-)^4 \frac{1}{\left[k_-^2 - (k_\omega^-)^2\right]^{1/2} h_o} \frac{\rho^-}{\rho_o} = N_3^{-1} \qquad (2\text{-}53)$$

with N_3 taken such that $N_3 \gg 1$.

The conditions for (31) may be written as

$$R_o/a \ll 1,$$

$$\frac{\partial\Gamma(k_+;L)}{\partial k_+} \frac{\pi}{a} \ll 1. \qquad (2\text{-}54)$$

The condition for (41) once more is

$$k_\omega a \gg \pi. \qquad (2\text{-}55)$$

The conditions for (43), generalizing conditions (52), may be expressed as

$$\frac{(\partial^2\hat{g}/\partial k_2^2)_o k_\omega}{(\partial\hat{g}/\partial k_2)_o} (k_\omega a)^{-1/2} \ll 1,$$

$$\frac{\hat{g}_o}{(\partial\hat{g}/\partial k_2)_o\, k_c} (k_\omega a)^{-1/2} \ll 1 \qquad (2\text{-}56)$$

where a subscript 0 denotes evaluation at $k=k_\omega$ (i.e., $k_2=0$) and

$$k_c = \begin{cases} k_\omega & \text{if } k_\omega R_o \lesssim 1 \\ R_o^{-1} & \text{if } k_\omega R_o \gtrsim 1. \end{cases}$$

On assumption that $k_+ \gtrsim 2k_\omega$, these conditions reduce to the conditions (52) together with a further condition on the dome-cover impedance z at wave number ω/c, where we set $z(k=\omega/c,\omega) \equiv z_\omega$:

$$(|z_\omega|/\rho c)(k_\omega a)^{-1/2} \ll 1. \qquad (2\text{-}57)$$

2.4.4.3 Nonrigid Boundary (Without Dome)

The condition for Eqs. (25)-(28) may be expressed as the condition that $I_o(k)$ be negligible in contribution to Q_1 for $k < k_+$, except possibly for $(k-k_n)a \lesssim n_o\pi$, where k_+ is as specified at (48) and k_- is defined, by reference to Eq. (1-69), as satisfying

$$(k_o/k_-)^4 \frac{1}{\left[k_-^2-(k_\omega^+)^2\right]^{1/2} h_o} \frac{\rho}{\rho_o} = N_3^{-1} \; (\ll 1). \qquad (2\text{-}58)$$

The conditions for (31) may be written as

$$R_o/a \ll 1, \qquad (2\text{-}59)$$

$$\frac{\partial\Gamma(k_+,-L)}{\partial k_+}\frac{\pi}{a} \ll 1.$$

The condition for (41) is again

$$k_\omega a \gg \pi.$$

For Eqs. (43), conditions (56) again roughly apply. In this instance, these reduce to

$$(|z_\omega|/\rho c)(k_\omega a)^{-1/2} \ll 1, \qquad (2\text{-}60)$$

$$(\rho c/|z_\omega|)(k_\omega a)^{-1/2} \ll 1 \quad \text{if} \quad k_\omega R_o \lesssim 1 \qquad (2\text{-}61)$$

$$(\rho\omega R_o/|z_\omega|)(k_\omega a)^{-1/2} \ll 1 \text{ if } k_\omega R_o \gtrsim 1. \qquad (2\text{-}62)$$

2.5 Average-Pressure Spectrum on a Rigid Boundary and Its Relation to Dome Effectiveness; Scaling Laws

We return once more to the question of the driving spectrum $P_o(k,\omega)$ [or $I_o(k,\omega)$] due to a turbulent boundary layer and the corresponding frequency spectrum $Q_c(\omega)$ of average pressure on a circular area πR_o^2 of a rigid boundary, and relate this question to the effectiveness of a dome.

2.5.1 Character of Contributions from High and Low Wave Numbers

Suppose $\omega R_o/U_\infty \gg 1$. We recall that

$$Q_o(\omega)=\int_o^\infty dkk\ [2J_1(kR_o)/kR_o]^2\ I_o(k,\omega) \tag{2-63}$$

[Eq. (1-27) with $M \doteq 1$]. Below its peak, which lies somewhat above $k = \omega/U_\infty$ (see Figure 1-3), $I_o(k,\omega)$, for fixed ω, presumably decreases monotonically* with decreasing k, while the envelope of the averaging factor $[2J_1(kR_o)/kR_o]^2$, where $k \gtrsim \pi R_o^{-1}$, decreases monotonically with increasing k. Hence, two intervals of k may be distinguished from which the contributions to the integral of (63) together will greatly exceed those from the rest of the integration interval. In particular, setting

$$Q_o = Q_{o-} + Q_{o+}, \tag{2-64}$$

where Q_{o-} derives from $k < \omega/U_\infty$ and Q_{o+} from $k > \omega/U_\infty$, we have approximately

$$Q_{o-}(\omega) \simeq \int_o^{mR_o^{-1}} dkk[2J_1(kR_o)/kR_o]^2 I_o(k,\omega), \tag{2-65a}$$

$$Q_{o+}(\omega) \simeq (4/\pi)R_o^{-3}\int_{\omega/U_\infty}^\infty dkk^{-2}I_o(k,\omega) \tag{2-65b}$$

[see Eq. (1-31)], where the minimum permissible m may be considered fixed at a moderate multiple of unity. The point pressure spectrum $P_o(\omega)$ may be written formally without approximation as

$$P_o = P_{o-} + P_{o+}, \tag{2-66}$$

*A local increase may occur again at the very low wave numbers $k \sim \omega/c$, as noted below.

$$P_{o-} = \int_{0}^{\omega/U_\infty} dkkI_o(k,\omega), \tag{2-67a}$$

$$P_{o+} = \int_{\omega/U_\infty}^{\infty} dkkI_o(k,\omega). \tag{2-67b}$$

In view of (65b), (67b), and the character of I_o, we have roughly

$$Q_{o+}(\omega) \sim (\omega R_o/\eta U_\infty)^{-3} P_{o+}(\omega), \tag{2-68}$$

where ηU_∞ is an effective convection velocity that declines slowly with increasing frequency but may be taken $\sim 0.6U_\infty$ in the high-frequency range of main interest. Also, if $I_o(k,\omega)$ varies not too rapidly in the interval $0 < k < mR_o^{-1}$, by extracting I_o from under the integral in (65b) we may usefully write

$$Q_{o-}(\omega) \sim 2R_o^{-2} I_o(\sim R_o^{-1},\omega), \tag{2-69}$$

where $I_o(\sim R_o^{-1},\omega)$ denotes the appropriately weighted average of I_o over the interval $0 < k < mR_o^{-1}$.*

As seen from (68) and (69), the average-pressure spectrum on a large area $(\omega R_o/U_\infty \gg 1)$ is expected to vary with R_o as R_o^{-3} if $Q_{o+} \gg Q_{o-}$, i.e., roughly if

$$P_{o+}(\omega) \gg (R_o/\eta U_\infty) \left[(\omega/\eta U_\infty)^2 I_o(\sim R_o^{-1},\omega)\right]; \tag{2-70}$$

it is expected to vary rather as R_o^{-2} if the inverse inequality holds. The latter result as we have noted, accords better with measurement, though the former perhaps accords better with our present understanding of $I_o(k,\omega)$ for $k \ll \omega/U_\infty$. [Even if $Q_{o-} \gtrsim Q_{o+}$, it may well be true that $P_{o+} \gtrsim P_{o+}$, so that P_{o+} may be identified with the total point spectrum $P_o(\omega)$ in (68) and (70).]

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*In Appendix 5 it is shown that if $I_o(k,\omega)$ varies substantially as k^n up to some $k > mR_o^{-1}$, then $Q_{o-}(\omega)$ varies with R_o as $R_o^{-(2+n)}$ if $0 < n < 1$ and as R_o^{-3} if $n > 1$.

From the formal expression of $I_o(k,\omega)$ in terms of the boundary-layer velocity field in the general case of compressible flow, it appears, contrary to the assumption in (69), that some anomalous behavior of I_o, perhaps a pronounced peak, occurs in the neighborhood of $k = \omega/c$ (see Appendix 5, Section A5.1, and Ref. 14). If such a peak occurs, it will be reflected in an additional contribution to the average-pressure spectrum Q_o (via Q_{o-} in (65a)) that is independent of R_o so long as $\omega R_o/c \lesssim 1$, i.e., R_o less than roughly a quarter wave length of sound. In this event the area dependence of Q_o for $\pi U_\infty/\omega << R_o \lesssim c/\omega$ would be roughly of the form

$$Q_o(\omega) = A(\omega)R_o^{-2} + B(\omega) + C(\omega)R_o^{-3}, \qquad (2\text{-}70.1)$$

where the first and third terms correspond to (69) and (68). Observed spectra for different transducer size, as well as measured narrow-band spatial correlations, place experimental constraints on A, B, and C. Hereafter we ordinarily ignore the possible radiation contribution $B(\omega)$.*

In Appendix 5 the relation between Q_{o-} and Q_{o+} and other quantities are examined under various assumptions. Also, two different explicit assumed forms for $I_o(k,\omega)$, one of which has commonly been used in dome analyses, are considered. This one accords with the observed variation as R_o^{-2} and with the gross time decay of the spatial pressure correlation in the appropriate convected frame, but probably not with the frequency dependence of the area scale; neither form has any theoretical basis.

The extent to which a dome is capable of reducing flow noise, and in particular the possible success of a fluid sheath, is critically dependent on this question of the relative contributions to $Q_o(\omega)$ due to wave number components with $k \gtrsim \omega/U_\infty$ and with $k \lesssim mR_o^{-1}$. This dependence results because the former component in the transmitted pressure spectrum $Q(\omega,y)$ is strongly attenuated for dome size a not too small (and $U_\infty << c^-$), whereas, except for relatively small R_o, the latter is not. In the approximations of (37) - (38), the former, attenuated contribution is contained in $Q_+^\infty(y) + Q_+^-(y)$ and the latter,

* A re-examination of observed data to set an upper limit on the possible effect of the value of B in reducing the apparent ratio A/C appears desirable.

relatively unattenuated one in $Q_-(y)$. This point will be made more explicitly in the course of drawing estimates and conclusions for the individual configurations.

2.5.2 Scaling with Boundary-Layer Parameters

We consider briefly also the probable scaling laws for the direct boundary-layer pressure*. We shall refer to these later in considering their modification by a dome or by a non-rigid boundary. At high frequency, perhaps where $\omega\nu/v_*^2 \gtrsim 0.1$, with reference to wave numbers $k > \omega/U_\infty$ corresponding to the part of the spectrum generated by convection, we have reason to expect the pertinent length and velocity scales to be proportional, respectively, to ν/v_* and v_*, which are characteristic of the portion of the boundary layer just outside the viscous sublayer. In such case, the wave number-frequency spectrum $P_o(\vec{k},\omega)$ of the point pressure has the form

$$P_o(k,\omega) = \rho^2\nu^3F_+(\vec{k}\nu/v_*^2, \omega\nu/v_*^2), \quad (k \gtrsim \omega/U_\infty) \tag{2-71}$$

where F_+ is some function of the indicated dimensionless variables and ρ is the fluid density. Hence, after integration over the angle of $\vec{k}$, we have likewise

$$I_o(k,\omega) = \rho^2\nu^3G_+(k\nu/v_*,\omega\nu/v_*^2), \quad (k \gtrsim \omega/U_\infty) \tag{2-72}$$

where the function G_+ (and similar functions introduced below) are related to F_+. Eq. (67b) then yields the form of $P_{o+}(\omega)$:

$$P_{o+}(\omega) = \rho^2\nu v_*^2\, H_+(\omega\nu/v_*^2). \tag{2-73}$$

Eq. (65b) (with U_∞ more appropriately replaced by a multiple of v_*) yields in turn the form

$$Q_{o+}(\omega) = (\omega R_o/v_*)^{-3}\rho^2\nu v_*^2\, L_+(\omega\nu/v_*^2). \tag{2-74}$$

---

*These points are discussed further in Ref. 16.

In some other domain of frequency and wave number, the pertinent length and velocity scales may well be those characteristic of the boundary layer as a whole, namely δ_* and U_∞, rather than the "inner" parameters ν/v_* and v_*.* This presumption appears likely to be valid at least in the domain $k \lesssim \delta_*^{-1}$. In such a domain the wavenumber-frequency spectrum, contrary to (71), has the form

$$P_o(\mathbf{k},\omega) = \rho^2\delta_*^3U_\infty^3F_-(\mathbf{k}\delta_*,\omega\delta_*/U_\infty) \quad (k \lesssim \delta_*^{-1}) \qquad (2\text{-}75)$$

whence also

$$I_o(k,\omega) = \rho^2\delta_*^3U_\infty^3G_-(k\delta_*,\omega\delta_*/U_\infty). \qquad (2\text{-}76)$$

From (76), in the low-frequency regime where $\omega\delta_*/U_\infty \lesssim 1$ we obtain the conventional "outer" form

$$P(\omega) = \rho^2\delta_*U_\infty^3H(\omega\delta_*/U_\infty) \qquad (\omega\delta_*/U_\infty \lesssim 1) \qquad (2\text{-}77)$$

and likewise, from (63), for nonvanishing radius

$$Q_o(\omega) = (\omega R_o/U_\infty)^{-2}\rho^2\delta_*U^3N(\delta_*/R_o,\omega\delta_*/U_\infty). \qquad (2\text{-}78)$$

If I_o, given by (76), varies little for $k < mR_o^{-1}$, which may be so for $\omega\delta_*/U_\infty \lesssim 1$ only if $R_o \gg \delta_*$, Eq. (78) reduces to the simpler form

$$Q_o(\omega) = 2(\omega R_o/U_\infty)^{-2}\rho^2\delta_*U^3\hat{G}_-(\omega\delta_*/U_\infty), (\omega\delta_*/U_\infty \lesssim 1, R_o \gtrsim \delta_*) \qquad (2\text{-}78.1)$$

where the function $\hat{G}_-$ is identified as a suitable average over k of the function $G_-(k\delta_*,\omega\delta_*/U_\infty)$ of Eq. (76).

In the higher-frequency regime ($\omega\delta_*/U_\infty \gtrsim 1$), accepting (75), we nevertheless cannot infer the form of $P_{o-}(\omega)$ without knowledge of the form of I_o in the range $\delta_*^{-1} \lesssim k < \omega/U_\infty$. Hence, we remain

*The difference between assumption of velocity scales proportional to v_* and to U_∞ is relatively minor, since v_*/U_∞ varies only weakly with Reynolds number, expecially where the latter is very high.

uncertain whether $P_{o-}(\omega)$, like $P_{o+}(\omega)$, scales for $\omega\nu/v_*^2 \gtrsim 0.1$ as the "inner" form (73) or as the "outer" form (77), or whether its form essentially contains both inner and outer sets of parameters. Since the relative importance of P_{o+} and P_{o-} is also uncertain, it is uncertain how their sum, the total point spectrum $P(\omega)$, scales with these parameters (see also Ref. 16).

On the other hand, for $R_o \gtrsim \delta_*$, form (76) alone permits inference of the form (78) for Q_{o-}, which, under the stated assumptions, reduces to form (78.1). If $\omega\nu/v_*^2 \gtrsim 0.1$ holds in addition, we infer from (74) for Q_{o+} and (78.1) for Q_{o-} a form for the total Q_o:

$$Q_o(\omega) = 2(\omega R_o/U_\infty)^{-2}\rho^2\delta_* U_\infty^3 \hat{G}_-(\omega\delta_*/U_\infty) + (\omega R_o/v_*)^{-3}\rho^2 \nu v_*^2 L_+(\omega\nu/v_*^2) \quad (\omega\nu/v_*^2 \gtrsim 0.1, R_o \gtrsim \delta_*). \tag{2-79}$$

The assumption that form (76) holds for I_o in some interval of low k is probably inconsistent with the belief that I_o in this interval may be taken equal to its value at $k = 0$. In fact, we know that for the pressure in a turbulent boundary layer, I_o vanishes at $k = 0$ apart from effects of compressibility, viscosity, and inhomogeneity in the direction of flow. The first of these three effects involves an additional parameter, viz. the velocity of sound, and hence at least if the value of I_o at $k = 0$ is largely due to compressibility, I_o near $k = 0$ will not have the form (76). At the same time, the factor $(\omega R_o/U_\infty)^{-2}$ in the result (79) accords fairly well with experiments (e.g., see Ref. 3, but cf. Ref. 17).* It is conjectured, therefore, that $I_o(k)$ does not differ greatly from its average value over most of the range $0 < k < mR_o^{-1}$, where $\pi U_\infty/\omega \ll R_o \lesssim c/\omega$, but that this average value is substantially larger than $I_o(0)$.

*The same factor R_o^{-2} would be obtained whatever the cause of a nonvanishing I at $k = 0$, but the same scaling with ω and U_∞, in general, would not.

2.6 Estimates for Various Configurations

2.6.1 Fluid Dome

2.6.1.1 Contribution from High Wave Numbers

Regarding first the contribution $Q_+ = Q_+^{\infty} + Q_+^{-}$ to $Q(\omega,y)$, it is useful to define the modification factor $\Lambda(k,y)$ by which the frequency-wave number spectrum of average pressure on the area πR_o^2 at depth y differs from that due to the excitation pressure (excluding the acoustic) on an infinitesimal area in the outer surface, with reference only to excitation wave numbers $k > k_+$:

$$Q_+(y) = \int_{k_+}^{\infty} dk k I_o(k) \Lambda(k,y) \tag{2-80}$$

[cf. Eq. (26)]. This factor includes both the effects of area-averaging over πR_o^2 and cushioning by the intervening medium. We take $k_+ = \omega/U_\infty$.

As implied by the approximation and discussion in the paragraph containing Eq. (32), $\Lambda(k,y)$ contains a part that oscillates rapidly with a period in k that is short compared to any interval of significant change in any other factor, including $I_o(k)$. We may then replace $\Lambda(k,y)$ in 80) by its local wave-number average, say $\tilde{\Lambda}(k,y)$. In the approximations of Eqs. (38), then,

$$\tilde{\Lambda}(k,y) = \Lambda_\infty(k,y) + \Lambda_-(k,y), \tag{2-81}$$

where

$$\Lambda_\infty(k,y) = \Lambda_o(k)|\Gamma(k,y)|^2, \tag{2-82}$$

$$\Lambda_-(k,y) = 8(R_o a)^{-2} \times \begin{cases} k\left|\sum_{n=o}^{n_-}(-)^n T_n\right|^2 & \text{(rigid)} \\ k^{-1}\left|\sum_{n=o}^{n_-}(-)^n S_n\right|^2 & \text{(free)} \end{cases} \tag{2-83}$$

$$\Lambda_o(k) = 4J_1^2(kR_o)/(kR_o)^2. \tag{2-84}$$

Here $\Lambda_\infty(k,y)$ represents the limiting value of the modification factor $\bar{\Lambda}(k,y)$ (or Λ) for an infinite dome ($a \to \infty$ at fixed k, R_o, L). Similarly $\Lambda_o(k)$ represents the value pertinent to the same circular area πR_o^2 in a rigid surface exposed to the exciting pressure fluctuations and merely reflects the area averaging. For $\omega R_o/U_\infty >> 1$, with assumption that $I_o(k)$ for $k > \omega/U_\infty$ is substantially unchanged over an interval R_o^{-1} in k, as at Eq. (65b) we may appropriately employ a local k-average also in place of Λ_o in (84), say $\bar{\Lambda}_o$:

$$\bar{\Lambda}_o = (4/\pi)(kR_o)^{-3}. \qquad (\omega R_o/U_\infty >> 1) \qquad (2\text{-}85)$$

In the opposite case where $kR_o << 1$ for all significant k, or roughly where $\omega R_o/U_\infty << 1$, we naturally have

$$\bar{\Lambda}_o \simeq 1 \qquad (\omega R_o/U_\infty << 1) \qquad (2\text{-}86)$$

as at Eq. (67b)

If the additional condition (51) is satisfied, approximation (41) (with $\Sigma_r = 0$) may be employed in (83) as previously in (38c), and if conditions (52) are satisfied, approximations (43) may also be employed. Though the latter conditions can apply only for a diaphragm radius a very large compared to the sound wave length, especially if the thickness L is not a substantial fraction of a wave length, this affords a suitably simple regime to examine. As an inessential further simplification, we assume a rigid inner surface, whence (45) applies. Then Eqs. (43) yield for the finiteness term Λ_- in the modification factor $\bar{\Lambda}$ of (81) at the inner surface

$$\Lambda_-(k,y) \approx 16\pi(ka)^{-3}[J_1(k_\omega R_o)/k_\omega R_o]^2(k_\omega L)^2[1-(k_\omega/k)^2]^{-2} \times [M^2(b)+M^2(1-b)] \times \begin{cases} 1 & \text{(rigid wall)} \\ (k_\omega/k)^2 & \text{(free wall)} \end{cases} \qquad (2\text{-}87)$$

If the inner surface is instead a pressure-release one, by (46) we have only to replace L^2 in the numerator by $(y+L)^2$ and likewise in Eq. (91), below; similarly, if the inner medium is infinite, by (46.1) we have to replace L^2 by $(1/4)y^2$.* Thus, the contribution of Λ_- to the average-pressure spectrum $Q(\omega,y)$, in the regime of (87), is independent of depth $(-y)$ if the inner surface is rigid, increases quadratically with distance from that surface if it is pressure release, and increases quadratically with distance from the outer surface if the inner medium is infinite.

According to (87), Λ_- for the case of a free wall is smaller than for that of a rigid wall by a factor $\sim(k_\omega/k)^2$, i.e., the square of the ratio of the wave length of excitation to that of sound. If the boundary conditions were mixed, which perhaps would best simulate actual configurations, a contribution of the type of the rigid case presumably remains and is the dominant term, though with a reduced numerical coefficient. The significance of this strong dependence on the lateral boundary condition satisfied by the fluid diaphragm in the envisaged application is subject to some doubt, however, inasmuch as the boundary condition in the present model was artifically extended outside the fluid diaphragm indefinitely into the region of the outer fluid. For this same reason the model itself may have doubtful quantitative applicability if the thickness L of the diaphragm is too small a fraction of a wave length.**

We proceed to obtain explicit estimates of the several contributions to the interior average-pressure spectrum $Q(\omega,y)$. In the present case of a fluid sheath $(\mathfrak{z} = 0)$ with rigid inner surface $(\hat{\mathfrak{K}} = 0)$, we recall from Eq. (1-62) that

$$\left|\Gamma(k,y)\right|^2 = \exp[-2(k^2-k_\omega^2)^{1/2}L]\,\mathrm{ch}^2[(k^2-k_\omega^2)^{1/2}(y+L)]; \qquad (2\text{-}88)$$

The contribution $Q_+^\infty(\omega,y)$ to $Q(\omega,y)$, deriving from Λ_∞ in (81), is estimated very roughly as related to the high-k contribution $Q_{o+}(\omega)$ to the spectrum of averaging driving pressure $Q_o(\omega)$ on an equal area of the outer surface [Eq. (65b)] by

*These replacements are approximately correct even though the surface of the element itself remains rigid, provided $R_o \lesssim k_\omega^{-1}$ [cf.(99.2)].

**The case of the diaphragm will be treated more adequately in a later report.

$$Q_+^{\infty}(\omega,y) \sim \left|\Gamma(\omega/\eta U_\infty,\omega,y)\right|^2 Q_{o+}(\omega) \qquad (2\text{-}89)$$

(in which, however, η should be regarded as slightly larger than it was in Eq. (68) on account of the weighting factor $\left|\Gamma(k,\omega,y)\right|^2$ in the integral over k in (80)).

By (1-62),(1-63), and (1-63.1) we have approximately (for $U_\infty \ll c$)

$$\left|\Gamma(\omega/\eta U_\infty,\omega,y)\right|^2 \simeq e^{-2\omega L/\eta U_\infty} ch^2[\omega(y+L)/\eta U_\infty] \qquad (2\text{-}90)$$

for a rigid inner surface.

The contribution $Q_+^-(\omega,y)$, deriving from Λ_- in (81), is estimated from (80) and (87) for a rigid inner surface, with $k_\omega R_o \lesssim 1$, as

$$\begin{aligned} Q_+^-(\omega,y) &\simeq (k_\omega L)^2 f(k_\omega a) a^{-3} \int_{\omega/U_\infty}^{\infty} dk k^{-2} I_o(k,\omega) \\ &\simeq (\pi/4)(k_\omega L)^2 f(k_\omega a)(R_o/a)^3 Q_{o+}(\omega), \qquad \text{(rigid wall)} \end{aligned} \qquad (2\text{-}91)$$

where $f(k_\omega a)$ is the positive oscillating function with period π defined by

$$f(k_\omega a) = 4\pi\ [M^2(1-b)+M^2(b)] \equiv \psi(b) \qquad (2\text{-}91.1)$$

$\psi(b)$ is shown in Figure 2-3; it has a peak value 3.44 and a value somewhat less than half as large when averaged over $k_\omega a$. In the case of free walls the estimate (91) is reduced by a factor of the order of $(\eta U_\infty/c)^2$. The second form in (91) relates Q_+^- to the high-wave number part Q_{o+} of the spectrum of driving pressure over an area of radius R_o, where R_o is understood to be such that $\omega R_o/U_\infty >> 1$.

We give the expression corresponding to (91) outside the regime where (87) applies, i.e., when conditions (52) are not satisfied, but where the condition $\omega/U_\infty >> (L^{-2} + k_\omega^2)^{1/2}$ still holds (and, for simplicity, $k_\omega R_o \lesssim 1$). From (83) and (23) we obtain

$$Q_+^-(\omega,-L) \approx 2 \left| \sum_{n=0}^{n_-} (-)^n k_n^{1/2} \Gamma(k_n,\omega,-L) \right|^2 a^{-2}$$

$$\times \int_{\omega/U_\infty}^{\infty} dk\, k^{-2} I_o(k,\omega) \qquad \text{(rigid wall)} \qquad (2\text{-}92)$$

$$\simeq (\pi/2) a \left| \sum_n (-)^n k_n^{1/2} \Gamma(k_n,\omega,-L) \right|^2 (R_o/a)^3 Q_{o+}(\omega)$$

by use of (65b).

2.6.1.2 Contribution from Low Wave Numbers

Unless $I_o(k,\omega)$ is negligible for wave numbers $k \lesssim \omega/U_\infty$, the contribution to $Q(\omega,y)$ from $Q_-(\omega,y)$ of (38a) must also be included. The sum may presumably be confined to $k_n \lesssim mR_o^{-1}$ (for $R_o > k_+^{-1}$), where m is a fixed number exceeding unity by a moderate factor, on account of attenuation by the area-averaging factor $[2J_1(k_nR_o)/k_nR_o]^2$. If wave numbers $k < mR_o^{-1}$ actually predominate in the wave-number spectrum of the driving average-pressure spectrum $Q_o(\omega)$, as suggested by the measured R_o-dependence of the latter (see Section 5), we must have $Q_-(\omega,y)$ of the same order of magnitude as $Q_o(\omega)$. This conclusion holds unless $R_o << mk_\omega^{-1}$ and $\pi y/R_o >> 1$, since the factor $|\Gamma(k_n,y)|^2$ in

(38a) attenuates only for k_n such that $k_n > k_\omega$ and $(k_n^2 - k_\omega^2)^{1/2} y \gtrsim 1$. The fluid dome, in such event, will be ineffective relative to flush mounting [see Eq. (94) below].

We know too little about the wavenumber spectrum $I_o(k,\omega)$ of turbulent pressure for $k < \omega/U_\infty$ to be able to estimate $Q_-(\omega,y)$ reliably, even in terms of $Q_{o-}(\omega)$, in the parameter regime where these differ appreciably. For indicative estimates, however, we assume that $I_o(k,\omega)$ varies only moderately throughout the pertinent interval $0 < k \lesssim K_c$, (except possibly for a negligible subinterval*), where the rough cutoff K_c is defined by

$$K_c = \text{lesser of } mR_o^{-1},\ (k_\omega^2 + |y|^2)^{1/2}.$$

Provided conditions (51) and the first of (52) are satisfied, the sum (38a) may be approximated by an integral:

$$Q_-(\omega,y) \simeq \int_0^\infty dkk[2J_1(kR_o)/kR_o]^2 I_o(k,\omega) \left|\Gamma(k,\omega,y)\right|^2 \qquad (2\text{-}93)$$

We assume a dome sufficiently large that this approximation may be used. For definiteness, we consider a depth $y = -L$ at a rigid inner surface, except as later noted.

First, if $mR_o^{-1} < (k_\omega^2 + 1/64L^2)^{1/2}$, from (1-62) we have $1 \geq |\Gamma|^2 > e^{-1/4}$ for $k < mR_o^{-1}$; (93) and (65a) then yield

$$Q_-(\omega,-L) - \int_0^{mR_o^{-1}} dkk[2J_1(kR_o)/kR_o]^2 I_o(k,\omega) \qquad (2\text{-}94)$$

$$= Q_{o-}(\omega) . \qquad [R_o >> (k_\omega^2 + 1/64L^2)^{-1/2}]$$

This result does not depend on assumption of slowly varying $I_o(k)$, except as Eq. (38a) does, and holds, in particular, when R_o exceeds about a wave length.

---

* The possibility that $I_o(0,\omega) << I_o(k,\omega)$ for most $k < K_c$ thus need not be excluded.

Next, if $k_\omega L \gtrsim 1$, corresponding to a thickness greater than about a sixth of a wave length, and $k_\omega \lesssim mR_o^{-1}$, we find

$$Q_-(\omega,-L) \simeq \int_o^{k_\omega} dkk[2J_1(kR_o)/kR_o]^2 I_o(k,\omega)$$

$$\sim 2R_o^{-2} I_o(\sim k_\omega) T(k_\omega R_o) \tag{2-95}$$

$$\sim FT(k_\omega R_o) Q_{o-}(\omega)$$

by use of (69), where the function T is defined by

$$T(x) = 2 \int_o^x dzz^{-1} J_1^2(z), \tag{2-95.1}$$

$I_o(\sim k_\omega)$ denotes the suitably weighted coverage of $I_o(k)$ over the interval $0 < k < k_\omega$, and F represents the ratio of $I_o(\sim k_\omega)$ to the weighted average of I_o over the interval pertinent to Eq. (69):

$$F = I_o(\sim k_\omega)/I_o(\sim R_o^{-1}). \tag{2-95.2}$$

If $k_\omega R_o \lesssim 1$, we have

$$T(k_\omega R_o) \simeq (1/4)(k_\omega R_o)^2,$$

whence (95) yields

$$Q_-(\omega,-L) \sim (1/2) k_\omega^2 I_o(\sim k_\omega) \sim (1/4)(k_\omega R_o)^2 F Q_{o-}(\omega). \tag{2-96}$$

In this regime the effect of the sheath is roughly equivalent to averaging the pressure spectrum over an area of radius $\lambda/\pi (=2/k_\omega)$ in place of the actual (smaller) area of radius R_o. If $I_o(k)$ increases appreciably from $k \sim k_\omega$ to $k \sim R_o^{-1}$, so that $F \ll 1$, the factor of reduction by the sheath, from (96), will be larger.

If $k_\omega \gtrsim mR_o^{-1}$, Eq. (95) with $F \doteq 1$ again holds, $T(k_\omega R_o) = 1$, and (95) reduces to (94).

If $k_\omega \lesssim R_o^{-1}$ and $\pi L/R_o \ll 1$, corresponding to a thin sheath, by expansion in the parameter L/R_o and appropriate approximation, assuming still a rigid inner surface, we estimate

$$Q_-(\omega,-L) \simeq 2R_o^{-2} I_o(\sim R_o^{-1}) \left\{1-2\pi^{-1}(L/R_o)[\ln(R_o/L)-0.6]\right\}$$

$$\simeq \left\{1-2\pi^{-1}(L/R_o)[\ln(R_o/L)-0.6]\right\} Q_{o-}(\omega), \qquad (2\text{-}97)$$

where $I_o(\sim R_o^{-1})$ denotes a suitably weighted average over $0 < k < mR_o^{-1}$. In the range in question, where $\pi L/R_o \ll 1$, according to (97) $Q_-(\omega,-L)$ is smaller than $Q_{o-}(\omega)$ by less then 10%.

Again, if $R_o \lesssim (k_\omega^2 + L^{-2})^{-1/2}$, i.e. roughly if the diameter $2R_o$ is less than a quarter wave length and less than 1.4 times the thickness L, we have the estimate

$$Q_-(\omega,-L) \simeq \int_o^{k_\omega} dkkI_o(k,\omega) + \int_{k_\omega}^{\infty} dkkI_o(k,\omega)\exp[-2(k^2-k_\omega^2)^{1/2}L]$$

$$\simeq 2R_e^{-2} I_o(\sim R_e^{-1}) \qquad (2\text{-}98)$$

$$\sim H(R_o/R_e)^2 Q_{o-}(\omega)$$

where R_e is an effective radius defined by

$$R_e^{-2} = (1/4)[k_\omega^2 + (1/2)L^{-2}] \doteq (1/4)[(2\pi)^2\lambda^{-2} + (1/2)L^{-2}]; \qquad (2\text{-}99)$$

$I_o(\sim R_e^{-1})$ denotes the appropriate weighted average of $I_o(k)$, determined, on account of the exponential factor in (98), mainly by $I_o(k)$ for $0 < k \lesssim (k_\omega^2 + L^{-2})^{1/2}$ and H, similarly to F in (95), represents the ratio of $I_o(\sim R_e^{-1})$ to the similar average $I_o(\sim R_o^{-1})$ of Eq. (69):

$$H = I_o(\sim R_e^{-1})/I_o(\sim R_o^{-1}) \quad (2\text{-}99.1)$$

Where the approximation (98) holds at all we have $R_e > R_o$.

According to Eq. (98), the spectrum of average pressure for a shielded element of radius $R_o \lesssim R_e$ is independent of R_o and has a value corresponding to a flush element of averaging radius R_e if $H \simeq 1$ (or smaller than this by a factor H, in general). For example, if $L \lesssim \lambda/9$, the contribution to Q_- in (98) from the second term in (99) is the larger, and the effective averaging radius is $\simeq 2.8L$. If $L \gtrsim \lambda/9$, on the other hand, the first term is the larger, and the effective radius is $\simeq \lambda/\pi$. For an element with given radius R_o, then, the sheath reduces the spectrum of average pressure only to the extent that the relations

$$L >> R_o/2.8 \text{ and } R_o << \lambda/\pi$$

are satisfied.

Though the domains of validity of approximations (94) and (98) do not overlap, there being a transition region between, we may crudely regard (94) as applying if $R_o \gtrsim R_e$ and (98) if $R_o \lesssim R_e$.

The effect of the sheath is shown graphically in Figure 2-4 as a function of L at fixed λ with R_o as a parameter. The form of dependence is shown for three values of R_o, say $R_{o1} < R_{o2} < R_{o3}$, which are introduced as arguments in the respective spectra $Q_-(\omega,-L)$. It is supposed that $\lambda >> R_{o1} >> U_\infty/\omega$ and $R_{o2} \lesssim \lambda/\pi \lesssim R_{o3}$. For $Q_-(\omega,R_{o1},-L)$, Eq. (98) then holds over the range $L \gtrsim R_{o1}/2$. In the

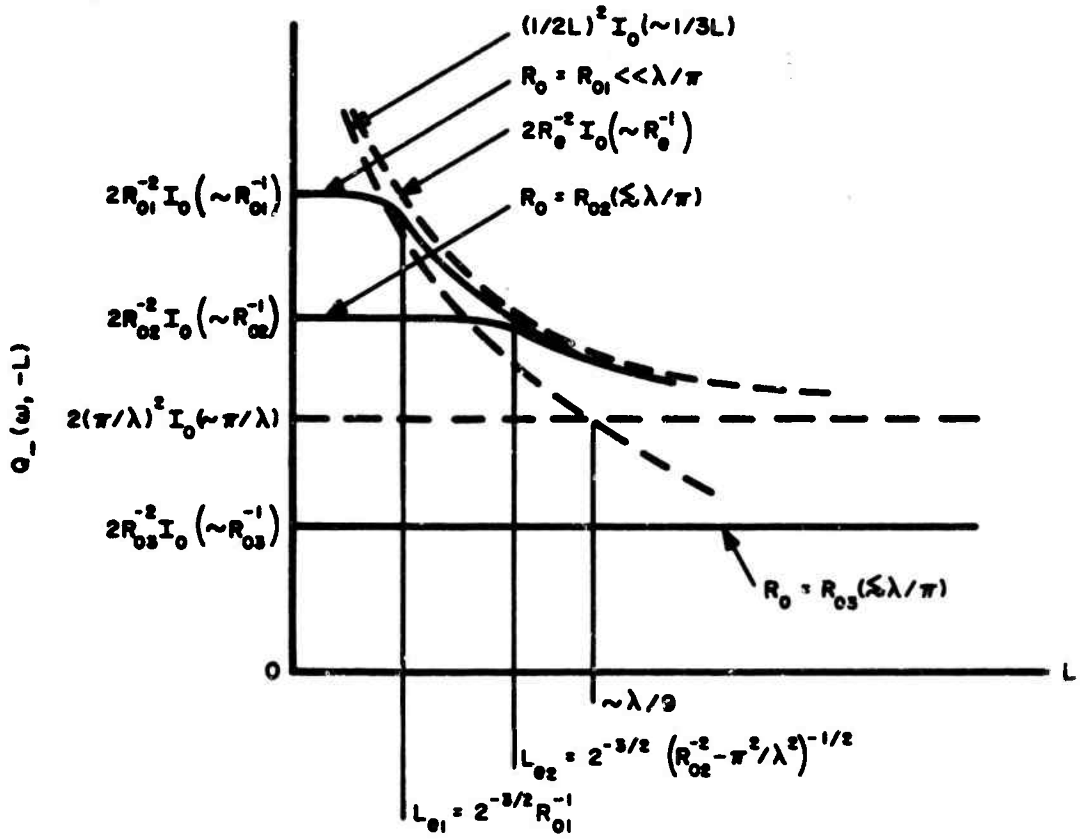

FIGURE 2-4. AVERAGE-PRESSURE SPECTRUM $Q_{-}(\omega,-L)$ WITHIN FLUID DOME DUE TO LOW EXCITATION WAVE NUMBERS VS. DOME DEPTH L FOR SEVERAL ELEMENT RADII, $R_{o3} > R_{o2} > R_{o1} \gg U_{\infty}/\omega$.

range $L > L_{e2}$, where L_{e2} is defined by $R_e(\lambda, L=L_{e2})=R_{o2}$ with R_e given by (99), the Q_- spectra for R_{o1} and R_{02} coincide, since the effective radius there is R_e for both. At $L = 0$, on the other hand, $Q_-(\omega,R_{o2},0)/Q_-(\omega,R_{o1},0) \doteq Q_{o-}(\omega,R_{o2})/Q_{o-}(\omega,R_{o1}) = (R_{o1}/R_{o2})^2$. The assumed condition on R_{o3} makes $R_{o3} > R_e$ for all L; hence R_{o3} is the effective radius for all L, and $Q_o(\omega,R_{o3},-L)$ is independent of L.

It has been assumed above that the exciting pressure spectrum $I_o(k,\omega)$ varies smoothly and moderately in the low-wave-number range in question. As noted in Section 2.5 and Appendix 5, however, I_o may behave anomalously near $k = k_\omega \doteq \omega/c$. If it has a significant peak there, but one that is broad relative to the spacing of the modal wave numbers k_n, there will be an additional contribution to $Q_-(\omega,-L)$ which by (93), since $\Gamma(k_\omega,\omega) = 1$, is just the same as the corresponding contribution to the spectrum $Q_o(\omega)$ for an exposed area and, like the latter, is independent of R_o for $k_\omega R_o \lesssim 1$.*

All the estimates of $Q_-(\omega,-L)$ above referred to a rigid inner surface. With regard to dependence on the impedance assumed at this surface, suppose the latter is transparent (infinite inner medium) but the element is still regarded as rigid. Then for wave number k, if $kR_o \lesssim 1$, the corresponding average pressure on the element is reduced relative to the rigid case by a factor

$$\simeq (1/2)[1+\xi_F(k_\omega^2-k^2)^{1/2}/k_\omega] \tag{2-99.2}$$

[with $(k_\omega^2-k^2)^{1/2} \rightarrow i(k^2-k_\omega^2)^{1/2}$ when $k > k_\omega$], where $\xi_F(\pi R_o^2\rho c)$ denotes the radiation impedance of the rigid element in free space and is

*If such a peak exists but is narrow relative to the spacing of the k_n, this contribution to $Q_-(\omega,y)$ is identified rather as a term $(\pi/a)\, k_\mu[2J_1(k_\mu R_o)/k_\mu R_o]^2 I_o(k_\mu,\omega)$ in the sum (38a), where μ denotes the mode such that k_μ lies nearest k_ω. Such a contribution would have a resonant character with peaks at frequencies $\omega_\mu = ck_\mu$.

given for $k_\omega R_o < 0.5$ by $\xi_F \doteq \xi_1 - i\xi_2 = 0.25(k_\omega R_o)^2 - i0.614 k_\omega R_o$ [cf. Eq. (140) further on].* When $R_o \lesssim R_e/2$, we then find in place of (98)

$$Q_-(\omega,-L) \simeq 2I_o(\sim R_e^{-1})(1/4) \left\{(1/4)[1+(4/3)\xi_1+(1/2)(\xi_1^2+\xi_2^2)]k_\omega^2 + (1/8)[1+2\xi_2(k_\omega L)^{-1}+(3/2)(\xi_1^2+\xi_2^2)(k_\omega L)^{-2}]L^{-2}\right\}, \quad (2\text{-}99.3)$$

which reduces to Eq. (98) when $k_\omega R_o \to 0$. The ratio Q_-/Q_{o-} may be formed as in the second form of (98) by writing $2I_o(\sim R_e^{-1}) \simeq HR_o^2 Q_{o-}$. It is more useful, however, to consider Q_-/Q'_{o-}, where Q'_{o-} denotes the result for a flush element, not in a rigid baffle as Q_{o-}, but in a transparent baffle similar to the inner surface assumed in deriving Q_- in (99.3); for this purpose, one would need to solve the acoustic problem yielding Q'_{o-}.** For $R_o \gtrsim R_e/2$, the value of $Q_-(\omega,-L)$ complementing Eq. (99.3) is also not so trivially obtainable.

2.6.1.3 Summary Discussion

Consideration of (91) for Q_+^- indicates that the effect of a fluid sheath relative to flush mounting, so far as the excitation from high wave numbers ($k > \omega/U_\infty$) is concerned, is somewhat of the nature of averaging the fluctuating pressure over the entire exposed area of the sheath, instead of over the area of the transducer itself.

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* Similarly, if the inner surface instead approaches pressure release, for wave numbers $k \lesssim R_o^{-1}$ the factor $|\Gamma(k,\omega,L)|^2$ in Eq. (93) is replaced by

$$\left|[\xi_o(k_\omega^2-k^2)^{1/2}/k_\omega]\exp[i(k_\omega^2-k^2)^{1/2}L]\right|^2$$

$[(k^2-k_\omega^2)^{1/2} \to i(k^2-k_\omega^2)^{1/2}$ when $k > k_\omega]$, where $\xi_o(\pi R_o^2 \rho c)$ denotes the radiation impedance of the element in a pressure-release surface. A result analogous to (99.3) can again be derived.

** For wave numbers $k \lesssim R_o^{-1}$ only, the contribution to Q'_{o-} is modified relative to Q_{o-} by the factor (2-99.2) but with the sign of ξ_F reversed.

There is the additional component Q_+^∞ in the transmitted pressure that is averaged only over the transducer area, but provided $\omega L/U_\infty >> 1$, it will be smaller than Q_+^- unless a is very large and at any rate very small relative to Q_{o+}. [see Eq. (89)]. On the other hand, the effect of the sheath so far as the excitation from low wave numbers ($k \lesssim mR_o^{-1}$) is concerned is not substantial unless the sheath thickness is roughly at least half the radius of the area (transducer) in question and this radius is roughly at most a quarter of the sound wave length.

2.6.1.4 Spectra of Pressure Averaged Over the Active Area of Arrays of Shielded and Flush Elements

In practical application, not only the average (noise) pressure on an individual area beneath the dome or sheath is of concern, but still more the average pressure on an array of such areas. Thus the effect of the sheath on the correlation of the pressure between areas (elements) must also be considered. Therefore, imagine an array of elements of radius r_o with center spacing $D(> 2r_o)$ and suppose $r_o << R_e$ [see Eq. (99)], corresponding to the regime where the sheath appreciably reduces $Q_-(\omega,-L)$. The effective radius R_e was seen to correspond to the largest area over which the average-pressure spectrum is roughly independent of averaging area; for much larger areas the pressure may be regarded as roughly constant over each partial area $\sim \pi R_e^2$, and the spectra of force on these areas added powerwise to obtain the spectrum of total force, and thence average pressure, on the whole area in question. Explicitly, the average-pressure spectrum on a large interior area is $2\pi A^{-1} I_o(0)$ by Eq. (93)*, since $|\Gamma(0,-L)|^2 = 1$; but, by definition of the correlation area A_e, this spectrum is given by $(A_e/A)Q_-$, where Q_- is the pressure spectrum at an interior point; the latter by (98), is given by $Q_- \simeq 2R_e^{-2} I_o(\sim R_e^{-1})$; hence, on assumption that $I_o(\sim R_e^{-1}) \simeq I_o(0)$, we have $A_e \simeq \pi R_e^2$, where R_e is still defined by (99). (If we consider an area at $y = -L$ in an infinite inner medium instead of at a rigid inner surface, the spectrum of average pressure on a large interior area is smaller by a factor 4, since $|\Gamma(0,-L)|^2 = 1/4$, but Q_- is smaller by the same factor, and the correlation area naturally is still $A_e = \pi R_e^2$.)

*For a large area πR^2, if I_o is constant for most $k < mR^{-1}$ but $I_o(\sim R^{-1}) \neq I_o(0)$, the argument k=0 should be replaced by $I_o(\sim R^{-1})$.

Consider first, for comparison, an array of N flush elements of radius R_o, where $\omega R_o/U_\infty >> \pi$ so that $Q_{o-} \propto R_o^{-2}$. We represent the active area of the array as $N(\pi R_o^2)=A$. The spectrum of pressure averaged over the total active area, say Q_{o-}^A, is given for center spacing $D_o \gtrsim \delta_*$, by

$$Q_{o-}^A(\omega) \simeq Q_{o-}(\omega)/N \simeq 2\pi A^{-1} I_o(\sim R_o^{-1}) \tag{2-100}$$

To the extent that $I_o(\sim R_o^{-1})$ is independent of R_o, Q_o^A thus does not depend explicitly on the size of the elements (where $R_o >> \pi U_\infty/\omega$) but varies inversely as the total active area A.

Now consider an array of N' shielded elements of radius r_o and center spacing D such that the active area $A = N'(\pi r_o^2)$ is the same. We define $A_o = \pi r_o^2$, $A_e = \pi R_e^2$. Two opposite limiting conditions are distinguished: (a) $D^2 >> A_e$ (loosely packed array); (b) $D^2 << A_e$ (tightly packed array). In the case of loose packing contributions of the elements to the spectrum of total force on the array are nearly statistically independent. The spectrum of pressure averaged over the total active area, say Q_-^A, then becomes

$$Q_-^A(\omega,-L) \simeq Q_-(\omega,-L)/N' \tag{2-101}$$

$$\simeq \begin{cases} 2\pi A^{-1} I_o(\sim R_e^{-1})(r_o^2/R_e^2) \simeq H(r_o^2/R_e^2) Q_{o-}^A(\omega) & \text{for } r_o << R_e \\ 2\pi A^{-1} I_o(\sim r_o^{-1}) \simeq Q_{o-}^A(\omega) & \text{for } r_o >> R_e \end{cases}$$

$$(D^2 >> \pi R_e^2) \tag{2-101.1}$$

from (98) and (100). In the case of tight packing, the spectrum of pressure averaged over the total active area differs little from that averaged over the total (active plus dead) area $A_T \doteq A(D^2/A_o)$. Hence,

$$Q_-^A(\omega,-L) \simeq 2\pi A_T^{-1} I_o(\sim A_T^{-1/2}) \simeq 2\pi A^{-1} I_o(\sim A_T^{-1/2})(\pi r_o^2/D^2)$$

$$\simeq H_T(\pi r_o^2/D^2) Q_{o-}^A(\omega) \tag{2-102}$$

$$(D^2 << \pi R_e^2 << A_T)$$

by reference to Eq. (93) for a circular area $A_T \equiv \pi R_T^2$ (or its equivalent for another shape), where*,**

$$H_T = I_o(\sim A_T^{-1/2})/I_o(\sim R_o^{-1}). \qquad (2\text{-}102.1)$$

We recall, however, that (93) constitutes an adequate approximation to the corresponding sum over modes of (38a), and the latter in turn an adequate approximation only where a range of excitation wave numbers k that is equal to many modal spacings (i.e., $\gg \pi/a$) is in question and where $[2J_1(kR_o)/kR_o]^2$, in particular, varies only moderately over an interval π/a in k. This condition applied to (102) requires that the array extent be small relative to the dome size, $R_T \ll a$. In application, we will have $R_T \sim a$; we may hope the results will be still roughly applicable if $R_T \lesssim a$.

If we consider an area at y = -L within an infinite inner medium instead of at a rigid inner surface, $Q_-(\omega,-L)$ and likewise $Q_-^A(\omega,-L)$, given previously by Eqs. (101)-(102), are equal to 1/4 of their previous values.

We also want to consider arrays composed of adjoining independent domes housing arrays of the type considered above.*** In this case we may reasonably assume the spectra of noise pressure on elements in separate dome sections are uncorrelated. Eqs (100)-(102) then remain valid for spectra of pressure averaged over the active area of the elements in all the sections taken together, provided A in (100) is now interpreted

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* If $H_T \lesssim 1$, there can be a reduction in average pressure due to a tight packing effect in the case of the flush array also, requiring Eq. (100) to be modified, as discussed further on.

** In the further regime where $\pi R_e^2 \gtrsim A_T$ we have $Q_-^A(\omega,-L) \simeq Q_-(\omega,-L) \simeq 2R_e^{-2} I_o(\sim R_e^{-1}) \simeq (A/\pi R_e^2) Q_{o-}^A(\omega)$. Even for $\omega = 0$ the former condition is satisfied only if $L \gtrsim R_T/2.8$.

*** In such a configuration the present circular geometry assumed for each section is clearly inappropriate, but the basic relations and conclusions remain valid.

as referring to this grand total active area (but with A_T still referring to a single section).

The probability that $I_o(k)$ is much smaller near k=0 than $I_o(\sim R_e^{-1})$ is sufficient to merit further attention. Suppose, therefore, $I_o(k)$ in the region of low wave numbers may be regarded crudely as given by

$$I_o(k) = \begin{cases} I' \text{ for } k < k_c \\ I'' \text{ for } k > k_c \end{cases} \qquad (2\text{-}103)$$

where I' and I" are constants with I" > I'. The pertinent product $I_o(k) \left|\Gamma(k,-L)\right|^2$ of Eq. (93) then behaves similarly but cuts off rather sharply where $k \sim R_e^{-1}$ (see Figure 2-5).

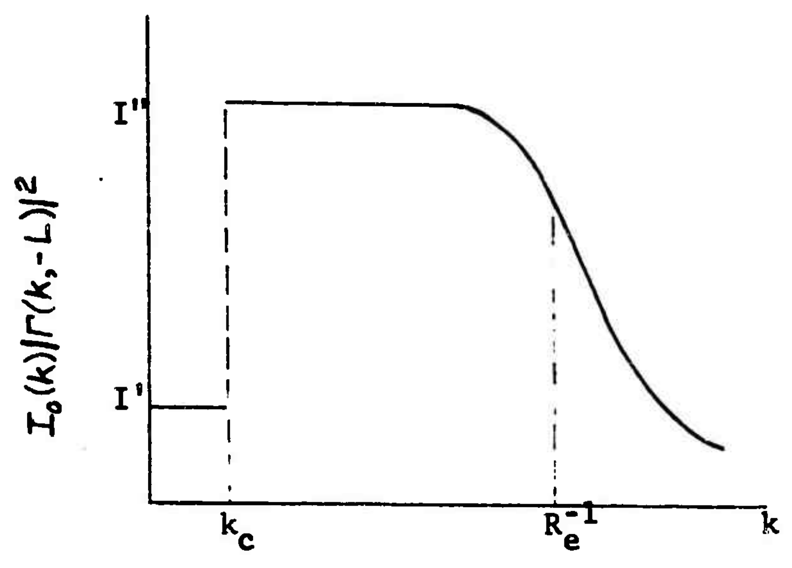

FIGURE 2-5. EXCITATION-RESPONSE PRODUCT $I_o(k) \left|\Gamma(k,-L)\right|^2$ FOR A HYPOTHETICAL STEP DEPENDENCE OF I_o, AND Γ FOR A FLUID DOME.

$I_o(k)$ as given by (103) may be regarded as composed of a sum $I_o = I_1 + I_2$ of a spectrum $I_2(k) = I''$ for all k and a (negative) spectrum $I_1(k) = -(I''-I')$ for $k < k_c$, $I_1(k)=0$ for $k > k_c$. Two pertinent correlation scales, R_e and k_c^{-1} ($\geq k_\omega^{-1}$) now enter. We must then distinguish loose or tight packing relative to the scale R_e from that relative to the scale k_c^{-1}.

We find in the several combinations of interest, assuming $A_T \gtrsim k_c^{-2}$:

1) Loosely packed relative to both scales, element small relative to both scales; $D \gg k_c^{-1}$, $r_o \ll R_e$:

$$Q_-^A(\omega,-L) \simeq 2\pi A^{-1} r_o^2 [\left\{ R_e^{-2} - (1/4)k_c^2 \right\} I'' + (1/4) k_c^2 I'] \qquad (2\text{-}103.1)$$

2) Loosely packed relative to small scale, tightly relative to large, element small relative to both; $k_c^{-1} \gg D \gg R_e$, $r_o \ll R_e$:

$$Q_-^A(\omega,-L) \simeq 2\pi A^{-1} r_o^2 [R_e^{-2} I'' - \pi D^{-2}(I''-I')] \qquad (2\text{-}103.2)$$

3) Tigh..y packed relative to both scales: $D \ll R_e$:

$$Q_-^A(\omega,-L) \simeq 2\pi A^{-1}(\pi r_o^2/D^2) I' \qquad (2\text{-}103.3)$$

4) Loosely packed relative to both scales, element large relative to small scale only; $D \gg k_c^{-1}$, $k_c^{-1} \gg r_o \gg R_e$:

$$Q_-^A(\omega,-L) \simeq 2\pi A^{-1}[I'' - (1/4)(k_c r_o)^2 (I''-I')] \qquad (2\text{-}103.4)$$

5) Loosely packed relative to small scale, tightly relative to large, element large relative to small scale only; $k_c^{-1} \gg D \gg R_e$, $k_c^{-1} \gg r_o \gg R_e$:

$$Q_-^A(\omega,-L) \simeq 2\pi A^{-1}[I'' - (\pi r_o^2/D^2)(I''-I')]. \qquad (2\text{-}103.5)$$

If the constancy of $I_o(k)$ for $k > k_c$ is regarded as somewhat relaxed, I" in cases 1) and 2) are to be regarded as $I_o(\sim R_e^{-1})$ and in cases 4) and 5) as $I_o(\sim r_o^{-1})$.

From cases 4) and 5) the similarly generalized results for a flush array may be inferred by letting $R_e \to 0$. With r_o replaced by R_o and with $I'' = I_o(\sim R_o^{-1})$, these become

4') $D \gg k_c^{-1}$, $R_o \ll k_c^{-1}$:

$$Q^A_{o-}(\omega) \simeq 2\pi A^{-1}[I_o(\sim R_o^{-1}) - (1/4)(k_c R_o)^2 \left\{I_o(\sim R_o^{-1}) - I'\right\}] \quad (2\text{-}103.6)$$

5') $D \ll k_c^{-1}$, $R_o \ll k_c^{-1}$:

$$Q^A_{o-}(\omega) \simeq 2\pi A^{-1}[I_o(\sim R_o^{-1}) - (\pi R_o^2/D^2) \left\{I_o(\sim R_o^{-1}) - I'\right\}] \quad (2\text{-}103.7)$$

Once more, if we have a number of independent dome sections, Eqs. (103.1)-(103.7) still apply to the spectra of pressure averaged over the total active area, with A interpreted as the new total area.

The formulas above remain roughly correct when the stated extreme inequality conditions are satisfied only roughly as inequalities. Eq. (103.6) and (103.7) generalize (100), Eqs. (103.1) and (103.2) generalize (101), (103.5) generalizes (101.1), and (103.3) generalizes (102). In order of magnitude, however, (100)-(102) remain correct provided the factors H and H_T are retained in general form.* We recall, in the case of several dome sections, that $I_o(\sim A_T^{-1/2})$ in H_T refers to an area A_T of the order of the area of a single section.

To compare dome-shielded and flush arrays, it suffices to consider a single dome section. Let the center spacing of the flush array now be D_o and its active area be A_o, which may differ from the spacing D and active area A of the shielded array.

*We note, however, the substantial reduction possible, according to (103.7) or (103.5), by tight packing, if $I_o(k)$ should have the dip hypothesized.

Suppose the total area A_T of the flush and shielded arrays is fixed independently of flow-noise considerations, and so also the maximum array factor $\pi R_o^2/D_o^2$ of the flush array and the minimum element radius r_o and minimum spacing D in the shielded array. We then have $A_T = A_o(D_o^2/\pi r_o^2) = A(D/\pi r_o^2)$. In such case, since the active areas of the flush and shielded arrays are not assumed equal, in place of Eqs. (100)-(102), we rewrite the spectra of pressure averaged over total active area for the flush and shielded arrays, for fixed total area A_T, as

$$Q_{o-}^{A_o}(\omega) \simeq 2\pi(A_o/A_T)^{-1}A_T^{-1}I_o(\sim R_o^{-1}), \qquad \text{(2-103.7a)}$$

$$Q_-^A(\omega,-L) \simeq \begin{cases} \left.\begin{array}{ll} H(D/D_o)^2(R_o^2/R_e^2)Q_{o-}^{A_o} & \\ \quad = H(A_o/A)(r_o^2/R_e^2)Q_{o-}^{A_o}, & (r_o \ll R_e) \\ (D/D_o)^2(R_o/r_o)^2 Q_{o-}^{A_o} & \\ \quad = (A_o/A)Q_{o-}^{A_o}, & (r_o \gg R_e) \end{array}\right\} & (D^2 \gg \pi R_e^2) \\ H_T(\pi R_o^2/D_o^2)Q_{o-}^{A_o} = H_T(A_o/A_T)Q_{o-}^{A_o}. & (D^2 \ll \pi R_e^2 \ll A_T) \end{cases}$$

(2-103.7b)

Suppose also that, within the given restrictions, the incident signal pressure, in active as well as passive operation, is nearly independent of element radius or spacing. By this supposition, the signal-to-noise ratio varies inversely as the spectrum of noise pressure averaged over the active area of the array.* According to (103.7a), so far as determinined by low

*Thus, in the regime considered, the power transmitted by a cavitation-limited active array is recognized as roughly independent of active area, in consequence of the effect of mutual coupling.

excitation wave numbers, for a flush array the signal-to-noise ratio is maximized by maximizing the array factor A_o/A_T. According to (103.7b), for a shielded array the ratio is maximized for fixed r_o, so far as the higher frequency range where $\pi R_e^2 \lesssim D^2$ is concerned, by minimizing D. Figure 2-6 shows the factor by which the noise ($\hat{Q}$) is reduced relative to that for a flush array of the same total area ($\hat{Q}_o$) (with element radius $R_o \gg \pi U_\infty/\omega$) as a function of R_e^{-2}, i.e., of frequency on a certain nonlinear scale. It is assumed that $H_T \sim H \sim 1$. Dashed lines show the same factor for a larger D, and again for a smaller r_o. Figure 2-7 shows the ratio in question where it is assumed instead that $H_T \lesssim H$ when $R_e \lesssim A_T^{1/2}$ and H thus increases with R_e^{-2}.

Returning to the assumption that $H_T \sim H \sim 1$, we see that the reduction factor due to the fluid dome can be made only as low as the plateau value A_o/A_T in Figure 2-6, i.e., equal to the array factor for the flush comparison array. Thus if a 100% array factor were achievable in a flush array, the low-wavenumber noise would not be reduced at all by use of a fluid sheath.

If $\omega_m (=2\pi c/\lambda_m)$ is the highest frequency of concern, the same (maximum) noise reduction is attained for any $D \lesssim \lambda_m/\pi^{1/2}$, provided the sheath thickness is taken to make $R_e(\omega=\omega_m) \gtrsim D/\pi^{1/2}$; this condition yields

$$L \gtrsim (1/5)D(1-\pi D^2/\lambda_m^2)^{-1/2}. \qquad \text{(2-103.7c)}$$

The requisite thickness is thus minimized by minimizing D. (At fixed D the noise reduction is nearly independent of r_o at all frequencies such that $R_e \gtrsim r_o$.)

The minimum spacing D is constrained by the economic limitation on the number of elements to be employed. In an active array it is further constrained by its relation to the minimum element radius at which the elements radiate power with adequate efficiency.

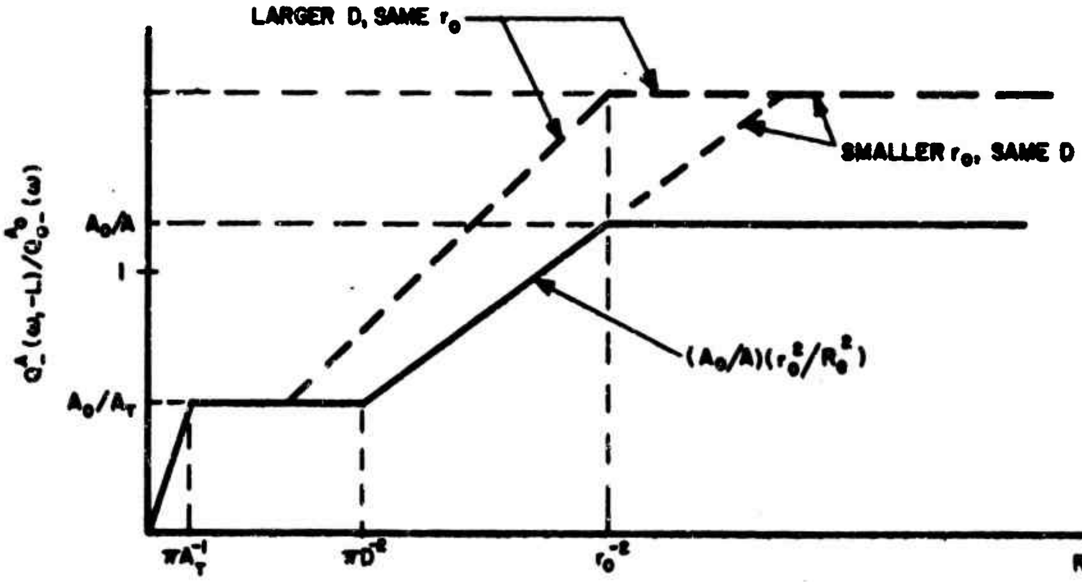

Figure 2-6. Ratio of spectrum of low-wavenumber flow-noise pressure averaged over active area for a shielded array to that for a flush array as a function of $[R_e(\lambda,L)]^{-2}$. [See Eq. (2-99).] The flush and shielded arrays have same total area A_T and active areas A_0 and A, respectively. Three curves (drawn with unrealistic square corners for simplicity) are shown, corresponding to two different radii r_0 and center spacing D for the shielded elements. I_0 ($\sim$k) is assumed roughly constant for the pertinent low-wavenumber ranges.

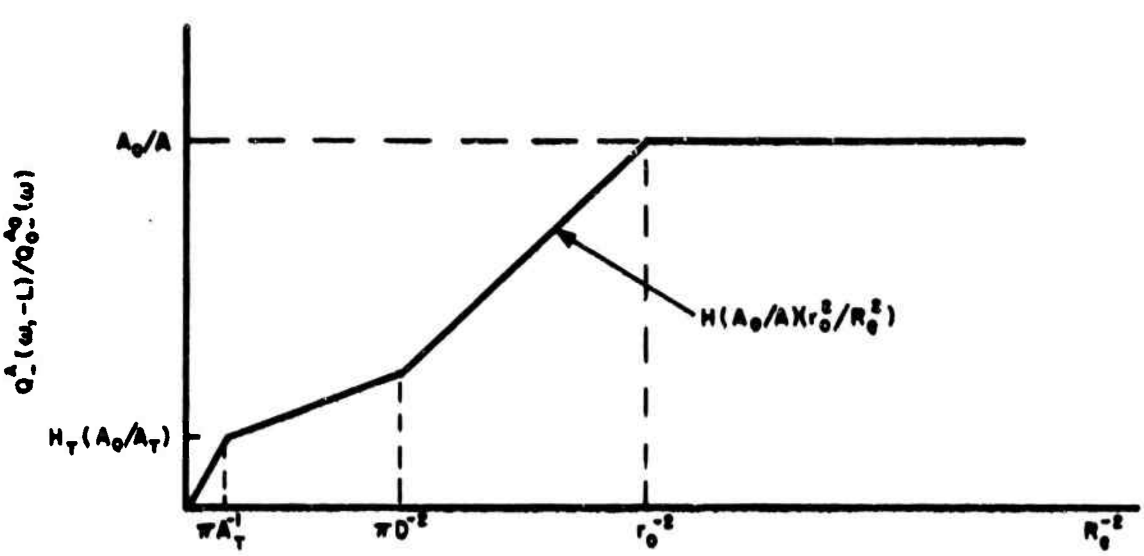

Figure 2-7. Reduction factor defined as in Figure 2-6. It is now assumed that the average I_0($\sim$k) (and hence H) decreases in some way as the effective upper limit on the wavenumber range in question decreases, so that $H_T \ll 1$.

Assuming, for example, a flush comparison array with array factor $\leq 25\%$, we may similarly compute the maximum frequency ω_c, say, below which the reduction factor for the shielded array corresponds to an improvement of signal-to-noise ratio by at least 6 db.* The result is shown as a function of L for two values of r_o in Figure 2-8. The value of ω_c approached asymptotically at large L is c/r_o; ω_c equals 0.86 of that value when $L = 1.4r_o$ or 0.71 of it when $L = r_o$. If c has the value for water, for example, at $L = r_o$ we have $\omega_c/2\pi = 13kc/2r_o$ with the diameter $2r_o$ expressed in inches.

Still assuming $H_T \sim H \sim 1$, we compare a specific typical array when shielded with the same array when flush mounted. Suppose $D = 7.75$ in., $r_o = 2.5$ in., $\omega/2\pi = 2.4$kc ($\lambda = 24$ in.). Then the array factor is $A/A_T = 0.327$. For any $L > 1.9$ in., we have $\pi R_e^2 \geq D^2$, whence $Q_-^A \simeq 0.327\ Q_{o-}^A$, i.e., the low-wavenumber noise is roughly 4.8 db below that for the same array if flush mounted. Now compare a shielded array with the same A_T but a 100% array factor ($A/A_T = 1$). For any L (including L=0 if such an array without dome were structurally possible) this array averages the noise over A_T and thus yields the same noise level as did the 32.7% array for $L > 1.9$ in.**

We consider crudely the array aspect of the noise contribution from high excitation wave numbers. As we saw, by taking $L \gg U_\infty/\omega$ (a modest requirement for most frequencies of interest) we can substantially obliterate the direct convective part Q_+^∞ (Eq. (88)), leaving the propagation overlap part Q_+^P. In a limited regime, roughly for very large $\omega a/c$, we saw also that the interior pressure spectrum is then conveyed mainly by wave numbers $k_n \simeq \omega/c$ [Eq. (87)].

We regard first an array of N flush elements of radius R_o, where $\omega R_o/U_\infty \gg \pi$, with cross-stream spacing $\gtrsim \delta_*$, yielding an active area $A = N(\pi R_o^2)$. For the high-wavenumber contribution to the spectrum of pressure averaged over the active area we have roughly

*This reduction refers to the case of a rigid inner surface at y = -L. If L refers instead to depth in an effectively infinitely deep inner medium, as noted before, there is an additional ~6-db reduction relative to flush mounting in a rigid baffle.

*This result neglects the effect of any difference in the impedance properties of the inner surface for the 100% and 32.7% arrays.

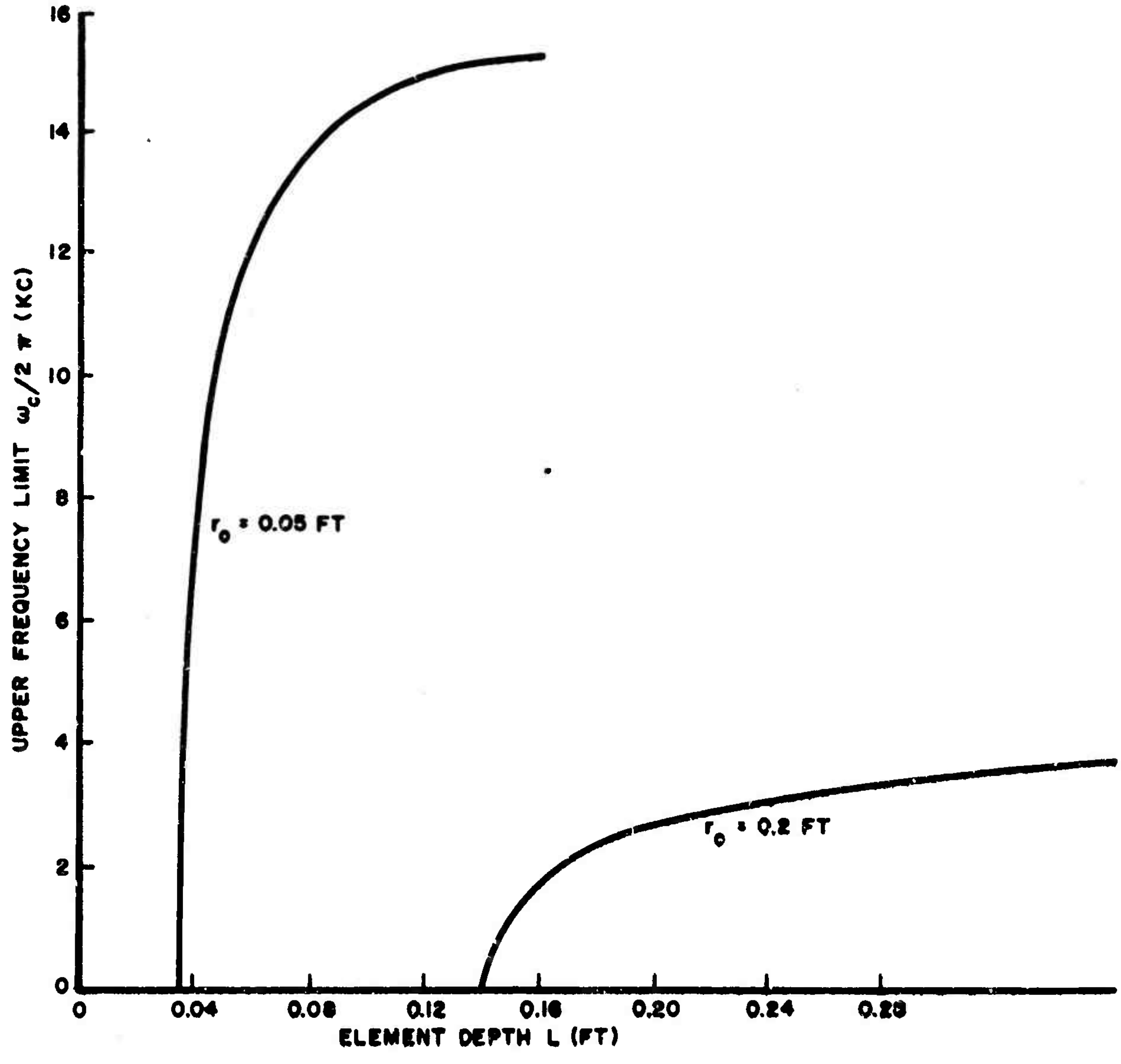

Figure 2-8. Upper frequency limit for 6-db reduction by a fluid dome of low-wavenumber flow-noise pressure averaged over active area of an array. Element radius r_0, dome thickness L, total active area fixed and less than 1/4 of array area, inner dome surface rigid.

$$Q^A_{o+}(\omega) \simeq Q_{o+}(\omega)/N \simeq (\pi R_o^2)^{1/2} A^{-1} s(\omega), \qquad (2\text{-}103.8)$$

where $s(\omega)$ is given from Eq. (65b) as

$$s(\omega) = (\pi R_o^2)^{3/2} Q_{o+}(\omega) \simeq 4\pi^{1/2} \int_{\omega/U_\infty}^{\infty} dk\, k^{-2} I_o(k,\omega) \qquad (2\text{-}103.9)$$

[cf. (100)]. Regarding now the shielded array, we again distinguish loose and tight packing. Considering the regime where Eqs. (87) and (91) are indicative, the packing criterion pertains to the relative magnitudes of the spacing D and k_ω^{-1} ($= c/\omega = \lambda/2\pi$). With N' loosely packed elements of radius r_o, i.e., for $D \gg k_\omega^{-1}$, with $r_o < k_\omega^{-1}$, we have a high wavenumber contribution to the spectrum of pressure averaged over the active area given by

$$Q^A_+(\omega,-L) \simeq Q^P_+(\omega,-L)/N' = \pi r_o^2 A^{-1} Q^P_+. \qquad (2\text{-}103.10)$$
$$(D \gg k_\omega^{-1})$$

Using the order-of-magnitude form

$$Q^P_+(\omega,-L) \sim (R_o/a)^3 Q_{o+}(\omega) \qquad (2\text{-}103.11)$$

based on Eq. (91) together with (103.8), we then obtain the relation of $Q^A_+(\omega,-L)$ to $Q^A_{o+}(\omega)$ for a flush array of the same active area

$$Q^A_+(\omega,-L) \sim \pi r_o^2 (\pi a^2)^{-3/2} A^{-1} s(\omega)$$
$$\sim (r_o^2 R_o/a^3) Q^A_{o+}(\omega). \qquad (2\text{-}103.12)$$

With tight packing, $D \ll k_\omega^{-1}$, Q^A_+ will be still smaller provided the total area $A_T \gg k_\omega^{-1}$, since the interior pressure spectrum falls off for $k < k_\omega$, resulting in cancellation [cf. (102) or (103.3)]. We consider crudely the average pressure for an area A_T by replacing $k_\omega R_o$ by $\pi^{-1/2} k_\omega A_T^{1/2}$ as the argument of the area-averaging factor $[2J_1(z)/z]^2$ in Eq. (87), and consider the resulting factor as $\sim (k_2 A_T^{1/2})^{-3}$. We may then suppose, in order of magnitude, that

$$Q_+^A(\omega,-L) \sim (k_\omega A_T^{1/2})^{-3} Q_+^P(\omega,-L)$$

$$\sim (k_\omega a)^{-3} A_T^{-3/2} s(\omega) \sim (k_\omega a)^{-3} (\pi r_o^2/D^2)^{3/2} A^{-3/2} s(\omega)$$

$$\sim (\omega a/c)^{-3} (A/A_T)(\pi R_o^2/A_T)^{1/2} Q_{o+}^A(\omega) \qquad (2\text{-}103.13)$$

$$(D << k_\omega^{-1})$$

by use of (103.8) [cf.(103.12)]. ($\omega a/c >> 1$ where this result has validity.)

The crude estimates (103.12) and (103.13) are based on results [e.g. Eq. (87)] valid only if the averaging area is small and far from the dome periphery; hence (103.12) and especially (103.13) require $A_T << \pi a^2$, i.e., the total array area in each dome section must be small relative to area of the section. This condition is not well satisfied in cases of interest. Furthermore, the results depend sensitively on the lateral boundary conditions. Nevertheless, it is clear that $Q_+^A(\omega,-L)$ for a shielded array can be made much smaller than $Q_{o+}^A(\omega)$ for a flush array of the same active area by taking the sheath area (πa^2) large relative to the area of individual elements and somewhat larger than the sound wave lengths of concern.

We may now consider the probable noise reduction achievable by use of a fluid sheath of small thickness. The conclusions may be presumed applicable also to an actual elastic-solid sheath, provided the transverse sound velocity in the latter greatly exceeds U_∞, meeting the order-of-magnitude assumptions made here regarding c. If the contribution Q_- from low excitation wave numbers in fact predominates, as indicated by the observed area dependence on large flush transducers, present analysis shows that a fluid dome permits significant noise reduction only if $L \gtrsim r_o$, where r_o is the radius of the shielded elements employed (Figure 2.8).* If instead the

* If, however, the wave-number spectrum of excitation contains a spike at $k \simeq \omega/c$, as speculated before, this contribution will not be reduced by the sheath independently of L.

convective contriution from high excitation wave numbers predominated in the average-pressure spectrum on large flush elements, significant noise reduction could be achieved, provided only that $L \gg U_\infty/\omega$ and that the lateral size of the sheath (or each section thereof) is large compared to individual element size.

2.6.2 Covered Dome

When the dome has a cover (plate or membrane), a more or less sharp resonance in $\Gamma(k_n, y)$ occurs if $k_n \simeq k_r$. This resonance may conceivably be sharp enough that, even if $I_o(k)$ is relatively small for $k < k_+$, nevertheless $Q_-(y)$ in (26) contains a significant contribution from $k \simeq k_r$; in the approximation of (38a), the terms with $k_r - \delta k < k_n < k_r + \delta k$, if no others, may then need to be retained. (As we have seen, however, $Q_-(y)$ is indicated by the observed area dependence of noise on flush-mounted transducers to be the dominant contribution even in the absence of resonances.) Likewise the k_n in this interval make a pronounced contribution to the sum in $Q_+^-(y)$ in (38c); thus, in approximation (41), Σ_r, which was set equal to zero for the fluid dome, now contributes.* The regions of the (k,k_n)-plane embraced in this approximation, including the non-resonant contribution to $Q_-(y)$, are shown in Figure 2-9 (cf. Figure 2-2). The relevant regions are crosshatched.

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* Consistently with Eqs. (37)-(38) and (41), representing Eq. (23) by $T_n(k,y) = t_n(k)\Gamma(k_n,y)$ (and similarly for S_n), we may neglect $t_n - t_{n-1}$ in Σ_r and write for use in (41) and (38c)

$$\Sigma_r (-)^n T_n \simeq \Sigma_r' \, t_n[\Gamma(k_n,y) - \Gamma(k_{n+1},y)]$$

where the prime on the sum on n on the right denotes restriction to even n only.

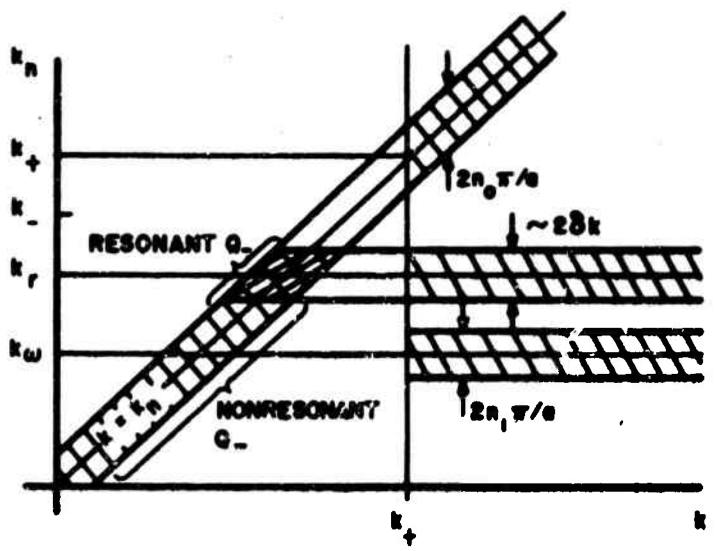

Figure 2-9. Regions of (k,k_n) Plane Retained in a Specified Approximation for Large $\omega a/c$.

2.6.2.1 Contribution from High Wave-Numbers

Referring to $Q_+(y)$ only, we may again define a spectrum modification factor $\Lambda(k,y)$ by Eq. (80), and thence the local average $\bar{\Lambda}(k,y)$, whence Eqs. (81)-(84) again apply, provided the requisite conditions of Section 2.4.3 for Eqs. (38b)-(38c) are satisfied. If the conditions for (43) are also satisfied, a suitably modified form of Eq. (87) applies with a contribution from Σ_r of (41) included. (The crosshatched strip centered at $k_n = k_\omega$ in Figure 2-9 then contracts to vanishing width.)

Further and alternative approximations also merit mention. If the resonance is sufficiently broad (relative to π/a) and damped that, in addition to (56) the appropriate one of the following conditions is satisfied:

$$k_r^{1/2}(\delta k)^{-2}a^{-3/2}k_\omega^{-1}\hat{g}_r[(\partial\hat{g}/\partial k_2)_o]^{-1} \ll 1 \qquad \text{(rigid)}$$

(2-104)

$$k_r^{3/2}(\delta k)^{-2}a^{-3/2}k_\omega^{-2}\hat{g}_r[(\partial\hat{g}/\partial k_2)_o]^{-1} \ll 1, \qquad \text{(free)}$$

where $\hat{g}_r = \Gamma(k_r,y)$ and it is assumed that $k \gtrsim 2k_r$, then in (41) the partial sum Σ_r, as estimated by use of Eq. (17), Appendix 2, is small compared to Σ_ω as given in Eqs. (43), and the resonance contribution in $Q_+^-(y)$ or $\Lambda_-(k,y)$ may be neglected.* In this extreme limit, which would apply for sufficiently large a, the resonance behavior of $g(k_n, k_{2n},y)$ [$\equiv\Gamma(k_n,y)$] is entirely irrelevant to the part of the force spectrum deriving from $k > k_+$, in consequence of the alternation of signs in Σ_r and the assumed close spacing of the k_n such that contributions from neighboring n cancel one another except where $k_n \simeq k_\omega$.

Similarly, if the resonance is sufficiently broad that

$$\delta ka/\pi \gg 1 \qquad (2\text{-}105)$$

* We recall that $\hat{g}_r$ may be considered roughly proportional to $(\delta k)^{-1}$ [see Eq. (1-38)].

and, in addition, roughly

$$k_r a/\pi >>1, \ k_r L/\left|k_r^2-k_\omega^2\right|^{1/2} a << 1 \qquad (2\text{-}106)$$

(cf. 50)), the contribution of (33), if assumed to derive appreciably only from k_n and k both near k_r as suggested in the paragraph following Eqs. (43), may be approximated alternatively to (34) by the corresponding form for an infinite dome:

$$Q_-(\omega,-L) \simeq \int_{k_r-\delta k}^{k_r+\delta k} dkk[2J_1(kR_o)/kR_o]^2 I_o(k)\left|\Gamma(k,y)\right|^2 \qquad (2\text{-}107)$$

When on the contrary, the spacing of the k_n is rather broader than the resonance peak in $\Gamma(k,y)$, i.e. $\delta ka \lesssim \pi$, it may suffice to include only one pair of terms, or even one term, in Σ_r in (41) and in (38a). When $\delta ka << \pi$ (but the other assumed conditions for (37)-(38) still apply), no terms at all may contribute appreciably, except that as the frequency ω and hence the resonant wave number $k_r(\omega)$ is varied, a succession of single terms n will contribute over the successive narrow frequency intervals where their respective k_n's satisfy roughly

$$k_r(\omega) - \delta k < k_n < k_r(\omega) + \delta k. \qquad (2\text{-}108)$$

Let us consider the resonance properties in the vicinity of the frequency, ω_ν say, at which the ν-th mode, of wave number k_ν, is in resonance $\left[k_\nu \simeq (\pi/a)(\nu + s)\right.$, where, in the present model, $s = 1/4$ for a rigid wall and $s = 3/4$ for a free wall$\left.\right]$. Thus we have $\omega_\nu = \omega_r(k_\nu)$, with ω_r given by Eq. (1-56) or equivalently $k_r(\omega_\nu) = k_\nu$, with k_r given by Eq. (1-38.1).* We denote also other functions of k

* If we had $k_r(\omega) \lesssim s\pi/a$ up to the highest frequency of interest, there could be no mode ν in resonance in this frequency range. Since, however, $k_r(\omega) \gtrsim k_\omega$ $(=\omega/c)$, independently of k_o, i.e., no matter how stiff the isolated plate, this condition would imply $\omega a/c \lesssim s\pi$, i.e., a lateral dome radius a small compared to a sound wave length in the fluid.

evaluated at $k = k_\nu$ by a subscript ν. Assuming that $k_r \sim k_o$ in accord with Eq. (1-52), by Eq. (1-4.1) the frequency spacing $\Delta\omega_\nu$ of resonances in the vicinity of the ν-th resonance is given crudely in terms of the wave-number spacing $\Delta k_r \simeq \pi/a$ by

$$\Delta\omega_\nu/\omega_\nu \sim \mu\pi/k_\nu a \simeq \mu/(\nu+s) \qquad (2\text{-}108.1)$$

where $\mu = 2$ for a plate and $\mu = 1$ for a membrane, I.e., the logarithmic spacing is roughly inversely proportional to mode number ν. As for the absolute spacings, we have

$$\Delta\omega_\nu \sim \begin{cases} 2a_o(\pi/a)^2(\nu + s) & \text{(plate)} \\ c_o\pi/a & \text{(membrane)}. \end{cases}$$

To test whether resonance is broad enough in wave-number that more than one k_ν can be near resonance at the same ω, we must consider whether $\delta k_\nu a/\pi \gtrsim 1$, where δk_ν for a plate is given by Eq. (1-42) evaluated at $k_r \simeq k_\nu$ and $\omega \simeq \omega_\nu$, whence

$$\delta k_\nu a/\pi \sim (1/5)(\nu+s)[\zeta+(\beta/\omega_\nu)(1+2/k_\nu h)^{-1}]. \qquad (2\text{-}109)$$

At fixed damping, this dimensionless resonance breadth is thus roughly proportional to mode number.

In the vicinity of the resonance at $\omega = \omega_\nu$, we may write $\Gamma(k_\nu,\omega,y)$ for use in $Q_-(\omega,y)$ by Eq. (1-54) as

$$\Gamma(k_\nu,\omega,y) \simeq \frac{b_\nu}{(\omega/\omega_\nu)-1+i\epsilon_\nu} \qquad (2\text{-}110)$$

where by Eqs. (1-56), (1-58)

$$\omega_\nu \simeq \omega_{o\nu}(1+2/K_{2\nu}h)^{-1/2}, \quad [K_{2\nu} = \left\{k_\nu^2-(\omega_\nu/c)^2\right\}^{1/2}] \qquad (2\text{-}111)$$

$$b_\nu = B(k_\nu)/\omega_\nu \simeq (k_\nu h)^{-1}(1+2/k_\nu h)^{-1}\exp[-(K_{2\nu}L+\hat{\alpha}],$$
$$\times \operatorname{ch}[K_{2\nu}(y+L)+\hat{\alpha}], \tag{2-112}$$

$$\epsilon_\nu = \delta\omega(k_\nu)/\omega_\nu \simeq (1/2)[\zeta + (\beta/\omega_\nu)(1+2/k_\nu h)^{-1}], \tag{2-113}$$

and for a low-impedance inner surface b_ν must be modified as stated at (1-44). The peak value of Γ, namely $\Gamma(k_\nu,\omega_\nu,y) \simeq b_\nu/\epsilon_\nu$ contains a factor $(k_\nu h)^{-1} \ll (\nu+s)^{-1}$ that decreases with increasing mode number ν; in addition, however, the exponential factor decreases to the extent of an increment $\approx -\pi(L/a)(k_\nu/K_{2\nu})$in its argument per unit increase in ν.

To fix ideas, we consider again the example of a steel plate of thickness 0.25 in. (and unstiffened by resonators) with water loading both sides and assume a dome radius a = 0.5 meter. We then have

$$a_o = 8.5\ m^2/sec,\ h = 0.049\ m.$$

For example, we consider the properties of resonances with frequencies ω_ν in the neighborhood of 3 kc. We take $k_\nu \simeq (\nu + 1/4)\pi/a$ (appropriate for rigid walls but nearly correct in any case when $\nu >> 1$). From (112) we find by use of the approximation (1-57) that the resonances of modes $\nu = 8$ and $\nu = 9$ fall at

$$\omega_8/2\pi \simeq 2700\text{cps and } \omega_o/2\pi \simeq 3460 \text{ cps} \tag{2-114}$$

(For comparison, these modal frequencies for the unloaded plate would be ω_{o8} = 3630 cps and ω_{o9} = 4570 cps.) The corresponding resonant wave numbers k_ν are

$$k_8 = 51.8m^{-1} \text{ and } k_9 = 58.1m^{-1}.$$

At the frequencies ω_8 and ω_o, the wave numbers of sound in the water are

$$\omega_8/c = 11.3m^{-1} \text{ and } \omega_o/c = 14.5m^{-1}.$$

In Eqs. (112)-113) the quantity $k_\nu h$ for ν=9 is given by k_9h=2.85. With reference to the exponential attentuation of the resonant mode with depth as given by (110) and (112), for $\nu = 9$ we have $K_{2\nu} = 50.6m^{-1}$, whence the exponential factor in $|\Gamma|^2$ at $y = -L$ (with $\hat{\alpha} \simeq 0$) is only 0.01 for a depth as shallow as L = 4.6 cm. For a thinner or less stiff dome cover the attenuation of the modes resonant in the same frequency range would be still more rapid.

2.6.2.2 Estimated Spectrum (Excluding Low Non-resonant Wave-Numbers)

As an example for estimation, consider the regime where Eqs. (43) hold for Σ_ω, but $2\delta ka \lesssim \pi$, so that Σ_r and the resonant part of Q_-, asy Q_-^r, may be approximated by retention of the single n nearest resonance, say $n = \nu$ (ν thus changes with ω). Collecting previous equations, we then have

$$Q(y) \simeq Q_-(y) + \int_{k_+}^{\infty} dkkI_o(k) \tilde{\Lambda}(k,y), \tag{2-115}$$

$$Q_-(y) = Q_-^r(y) + Q_-^p(y), \tag{2-116}$$

$$Q_-^r(y) \simeq (\pi/\varepsilon)k_\nu[2J_1(k_\nu R_o)/k_\nu R_o]^2 I_o(k_\nu) \left|\Gamma(k_\nu,y)\right|^2, \tag{2-117}$$

$$\tilde{\Lambda}(k,y) \simeq \Lambda_\infty + \Lambda_-, \tag{2-118}$$

$$\Lambda_\infty = [2J_1(kR_o)/kR_o]^2 \left|\Gamma(k,y)\right|^2 \tag{2-119}$$

Rigid-wall case:

$$\Lambda_- \simeq 8R_o^{-2}a^{-2}k \left| (-)^\nu \frac{J_1(k_\nu R_o)}{k_\nu} \frac{k_\nu^{1/2}}{k^2-k_\nu^2} \Gamma(k_\nu,y) + \Sigma_\omega \right|^2, \tag{2-120}$$

$$\Sigma_\omega \simeq (2\pi)^{1/2} a^{-1/2} \frac{J_1(k_\omega R_o)}{k_\omega} \frac{k_\omega}{k^2-k_\omega^2} \left|\frac{\partial g}{\partial k_2}\right| (-)^{j+1} [M(b)-iM(1-b)]$$

(2-121)

Free-wall case:

$$\Lambda_- \simeq 8R_o^{-2} a^{-2} k^{-1} \left| -(-)^\nu \frac{J_1(k_\nu R_o)}{k_\nu} \frac{k_\nu^{3/2}}{k^2-k_\nu^2} \Gamma(k_\nu,y) + \Sigma_\omega \right|^2, \quad (2\text{-}122)$$

$$\Sigma_\omega \simeq (2\pi)^{1/2} a^{-1/2} \frac{J_1(k_\omega R_o)}{k_\omega} \frac{k_\omega^2}{k^2-k_\omega^2} \left|\frac{\partial g}{\partial k_2}\right| (-)^{j+1} [M(b)-iM(1-b)]. \quad (2\text{-}123)$$

Consideration of the non-resonant part, $Q_-^p(y)$, of $Q_-(y)$ will be deferred to the following subsection.

Assuming once more that $k_+ \simeq \omega/U_\infty$, we estimate crudely the various contributions to $Q(\omega,y)$. We denote the contribution in (115) due to Λ_∞ in (118) by $Q_+^\infty(\omega,y)$, calling it now the direct convective part; we denote the contribution in (115) due to the squared (pure) resonance term of Λ_- in (120) [or (122)] by $Q_+^r(\omega,y)$, calling it the resonance overlap part, and that due to the pure non-resonance term $\Sigma_\omega{}^2$ in (120) by $Q_+^p(\omega,y)$, calling it the propagating overlap part; finally, we now call the contribution $Q_-^r(\omega,y)$ in (116) the direct resonance part. We do not estimate explicitly the contribution due to the cross-term of Λ_- in (120), since at most its magnitude will lie between those of the two squared terms.

In connection with Eq. (119), we recall Eqs. (85), (86), and (65b). The estimate (89) once more applies to $Q_+^\infty(\omega,y)$, but $|\Gamma|^2$ is different from what it was for the fluid dome. For a rigid inner surface, assuming $\omega/U_\infty >> k_r$, we have from (1-59)

$$\left|\Gamma(\omega/\eta U_\infty,\omega,y)\right|^2 \simeq 4(\omega/\eta U_\infty k_o)^{-2n}(\omega h/\eta U_\infty)^{-2}$$

$$\times \exp(-2\omega L/\eta U_\infty)\, ch^2[\omega(y+L)/\eta U_\infty] \tag{2-124}$$

in which $n = 4$ for a plate and $n = 2$ for a membrane and the effective η will be slightly larger (near unity) than that in (90).

The other non-resonance contribution Q_+^p in the present approximation is again given by (91) in the case of rigid lateral walls and a rigid inner surface and, as before, by (91) with L^2 replaced by $(y + L)^2$ in the case of a pressure-release inner surface. In the case of free lateral walls a factor $(\eta U_\infty/c)^2$ must be adjoined. For a mixed impedance condition at the inner surface or an infinite inner medium, L^2 is replaced by a factor containing the dome impedance for $k = k_\omega$ [Eqs. (44), (46.1)]. The correctness of the approximation (91) in this instance requires, we recall, that condition (57) as well as conditions (52) be satisfied.

The direct resonance contribution, for $k_\nu R_o \lesssim \pi$, is estimated from (117) by insertion of Eq. (110) as

$$Q_-^r(\omega,y) \simeq (\pi/a)k_\nu I_o(k_\nu,\omega) \frac{b_\nu^2}{[(\omega/\omega_\nu)-1]^2+\epsilon_\nu^2}$$

$$\sim \pi \frac{b_\nu^2}{[(\omega/\omega_\nu)-1]^2+\epsilon_\nu^2} (R_o/a)k_\nu R_o Q_{o-}(\omega), \tag{2-125}$$

where the second form follows from Eq. (2-69) on assumption that $I_o(k_\nu,\omega) \sim I_o(\sim R_o^{-1},\omega)$. ($Q_-^r$ is independent of R_o in approximation (125) notwithstanding the explicit introduction of R_o in the second expression.) As noted before, the magnitude of Q_-^r depends on the excitation spectrum in the obscure range of wave numbers $k \ll \omega/U_\infty$.

The resonance overlap contribution, for $k_\nu R_o \lesssim \pi$ and $k_\nu \ll \omega/U_\infty$, is similarly estimated from (115), (120), and (110) in the case of rigid walls, as

$$Q_+^r(\omega,y) \sim \frac{\pi}{2} \frac{b_\nu^2}{[(\omega/\omega_\nu)-1]^2+\epsilon_\nu^2} k_\nu a (R_o/a)^3 Q_{o+}(\omega) \qquad (2\text{-}126)$$

(which is independent of R_o). In the case of free walls, an added factor $(\eta k_\nu U_\infty/\omega)^2$ is required [cf. (87)].

We reiterate that for sufficiently large δka many terms n must be retained in the resonance sum Σ_r in Λ_- (rather than only $n = \nu$), and in the limit these contributions cancel, leaving to lowest order in (120), 122) only the non-resonant parts.

Eqs. (125) and (126) yield as the order of magnitude of the ratio of direct resonance to resonance-overlap contributions

$$Q_-^r/Q_+^r \sim 2(a/R_o)(Q_{o-}/Q_{o+}) . \qquad (2\text{-}127)$$

In the case of free walls the ratio (127) contains the additional factor $(\omega/\eta U_\infty k_\nu)^2$ (> 1). On acceptance of the observational evidence that $Q_{o-} \gtrsim Q_{o+}$ for $\omega R_o/U_\infty \gg 1$, we infer $Q_-^r \gg Q_+^r$.

2.6.2.3 <u>Contribution from Low Wave-Numbers</u>

The contribution $Q_-(\omega,y)$ was considered in detail for the case of the fluid dome in Section 2.6.1.2 on assumption that $I_o(k)$ varies little with k for $0 < k \lesssim mR_o^{-1}$ with $m \gtrsim 2\pi$, and does not become much larger until $k \gtrsim \omega/U_\infty$. In the present case, approximation of the sum (38a) by the integral (93) will ordinarily fail for k_n near resonance unless the damping is high or the level spacing ($\simeq \pi/a$) very small. If the interval of k where $|\Gamma(k,y)|^2 \gg 1$ near resonance is $\ll \pi/a$, however, at most a single k_n will be significantly nearly resonant at any given frequency. In such case $Q_-(\omega,y)$ may be reasonably approximated as a sum of a single-mode resonant term n = ν and an integral representing the remaining modes and properly excluding some interval of k near k_ν. The single-mode part was considered already in the preceding sub-section. The crudity of our model does not justify a thorough treatment of this separation. We must inquire, however, broadly how the non-resonant contribution to Q_- differs from that for a fluid dome and whether, in particular, it can be made much smaller by proper choice of dome-cover parameters.

$Q_-(\omega,-L)$ can be made appreciably smaller than its value for a fluid dome only by making the absolute value of the dome-cover impedance large enough that $|\Gamma(k,-L)|^2 \ll 1$ for most $k < k_\omega$, as well as for k moderately larger than k_ω. Obviously, however, to do so will reduce signal transmission as surely as it will reduce noise and therefore affords no advantage. Specifically, if a signal is incident from an angle θ from the normal, it corresponds to a wave-number component $(\omega/c)\sin\theta$ parallel to the dome face and will be attenuated in accord with the value of $|\Gamma((\omega/c)\sin\theta,-L)|^2$. Even if a "window" is arranged such that this value is not small for some particular value of θ, making $|\Gamma(k,-L)|^2 \ll 1$ for most $k < k_\omega$ will imply poor transmission of the signal for some other value of θ. We proceed to indicate how Q_- is modified by the dome cover, though for the reason above one would not wish this modification to be great.

From Eq. (1-12) with $\rho^+ = \rho^-$, $c^+ = c^-$, negligible damping, and a rigid inner surface ($\alpha = 0$) assumed, we have

$$\Gamma(k,y) = \frac{\cos[(k_\omega^2-k^2)^{1/2}(y+L)]}{\exp[-i(k_\omega^2-k^2)^{1/2}L]+(k_\omega^2-k^2)^{1/2}h\sin[(k_\omega^2-k^2)^{1/2}L](q^2-1)} \qquad (2\text{-}128)$$

The value of $|\Gamma(k,y)|^2$ near a non-pressure-release inner surface ($y = -L$) is seen to be appreciably smaller than unity for most $k < k_\omega$ unless

$$k_\omega h \lesssim 1$$

[we recall that $h = \sigma/\rho = h_o(\rho_o/\rho)$].

We consider first a dome cover with low stiffness, since this type yields a simpler analysis and has some advantage for noise attenuation. For low stiffness, we have $k_o \gg k_\omega$, i.e., free-wave length of the isolated cover small compared to wave length of sound. Since the resonant wave length k_r of the loaded plate satisfies $k_r > k_o$, the resonant contribution to Q_- is then strongly damped within the dome. Moreover, since $q \ll 1$ for $k < k_\omega$, the stiffness part $(\propto q^2)$ of the denominator of $|\Gamma|^2$ in (128) is small and may be neglected.

Since the sharp resonance near $k = k_r$ is far removed if $k_o \gg k_\omega$, we may, for reasonable dome size, employ the integral approximation (93). Omitting the resonant part, in any event, we separate Q_- as

$$Q_- = Q_-^t + Q_-^a, \qquad (2\text{-}129)$$

where Q_-^t derives from $k < k_\omega$ (waves propagating into the interior) and Q_-^a from $k > k_\omega$ (waves attenuating). We assume that $(k_\omega^2+L^{-2})/k_o^2 \lesssim 1$ and at present, in order to neglect area averaging as at (98), that $R_o \lesssim K_e^{-1}$, where

$$K_e \simeq \text{lesser of } (k_\omega^2+L^{-2})^{1/2}, (k_\omega^2+8/hL)^{1/2} \qquad (2\text{-}129.1)$$

For $k > k_\omega$, we recall from (1-13), the form complementing (128) is

$$\Gamma(k,y) = \frac{\mathrm{ch}[(k^2-k_\omega^2)^{1/2}(y+L)]}{\exp[k^2-k_\omega^2)^{1/2}L]-(k^2-k_\omega^2)^{1/2}h\,\mathrm{sh}[(k^2-k_\omega^2)^{1/2}L](q^2-1)} \qquad (2\text{-}130)$$

From this, assuming conditions (51), the first of (52), and (57) for use of (93), we readily approximate Q_-^a as

$$Q_-^a(\omega,-L) \simeq \int_{k_\omega}^{\infty} dkk[2J_1(kR_o)/kR_o]^2 I_o(k) \left|\Gamma(k,-L)\right|^2$$
$$\sim I_o(\sim K_e)(4L^2)^{-1}(1+h/4L)^{-2}, \qquad (2\text{-}131)$$

a form that properly reduces when $h = 0$ to the corresponding fluid-dome form given by the second term of (99) in (98). Here $I_o(\sim K_e)$ refers to the appropriately weighted interval above $k = k_\omega$. From (123) we may likewise express Q_-^t as

$$Q_-^t(\omega,-L) = \int_0^{k_\omega} dkk[2J_1(kR_o)/kR_o]^2 I_o(k) \left|\Gamma(k,-L)\right|^2$$
$$\sim L^{-2} I_o(\sim k_\omega) \int_0^{k_\omega L} dxx(1-Ax\,\sin 2x + A^2x^2\sin^2 x)^{-1}, \qquad (2\text{-}132)$$

where $A \equiv h/L$ (and $(k_\omega^2-k^2)^{1/2}L = x$ is the variable of integration). $I_o(\sim k_\omega)$ refers to an interval below $k = k_\omega$. Combining (131) and (132), we may write

$$Q_-(\omega,-L) \sim H'(R_o/S_e)^2 Q_{o-}(\omega), \qquad (2\text{-}133)$$

where

$$S_e^{-2} = (1/4)k_E^2 + (1/8)L^{-2}(1+h/4L)^{-2}, \qquad (2\text{-}133.1)$$

$$k_E^2 = 2L^{-2} \int_0^{k_\omega L} dxx(1-Ax\,\sin 2x + A^2x^2\sin^2 x)^{-1} \qquad (2\text{-}134)$$

and

$$H' = I_o(\sim K_e)/I_o(\sim R_o^{-1}) \qquad (2\text{-}134.1)$$

with $I_o(\sim K_e)$ redefined to take account of the interval in (132) as well as (131) and $I_o(\sim R_o^{-1})$ the quantity in (69). Approximate evaluation of this integral for certain regimes yields

$$
k_L^2 \simeq \begin{cases} k_\omega^2 + (h/L)[-(1/2)k_\omega^2 \cos 2k_\omega L + (1/2)k_\omega L^{-1} \sin 2k_\omega L \\ \qquad + (1/4)L^{-2}(\cos 2k_\omega L - 1)] & \text{if } h/L \ll 1, \\ 2.8(L/h)^{1/2} L^{-2} & \text{if } h/L \gtrsim 2.5 \text{ and } k_\omega L < \pi \\ [2.8(L/h)^{1/2} + 2k_\omega/h]L^{-2} & \text{if } h/L \gtrsim 2.5 \text{ and } k_\omega L \gtrsim \pi. \end{cases} \tag{2-135}
$$

The first approximate form for k_E^2 properly yields the first term of (99) in (98) when $h/L = 0$. The last form given, which is rough, reflects the fact that, apart from the transmission window where $k \simeq k_\omega$, additional windows occur where $(k_\omega^2 - k^2)^{1/2} L \simeq s\pi$ for integer s. The condition $R_o \lesssim (k_\omega^2 + L^{-2})^{-1/2}$ [or alternatively $R_o \lesssim (k_\omega^2 + 8/hL)^{-1/2}$], we recall, has been assumed in deriving the form (133). Likewise, the contribution from a mode or modes near resonance must be added if appreciable.

In Eq. (133), S_e, unlike R_e in (98), is not generally interpretable as a rough correlation distance of the interior pressure field, since it reflects not only an upper wave-number cutoff on $|\Gamma(k,-L)|^2$ but also the substantial variation in $|\Gamma|^2$ in the pertinent range of k. With reference to arrays of elements, similarly to (101) we now have in the case of loose packing*

$$
\begin{aligned}
Q_-^A(\omega,-L) &\simeq Q_-(\omega,-L)/N' \\
&\simeq 2\pi A^{-1} I_o(\sim K_e)(r_o^2/S_e^2) \simeq H'(r_o^2/S_e^2) Q_{o-}^A(\omega) \\
&\qquad \text{for } r_o \ll K_e^{-1} \\
&\qquad (\text{loose packing}: D \gg K_e^{-1})
\end{aligned} \tag{2-136}
$$

*The fact that K_e does not reduce exactly to R_e^{-1} in the limit h=0, despite its role in Eqs. (134.1), (136), (137) similar to that of R_e^{-1} in (98), (101), (102) results from our rather loose use of the notation $I_o(\sim k)$ and the nonuniform expression of extreme inequalities.

Similarly to (102), we have in the case of tight packing, assuming a large total area ($A_T \gtrsim h/k_\omega$),

$$Q_-^A(\omega,-L) \simeq 2\pi A_T^{-1}\left|\Gamma(k\sim A_T^{-1/2})\right|^2 I_o(\sim A_T^{-1/2})$$

$$\simeq H_T(\pi r_o^2/D^2)\left|\Gamma(0,-L)\right|^2 Q_{o-}^A(\omega) \qquad (2\text{-}137)$$

$$(\text{tight packing:} \quad D \ll K_\omega^{-1}),$$

with H_T as in (102) and $|\Gamma(0)|^2$ from (128) given by

$$|\Gamma(0,-L)|^2 = |\exp(-ik_\omega L)-k_\omega h \sin k_\omega L|^{-2}.$$

We mention briefly the opposite case of a stiff dome cover such that $k_o \ll k_\omega$. In this case, the resonant $k(=k_r)$ lies just above k_ω, and is not rapidly damped in the dome interior. If the corresponding interval where $|\Gamma|^2 > 1$ is smaller than the modal spacing($\simeq \pi/a$), the contribution to Q_- has a single-mode character and becomes substantial periodically in frequency as discussed earlier. In addition to the windows near $(k_\omega^2-k^2)^{1/2}L = s\pi$ where $|\Gamma|^2 \sim 1$ (even if $k_\omega h \gtrsim 1$ or $q^2 k_\omega h \gtrsim 1$) there is another where $k = k_o$ so that $q^2 -1=0$. The condition $k_o \ll k_\omega$ implies relatively high signal attenuation for fixed $k_\omega h$, since $q^2 \gg 1$ in (128) over much of the interval $0 < k < k_\omega$. The noise from $k < k_\omega$ is also somewhat more attenuated on this account.

2.6.3 Flush Element, Non-Rigid Boundary

The discussion of Section 2.5.2.1-2 on the covered dome applies virtually without change where the average pressure spectrum refers to an area on the outside of a flow-bounding plate or membrane rather than one interior to a dome, with the previous response coefficients $\Gamma(k,\omega,y)$ understood to be replaced appropriately in accord with the differences between corresponding terms of Eqs. (37) and (39).

In practical application, however, we are interested not in the average pressure on an imagined area of the non-rigid boundary, but rather in the average pressure on an area of a plug cut from the surrounding plate and representing a transducer. The modification of the high-wavenumber component of response pressure Q_{1+}^{∞} of Eqs. (39) and (40) by the term $\Gamma_1(k,\omega)$ in (40) due to the finite impedance of the bounding surface, which was derived under the former supposition of an intact baffle, has no validity for the plug problem if R_o is large compared to the wave length $2\pi k$ in question, that is, if $kR_o >> 1$, as is so if $k >\omega/U_\infty$(i.e., $k_+ \simeq \omega/U_\infty$) and $\omega R_o/U_\infty >> 1$. For this part of the response is determined locally by the stiffness, and this is not the same for the cut-out transducer element as it was for the surrounding plate. Moreover, if $k >> k_{r1}$, the contribution due to this term in (40) is quite small in any case (see Eq. (1-69)). In modifying the results to apply to the cut-out situation, we shall therefore merely neglect this contribution.

If, on the other hand, the wave numbers k_{n-} of the remaining appreciable contribution to Q_{1-} of (34) and Q_{1+} of (38c) are such that $k_nR_o \lesssim 1$, (i.e., if $k_rR_o \lesssim 1$, since no k_n much greater than the resonant wave-number is important), these contributions correspond to wave lengths at least somewhat larger than the size of the area and, with certain modifications, represent the desired result also for the cut-out situation. In particular, the plate motion for given driving force will not be substantially altered even near the cut where the plug is inserted. We assume that the alteration in the force on the plate due to altered motion of the plug area, and the consequent alteration in the acoustic force on the plug area due to the plate, are also negligible; this is probably true for moderate $k_\omega R_o$. Then on neither account is the plate motion for $k_n \lesssim R_o^{-1}$ appreciably altered by cutting and inserting the plug.

We assume that the inserted plug (transducer) is effectively inflexible and that it responds to the integrated net force over its outer face at the frequency in question.* Regarding the transducer as rigid (immobile) let $\hat{f}$ denote the force exerted on it by the acoustic field due to the baffle vibrations. By superposition of fields, this force is given by the difference between the force f_B that would be exerted on the same area if it were part of the continuous baffle partaking of its computed local velocity, say u_B, and the force on the transducer if it had the same velocity u_B with the surrounding baffle not otherwise excited, i.e.,

$$\hat{f} = f_B - z_R u_B, \tag{2-138}$$

where z_R is the well known radiation impedance of a piston of radius R_o in the actual baffle, written in terms of a dimensionless function** $\xi = \xi(\omega R_o/c)$ as

$$z_R = \rho c(\pi R_o^2)\xi \tag{2-139}$$

The spectrum contribution $Q_{1-} + Q_{1+}^{-}$ of Eq. (39) is identified as the spectrum of $(\pi R_o^2)^{-1} f_B$; the desired average-pressure spectrum for the cut-out situation is rather the spectrum of $(\pi R_o^2)^{-1} \times (f_B - z_R u_B)$, where the contribution to u_B from each mode can be obtained immediately from the solution of the simple acoustic problem. This procedure, we repeat, is valid only for low-wave number modes n such that $k_n R_o \lesssim 1$, for which the velocity contribution u_{Bn} to u_B is substantially constant over the area of the plug; the effect of boundary non-rigidity on the average pressure conveyed by high-wave number modes is simply neglected.

---

*The transducer may still have a non-infinite [illegible]ernal impedance z_I. Since this will be true also when the b[illegible]ing baffle is rigid, the corresponding reduction factor for the net force, $z_I/(z_I+z_R)$, with z_R the radiation impedance of the transducer in the baffle, will apply in both cases and may be regarded as compensated for in the calibration of the transducer; accordingly this factor is ignored.

**As $\omega R_o/c \to \infty$, we have $\xi \to 1$ and as $\omega R_o/c \to 0$, we have $\xi \to 0$.

For mode n the ratio of velocity to pressure in the acoustic field just outside the surface is given by

$$\pi R_o^2 u_{Bn}/f_{Bn} = k_{2n}/\rho\omega, \ [k_{2n} = (k_\omega^2 - k_n^2)^{1/2}]$$

whence

$$f_{Bn} - z_R u_{Bn} = f_{Bn}(1 - \xi k_{2n}/k_\omega) \tag{2-140}$$

[e.g. see Appendix 1, Eq. (8.2)]. Hence, for the lower group of modes the intended modification is achieved by applying to the response coefficient $\Gamma_1(k_n)$ in the definition of T_{1n} and S_{1n} at Eq. (23) the additional factor

$$(1 - \xi k_{2n}/k_\omega)$$

in which k_{2n} is replaced by iK_{2n} when $k_n > \omega/c$. Let T_{1n} and S_{1n} so modified be denoted as $\hat{T}_n$, $\hat{S}_n$.

Introducing the prescribed modification of $\Gamma_1(k_n)$ into the approximate Eq. (39), omitting $\Gamma_1(k)$ in Eq. (40), and identifying the contributions from the unity part of $1 + \Gamma_1$ in Eqs. (35) and (40) as representing the average-pressure spectrum $Q_o(\omega)$ for a rigid boundary, we have for the approximate spectrum $\hat{Q}(\omega)$ for the non-rigid boundary

$$\hat{Q} \simeq Q_o + \delta\hat{Q}_- + \delta\hat{Q}_+, \tag{2-141}$$

where

$$\delta\hat{Q}_-(\omega) \simeq (\pi/a)\sum_{n=0}^{n_-} k_n[2J_1(k_nR_o)/k_nR_o]^2 I_o(k_n) \times \left[\left|1+\Gamma_1(k_n)(1-\xi k_{2n}/k_\omega)\right|^2 - 1\right], \tag{2-142}$$

$$\hat{Q}_+(\omega) \simeq 8R_o^{-2}a^{-2}\int_{k_+}^{\infty} dkkI_o(k) \times \begin{cases} k\left|\sum_{n=0}^{n_-}(-)^n\hat{T}_n\right|^2 \\ k^{-1}\left|\sum_{n=0}^{n_-}(-)^n\hat{S}_n\right|^2. \end{cases} \quad (2\text{-}143)$$

With regard to the consequent modifications in the further approximations given in Eqs. (43) the previous $\hat{g}_1(k,k_2,\omega)[\equiv \Gamma_1(k,\omega)]$ must be replaced by $\hat{\hat{g}}(k,k_2,\omega)\equiv\Gamma_1(k,\omega)$ $\times(1-\xi k_2/k_\omega)$, whence the required derivative is given, instead of by (47), by

$$(\partial\hat{\hat{g}}/\partial k_2)_o = (z_\omega+\xi\rho c)/\rho\omega = k_\omega^{-1}(\xi+z_\omega/\rho c). \quad (2\text{-}144)$$

Likewise, condition (61) for use of (43) is replaced by

$$\left|(z_\omega/\rho c)+\xi\right|^{-1}(k_\omega a)^{-1/2} \ll 1 \quad (2\text{-}145)$$

(condition (62) is now irrelevant since $k_\omega R_o \lesssim 1$ here by assumption); condition (60) is unchanged. In these expressions z_ω is understood to represent $z(k,\omega)$ at $k = k_\omega$; this value, for the thin plate (n=4) or membrane (n=2), is given by

$$\begin{aligned} z(k_\omega,\omega) &\simeq i\sigma\omega[(k_\omega/k_o)^n-1] + \sigma\omega[(k_\omega/k_o)^n\zeta+\beta/\omega] \\ &\equiv iX_\omega + R_\omega. \end{aligned} \quad (2\text{-}146)$$

2.6.3.1 Estimated Spectrum (Excluding Low Non-Resonant Wave Numbers)

As an apt example for further consideration, analogous to the covered-dome example of Eqs. (115)-(123), consider a parameter regime where, with the modifications above, Eq. (43) holds for Σ_ω, but the resonance half width δk_1 satisfies $\delta k_1 a \lesssim \pi$ so that Σ_r and the resonant part of $\delta\hat{Q}_-$, say $\delta\hat{Q}_-^r$ may be approximated by retention of the single mode nearest resonance at frequency ω, say $n = \nu$. For validity of the modification for plug cut-out, we must assume also that $k_\nu R_o \lesssim 1$ (nevertheless, we retain $J_1(k_\nu R_o)$ without approximation). The acoustic increment, by (141), is given by

$$\hat{Q}-Q_o = \delta\hat{Q}_- + \int_{k+}^{\infty} dkkI_o(k)\hat{\Lambda}_-(k), \qquad (2\text{-}147)$$

where, by collection of previous expressions,

$$\delta\hat{Q}_- = \delta\hat{Q}_-^r + \delta\hat{Q}_-^p,$$

$$\delta\hat{Q}_-^r \simeq (\pi/a)k_\nu[2J_1(k_\nu R_o)/k_\nu R_o]^2 I_o(k_\nu)[\,|1+\Gamma_1(k_\nu)(1-i\xi K_{2\nu}/k_\omega)|^2-1]$$

(2-148)

Rigid-wall case:

$$\hat{\Lambda}_- \simeq 8R_o^{-2}a^{-2}k\left|(-)^\nu \frac{J_1(k_\nu R_o)}{k_\nu}\frac{k_\nu^{1/2}}{k^2-k_\nu^2}\Gamma_1(k_\nu)(1-i\xi K_{2\nu}/k_\omega)+\hat{\Sigma}_\omega\right|^2 \qquad (2\text{-}149)$$

$$\hat{\Sigma}_\omega \simeq (2\pi)^{1/2}a^{-1/2}\frac{J_1(k_\nu R_o)}{k_\omega}\frac{(z/\rho c)+\xi}{k^2-k_\omega^2}(-)^{j+1}[M(b)-iM(1-b)] \qquad (2\text{-}150)$$

Free-wall case:

$$\hat{\Lambda}_- \simeq 8R_o^{-2}a^{-2}k^{-1}\left|-(-)^\nu \frac{J_1(k_\nu R_o)}{k_\nu}\frac{k_\nu^{3/2}}{k^2-k_\nu^2}\Gamma_1(k_\nu)(1-i\xi K_{2\nu}/k_\omega)+\hat{\Sigma}_\omega\right|^2$$

(2-151)

$$\hat{\Sigma}_\omega \simeq (2\pi)^{1/2}a^{-1/2}\frac{J_1(k_\omega R_o)}{k_\omega}\frac{k_\omega[(z/\rho c)+\xi]}{k^2-k_\omega^2}(-)^{j+1}[M(b)-iM(1-b)].$$

(2-152)

Consideration of the non-resonant part, $\delta\hat{Q}_-^p$ of $\delta\hat{Q}_-$ will be deferred to the following subsection.

As with the dome, we denote the contribution in (147) due to the squared resonance term of $\hat{\Lambda}_-$ in (149) [or (151)] by $\delta\hat{Q}_+^r(\omega)$, called the resonance overlap part, and that due to the squared non-resonance term $|\hat{\Sigma}_\omega|^2$ in (149) by $\delta\hat{Q}_+^p(\omega)$, called the propagating overlap part, and call $\delta\hat{Q}_-$ in (147) the direct low-wavenumber part, omitting further reference to the interference term in (149).

Again taking $k_+ \simeq \omega/U_\infty$, we estimate crudely the various contributions to $\hat{Q}-Q_o$. The non-resonance contribution, $\delta\hat{Q}_+^p$, for rigid walls, in approximation (150) is estimated roughly in terms of the high-wavenumber average-pressure spectrum $Q_{o+}(\omega)$ for a flush element of radius $R_o(\gg U_\infty/\omega)$, by use of (65b), as

$$\delta\hat{Q}_+^p(\omega) \simeq \left|z_\omega/\rho c + \xi\right|^2 f(k_\omega a)a^{-3}\int_{\omega/U_\infty}^{\infty} dk k^{-2} I_o(k,\omega)$$

$$\simeq (\pi/4)\left|z_\omega/\rho c+\xi\right|^2 f(k_\omega a)(R_o/a)^3 Q_{o+}(\omega) \quad \text{(rigid wall)}$$

(2-153)

[cf.(91)]. In the case of free walls, estimate (153) is reduced by a factor of the order of $(\eta U_\infty/c)^2$. The boundary-impedance ratio $z_\omega/\rho c$, by (146), may be written in terms of $h(\equiv \sigma/\rho)$ as

$$X_\omega/\rho c = k_\omega h[(k_\omega/k_o)^n-1],\ R_\omega/\rho c=k_\omega h[(k_\omega/k_o)^n\zeta+ \beta/\omega]. \quad (2\text{-}153.01)$$

If the boundary may be approximated as rigid so far as the piston radiation impedance is concerned, then, under the present assumption that $k_\omega R_o \lesssim 1$, we may approximate the piston-impedance functions by

$$\xi \equiv r-ix \simeq (1/2)(k_\omega R_o)^2-i(8/3\pi)k_\omega R_o. \quad (2\text{-}153.02)$$

If $k_\omega/k_o \ll 1$, i.e., if at the wave length of sound the boundary impedes as a mass (the wave length of free waves in it being shorter), and if the plate damping is moderate, we have from (153.01) and (153.02)

$$|z_\omega/\rho c+\xi|^2 \simeq (\omega/c)^2[(h+8R_o/3\pi)^2+R_o^2] \qquad (2\text{-}153.1)$$

for use in (153) or in the condition (145). The latter in such case, however, is satisfied only for very large a. Similarly, condition (60) becomes approximately $(\omega h/c)(k_\omega a)^{-1/2} << 1$. On account of conditions (60) and (145), estimate (153) is justified only in a limited regime. In this regime, at least, the non-resonance acoustic average pressure on the plug is similar to the direct pressure averaged over the entire plate area πa^2 demarked by the bounding structural members.

The direct resonance contribution $\delta\hat{Q}_-^r$ will be estimated with neglect of the interference term in (148), which is linear in Γ_1, on assumption that $|\Gamma_1|^2 >> 1$ near resonance. In the vicinity of resonance at $\omega = \omega_\nu$, we may write $\Gamma_1(k_\nu,\omega)$ in accordance with (1-66) as

$$\Gamma_1(k_\nu,\omega) \simeq \frac{b_{1\nu}}{\omega/\omega_{1\nu}-1 + i\epsilon_{1\nu}}, \qquad (2\text{-}154)$$

where, as described in Section 1.5, by modification of (111)-(113)

$$\omega_{1\nu} \simeq \omega_{o\nu}(1+1/K_{2\nu}h)^{-1/2} \quad [K_{2\nu}=\left\{k_\nu^2-(\omega_{1\nu}/c)^2\right\}^{1/2}] \qquad (2\text{-}155)$$

$$b_{1\nu} \simeq (2k_\nu h)^{-1}(1+1/k_\nu h)^{-1}, \qquad (2\text{-}156)$$

$$\epsilon_{1\nu} \simeq (1/2)[\zeta+(\beta/\omega_{1\nu})(1+1/k_\nu h)^{-1}]. \qquad (2\text{-}157)$$

Since $(K_{2\nu}/k_\omega)^2 = (k_\nu/k_\omega)^2-1$, we may express one factor in $\delta\hat{Q}_-$ from (148) by use of (155) as

$$|1-i\xi K_{2\nu}/k_\omega|^2 \simeq [1-x\{(k_\nu/k_\omega)^2-1\}^{1/2}]^2+r^2[(k_\nu/k_\omega)^2-1]$$

If $k_\omega/k_\nu << 1$ at the ν pertinent to the interesting range of frequency (as true if $k_\omega/k_o << 1$ as supposed at (155)) and if $k_\nu R_o < 1$ (as weakly assumed already), this becomes

$$\left|1-i\xi K_{2\nu}/k_{\omega}\right|^2 \simeq (1-8k_\nu R_o/3\pi)^2.$$

In this event we may estimate from (148)

$$\delta\hat{Q}^{r}_{-}(\omega) \simeq (\pi/a)k_\nu I_o(k_\nu,\omega) \left.\frac{b^2_{1\nu}}{[(\omega/\omega_{1\nu})-1]^2+\epsilon^2_{1\nu}}(1-8k_\nu R_o/3\pi)^2\right|$$

$$\sim \pi \frac{b^2_{1\nu}(1-8k_\nu R_o/3\pi)^2}{[(\omega/\omega_{1\nu})-1]^2+\epsilon^2_{1\nu}} (R_o/a)k_\nu R_o Q_{o-}(\omega), \qquad (2\text{-}158)$$

where the second form follows from (69) on assumption that $I_c(k_\nu) \sim I_o(\sim R_o^{-1})$ [cf. (125)].

Similarly, the resonance overlap contribution, for $k_\nu \ll \omega/U_\infty$, in the case of rigid walls is estimated from (149) as

$$\delta\hat{Q}^{r}_{+}(\omega) \sim \frac{\pi}{2} \frac{b^2_{1\nu}(1-8k_\nu R_o/3\pi)^2}{[(\omega/\omega_{1\nu})^2-1]^2+\epsilon^2_{1\nu}} k_\nu a(R_o/a)^3 Q_{o+}(\omega) \qquad (2\text{-}159)$$

[independent of R_o, cf.(126)]. In the case of free walls, an added factor $(\omega/\eta U_\infty k_\nu)^{-2}$ is required.

As a numerical example for resonance properties, we consider as in Section 2.5.2 a steel plate of thickness 0.25 in. (but now with water loading one side only) and a dome radius a=0.5m.* With reference to the 3-kc region, the resonances $\nu=8$ and $\nu=9$ fall at frequencies

$$\omega_8/2\pi \simeq 3060 \text{ cps and } \omega_9/2\pi \simeq 3910 \text{ cps}.$$

*In the present case of flush mounting in a plate, as opposed to a dome, it would have more practical utility to consider a model with rectangular rather than circular boundary.

[cf. (114)]. As before, the quantity $k_\nu h$ of (156)-(157) for $\nu=9$ has the value $k_9 h=2.85$.

2.6.3.2 Contribution from Low Wave Numbers

We refer now to the sum (142) for $\delta\hat{Q}_-$. The initial discussion of Section 2.5.2.3 again applies. If the modal spacing ($\sim\pi/a$) is small enough, the contributions to $\delta\hat{Q}_-$ from modes removed from resonance may be approximated by an integral analogous to (93). Near resonance this approximation will not ordinarily suffice, depending also on the degree of damping. In the preceding subsection the resonant contribution to $\delta\hat{Q}_-$ was considered on the simplifying assumption that at most one mode is near resonance ($k_n \simeq k_r$) at any given frequency. In any case, modes with k_n substantially greater than k_r contribute negligibly, since $|\Gamma_1(k_n)|$ in that range is small, as noted earlier.

No single approximation can be written for the entire sum (142) in the general case. Moreover, the form (142) was itself a valid representation of $\delta\hat{Q}_-$ only when $k_n R_o \lesssim 1$ for all n, i.e., $k_r R_o \lesssim 1$. In a certain opposite limit, however, it is possible to differentiate clearly between the contributing modes at resonance and those removed from resonance, and to write a rough explicit approximation to the partial sum contributed by the latter. The limit envisaged is that where $k_o(\omega) R_o >> 1$, whence also $k_r R_o >> 1$, $k_o(\omega)$ here being the resonant wave-number of the free plate. In this limit the contribution to average pressure from modes as high as those near resonance is reduced greatly by area averaging.

We assume the modal spacing is sufficiently small that $\delta\hat{Q}_-$ may be expressed as an integral over k ($\lesssim k_r$) rather than a sum over modes. The form (142) or the related integral, however, is inapplicable when $k_n R_o \gtrsim 1$, as will be so for some contributing n in the present limit. This inapplicability, we recall, arises from three distinguishable effects: (1) the velocity of the uncut plate varies appreciably over the area where the rigid plug is inserted in the cut plate, so that the changed radiation reaction on this area when the plug is inserted cannot be expressed as due to eliminating a rigid motion; (2) elimination of motion of the plug-area of the intact plate by insertion of a stationary plug eliminates the acoustic field due to this area and hence eliminates the corresponding pressure on the plate, thereby affecting its motion; (3) the response of the cut and plugged plate to a given excitation pressure (of wave number $k_n \gtrsim R_o^{-1}$) on its surface, excluding the plug area, differs appreciably from the response of the uncut plate to the same excitation, excluding the plug area, and therefore the acoustic field and force on the plug area due to vibration of the surrounding plate also differ from the cut to the uncut case.*

If $k_\omega R_o << 1$, however, the effect (1) is negligible since the radiation reaction on the plug area is then negligible. Effect (2) also becomes negligible in this limit. If $k_n R_o >> 1$, the radiation reaction, though not computed here, is probably small independently of $k_\omega R_o$. Effect (3) is negligible if $k_\omega R_o << 1$ and $h\left|1-q^2(k_n)\right|/R_o << 1$, as implied by the preceding footnote. In fact, in this limit the acoustic force just cancels the force due directly to boundary-layer pressure fluctuations ($\delta\hat{Q}_- = -Q_{o-}$), leaving the plug an area of pressure release. In any case, those k_n such that $k_n R_o \gtrsim \pi$ make contributions that decrease as k_n increases on account of the area-averaging effect. Hence, one may still use (142) (or the integral form) for crude estimates, with ξ still referring to a rigid piston but with ξ set to zero for, say, $k_n > \pi R_o^{-1}$. We make a crude estimate only for the case where $k_\omega R_o << 1$ (but $k_o(\omega) R_o >> 1$), i.e., the plug radius times 2π lies between and well removed from the (long) sound wave length ($2\pi/k_\omega$) and the (short) free-wave length in the plate. This is a very limited regime.

Assuming conditions (51) and (57) for use of the integral approximation and taking $\xi = 0$, we rewrite the non-resonant part of (142) as

*The difference in plate response referred to under point (3) vanishes, however, in the limit of vanishing plate impedance (pressure-release boundary).

$$\delta\hat{Q}^P_-(\omega) \simeq \int_0^{mR_o^{-1}} dkk[2J_1(kR_o)/kR_o]^2 I_o(k) \; [|1+\Gamma_1(k)|^2-1]. \quad (2\text{-}160)$$

Neglecting damping, we rewrite Eqs. (1-16) and (1-17) for Γ_1 as

$$\Gamma_1(k) = \begin{cases} -1/[1-i(k_\omega^2-k^2)^{1/2}h(1-q^2)] & (k < k_\omega) \\ -1/[1+(k^2-k_\omega^2)^{1/2}h(1-q^2)] & (k > k_\omega) \end{cases} \quad (2\text{-}161)$$

By virtue of the assumption that $k_o(\omega)R_o \gg 1$, we have $q^2 \doteq (k/k_o)^n \ll 1$ for all $k < mR_o^{-1}$. To approximate better when q^2 does not well satisfy this condition, however, we do not neglect q^2, but regard it as set equal to a constant value, say $q_o^2 \doteq q^2(k=R_o^{-1})=(k_oR_o)^{-n}$ appropriate for a value of k near the peak of the factor $k[2J_1(kR_o)/kR_o]^2$ in (160). To perform the integration in (160) crudely and simply, as usual neglecting variation in $I_o(k)$, we replace $J_1(kR_o)$ by

$$\begin{cases} (1/2)kR_o \text{ for } 0 < kR_o < 1 \\ 1/2 \text{ for } 1 < kR_o < t = 2.88 \\ 1/(\pi kR_o)^{1/2} \text{ for } t < kR_o, \end{cases} \quad (2\text{-}162)$$

the last of which yields correctly the asymptotic value of the average of $J_1^2(kR_o)$ over several periods. Here t is assigned that value which yields from (160) in the case of a pressure-release surface ($\Gamma_1 = -1$) the desired result

$$\delta\hat{Q}^P_- \simeq -I_o\int_0^{mR_o^{-1}} dk[2J_1(kR_o)/kR_o]^2 \simeq -2R_o^{-2}I_o,$$

obtained without approximation of $J_1(kR_o)$.*

*Explicitly, t satisfies $(1/2) + \ln t + 4/\pi t = 2$.

Approximating (160) by use of (162) and neglecting $k_\omega R_o$, we find

$$\delta \hat{Q}^p_-(\omega) \simeq (1/2) Q_{o-}(\omega) \left\{ \epsilon^{-2} [-3 \ln(1+\epsilon) + \epsilon(3+2\epsilon)/(1+\epsilon)] + \frac{(t-1)\epsilon}{(1+\epsilon)(1+t\epsilon)} - \ln\left[\frac{1+t\epsilon}{t(1+\epsilon)}\right] + \frac{4}{\pi} \frac{1}{t(1+t\epsilon)} \right\} , \tag{2-163}$$

where

$$\epsilon = (h/R_o)(1-q_o^2) \tag{2-164}$$

and the pertinent average $I_o(\sim R_o^{-1})$ has been presumed nearly equal to $I_o(\sim R_o^{-1})$ of (69). By construction, this yields in the limit of a massless plate

$$Q_{o-} + \delta \hat{Q}^p_- = 0 \quad (\epsilon = 0)$$

as the net average-pressure spectrum. The result obtained from (163) to first order in ϵ cannot be relied on, since only a contribution from the range $k > tR_o^{-1}$ fails to cancel, and this part has been inadequately approximated. In the opposite limit of a massive plate, (163) yields

$$Q_{o-} + \delta Q^p_- \simeq Q_{o-} [1 - \epsilon^{-1}(1.7 - 1.5\epsilon^{-1} \ln \epsilon)] \quad (\epsilon \gg 1) \tag{2-165}$$

to the given order in ϵ^{-1}. In this limit the high wave numbers ($k > tR_o^{-1}$) contributed little, so that the approximations entailed are passably justified.

Under present assumptions, we note, the parameter ϵ that measures the unyielding quality of the plate, i.e., the extent to which it preserves the net driving force on the plug without canceling it by the acoustic force generated by its own motion in response to the boundary-layer pressure fluctuations, is the equivalent water thickness of the plate (reduced by the factor $1-q_o^2$) in units of the radius of the plug in question. In practice, the effective equivalent thickness in the frequency range of operation must be large lest signals be canceled by plate vibration (as we compute herefor the noise) especially near grazing incidence, at which $k \approx k_{\omega}$, making $\Gamma_1 = -1$ in (161). If resonating devices, which are effective in a limited frequency range, are used for this purpose, they may be regarded as implying a frequency dependent effective thickness $h(1-q_o^2)$ having a sharp resonance. In some noise measurements, on the other hand, as opposed to actual operations with a viable configuration, the parameter ϵ may not be large, and hence acoustic cancellation may affect the apparent noise.

2.6.4 Effect of Dome and Non-Rigid Boundary on Scaling Law of Average-pressure Spectrum due to Turbulent Flow

We consider how the scaling of the average-pressure spectrum differs within a dome from the scaling for a similar area exposed to the flow, and also how the scaling in the latter instance is altered by the acoustic field generated if the boundary is non-rigid. We are able to make the comparison only separately between the respective contributions due to high ($k \geq \omega/U_\infty$) and to low wave numbers. Which comparison is most pertinent then depends on which contribution predominates for the regime in question. As we have noted, however, judging by the observed area dependence, low wave numbers apparently predominate for an exposed area such that $\omega R_o/U_\infty \gg 1$ and, if so, also for a dome-shielded area.

We examine only the dependence on flow velocity U_∞ corresponding to given dependence on ω, starting with the fluid dome. The contribution $Q_+^\infty(\omega,-L)$ as estimated by (89) is related to Q_{o+} by the form*

*The argument ω/U_∞ of the function f_1 is rendered dimensionless by L. We consistently omit in the following any dependence other than on U_∞ and ω.

$$Q_+^{\infty} \propto\!\!\!\!\!\!\sim f_1(\omega/U_\infty)Q_{o+}, \qquad (2\text{-}166)$$

If Q_{o+} has the form (74) (with $v_* \lesssim U_\infty$), we see that, in a range of given dependence on ω, Q_+ depends on U_∞ slightly differently from Q_{o+}. Likewise, in the regime of (91), for rigid walls Q_+^p has the form

$$Q_+^p \propto\!\!\!\!\!\!\sim \omega^2 f(\omega a/c)Q_{o+} \qquad (2\text{-}167)$$

(where f is the specific function defined earlier). It thus differs substantially from Q_{o+} in its U_∞-dependence for fixed ω-dependence. (Unlike Q_{o+}, we recall, for $\omega R_o/c \lesssim 1$ it is also independent of R_o.) As for the contribution from low wave numbers, in the approximation of (98) we have

$$Q_- \propto\!\!\!\!\!\!\sim [(\omega/c)^2+(1/2)L^{-2}]Q_{o-}.$$

Hence at frequencies low enough that $\lambda(=2\pi c/\omega) >> 9L$ we have Q_- for the shielded area varying with ω, U_∞(and other flow parameters) exactly as Q_{o-} for the exposed area. At frequencies high enough that $\lambda << 9L$, on the other hand, Q_- contains an additional factor ω^2; in regions of equivalent ω dependence (which correspond to different ranges of ω) Q_- thus increases with U_∞ more rapidly than Q_{o-} by a factor U_∞^2, on assumption that Q_{o-} depends on ω and U_∞ only via ω/U_∞ as in (78) or the first term in (79). If we compare the Q_- and Q_{o-} spectra at equal ω, we have the curve for $Q_-(\omega,-L)$ below and parallel to that for $Q_{o-}(\omega)$ up to $\omega \sim c/\sqrt{2}L$ and then approaching it (i.e., decreasing less rapidly than Q_{o-}) according to

$$d\log Q_-/d\log\omega \simeq d\log Q_o/d\log\omega + 2 \qquad (2\text{-}169)$$

where $\omega \gtrsim c/\sqrt{2}L$.

We proceed to the case of a covered dome. According to (124) we have

$$Q_+^\infty \lesssim U_\infty^4 f_2(\omega/U_\infty) Q_{o+} \qquad (2\text{-}170)$$

At fixed ω-dependence, Q_+^∞ thus tends to increase more rapidly with U_∞ than Q_{o+}. With regard to Q_+^p in a previously defined regime, estimate (91) applies also for the covered dome and hence also (167) above. The resonance contributions Q_+^r and Q_-^r bear little resemblance to Q_{+o} and Q_{-o} in their dependence on ω, though differing by a simple resonance factor in the single-mode approximation of (125) and (126). To the extent that Q_+^r and Q_-^r are important and have the single-mode form, Q_- will differ from the spectrum Q_o in having bumps where successive modal wave-numbers k_ν become resonant, the non-dimensional frequencies at these resonances will differ for different values of the boundary-layer parameters employed in defining the non-dimensional frequency variable. As for the nonresonant contribution from low wave numbers, the behavior will be similar to that discussed above in the case of the coverless dome, provided $h/L \ll 1$, but some additional modulation dependent on the variable $\omega L/c$ will be introduced in accord with the first form of Eq. (135) in the regime where this equation applies.

Statements similar to those above for the domed dome apply to the acoustic contribution to the average-pressure spectrum for an area in an exposed non-rigid surface [see estimates (153)-(165)]. With reference to the nonresonant contribution from low wave numbers, however, in the regime of the approximation (165) (for which, among other things, it is required that $\omega R_o/c \lesssim 1$) the augmented spectrum has roughly the same dependence on ω, U_∞, and other flow parameters as Q_{o-}.

LIST OF NOTATION

a	lateral radius of dome or plate (Fig. 2-1)
a_o	plate constant of Eq. (1-3)
$a_s(k)$	overlap of s-th modal function with plane wave, Eq. (2-8)
A_o	area of pressure averaging [e.g., Eq. (2-9)]
A	dome cross-sectional area, Sec. 2.2; active area of array, Sec.2.6.1.4.
$A(\omega)$	coefficient in Eq. (2-70.1)
A_e	$=\pi R_e^2 \sim$ correlation area of pressure on face shielded by fluid dome
A_T	total (active-plus-dead) area of array
A_{TM}	maximum permissible value of A_T
A_{TO}	minimum permissible value of A_T
b	number describing value of sound wave number in fluid relative to adjacent modal wave-numbers, Eq. (2-43.1)
$B(k)$	numerator in frequency-resonance form for interior acoustic-response coefficient, Eq. (1-54)
$b(\omega)$	numerator in wave-number-resonance form for same, Eq. (1-38)
$B_1(k)$	numerator in frequency-resonance form for acoustic-response coefficient on outside of plate, Eq. (1-66)
$b_1(\omega)$	numerator in wave-number-resonance form for same, Eq. (1-65)
b_n	modal expansion coefficient of driving pressure, Eq. (2-2)
$B(k,y)$	(ω suppressed) averaging-response function for pressure spectrum at excitation wave-number k and depth y, Eq. (2-17a)
$B_1(k)$	same at outer face of plate, Eq. (2-17b)
$B_n(k,y)$	modal contributions to B(k,y), Eq.(2-18)

$B_{1n}(k)$	same to $B_1(k)$, Eq. (2-18)
$B_+(k,y)$	contribution to $B(k,y)$ from modes with $k_n > k_+$, Eq. (2-25)
$B_-(k,y)$	same from modes with $k_n < k_-$, Eq. (2-25)
$B_{1+}(k)$	contribution to $B_1(k)$ from modes with $k_n > k_+$, Eq. (2-25)
$B_{1-}(k)$	same from modes with $k_n < k_-$, Eq. (2-25)
$B_{10}(k)$	defined at Eq. (2-25)
$B(n)$	asymptotically interpolated form of $B_n(k,y)$ for non-integer n
$B(\omega)$	coefficient in Eq. (2-70.1)
b_ν	$=B(k_\nu)/\omega_\nu$, Eq. (2-110)
$c^+(c^-)$	sound speed in outer (inner) fluid
c	$=c^{\pm}$ if $c^+ = c^-$, or c^+ if inner fluid is absent ($\rho^-=0$)
c_o	velocity of free waves in membrane, Eq. (1-4)
c_t	shear-wave speed in plate material
c_{mn}, c_n	modal overlap coefficient in circular geometry, Eq. (2-10.1)
$C_n(ka)$	$=N_n c_n(k)$
$C(\omega)$	coefficient in Eq. (2-70.1)
$D(k)$	real part of denominator of $\Gamma(k,\omega,y)$ at Eq. (1-37)
D	center spacing of elements in square-celled array
D_M	$=(A_{TM}/A)\pi r_{om}^2$ = maximum permissible spacing
D_m	$=(A_{TO}/A)\pi r_{om}^2$ = minimum permissible spacing
E	Young's modulus for plate material
e_n	$= 2\left[\pi x_n J_i^2(x_n)\right]^{-1}$, Eq. (2-23.1)
$F_{\pm}(z_1,z_2)$	undetermined functions of dimensionless arguments z_1,z_2 [Eqs. (2-71), (2-75)]
$f(k_\omega a)$	$= 4\pi\left[M^2(1-b)+M^2(b)\right]$, a certain periodic function of $k_\omega a$, Eq. (2-91.1)
F	defined at Eq. (2-95.1)

$\hat{f}$	acoustic force on rigid element due to plate vibration
f_B	acoustic force on plate element of same area
f_{Bn}	same for plate vibration in mode n
f_1, f_2	certain functions of an indicated variable, Eqs. (2-166), (2-170)
$g_n(y)$	depth-dependence function of pressure field in mode n, Eq. (2-4)
g_{1n}	analogous coefficient for pressure on outer surface of plate
$\hat{g}(k,k_2,y)$	(ω suppressed) functional form representing $\Gamma(k,y)$ [Eq. (2-43)]
$(\partial\hat{g}/\partial k_2)_o$	indicated partial derivative evaluated at $k_2=0$
$\hat{g}_1(k,k_2)$	functional form representing $\Gamma_1(k)$
$G_{\pm}(z_1,z_2)$, $\hat{G}_-(z)$	undetermined functions of dimensionless arguments z_1, z_2, z [Eqs. (2-72), (2-76), (2-78.1)]
$\hat{g}_r$	$= \Gamma(k_r,y)$ (ω suppressed), conditions (2-104)
$\hat{\hat{g}}(k,k_2)$	$= \Gamma_1(k,\omega)(1-\xi k_2/k_\omega)$ [Eq. (2-144)]
h_o	plate thickness
$h^{\pm}$	$= \sigma/\rho^{\pm}$, thickness of outer (inner) fluid having same mass per unit area as the plate or membrane
h	$= h^{\pm}$ if $\rho^+ = \rho^-$, or h^+ if $\rho^- = 0$
$H_+(z)$, $H(z)$	undetermined functions of dimensionless argument z [Eqs. (2-73), (2-77)]
H	defined at Eq. (2-99.1)
H_T	defined at Eq. (2-102.1)
H'	defined at Eq. (2-134.1)
$I(k,\omega)$	wave-number-frequency spectrum of some pressure; k is magnitude of two-dimensional wave vector parallel to surface bounding flow

$I_o(k,\omega)$	same for pressure due to a turbulent boundary layer on a plane rigid wall [Eq. (1-28)]
$I_o(\sim k',\omega)$	variously defined weighted average of $I_o(k,\omega)$ over interval $0 < k \lesssim k'$
I'	constant wavenumber-frequency spectrum for $k < k_c$ in sample computation [Eq. (2-103)]
I''	same for $k > k_c$
j	mode with wave number next below that of sound in fluid, defined by (2-42.1)
$\mathbf{k}$	two-dimensional wave vector
$k_o(\omega), \hat{k}_o(\omega)$	free-wave number in isolated plate (k_o must be distinguished from k_n for $n = 0$)
$k_\omega^\pm$	$= \omega/c^\pm$, sound wave number in outer (inner) fluid
k_ω	$= k_\omega^\pm$ if $c^+ = c^-$, or k_ω^+ if $\rho^- = 0$
$k_2^\pm$	y-component of wave number of pressure wave in outer (inner) fluid with (x,z)-component k, where $k < \omega/c^\pm$, Eq. (1-7.1)
k_2	$= k_2^\pm$ if $c^+ = c^-$, or k_2^+ if $\rho^- = 0$
$K_2^\pm$	magnitude of imaginary y-component of sound wave-number of pressure wave in outer (inner) fluid with (x,z)-component k, where $k > \omega/c^\pm$, Eq. (1-7.2)
K_2	$= K_2^\pm$ if $c^+ = c^-$, or K_2^+ if $\rho^- = 0$
$k_m(\omega)$	hypothetical wave-number such that $I_o(k,\omega)$ is negligible for $k < k_m(\omega)$
k_{mn}, k_n	modal wave-number (defined with regard to (x,z) plane)
$k_r(\omega)$	surface wave number of resonance for coupled dome system at frequency ω ($k_r > \omega/c$)
$\delta k(\omega)$	half width in wave number of a resonance
$K_{2r}(\omega)$	K_2 evaluated at $k=k_r$

$k'_r(\omega)$	wave number of radiatively damped resonance for coupled dome system at frequency ω ($k'_r < \omega/c$) [Eq.(1-53.1)]
$k'_{2r}(\omega)$	k_2 evaluated at $k=k'_r$
$k_{r1}(\omega)$	wave number of resonance for coupled plate-fluid system at ω (fluid on one side only)
$\delta k_1(\omega)$	half width of this resonance at k_{r1}
$K_{2r1}(\omega)$	K_2 evaluated at $k=k_{r1}$
$k'_{r1}(\omega)$	wave number of radiatively damped resonance for plate-fluid system at ω
k_+, k_-	cutoff wave-numbers such that, on account of the smallness of $I_o(k)$ for $k < k_+$ and of $\Gamma(k,y)$ or $\Gamma_1(k_n)$ for modes with $k_n > k_-$, wavenumber pairs (k,k_n) having $k < k_+$ and $k_n > k_-$ (simultaneously) are negligible
k_c	wave number of rapid increase in wave-number spectrum in sample computation Eq. (2-103)
k_u	$= \omega/\eta U_\infty$ [Eq. (1-60)]
$k_\nu(\omega)$	modal wave-number nearest resonance
$K_{2\nu}$	K_2 evaluated at $k=k_\nu$
k_E	inverse distance defined by Eq. (2-134)
K_e	wave-number defined by Eq. (2-129.1)
$M(k,\omega)$	$=\lvert\Gamma(k,\omega)\rvert^2$, response function, Eq (1-18)
$M_L(k,\omega)$	$=\lvert\Gamma(k,\omega,-L)\rvert^2$, pressure response function at inner dome surface
$M_1(k,\omega)$	$= \lvert 1 + \Gamma_1(k,\omega)\rvert^2$, total pressure response function at outer surface of plate
$M(b)$	certain function defined for $0 \leq b < 1$ by App. 2, Eq. (26), and graphed in Figure 2-3
m	number large relative to unity (e.g., $\sim 2\pi$), used in mR_o^{-1}, a wave-number cutoff for the area-averaging factor [Eq.(2-65a)]

n	index for radial mode number; sometimes also represents parameter with values n=4 for plate, n=2 for membrane
N_{mn} [N_n]	normalizing coefficient for $J_m(k_{mn}R)$ [$J_o(k_nR)$]
n_o, n_2, n_r	integers $\gg 1$ [Eqs. (2-28), (2-42a), (2-42b)]
$n_{\pm}$	mode numbers defined by Eq. (2-29)
N_2, N_3	numbers $\gg 1$ [Eqs. (2-48), (2-53)]
N_1	number $\gtrsim 1$ [Eq. (2-48)]
$N(z_1, z_2)$	undetermined function of the dimensionless arguments z_1, z_2 [Eq. (2-78)]
N	number of elements in flush array [Eq. (2-100)]
N'	Same in shielded array [Eqs. (2-101), (2-101.1)]
$p(\overline{R}, t)$	excitation pressure wave, Eq. (1-7)
p_o	amplitude of $p(\overline{R}, t)$
$p^{\pm}(\overline{r}, t)$	pressure wave in outer (inner) fluid
$p_o^{\pm}$	amplitude of $p^{\pm}(\overline{r}, t)$
$P(\mathbf{k}, \omega)$	wavenumber-frequency spectrum of some pressure; $\mathbf{k}$ is wave vector parallel to surface bounding flow
$P_o(\mathbf{k}, \omega)$	same for pressure due to turbulent boundary layer
$P_1(\mathbf{k}, \omega)$	same for total pressure on outside of plate
$P(\omega)$	frequency spectrum of some pressure
$P_o(\omega)$	same for pressure due to turbulent boundary layer
$P(\omega, \overline{R}_1, \overline{R}_2, y)$	cross-spectral density of pressure between points $\overline{R}_1$, $\overline{R}_2$ at depth -y within dome [Eq. (2-7)]
$P_{o\pm}(\omega)$	part of $P_o(\omega)$ due to wave numbers $k \gtrless \omega/U_\infty$ Eq. (2-63)
$q_r(\omega)$	$q(k, \omega)$ at $k = k_r(\omega)$
$q_r'(\omega)$	$q(k, \omega)$ at $k = k_r'(\omega)$
$q_{r1}(\omega)$	$q(k, \omega)$ at $k = k_{r1}(\omega)$
q_o	$q(k, \omega)$ at $k \simeq R_o^{-1}$ [Eq. (2-164)]
$q(k, \omega)$	$(k/\hat{k}_o)^{n/2}$ (plate n=4, membrane n=2) [Eq. (1-2)].

$Q(\omega)$	frequency spectrum of some pressure averaged over circular area of radius R_o [Eq. (1-27)]
$Q_o(\omega)$	same for pressure due to turbulent boundary layer
$Q(\omega,y)$	same for acoustic pressure induced by the turbulent flow at depth -y within dome
$Q_1(\omega)$	same for total (turbulent plus acoustic) pressure at outside of plate
$Q_L(\omega)$	$= Q(\omega,-L)$ in case of laterally infinite dome
$Q_L^r(\omega)$	resonance part of $Q_L(\omega)$ [Eq. (1-32)]
$Q_L^{'}(\omega)$	nonresonance part of $Q_L(\omega)$ [Eq. (1-32)]
$Q_{L+}(\omega)$	part of $Q_L^{'}(\omega)$ from wave-numbers $k > \omega/U_\infty$ [Eq. (1-34)]
$Q(\mathbf{k},\omega,y)\,d^2\mathbf{k}$	contribution to $Q(\omega,y)$ from wavenumber element $d^2\mathbf{k}$
$Q_{o\pm}(\omega)$	part of $Q_o(\omega)$ due to wave numbers $k \gtrless \omega/U_\infty$ [Eq. (2-64)]
$Q_\pm(\omega,y)$ [$Q_{1\pm}(\omega)$]	part of $Q(\omega,y)$ [$Q_1(\omega)$] due to wave numbers $k > k_+$, $k < k_-$, respectively [Eq. (2-25)] or, nearly equivalently, to $k \gtrless \omega/U_\infty$ (convective and nonconvective parts)
$Q_+^\infty(\omega,y)$ [$Q_{1+}^\infty(\omega)$]	part of $Q_+(\omega,y)$ [$Q_{1+}(\omega)$] due to modal wave numbers $k_n > k_+$ (direct convective part) [Eqs. (2-37, (2-39)]
$Q_+^-(\omega,y)$ [$Q_{1+}^-(\omega)$]	part of $Q_+(\omega,y)$ [$Q_{1+}(\omega)$] due to $k_n < k_-$ (convective overlap part)
$Q_-^r(\omega,y)$	resonance contribution to $Q_-(\omega,y)$ (direct resonance part)
$Q_-^p(\omega,y)$	nonresonance contribution to $Q_-(\omega,y)$, when distinguishable from Q_-^r (direct propagating part)
$Q_+^r(\omega,y)$	resonance contribution to $Q_+(\omega,y)$ (resonance overlap part)
$Q_+^p(\omega,y)$	nonresonance contribution to $Q_+(\omega,y)$ (propagating overlap part)
$Q_-^t(\omega,y)$	part of $Q_-^p(\omega,y)$ from $k < \omega/c$ (transmitted) [Eq. (2-129)]

$Q_-^a(\omega,y)$	part of $Q_-^p(\omega,y)$ from $k > \omega/c$ (attenuated)
$\hat{Q}(\omega)$	frequency spectrum of total (turbulent plus acoustic) pressure averaged over outer surface of rigid circular plug of radius R_o severed from plate bounding flow [Eq. (2-141)]
$\delta\hat{Q}_-(\omega)$	acoustic part of $\hat{Q}(\omega)$ due to excitation wave numbers $k < k_-$
$\delta\hat{Q}_+(\omega)$	acoustic part of $\hat{Q}(\omega)$ due to $k > k_+$
$\delta\hat{Q}_-^r(\omega)$	resonance contribution to $\delta\hat{Q}_-(\omega)$ [Eq. (2-148)]
$\delta\hat{Q}_-^p(\omega)$	nonresonance contribution to $\delta\hat{Q}_-(\omega)$, when distinguishable from $\delta\hat{Q}_-^r(\omega)$
$\delta\hat{Q}_+^r(\omega)$	resonance contribution to $\delta\hat{Q}_+(\omega)$
$\delta\hat{Q}_+^p(\omega)$	nonresonance contribution to $\delta\hat{Q}_+(\omega)$, when distinguishable from $\delta\hat{Q}_+^r(\omega)$
$Q_{o-}^A(\omega)$	frequency spectrum of pressure due to turbulent boundary layer from $k < k_-$ averaged over active area A of array of flush elements of radius R_o
$Q_{o+}^A(\omega)$	same from $k > k_+$
$Q_-^A(\omega,y)$	spectrum of acoustic pressure induced by turbulent flow from $k < k_-$ averaged over active area A of array of shielded elements of radius r_o at depth -y
$Q_+^A(\omega,y)$	same from $k > k_+$
$\bar{r}$	position vector (x,y,z) with x directed downstream and y directed normally to the boundary into the region of flow
$\bar{R}$	planar position vector (x,z)
$R_o, r_o, R_{o1}, R_{o2}, R_{o3}$	radii of circular averaging areas
R_e	effective averaging radius defined by Eq. (2-99)
R_T	linear dimension of array, $\sim (A_T/\pi)^{1/2}$

r_{om}	minimum permissible radius r_o of shielded element
R	real part of $z(k,\omega)$, Eq. (1-36)
R_ω	real part of $z(k_\omega,\omega)$, Eq. (2-146)
$S_n(k,y)$	(ω suppressed) factor in $B_n(k,y)$ in case of free lateral wall, Eq. (2-23)
$S_{1n}(k)$	same in $B_{1n}(k)$
$\hat{S}_n(k)$	$S_{1n}(k)$ with added factor $1-\xi k_{2n}/k_\omega$ [Eq. (2-140)]
s	mode with wave number next below specified excitation wave-number k, defined by Eq. (2-29)
$s(\omega)$	defined by Eq. (2-103.9)
S_e	length defined by Eq. (2-133.1)
$T_n(k,\omega)$	(ω suppressed) factor in $B_n(k,y)$ in case of rigid lateral wall, Eq. (2-23)
$T_{1n}(k)$	same in $B_{1n}(k)$
$\hat{T}_n(k)$	$T_{1n}(k)$ with added factor $1-\xi k_{2n}/k_\omega$ [Eq. (2-140)]
T	tension in isotropic membrane [Eq. (1-4)]
$T(k_\omega R_o)$	function defined by Eq. (2-95.1)
t	time
U_∞	asymptotic mean flow velocity over plane boundary
u(y)	mean flow velocity at distance y from boundary
u_B	local velocity (y direction) of plate [Eq. (2-138)]
u_{Bn}	same for mode n
v_*	friction velocity for a turbulent boundary layer ($\sim 0.03 U_\infty$ for typical Reynolds numbers)
X	imaginary part of $z(k,\omega)$, Eq. (1-36)
X_ω	imaginary part of $z(k_\omega,\omega)$, Eq. (2-146)
x_{mn}, x_n	$=k_{mn}a$, $k_n a$, eigenvalues determined by lateral boundary conditions, e.g. Eqs. (2-20), (2-21)

$z(k,\omega)$	acoustic impedance of dome cover for parallel wave-number k and frequency ω
z_ω	$=z(k_\omega,\omega)$
$z_L(k,\omega)$	acoustic impedance of surface (y=-L) beneath dome
$z_R(\omega)$	radiation impedance of piston of radius R_o in given plate [Eq. (2-138)]
$\alpha(k,\omega)$	phase angle of pressure wave interior to dome, determined from z_L by Eq. (1-11)
$\hat{\alpha}(k,\omega)$	magnitude of imaginary phase angle, determined from z_L by Eq. (1-11)
β	frequency parameter expressing viscous damping of dome cover [Eq. (1-1)]
β_n	modal expansion coefficient for interior pressure, Eq. (2-3)
$\Gamma(k,y)$	(ω suppressed) acoustic response coefficient for pressure wave (k,ω) at depth -y in dome, Eq. (1-8.1)
$\Gamma_1(k)$	same for wave (k,ω) at outer surface of plate
γ	phase angle of pressure wave interior to dome, determined from z_L by App. 1, Eq. (14.2)
$\hat{\gamma}$	magnitude of imaginary phase angle, determined from z_L by App. 1, Eq. (14.1)
γ_{ns}	coefficient expressing interior pressure response in mode n (normalized to y=0-) per unit exterior pressure excitation in mode s, Eq. (2-6)
ϵ_ν	half width in logarithmic frequency of resonance of ν-th mode, Eq. (2-113)
ϵ	reciprocal acoustic-cancellation factor for plug in plate, Eq. (2-164)
ζ	dimensionless hysteretic damping coefficient for dome cover or plate bounding flow [Eq. (1-1)]
$\boldsymbol{\zeta}$	$=(\zeta_1, \zeta_3)$ planar separation vector in spatial correlation functions

$\eta(\bar{R},t)$	displacement (y direction) of vibrating dome cover or plate at $\bar{R}$ at time t
η	coefficient yielding a variously defined effective mean flow velocity as ηU_∞
$\theta(k,\omega)$	angle defined by Eq. (1-11)
$\hat{\theta}(k,\omega)$	magnitude of imaginary angle, defined by Eq. (1-11)
$\theta_r'(k,\omega)$	$=\theta(k_r',\omega)$ [Eq. (1-53.1)]
$\Lambda(k,y)$	(ω suppressed) cushioning-area-averaging factor for depth-y and excitation wave number $k > k_+$, defined by Eq. (2-80)
$\check{\Lambda}(k,y)$	local wavenumber (k) average of $\Lambda(k,y)$
$\Lambda_\infty(k,y)$	part of $\check{\Lambda}$ due to $k > k_+$ (direct convective part) [Eq. (2-81)]
$\Lambda_-(k,y)$	part of $\check{\Lambda}$ due to $k < k_-$ (convective overlap part)
$\Lambda_o(k)$	averaging factor for excitation wave, Eq. (2-84)
$\hat{\Lambda}_-(k)$	local wavenumber average of response-area-averaging factor for pressure on plug in plate from a $k > k_+$, Eq.(2-147]
λ	$=2\pi c/\omega$, wave length of sound in fluid(s)
ν	mode nearest resonance
ξ	defined by Eq. (2-139)
ρ_o	mass density of plate
$\rho^\pm$	density of outer (inner) fluid
ρ	$=\rho^\pm$ if $\rho^+ = \rho^-$, or ρ^+ if $\rho^- = 0$
σ	mass per unit area of dome cover or boundary plate
σ_o	Poisson's ratio for plate material
Σ_ω	partial sum $\Sigma(-)^n T_n$ or $\Sigma(-)^n S_n$ from n such that $k_n \simeq k_\omega$ when $k_\omega a \gg \pi$ [Eq. (2-41)]
Σ_r	similar partial sum from n such that $k_n \simeq k_r$
$\hat{\Sigma}_\omega$	as Σ_ω with T_n, S_n replaced by $\hat{T}_n$, $\hat{S}_n$
τ	time separation in temporal correlation functions

$X_n(\bar{R})$	eigenfunction of two-dimensional Helmholtz equation [Eq. (2-1)]
$\psi(b)$	function defined by Eq. (2-91.1) and shown in Fig. 2-3
ω	angular frequency
$\omega_r(k)$	frequency of resonance of coupled dome system for planar wave-number k [Eq. (1-54)]
$\delta\omega(k)$	frequency half width of resonance
$\omega_{r1}(k)$	same as $\omega_r(k)$ for coupled plate-fluid system
$\delta\omega_1(k)$	half width of this resonance at ω_{r1}
ω_ν	frequency such that $k_r(\omega_\nu) = k_\nu$

REFERENCES

1. R.H. Kraichnan, J. Acous. Soc. Am. 28, 378 (1956).

2. I. Dyer, Second Symposium on Naval Hydrodynamics, August 25-29, 1958, ACR-38, ONR, Washington, D.C.

3. G.M. Corcos, J. Acous. Soc. Am. 35 192 (1963).

4. R.H. Kraichnan, J. Acous. Soc. Am. 29, 65 (1957).

5. G.M. Corcos and H.W. Liepmann, "On the Transmission Through a Fuselage Wall of Boundary Layer Noise," Douglas, Santa Monica Division, Rept. SM-19570, December 1955.

6. H.S. Ribner, "Boundary-Layer-Induced Noise in the Interior of Aircraft," UTIA Rept. 37, April, 1956.

7. D.H. Tack and R.F. Lambert, J. Aerospace Sci. 29, 311 (1962).

8. M. el Baroudi, G. Ludwig, and H. Ribner, "An Experimental Investigation of Turbulence-Excited Panel Vibration and Noise (Boundary-Layer Noise)," AGARD Rept. 465, April 1963.

9. D.E. Davies, "The Radiation Sound Field of a Rectangular Panel Set in an Infinite Wall and Excited by a Turbulent Boundary Layer," R.A.E. Tech. Note Structures 355, April 1964.

10. C.A. Mercer, J. Sound Vib. 2, 293 (1965).

11. L. Maestrello, J. Sound Vib. 2, 270 (1965).

12. J.E. Ffowcs Williams and R.H. Lyon, "The Sound Radiated from Turbulent Flows Near Flexible Boundaries," BBN Rept. No. 1054, Ncnr 2321(00), August 1963.

13. Sonar Dome Study Interim Engineering Report, NObsr 91145, Raytheon Co., October 1964, App. II (Confidential).

14. J.E. Ffowcs Williams, J. Fluid Mech. 22, 507 (1965).

15. A. Powell, J. Acoust. Soc. Am. 30, 1136 (1958).

16. D.M. Chase, "Dome Effectiveness and the Scaling and Area Dependence of Boundary-Layer Pressure Spectra Viewed with Regard to Characteristic Wavenumber Ranges," Proceedings 23rd Navy Symposium on Underwater Acoustics, November 1965 (Confidential).

17. W.W. Willmarth and F.W Roos, J. Fluid Mech. 22, 81 (1965).

18. O.M. Phillips, Proc. Roy. Soc. 234A, 327 (1956).

APPENDIX 1

DERIVATION OF ACOUSTIC RESPONSE FUNCTIONS FOR THE INFINITE DOME SYSTEM

The solution of the basic acoustic problem of the text will be given here (see Fig. (1-1)).The driving pressure p assumed to be applied to the plate or membrane is given by

$$p(\bar{R},t) = p_o e^{i\bar{k}\cdot\bar{R}-i\omega t}. \quad [R=(x,z)]. \tag{0}$$

Let $\phi^{\pm}(\bar{r},t)$ $[\bar{r}=(x,y,z)]$ be the velocity potentials for the acoustic field in the outer (+) and inner (-) fluids and $p^{\pm}(\bar{r},t)$ the corresponding acoustic pressures:

$$p^{\pm} = -\rho^{\pm}\partial\phi^{\pm}/\partial t. \tag{1}$$

The equation of motion of the membrane is

$$\sigma\ddot{\eta}-T(1-i\zeta)\nabla^2\eta + \sigma\beta\dot{\eta} = (p^- - p^+)_{y=o} - p, \tag{2}$$

where T is the tension (force/length) on the membrane and σ its mass per unit area. (Unspecified symbols are defined in the text at Eq. (1-3).) If a thin plate replaces the membrane, the equation becomes instead

$$\sigma\ddot{\eta} + \sigma a_o^2(1-i\zeta)\nabla^4\eta + \sigma\beta\dot{\eta} = (p^- - p^+)_{y=o} - p, \tag{3}$$

with $\sigma = \rho_o h_o$. The acoustic wave equations for outer and inner media are

$$\nabla^2\phi^{\pm} - (c^{\pm})^{-2}\ddot{\phi}^{\pm} = 0. \tag{4}$$

The boundary conditions that apply to η, ϕ^+, ϕ^- are the following:

$$\dot{\eta} = (\partial\phi^+/\partial y)_{y=o} \quad \text{(contact of plate and outer fluid)} \quad (5)$$

$$\dot{\eta} = (\partial\phi^-/\partial y)_{y=o} \quad \text{(contact of plate and inner fluid)} \quad (6)$$

$$\rho^-\left[\phi^-/(\partial\phi^-/\partial y)\right]_{y=-L} = z_L(k,\omega) \quad \text{(specified impedance at inner surface)} \quad (7)$$

$$(k^2=k_1^2+k_3^2)$$

ϕ^+ contains no traveling or exponentially decreasing wave in the -y direction. (8)

In view of Eq. (0) and condition (8) we may assume

$$\eta(\overline{R},t) = \eta_o e^{i\overline{k}.\overline{R}-i\omega t} \quad (8.1)$$

$$\phi^+(\overline{r},t) = \phi_0^+ \, e^{ik_2^+ y + i\overline{k}.\overline{R} - i\omega t} \quad (8.2)$$

$$\phi^-(\overline{r},t) = \phi_0^- \frac{\cos[k_2^-(y+L)+\alpha]}{\cos(k_2^-L+\alpha)} e^{i\overline{k}.\overline{R} - i\omega t}, \quad (9)$$

where, to satisfy Eq. (4),

$$(k_2^{\pm})^2 = (\omega/c^{\pm})^2 - k^2,$$

and, to satisfy Eq. (7),

$$\tan\alpha = i\omega\rho^-/k_2^- z_L.$$

The remaining Eqs. (2) or (3), (5), and (6) reduce to linear equations for η_0, ϕ_0^+ and ϕ_0^-, from which all desired quantities can be formed. Eqs. (2) and (3) may then be written in the generalized form

$$-i\omega z\eta_0 = -i\omega(\rho^-\phi_0^- - \rho^+\phi_0^+) - p_0, \tag{10}$$

where z denotes the impedance of the plate or membrane, namely

$$z(k,\omega) = \sigma[i\omega q^2(1-i\zeta) - i\omega + \beta] \tag{11}$$

with

$$q = \begin{cases} a_0 k^2/\omega & \text{(plate)} \\ c_0 k/\omega & \text{(membrane)}. \end{cases}$$

Eq. (10) and the reduced Eqs. (5) and (6) may be written

$$\begin{bmatrix} -i\omega z & i\rho^+\omega & -i\rho^-\omega \\ -i\omega & -ik_2^+ & 0 \\ -i\omega & 0 & k_2^- \tan(k_2^- L+\alpha) \end{bmatrix} \begin{bmatrix} \eta_0 \\ \phi_0^+ \\ \phi_0^- \end{bmatrix} = \begin{bmatrix} -p_0 \\ 0 \\ 0 \end{bmatrix} \tag{12}$$

From (12) we find, in particular, for the ratio of the interior to the driving pressure

$$\Gamma(k,\omega,y) = \frac{p^-(\bar{r},t)}{p(\bar{R},t)} = \frac{\cos[k_2^-(y+L)+\alpha]}{\cos(k_2^- L+\alpha) - i\sin(k_2^- L+\alpha)[(k_2^-\rho^+/k_2^+\rho^-) + (zk_2^-/\rho^-\omega)]}$$

(13a)

If $k > \omega/c^{\pm}$, the result is preferably written*

$$\Gamma(k,\omega,y) = \frac{ch[K_2^-(y+L)+\hat{\alpha}]}{ch(K_2^-L+\hat{\alpha})+sh(K_2^-L+\hat{\alpha})[(K_2^-\rho^+/K_2^+\rho^-)+i(zK_2^-/\rho^-\omega)]} \tag{13b}$$

where

$$K_2^{\pm} = [k^2-(\omega/c^{\pm})^2]^{1/2},$$

$$th\,\hat{\alpha} = -i\omega\rho^-/K_2^-z_L \qquad (\text{i.e. } \hat{\alpha} = -i\alpha) \tag{14}$$

as obtained from (13a) by setting $ik_2^{\pm} = -K_2^{\pm}$. If $\omega\rho^-/K_2^-|z_L| > 1$(with z_L pure imaginary), as for a pressure-release inner surface, the definition (14) is inconvenient; one may instead define

$$cth\,\hat{\gamma} = -i\omega\rho^-/K_2^-z_L,$$

and obtain in place of (13b)

$$(k,\omega,y) = \frac{sh[K_2^-(y+L)+\hat{\gamma}]}{sh(K_2^-L+\hat{\alpha})+ch(K_2^-L+\hat{\gamma})[(K_2^-\rho^+/K_2^+\rho^-)+i(zK_2^-/\rho^-\omega)]} . \tag{13c}$$

Though not necessary, we may also define

$$\cot\gamma = -i\rho^-\omega/k_2^-z_L$$

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* If $c^+ \neq c^-$, there will be a range of k where $k < \omega/c^+$ but $k > \omega/c^-$ or vice-versa; the corresponding suitable form of $\Gamma(k,\omega,y)$ will not be of sufficient use to be recorded here.

and obtain in place of (13a)

$$\Gamma(k,\omega,y) = \frac{\sin[k_2^-(y+L)+\gamma]}{\sin(k_2^-L+\gamma)+i\cos(k_2^-L+\gamma)\,[(k_2^-\rho^+/k_2^+\rho^-)+(zk_2^-/\rho^-\omega)]} \tag{13d}$$

Likewise, if the inner medium is infinitely thick, so that the impedance at any chosen depth y=-L is

$$z_L(k,\omega) = \omega\rho^-/k_2^-,$$

we may write, in place of (13),

$$\Gamma(k,\omega,y) = \begin{cases} \left[1+(k_2^-\rho^+/k_2^+\rho^-)+(zk_2^-/\rho^-\omega)\right]^{-1} e^{-ik_2^-y} & (k < \omega/c^{\pm}) \quad (13e) \\ \left[1+(K_2^-\rho^+/K_2^+\rho^-)+i(zK_2^-/\rho^-\omega)\right]^{-1} e^{K_2^-y} & (k > \omega/c^{\pm}) \quad (13f) \end{cases}$$

We may likewise find from (12) the ratio of the acoustic to the driving pressure on the outer surface, i.e.,

$$\Gamma_1(k,\omega) = \frac{p^+(y=0)}{P} = \frac{i\omega\rho^+\phi_0^+}{P_0}\,.$$

We write the result only for the case where there is no interior fluid (ρ^-=0), dropping the index +:

$$\Gamma_1(k,\omega) = \begin{cases} -[1+(zk_2/\rho\omega)]^{-1} & (k < \omega/c) \\ -[1+i(zK_2/\rho\omega)]^{-1} & (k > \omega/c) \end{cases} \tag{15}$$

APPENDIX 2

ACOUSTIC EXCITATION WITHIN BOUNDED FLUID BODIES OF LARGE LATERAL DIMENSION

1. Introduction

We present here an asymtotic solution for a group of acoustic boundary-value problems. The type of problem in question is pertinent to the study of models of sonar domes and sheaths (but not to boots on hydrophones where the area of the boot is not large compared to the active transducer area and not large compared to the wave length of sound in the outside medium). Some of the results obtained in the following sections will be outlined here first. Consider a semi-infinite circular cylinder of ideal fluid of radius a having either rigid or pressure-release lateral walls and terminated in a cross section (y = - L) of given impedance (see Fig. A2-1). At another cross section (y = 0) there is an applied pressure discontinuity of definite frequency ω and wave number k parallel to the cross section. Consider the resulting pressure transmitted acoustically to a point y on the axis between the cross sections (0 > y > - L). In an envisaged application the pressure source is due to a turbulent flow over the outside of the cross section y = 0. At the cross section y = 0, just inside the pressure source, there can be interposed a thin dynamic sheet (plate or membrane), provided it satisfies certain special boundary conditions at its periphery, r = a.

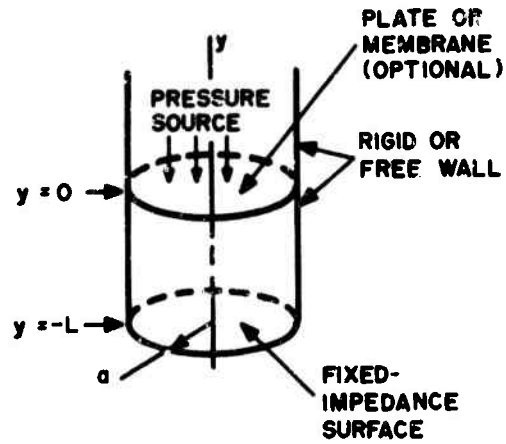
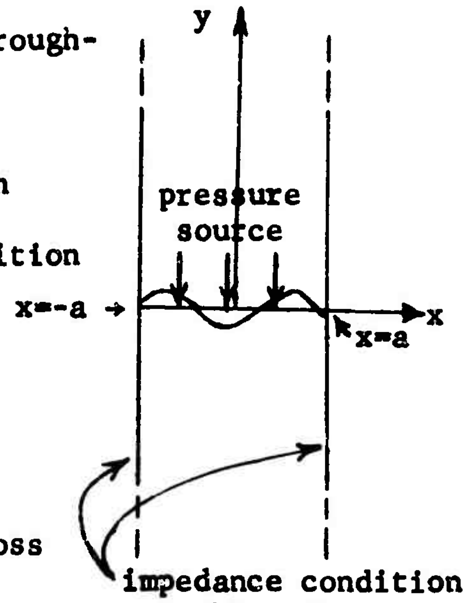


FIGURE A2-1. RUDIMENTARY FINITE DOME MODEL

If one lets the lateral size become infinite, i.e. $a \to \infty$, at fixed k^{-1} and k_ω^{-1}, where $k_\omega = \omega/c$ is the sound wave number in the fluid and $k > k_\omega$, then the acoustic pressure is conveyed by waves with the same wave number k as the pressure source parallel to the planes y = constant and is attenuated nearly exponentially with increasing $|y|$. If, on the other hand, one lets $a \to \infty$ at fixed k^{-1} and k_ω^{-1}, but with $|y|$ (and L) $\to \infty$, where $|y|$, however, increases more slowly than a (e.g. as a^δ, $0 < \delta < 1$), then the acoustic pressure in the limit is conveyed exclusively by waves with the wave number k_ω parallel to the source plane and hence with vanishing wave number in the y direction. The same is true if the circular cylinder is changed to a semi-infinite slab with faces normal to the wave vector of the planar pressure source. In the second type of limit in all cases the pressure varies with a (disregarding dependence on y) as $a^{-3/2}$. We proceed to the detailed calculations.

2. Infinite rectangular geometry; dipole pressure source without dome

Consider an infinite slab of ideal fluid subject to a fixed impedance condition on the two faces ($x = \pm a$) and a plane pressure source with time dependence $\exp(-i\omega t)$ on one side of a cross section (y = 0, - a < x < a) normal to the two faces (see figure). The pressure source, and hence all other quantities, are assumed independent of the third coordinate, z. Omitting time dependence throughout, let the applied pressure be P(x). Denote by p(x,y) the acoustic pressure field in the slab, in which the sound velocity is c. The boundary condition (b.c.) on pressure at y = 0 is then



(1) $$p(x,0+) + P(x) = p(x,0-),$$

i.e. there is a discontinuity -P(x) in p(x,y) across the plane y = 0. Continuity of fluid velocity at y = 0 implies the second b.c.

(2) $$(\partial p/\partial y)_{y=0+} = (\partial p/\partial y)_{y=0-} .$$

The solution can be expanded in normal modes for the x dependence. Taking account of (1) and (2) one has

(3) $$p(x,y) = \mp (1/2) \sum_{n=0}^{\infty} c_n \psi_n(x) \exp(ik_{2n}|y|)$$

for $y \gtrless 0$, respectively, where the $\psi_n(x)$ are a complete set of orthonormal functions satisfying the one-dimensional Helmholtz equation

(4) $$(d^2/dx^2 + k_n^2)\psi_n(x) = 0$$

and the given impedance conditions at $x = \pm a$; c_n is given by

(5) $$c_n = \int_{-a}^{a} dx\ \psi_n^*(x)\ P(x);$$

and

(6) $$k_{2n}^2 = k_\omega^2 - k_n^2\ , \quad k_\omega \equiv \omega/c.$$

By virtue of (4) and (6), the expression (3) properly satisfies the Helmholtz equation for the acoustic field (where $y \neq 0$):

(7) $$(\nabla^2 + k_\omega^2)\ p(x,y) = 0. \qquad (y \neq 0)$$

For n such that $k_{2n}^2 < 0$, one replaces ik_{2n} by $-(-k_{2n}^2)^{1/2}$.

2.1 Rigid walls

Suppose that the walls at $x = \pm a$ are rigid. Consider first a pressure source

(8) $$P(x) = p_o \cos kx.$$

Then

(9) $$\psi_n(x) = a^{-1/2} \cos k_n x, \qquad n > 0$$

$$\psi_o(x) = 2^{-1/2},$$

(9.1) $$k_n = n\pi/a,$$

and (5) yields

(10) $$c_n = 2\, p_o s_n a^{-1/2}\, k \sin ka \frac{(-)^n}{k^2 - k_n^2},$$

where

$$s_n = 1 \;(n \neq 0), \quad s_o = 2^{-1/2}.$$

Hence

(11) $$p(x,y) = 2\, p_o a^{-1}\, k \sin ka \left\{ \sum_{n=1}^{\infty} (-)^n f(n) + \frac{1}{2} f(0) \right\}$$

$$= p_o a^{-1}\, k \sin ka \sum_{n=-\infty}^{\infty} (-)^n f(n),$$

where

(12) $$f(n) = \mp (1/2) \frac{\cos k_n x}{k^2 - k_n^2} \exp(ik_{2n} |y|).$$

The infinite sum in (11) can be transformed in various ways by forming the function $\pi f(z) \csc \pi z$ for complex z and integrating it around one or another closed contour in the z plane (Sommerfeld-Watson method), bearing in mind that a branch cut connects the branch points $z = \pm k_\omega a/\pi$, at which $k_z = k_\omega$ and $k_{2z} = 0$. One such path yields a result that can be written (we assume $k > k_\omega$)

(13) $$p(x,y) = p_\infty(x,y) + \Delta p(x,y),$$

where

(13.1) $$p_\infty(x,y) = \mp (1/2) p_o \cos kx \exp\left[- (k^2 - k_\omega^2)^{1/2} |y|\right]$$

$$\Delta p(x,y) = \mp \pi^{-1} p_o ka^{-1} \sin ka \Big\{ i(\pi/2) \sum_{n=-m}^{m} (-)^n \frac{\cos k_n x}{k^2 - k_n^2} \sin\left[(k_\omega^2 - k_n^2)^{1/2} |y|\right]$$

$$- \text{P.V.} \int_0^{k_\omega a} du \left[\frac{1}{\sin u} \frac{\cos(xu/a)}{k^2 - (u/a)^2} \sin\left\{\left[k_\omega^2 - (u/a)^2\right]^{1/2} |y|\right\}\right.$$

$$\left. - \frac{1}{\text{sh} u} \frac{\text{ch}(xu/a)}{k^2 + (u/a)^2} \sin\left\{\left[k_\omega^2 + (u/a)^2\right]^{1/2} |y|\right\}\right] \tag{14}$$

$$+ \int_{k_\omega a}^{\infty} du \frac{1}{\text{sh} u} \frac{\text{ch}(xu/a)}{k^2 + (u/a)^2} \sin\left\{\left[k_\omega^2 + (u/a)^2\right]^{1/2} |y|\right\} \Big\}$$

where the integer m is defined by $m \leq k_\omega a/\pi < m + 1$, P.V. denotes principal value, and ch $\equiv$ cosh, sh $\equiv$ sinh. In Eq. (14), Δp is expressed as the sum of a number of real integrals, whose values are real, and a pure imaginary term proportional to a finite sum. At $y = 0$, clearly $\Delta p = 0$ and by (13)

$$p(x,0) = \mp (1/2) p_o \cos kx,$$

which satisfies the b.c. (1), in view of (8), as it must.

Imagine now that the distance between the walls, 2a, increases indefinitely while the lengths, k^{-1}, x,y remain fixed. The length k_ω^{-1} ($= c/\omega$) may be taken to be fixed or to increase as <u>a</u>. The real contributions to Δp clearly vanish in this limit. The number of terms in the sum in the imaginary part increases proportionally to $k_\omega a$, but on account of the alternating signs (considered further below) this contribution vanishes as well, leaving only the first term p_∞, in (13). This term is just the solution for the problem in the absence of walls, where the slab (and source plane) are infinite in the x direction. That is, the pressure at a given point is unaffected by reflections from the walls when these recede to infinity. **(See also Ref. 15.)**

Now imagine instead that as $a \to \infty$, so also $|y| \to \infty$, but that $|y|$ does so more slowly, e.g.

$$y/a \propto a^{-\delta} \quad \text{with} \quad 0 < \delta < 1. \tag{14.1}$$

In this limit one has rather $p_\infty/\Delta p \to 0$. In fact, consider for any fixed wave number k_M, such that $k_M > k_\omega$, that part $p_M(x,y)$ of $p(x,y)$ in Eq. (11) that derives from modes n with $k_n > k_M$:

$$p_M(x,y) \equiv 2\, p_o a^{-1}\, k \sin ka \sum_{n=N+2}^{\infty} (-)^n f(n). \tag{15}$$

where N is defined by $k_{N+1} < k_M < k_{N+2}$ (thus N increases as $\underline{a}$, since k_M is independent of $\underline{a}$). (This partial sum, if $(k-k_M)a \gg \pi$, includes those terms that principally contribute to $p_\infty(x,y)$.) Each term in the sum (15) decreases exponentially with $|y|$, and one recognizes that in the limit where $a \to \infty$ and $|y| \to \infty$ in accord with (14.1), one will have $p_M/p \to 0$ and

(16) $$p(x,y) \to 2\, p_o a^{-1}\, k \sin ka \left\{ \frac{1}{2} f(0) + \sum_{n=1}^{N+1} (-)^n f(n) \right\} .$$

Stated otherwise without reference to limiting processes, in some domain of values of $k, a, x,$ and y (a domain of considerable interest in the intended applications), the pressure at depth $|y|$ from the source plane is well approximated not by the result p_∞ for infinite $\underline{a}$ (dome size), but by the contribution from the lowest modes n up to some modal wave number limit, i.e. for $k_n < k_M$, the contribution from higher modes, including those with $k_n \simeq k$, being negligible on account of their rapid exponential attenuation with $|y|$.

The limiting case to be dealt with here is that where, in addition to the previously assumed limiting parameter relations, namely

$$1/ka \to 0, \quad x/a \to 0, \quad y/a \to 0,$$

and relation (14.1), we have also

$$1/k_\omega a \to 0,$$

i.e. the lateral extent (parallel to the source plane) is large compared to a sound wave length. In this case the sum (11) [or (16)] for $p(x,y)$ can be summed, independently of any parameters, to leading order in a^{-1}, as shown below.

Take N in Eq. (16) to be even. Let $f(n) \doteq T(k_n)$, in accord with (12). For n such that $T(k')$ is analytic over $k_{n-(1/2)} < k' < k_{n+(1/2)}$ (where Eq. (9.1) is extended to nonintegral n), we have, since

$k_{n+1} - k_n = \pi/a,$

(17) $$f(n) - f(n+1) = (1/2)\left[f(n-1/2) - f(n+3/2)\right] + O((\pi/a)^3 T'''(k_n)),$$

where a prime denotes d/dk_n and $O(\epsilon)$ a quantity of the order of ϵ. Hence, apart from a modification required for n such that $T(k')$ is not analytic for k' near k_n, we have

(18) $$\sum_{n=o}^{N+1} (-)^n f(n) = (1/2)\left[f(-1/2) - f(N+3/2)\right] + O((\pi/a)^3 \sum_{\substack{n=o \\ even}}^{N} T'''(k_n))$$

By (12), $T(k')$ is not analytic at $k' = \pm k_\omega$, having branch points there on account of the factor $\exp\left[i(k_\omega^2 - k_n^2)^{1/2}|y|\right]$. Therefore, in a certain range of n where $k_n \simeq k_\omega$ we do not use (17); this range is specified as running from $n = j + 1 - n_2$ to $n = j + n_2$, where the integer j is defined by

(18.1) $$k_j \leq k_\omega < k_{j+1} \, .$$

If the wave number range thus to be excluded is called Δk, we note that the number of modes $2n_2$ in this range increases indefinitely as $\Delta ka \to \infty$; we are free to let $\Delta k \to 0$ in this limit, however, and so, as $k_\omega a \to \infty$, we let

(19) $$n_2/k_\omega a \to 0, \quad \text{while} \quad n_2 \to \infty,$$

e.g. $n_2/k_\omega a \propto a^{-\epsilon}$ with $0 < \epsilon < 1$ (cf. (14.1)). If we apply (17) outside the stated range of n, we obtain instead of (18)

$$(20)\quad \sum_{n=o}^{N+1} (-)^n f(n) = (1/2)\left[f(-1/2) - f(N+3/2)\right] + S_- + S_+ + O((\pi/a)^3 \sum_{\substack{n=o\\ \text{even}}}^{N}{}' T_n''')$$

where

$$S_- = \sum_{\substack{n=j+1-n_2\\ \text{even}}}^{j-1} \left[f(n) - f(n+1)\right] - (1/2)f(j + 1/2 - n_2),$$

(20.1)

$$S_+ = \sum_{\substack{n=j+1\\ \text{even}}}^{j-1+n_2} \left[f(n) - f(n+1)\right] + (1/2)f(j + 1/2 + n_2),$$

we assume for definiteness that j is odd, and the prime on the sum over T_n''' is intended to signify that the interval $j+1-n_2 \leq n \leq j-1+n_2$ is omitted.

Consider the final term in (20). From explicit differentiation of (12) with respect to k_n it can be seen that the part of T_n''' of lowest order in a^{-1} (with x fixed) is

$$(20.2)\qquad \mp \frac{1}{2} \frac{\cos k_n x}{k^2 - k_n^2} \left[\frac{d}{dk_{2n}} \exp(ik_{2n}|y|)\right] \frac{d^3 k_{2n}}{dk_n^3},$$

in which we have

$$(20.3)\qquad d^3k_{2n}/dk_n^3 = -3k_\infty^2 k_n k_{2n}^{-5};$$

this part is of lowest order because for n near the lower limit, $j + 1 - n_2$, of the excluded interval, we have

$k_n \sim k_\infty - n_2\pi/a$, whence $k_{2n} \sim (2k_\infty n_2\pi/a)^{1/2}$, as $n_2/k_\infty a \to 0$.

The dependence on $\underline{a}$ and n_2 of the lowest-order part of the last term in (20) is thus given, in view of (20.2) and (20.3), as

$$\propto a^{-3} \sum_{n=o}^{j+1-n_2} s_n k_{2n}^{-5} \underset{\sim}{\propto} a^{-2} \int_0^{k_\infty - n_2\pi/a} dk_n (k_\infty^2 - k_n^2)^{-5/2}$$

(20.4)

$$\sim O(a^{-1/2}\ n_2^{-3/2})$$

where $s_n \sim O(a^o\ n_2^o)$ and factors unrelated to the dependence on $\underline{a}$ and n_2 in the limit $k_\infty a \to \infty$, $n_2 \to \infty$, $n_2/k_\infty a \to 0$ have been dropped. A contribution of the same order as (20.4) results from the sum over n on the upper side of the excluded interval. It will be shown below that $S_- + S_+ (\equiv S)$ in (20) is also of the order of $a^{-1/2}$ in $\underline{a}$ but is independent of n_2 in the limit $n_2 \to \infty$. Hence in the limit in question, according to (20.4), the contribution of the last term in (20) in fact vanishes relative to $S_- + S_+$. We therefore neglect this term.

The term $-(1/2)f(N + 3/2)$ in (20) we can write directly from (12). By definition of N and the limiting relation (14.1), it can be neglected. The term $(1/2)f(- 1/2)$ in (20) may be combined in computation of $p(x,y)$ with the term $-(1/2)f(0)$, which must be added to (20) to form (16). The difference $(1/2)\left[f(- 1/2) - f(0)\right]$ may be found in the pertinent limit of large $\underline{a}$ by a Taylor expansion of (12) about $n = 0$. The linear term in this expansion vanishes leaving the difference $\sim O(a^{-2})$ which is of no lower order than contributions being discarded.

In computing the remaining terms $S_- + S_+$ in (20), for generality we replace expression (12) for $f(n)$ by

(21) $$f(n) = h(k_n,x)\ g(k_{2n},y),$$

where $g(k_{2n},y) = \mp (1/2) \exp(ik_{2n}|y|)$ and suppress the x and y dependence. We define b, a function of $k_\omega a$ that described the position of k_ω between adjacent modal wave numbers, by

(22) $$k_\omega - k_j = b(\pi/a) \quad (0 \leq b < 1),\ \text{i.e.}\ b = (k_\omega a/\pi) - j.$$

Then

$$k_n = k_j - (j-n)\pi/a = k_\omega - (j - n + b)\pi/a.$$

The sums in (20.1) have increasingly many terms as $n_2 \to \infty$ but for all of them, as stated above, $k_n \to k_\omega$ as $a \to \infty$. Proceeding first with S_-, from (22) we have

(23) $$k_{2n} = (k_\omega^2 - k_n^2)^{1/2} \to \sqrt{2}\, k_\omega^{1/2}(\pi/a)^{1/2}(j - n + b)^{1/2}$$

to lowest order in π/a. Thus

$$f(n) - f(n+1) \to h(k_\omega)(\partial g/\partial k_2)_o\ (k_{2n} - k_{2,n+1}),$$

(24)

$$f(j + 1/2 - n_2) \to h(k_\omega)\left[g_o + (\partial g/\partial k_2)_o\ k_{2,j} + 1/2 - n_2\right],$$

where $g_o = g(k_2 = 0)$, $(\partial g/\partial k_2)_o = (\partial g/\partial k_2)_{k_2} = 0$. From (20.1), (23), (24), we find in the limit of $n_2 \to \infty$

(25) $$S_- \to -(1/2)\, h(k_\omega) g_o + \sqrt{2}\, h(k_\omega)(\partial g/\partial k_2)_o k_\omega^{1/2} (\pi/a)^{1/2} M(b),$$

where

(26) $$M(b) = \lim_{n_2 \to \infty} \left\{ \sum_{\substack{m=1 \\ \text{odd}}}^{n_2 - 1} \left[(m+b)^{1/2} - (m-1+b)^{1/2} \right] - (1/2)(n_2 + b - 1/2)^{1/2} \right\}.$$

Similarly, with regard to S_+, which involves a sum in which $K_{2n}^2 \equiv -k_{2n}^2 > 0$, we have

$$f(n) - f(n+1) \to h(k_\omega)(\partial g/\partial k_2)_o \cdot i\, (K_{2n} - K_{2,n+1}).$$

Proceeding as before, we now find

(27) $$S \to (1/2)\, h(k_\omega) g_o - i\sqrt{2}\, h(k_\omega)(\partial g/\partial k_2)_o\, k_\omega^{1/2} (\pi/a)^{1/2} M(1-b).$$

Thus $S_- + S_+$ is $\sim O(a^{-1/2})$. Hence, from (16), (20), (25), and (27), we finally obtain, to lowest order in a^{-1},

(28) $$p(x,y) \to 2\, (2\pi)^{1/2}\, p_o\, a^{-3/2} k\, k_\omega^{1/2}\, h(k_\omega)(\partial g/\partial k_2)_o \times \sin ka \left[M(b) - i\, M(1-b) \right],$$

where, in the present instance of Eq. (12),

(28.1) $$h(k_\omega) = \cos k_\omega x/(k^2 - k_\omega^2),$$

$$(\partial g/\partial k_2)_o = \mp (1/2)\, i|y| = -\, i(1/2)y.$$

For values of $k_\omega a$ such that the integer j defined by (18.1) is even instead of odd as assumed above and b is still defined as at (22), the value of S is found to be the negative of the value of S for the same b in the case where j is odd. In other words, if we define j rather as that <u>odd</u> integer such that

$$k_j \leq k_\omega < k_{j+2},$$

and b by

$$k_\omega - k_j = b(\pi/a), \quad 0 \leq b < 2,$$

then, if $S_o(b)$ denotes S as computed above [Eqs. (25), (27)] for $0 \leq b < 1$, one has for the full range $0 \leq b < 2$:

(29)
$$S(b) = \begin{cases} S_o(b), & 0 \leq b < 1 \\ -S_o(b-1), & 1 \leq b < 2; \end{cases}$$

$$S_o(b) = \sqrt{2}\, h(k_\omega)(\partial g/\partial k_2)_o\, k_\omega^{1/2} (\pi/a)^{1/2} \left[M(b) - i\, M(1-b) \right].$$

According to (29), $p(x,y)$ will be a continuous function of $k_\omega a$, as it must be, only if $S_o(1) = - S_o(0)$, or equivalently if $M(1) = - M(0)$. This equality in fact holds, as one can show from the definition (26).

To lowest order, then, $p(x,y)$ [Eq. (28)] has <u>a</u> dependence of the form

$$a^{-3/2}\, E(k_\omega a),$$

where $E(z)$ is a periodic function with period 2π (or, apart from a sign reversal, π). The function $M(b)$ was programmed for machine computation by Dr. H. Steinberg; results, with use of (29), are given in Fig. 2-3 of the text.

According to the result embodied in (28), in the limit considered, the principal contribution to the pressure at (x,y) derives from modes whose x-wave numbers k_n nearly coincide with the wave number of sound, k_ω, at the given frequency, and whose y-wave numbers, therefore, nearly vanish, producing a linear dependence on distance $|y|$ from the source plane.

Now consider, in place of (8), a pressure source

(30) $$\tilde{P}(x) = p_o \sin kx,$$

where we apply a tilde to quantities to distinguish this new case. Then in place of (9) and (9.1)

(31) $$\tilde{\psi}_n(x) = a^{-1/2} \sin\tilde{k}_n x \quad (\text{for all } n \geq 0), \quad \tilde{k}_n = (n + 1/2)\pi/a.$$

Proceeding, we find in place of (11)

$$\tilde{p}(x,y) = -2\, p_o a^{-1} k \cos ka \sum_{n=o}^{\infty} (-)^n \tilde{f}(n), \tag{32}$$

where

$$\tilde{f}(n) = \mp \frac{1}{2} \frac{\sin \tilde{k}_n x}{k^2 - \tilde{k}_n^2} \exp\,(i\tilde{k}_{2n}|y|). \tag{33}$$

As at (13), in the limit of $a \to \infty$ at fixed k^{-1}, k_∞^{-1}, x, and y, the solution is that obtained in the absence of walls:

$$\tilde{p}_\infty(x,y) = \mp (1/2) p_o \sin kx \exp\left[-(k^2 - k_\infty^2)^{1/2} |y|\right]. \tag{34}$$

In the modified limit treated above, where $|y| \to \infty$ as $a \to \infty$ as specified at (14.1), we may cut off the infinite sum in (32) at $n = N + 1$, say, as in (16). Introducing the expansion (17) for $\tilde{f}(n) - \tilde{f}(n+1)$ and proceeding as before, we obtain an expression of the form of (20). There is no term analogous to $(1/2)\, f(0)$ to be subtracted this time, as seen on comparing (11) and (32); however, the term $(1/2)\, \tilde{f}(-1/2)$ of (20) itself vanishes by (31) and (33), since

$$\sin k_{-1/2} x = \sin 0 = 0.$$

To lowest order in a^{-1}, therefore, we obtain, as at (28),

$$\tilde{p}(x,y) \to -2\ (2\pi)^{1/2} p_o a^{-3/2} k\ k_\omega^{1/2}\ h(k_\omega)\,(\partial g/\partial k_2)_o$$

$$(35) \qquad \times \cos ka \left[M(b) - iM(1-b) \right] ,$$

where by (33) we have in place of (28.1)

$$h(k_\omega) = \sin k_\omega x/(k^2 - k_\omega^2),$$

$$(36) \qquad (\partial \tilde{g}/\partial k_2)_o = -\, i(1/2)y.$$

The pressure corresponding to an excitation

$$\hat{P}(x) = p_o\ e^{ikx},$$

obtained by combining (28) and (35), can then be written

$$\hat{p}(x,y) \to -\ (2\pi)^{1/2} p_o \frac{kk_\omega^{1/2} y}{a^{3/2}(k^2 - k_\omega^2)} \left[M(1-b) + iM(b) \right]$$

$$(37) \qquad \times\ (\sin ka \cos k_\omega x - i \cos ka \sin k_\omega x).$$

2.2 <u>Free walls</u>

Now suppose that the walls at $x = \pm a$ are free (zero impedance). Consider first a pressure source

$$P(x) = p_o \cos kx.$$

Then, using the earlier notation for present quantities even if they differ from the earlier ones, we have

$$\psi_n(x) = a^{-1/2}\cos\tilde{k}_n x \qquad (n \geq 0)$$

with $\tilde{k}_n$ given at (31). The solution as $a \to \infty$ at fixed $|y|$, etc., is independent of the boundary condition at the wall and hence given again by (13.1). In the modified limit of concern, on the other hand, we obtain

$$p(x,y) \to -2\, p_o a^{-1} \cos ka \sum_{n=o}^{N+1} (-)^n f(n),$$

where

$$f(n) = \mp \frac{1}{2} \frac{\tilde{k}_n \cos\tilde{k}_n x}{k^2 - \tilde{k}_n^2} \exp(i\tilde{k}_{2n}|y|).$$

The term $(1/2)f(-1/2)$ again vanishes, this time on account of the factor $\tilde{k}_n$ in $f(n)$. We have

$$p(x,y) \to -2\,(2\pi)^{1/2}\, p_o a^{-3/2}\, k_\omega^{1/2}\, h(k_\omega)(\partial g/\partial \tilde{k}_2)_o$$

$$\times \cos ka \left[M(b) - iM(1-b)\right] \qquad (38)$$

with

$$h(k_\omega) = k_\omega \cos k_\omega x/(k^2 - k_\omega^2),$$

$$(\partial g/\partial \tilde{k}_2)_o = -i(1/2)y.$$

Now consider a pressure source

$$P(x) = p_o \sin kx.$$

We have

$$\psi_n(x) = a^{-1/2} \sin k_n x \qquad (n \geq 1)$$

with $k_n = n\pi/a$ as at (9.1). In the limit of concern

$$p(x,y) \rightarrow 2\, p_o a^{-1} \sin ka \sum_{n=1}^{N+1} (-)^n f(n),$$

where

$$f(n) = \mp \frac{1}{2} \frac{k_n \sin k_n x}{k^2 - k_n^2} \exp(ik_{2n}|y|).$$

The term $(1/2)f(-1/2)$ is $\sim 0(a^{-2})$ on account of the factors $k_n \sin k_n x$ in $f(n)$, and hence of the order of neglected terms. We have then

$$p(x,y) \rightarrow 2\,(2\pi)^{1/2}\, p_o a^{-3/2}\, k_\omega^{1/2}\, h(k_\omega)\,(\partial g/\partial k_2)_o \quad (39)$$

$$\times \sin ka \left[M(b) - iM(1-b) \right]$$

with

$$h(k_\omega) = k_\omega \sin k_\omega x/(k^2 - k_\omega^2),$$

$$(\partial g/\partial k_2)_o = -\, i(1/2)y \; .$$

For a pressure source

$$P(x) = p_o e^{ikx} ,$$

by combination of (38) and (39) we obtain

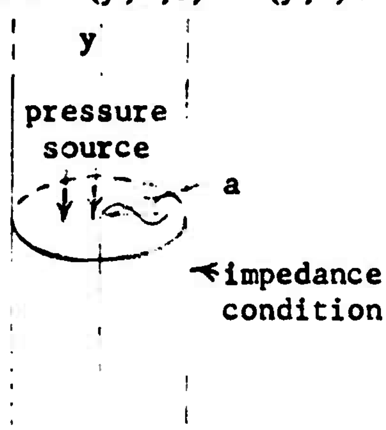
$$p(x,y) \rightarrow (2\pi)^{1/2} p_o \frac{k_\omega^{3/2}\, y}{a^{3/2}(k^2-k_\omega^2)}\Big[M(1-b) + iM(b)\Big] \tag{40}$$

$$\times\ (\cos k_\omega x \cos ka - i \sin k_\omega x \sin ka)$$

The amplitude of the pressure in the limit, according to (37) and (40), is smaller in the case of free walls than in the case of rigid walls by a factor k_ω/k.

3. <u>Cylindrical geometry; dipole pressure source without dome</u>

In place of the infinite slab we consider now an infinite circular cylinder of radius <u>a</u> with a pressure discontinuity applied over a cross section $(y = 0)$ as before and an impedance condition at the cylindrical surface. Let the cylindrical coordinates be $(y,r,\theta) \equiv (y,\vec{r})$. Consider a pressure source



$$P(r,\theta) = p_o e^{ikr\cos(\theta-\beta)}, \tag{41}$$

i.e. a plane (line) wave of definite wave number in direction β.

The resulting pressure field may be expanded in the form

$$p(r,\theta,y) = (2\pi)^{-1/2} p_o \sum_{m=-\infty}^{\infty} \sum_{n=o}^{\infty} C_{mn}\, J_m(k_{mn} r) e^{im(\theta-\beta)}\, g(k_{2mn}, y)\ , \tag{42}$$

where the C_{mn} are expansion coefficients corresponding to (41) which may be written explicitly, the k_{mn} are modal wave numbers that satisfy the b.c. at $r = a$, and

(43) $$g(k_{2mn}, y) = \mp (1/2)\exp(ik_{2mn}\, y) \text{ for } y \gtrless 0,\ k_{2mn} = (k_\omega^{\,2} - k_{mn}^{\,2})^{1/2}.$$

We shall consider only the average force, say $F(y)$, on a circular area of radius R_o centrally located in a cross section of the cylinder at y. Let any quantity q_{mn} for $m = 0$ be denoted simply by q_n. Then we have

(44) $$F(y) = \int_0^{2\pi} d\theta \int_0^{R_o} dr\ r\ p(r,\theta,y) = (2\pi)^{1/2} p_o \sum_n C_n\ k_n^{-1}\ R_o\ J_1(k_n R_o) g(k_{2n}, y).$$

The C_n are found to be given by

(45) $$C_n(ka) = 2(2\pi)^{1/2} \frac{ka\ J_o(x_n)\ J_1(ka) - x_n\ J_o(ka)\ J_1(x_n)}{\left[(ka)^2 - x_n^{\,2}\right]\left[J_o^{\,2}(x_n) + J_1^{\,2}(x_n)\right]},$$

where $x_n \equiv k_n a$.

3.1 Rigid walls

In this case the eigenvalues x_n are specified by

(46) $$J_1(x_n) = 0 \qquad (n \geq 0).$$

Eq. (45) then yields

$$C_n = 2(2\pi)^{1/2} \frac{ka\, J_1(ka)}{J_o(x_n)\left[(ka)^2 - x_n^2\right]} \tag{47}$$

$$= \frac{2\pi(-)^n\, ka\, J_1(ka)\,(e_n k_n a)^{1/2}}{(k^2 - k_n^2)a^2} ,$$

where e_n is defined by

$$J_o(x_n) = (-)^n (2/\pi e_n x_n)^{1/2} .$$

By virtue of the asymptotic form of J_1 in (46) and of J_o, we have

$$e_n \to 1 \quad \text{as} \quad n \to \infty ,$$

but for $n = 0$, since $k_o = 0$ and $J_o(0) = 1$, one must regard the product $e_o k_o$ as inseparable and given by $e_o k_o = 2/\pi a$. Eq. (44) thus becomes

$$F(y) = (2\pi)^{3/2} p_o a^{-1/2}\, kR_o\, J_1(ka) \sum_{n=o}^{\infty} (-)^n f(n), \tag{48}$$

where

$$f(n) = \frac{J_1(k_n R_o)}{k_n} \frac{(e_n k_n)^{1/2}}{k^2 - k_n^2} g(k_{2n}, y) \tag{49}$$

[cf. (11) and (12)].

In the limit of $a \to \infty$ with other lengths fixed, the solution becomes that in the absence of the wall, namely

$$p_{\infty}(r,\theta,y) = \mp (1/2)p_o \exp\left[ikr\cos(\theta-\beta)\right]\exp\left[-(k^2-k_{\omega}^2)^{1/2}|y|\right],$$

(50)
$$F_{\infty}(y) = \mp (1/2)p_o\left[2\pi k^{-1}R_o J_1(kR_o)\right] \exp\left[-(k^2-k_{\omega}^2)^{1/2}|y|\right],$$

the first square-bracketed factor in F_{∞} (times p_o) being merely the integral of the source pressure over the area πR_o^2.

In the limit of special concern [Eq. (14.1)] , on the other hand, as usual the sum in (48) may be terminated at some $n = N + 1$. Application of the previous procedure based on (17) in this case can be made only at large n, as will be seen below; in consequence, in addition to the contribution of lowest order in a^{-1} deriving from those n such that in the limit $k_n \simeq k_{\omega}$, we might find a second contribution of this order from those n such that in the limit $k_n \simeq 0$. The latter contribution did not occur previously and is associated with the circular geometry of the present case; in fact, however, it proves to vanish.

Consider those terms in the sum

$$\sum_{n=o}^{N+1} (-)^n f(n)$$

in (48) with $n > n_1$ for some even n_1 such that $1 << n_1 < N$. As $k_{\omega}a \rightarrow \infty$, we shall let

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(50.1) $$n_1 \to \infty \text{ but } n_1/k_\infty a \to 0,$$

just as for n_2 at (19). For $n > n_1$, then, we have

(51) $$e_n \to 1, \quad x_n \to (n + 1/4)\pi \quad ,$$

where the latter results from the asymptotic form of J_1 in (46). For these n_1 we may thus omit e_n in (49), whence $f(n)$ depends on n only via k_n, and we may define

$$f(n) \equiv T(k_n).$$

For non-integer n we may define $T(k_n)$ by use of the asymptotic form (51) for $k_n a (\equiv x_n)$.

Now, having $T(k_n)$ an analytic function and $k_{n+1} - k_n \to \pi/a$ for $n > n_1$, we may apply Eq. (17) to those n outside the intervals $0 \leq n \leq n_1$, $j + 1 - n_2 \leq n \leq j - 1 + n_2$. Analogously to (20), we have now

(52) $$\sum_{n=0}^{M+1} (-)^n f(n) \to S_1 + S_- + S_+ - (1/2)f(N+3/2) + O((\pi/a)^3 \sum_{\substack{n \\ \text{even}}}'' T'''(k_n)),$$

where

(53) $$S_1 = \sum_{\substack{n=0 \\ \text{even}}}^{n_1} \left[f(n) - f(n+1) \right] + (1/2)f(n_1+3/2)$$

S_-, S_+ are defined as at (20.1) (for odd j), and Σ'' denotes a sum over the intervals $n_1 + 2 \leq n \leq j - 1 - n_2$, $j + 1 + n_2 \leq n \leq N$. Again we may neglect the term $- (1/2)f(N+3/2)$ in the limit in question.

The last term in (52) requires consideration with regard to terms n such that $n \sim n_1$ as well as the terms such that $n \sim (k_\omega a/\pi) \pm n_2$. The contribution of the latter terms depends on $\underline{a}$ and n_2 just as previously determined at (20.4). As for the former terms, we may write (49) in the form

$$(54) \qquad T(k_n) = (8/3)k_n^{1/2}\, t(k_n), \qquad (n > n_1)$$

where the factor 8/3 is introduced for convenience. Then $t(k_n)$ at fixed n (or at fixed $n-n_1$) approaches a nonzero constant as $a \to \infty$. Likewise, by differentiation of (54) we obtain a result of the form

$$(55) \qquad T'''(k_n) = k_n^{-5/2}\, t_n + k_n^{-3/2}\, u_n + k_n^{-1/2}\, v_n + k_n^{1/2}\, w_n,$$

where $u_n \left[\equiv u(k_n) \right]$, v_n, and w_n, like t_n, approach constants at fixed n as $a \to \infty$. Consider, for example, the contribution of the first term of (55) to the residual sum in (52) with regard to its dependence on $\underline{a}$ and n_1 as $a \to \infty$:

$$(56) \qquad (\pi/a)^3 \sum_{\substack{n=n_1+2 \\ \text{even}}}^{j-1-n_2} k_n^{-5/2}\, t(k_n) \sim (1/2)(\pi/a)^2 \int_{n_1\pi/a}^{\infty} dk_n\, k_n^{-5/2}\, t(k_n)$$

$$\sim O((\pi/a)^2 (n_1\pi/a)^{-3/2}) \sim O(a^{-1/2}\, n_1^{-3/2}).$$

Similarly, the remaining terms in (55) are respectively

(57) $$O(a^{-3/2}\, n_1^{-1/2}),\quad O(a^{-5/2}\, n_1^{1/2}),\quad O(a^{-7/2}\, n_1^{3/2}).$$

By comparison we shall see below that the contribution S_1 defined at (53) is $\sim O(a^{-1/2})$, as was the contribution $S_- + S_+$ in (52), and is independent of n_1 in the pertinent limit. The terms in (57) are thus of higher order in a and will be neglected. As $n_1 \to \infty$, in accord with (50.1), the term (56) also vanishes relative to $S_1 + S_- + S_+$ [cf. (20.4)] and may also be neglected.

To evaluate S_1, write (49) as

$$f(n) = a^{-1/2}\, \frac{J_1(x_n R_o/a)}{x_n/a}\, \frac{g(k_{2n},y)}{k^2 - (x_n/a)^2}\, (e_n x_n)^{1/2} ,$$

with $k_{2n}^{\;2} = k_\omega^{\;2} - (x_n/a)^2$. In the limit, in view of (50.1) we have, in the pertinent range $n \le n_1 + 1$,

$$f(n) \to (1/2)R_o\, g(k_\omega,y)k^{-2}(e_n x_n)^{1/2}\, a^{-1/2} .$$

Hence by (53)

(58) $$S_1 \to (1/2)R_o\, g(k_\omega,y)k^{-2}G\, a^{-1/2} ,$$

where the constant G is defined by

$$G = \lim_{n_1 \to \infty} \left\{ \sum_{\substack{n=0 \\ \text{even}}}^{n_1} \left[(e_n x_n)^{1/2} - (e_{n+1} x_{n+1})^{1/2} \right] + (1/2)(e_{n_1+3/2} x_{n_1+3/2})^{1/2} \right\}$$

(59)

$$= \lim_{n_1 \to \infty} \left\{ \sum_{\substack{n=0 \\ \text{even}}}^{n_1} \left[(e_n x_n)^{1/2} - (e_{n+1} x_{n+1})^{1/2} \right] + (\pi^{1/2}/2)(n_1 + 7/4)^{1/2} \right\} .$$

Representing (49) as

(60) $$f(n) = h(k_n)\, g(k_{2n}, y)$$

in analogy with (21), we have also from (25) and (27)

(61) $$S \equiv S_- + S_+ \to \sqrt{2}\, h(k_\omega)(\partial g/\partial k_2)_o\, k_\omega^{1/2} (\pi/a)^{1/2} \left[M(b) - iM(1-b) \right]$$

with

$$h(k_\omega) \to \frac{J_1(k_\omega R_o)}{k_\omega} \frac{k_\omega^{1/2}}{k^2 - k_\omega^2} ,$$

(62)

$$(\partial g/\partial k_2)_o = -\, i(1/2)y.$$

Eqs. (48), (52) then yield

$$F(y) \to 4\pi p k_o^{1/2} R_o a^{-3/2} \cos(ka - 3\pi/4) \left\{ (1/2) R_o k^{-2} G\, g(k_\omega, y) \right.$$

(63)

$$\left. + (2\pi)^{1/2} k_\omega^{1/2} h(k_\omega)(\partial g/\partial k_2)_o \left[M(b) - iM(1-b) \right] \right\}.$$

with $h(k_\omega)$ and $(\partial g/\partial k_2)_o$ as given in (62) and

$$g(k_\omega, y) = \mp (1/2) \exp(ik_\omega |y|).$$

since, in the limit $ka \to \infty$,

(64) $$J_1(ka) \to (2/\pi ka)^{1/2} \cos(ka - 3\pi/4).$$

The envelope of the limiting form (63) for $F(y)$ varies as $a^{-3/2}$, just as in the cases of rectangular geometry in Sec. 2.

A numerical evaluation of the constant G of Eq. (59), however, indicates* that

(65) $$G = 0 .$$

This result could presumably be established exactly analytically. Thus, according to the result expressed in (63) with $G = 0$, for cylindrical geometry in the limit considered, the principal contribution to the (average) pressure near the axis of the cylinder at distance $|y|$ from the source plane derives exclusively from modes with $k_n = k_\infty$, just as in the case of rectangular geometry.

3.2 Free walls

In this case the eigenvalues x_n are specified by

(66) $$J_o(x_n) = 0 \quad (n \geq 0).$$

Eq. (45) then yields

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* From the computation it is directly inferred that $|G| \lesssim 10^{-5}$.

$$C_n = -2(2\pi)^{1/2} \frac{x_n J_o(ka)}{J_1(x_n)\left[(ka)^2 - x_n^2\right]}$$

(67)

$$= \frac{-2\pi(-)^n J_o(ka)\, e_n^{1/2}(k_n a)^{3/2}}{(k^2 - k_n^2)a^2},$$

where e_n is now defined by

$$J_1(x_n) = (-)^n (2/\pi e_n x_n)^{1/2},$$

whence again

$$e_n \to 1 \quad \text{as} \quad n \to \infty .$$

Eq. (44) thus becomes

(68) $$F(y) = -(2\pi)^{3/2} \rho_o a^{-1/2} R_o J_o(ka) \sum_{n=o}^{\infty} (-)^n f(n),$$

where

(68.1) $$f(n) = \frac{J_1(k_n R_o)}{k_n} \frac{e_n^{1/2} k_n^{3/2}}{k^2 - k_n^2} g(k_{2n}, y).$$

We may proceed as previously for rigid walls through Eq. (52). In place of (54), however, we have a summand of the form

$$T(k_n) = (8/3)\, k_n^{3/2}\, t(k_n). \quad (n > n_1)$$

Hence the terms corresponding to those of (56) and (57) are each of the order of the respective previous one multiplied by n_1/a. As for S_1, we write (68.1) as

$$f(n) = a^{-3/2} \frac{J_1(x_n R_o/a)}{x_n/a} \frac{g(k_{2n},y)}{k^2-(x_n/a)^2} e_n^{1/2} x_n^{3/2},$$

with $k_{2n}^2 = k_\omega^2 - (x_n/a)^2$. In the limit, in the range $n \leq n_1 + 1$ we have

$$f(n) \to (1/2)R_o\ g(k_\omega,y)k^{-2}\ e_n^{1/2}\ x_n^{3/2}\ a^{-3/2},$$

whence

$$S_1 \to (1/2)R_o\ g(k_\omega,y)k^{-2}F\ a^{-3/2},$$

where the constant F is defined by

$$F = \lim_{n_1\to\infty} \left\{ \sum_{\substack{n=o \\ \text{even}}}^{n_1} \left[(e_n x_n^3)^{1/2} - (e_{n+1} x_{n+1}^3)^{1/2} \right] + (\pi^{3/2}/2)(n_1 + 9/4)^{3/2} \right\}.$$

On the other hand,

(69) $$S \to \sqrt{2}\ h(k_\omega)(\partial g/\partial k_2)_o\ k_\omega^{1/2}\ (\pi/a)^{1/2} \left[M(b) - iM(1-b) \right]$$

with

$$h(k_\omega) \to \frac{J_1(k_\omega R_o)}{k_\omega} \frac{k_\omega^{3/2}}{k^2 - k_\omega^2},$$

(70)

$$(\partial g/\partial k_2)_o = -\ i(1/2)y.$$

Hence, whereas S is $\sim O(a^{-1/2})$ as in the case of the rigid wall, S_1 is $\sim O(a^{-3/2})$ and the leading residual term (corresponding to (56)) is $\sim O(a^{-3/2}\ n_1^{-1/2})$. Therefore, in this case the leading term is due only to S. Eqs. (68) and (69) thus yield

(71)

$$F(y) \to -(32\pi^3)^{1/2} p_o R_o a^{-3/2} k^{-1/2} \cos(ka-\pi/4) k_\omega^{1/2} h(k_\omega)(\partial g/\partial k_2)_o \left[M(b) - iM(\text{[illegible]}-b)\right]$$

since $J_o(ka) \to (2/\pi ka)^{1/2} \cos(ka - \pi/4)$, with $h(k_\omega)$ and $(\partial g/\partial k_2)_o$ as given by (70).

4. Generalization to certain interposed domes

With a view to sonar domes, we may imagine that a thin planar dynamic element, i.e. a plate or membrane of negligible thickness, is present at $y = 0-$, just beneath the source plane. We also generalize to the case where the lower fluid does not extend to $y = -\infty$, but is terminated at a plane $y = -L$ on which an impedance condition is given. In addition, since little complication results, we permit the fluids on the two sides of this dynamic sheet to have different densities and sound velocities, namely ρ^+, c^+ for $y > 0$ and ρ^- c^- for $y < 0$.

Now for certain boundary conditions at the periphery of the dynamic sheet, related to the impedance conditions given at the lateral walls of the fluids, the set of modal functions for the sheet and the sets of planar modal functions for both fluids will all coincide. (This point will be discussed more explicitly in another report). In such case the previous work can be trivially generalized to treat the new problem.

Let these modal functions and wave numbers again be denoted $\psi_n(x)$, k_n, respectively, where we assume the rectangular geometry for definiteness. An acoustic impedance can be defined for the sheet in each mode, say $z_n \equiv z(k_n,\omega)$.

Analogously to (3), we expand

$$p(x,y) = \sum_{n=o}^{\infty} c_n^{\pm} \psi_n(x)\, g(k_{2n}^{\pm},y) \quad \text{for} \quad y \gtrless 0, \tag{72}$$

respectively, where the $\psi_n(x)$ are orthonormal and satisfy (4), and $g(k_{2n},y)$ is given by

$$g(k_{2n}^{\pm},y) = \begin{cases} \exp(ik_{2n}^{+}y)\ , & y \geq 0 \\ \dfrac{\cos\left[k_{2n}^{-}(y+L)+\alpha\right]}{\cos\left[k_{2n}^{-}L+\alpha\right]} & y < 0 \end{cases} \tag{73}$$

$$k_{2n}^{\pm} = (k_{\omega}^{\pm} - k_n^{2})^{1/2}, \qquad k_{\omega}^{\pm} = \omega/c^{\pm}\ ,$$

where α is related to the impedance at $y = -L$, say z_L, by

$$\tan\alpha = i\omega\rho^{-}/k_{2n}^{-}z_L\ ,$$

and the function g is arbitrarily normalized to $g(k_{2n}^{\pm},0) = 1$. For k_n such that $k_{2n}^{\pm\,2} < 0$, it is convenient to rewrite (73) as

$$g(k_{2n}^{\pm},y) \equiv g(iK_{2n}^{\pm},y) = \begin{cases} \exp(-K_{2n}^{+}y) , & y \geq 0 \\ \dfrac{\mathrm{ch}\left[K_{2n}^{-}(y+L) + \hat{\alpha}\right]}{\mathrm{ch}\left[K_{2n}^{-}L + \hat{\alpha}\right]} , & y < 0 \end{cases}$$

(73.1)

$$K_{2n}^{\pm} = (k_n^{\,2} - k_\omega^{\pm\,2}) ,$$

$$\mathrm{th}\,\hat{\alpha} = - i\omega\rho^{-}/K_{2n}^{-} z_L \; . \qquad (\hat{\alpha} = - i\alpha)$$

The acoustic velocity field corresponding to the pressure (72) in mode n has a y component

$$v_{2n}^{\pm}(x,y) = (i\rho^{\pm}\omega)^{-1}\, c_n^{\pm}\, \psi_n(x)\, \partial g(k_{2n}^{\pm},y)/\partial y ,$$

Boundary condition (1) is now replaced by

(74) $$p(x,0+) + P(x) + \sum_{n=o}^{\infty} z_n v_{2n}^{-}(x,0) = p(x,0-) .$$

Likewise, (2) is replaced by

(75) $$\rho^{-}(\partial p/\partial y)_{y=o+} = \rho^{+}(\partial p/\partial y)_{y=o-} ,$$

whence, by substitution of (72),

(76) $$\rho^{-}\, c_n^{+} \left[\partial g(k_{2n}^{+},y)/\partial y\right]_{y=o} = \rho^{+}\, c_n^{-} \left[\partial g(k_{2n}^{-},y)/\partial y\right]_{y=o} .$$

We expand

(77) $$P(x) = \sum_{n=0}^{\infty} c_n \psi_n(x).$$

Substituting (72) and (77) into (74) and using (76), one obtains

(78) $$c_n^- = c_n \left\{ 1 - i \tan\theta_n \left[(k_{2n}^- \rho^+/k_{2n}^+ \rho^-) + (z_n k_{2n}^-/\rho^- \omega) \right] \right\}^{-1} ,$$

where $\theta_n = k_{2n}^- L + \alpha$. A similar result can be given for c_n^+. By (77) the present c_n is the same that of Eq. (3) and is again given by (5).

The limiting processes of Sec. 2 can be carried through once more mutatis mutandis, as will be shown. In (3), for example, c_n is replaced by $c_n^{\pm}$ and $\mp(1/2) \exp(ik_{2n}|y|)$ by $g(k_{2n}^{\pm},y)$. Similarly, the result for $a \to \infty$ at fixed y now becomes

$$p_{\infty}(x,y) = P(x)\ g(iK_2^{\pm},y)$$

with $K_2 = (k^2 - k_{\omega}^{\pm\,2})^{1/2}$ [cf. special cases (13.1) and (34)]. With regard to the second type of limit, however, where $|y| \to \infty$ while $y/a \to 0$, if $k_{\omega}^+ \neq k_{\omega}^-$, one must consider separately the groups of terms with $k_n \simeq k_{\omega}^+$ and $k_n \simeq k_{\omega}^-$, since k_{ω}^+ and k_{ω}^- are then distinct branch points. We assume rather once more that $c^+ = c^-$, whence $k_{\omega}^+ = k_{\omega}^-$, and also that $\rho^+ = \rho^-$. The index $\pm$ will be dropped from these quantities. In this case the product $c_n^- g(k_{2n}^-,y)$ in (72) for $y < 0$ may be written by (73) and (78) as

$$(79)\qquad c_n^{-}\, g(k_{2n},y) = c_n \frac{\cos(k_{2n}y + \theta_n)}{\cos\theta_n - i\,\sin\theta_n\left[1 + (z_n k_{2n}/\rho\omega)\right]}$$

$$\equiv c_n\, \hat{g}(k_n,k_{2n},y),$$

in which the new function $\hat{g}$ thus depends on k_{2n} in part via $\theta_n(k_{2n})$ and on k_n via z_n. All the work of Sec. 2 and 3 is once again applicable, the only required change being that the former $g(k_{2n},y)$ is replaced by the present $\hat{g}(k_n,k_{2n},y)$. In particular, with this change the results (28), (35), (38), (39), (63), and (71) still hold. [In (63), for example, $(\partial g/\partial k_2)_o$ becomes $(\partial\hat{g}(k_\omega,k_2,y)/\partial k_2)_{k_2=o}$, since the arguments k_n,k_{2n} of $\hat{g}$ are related by $k_n^2 + k_{2n}^2 = k_\omega^2$.]

Where, as here, a sheet of impedance z_n is interposed, there is generally some resonant wave number k_r for each frequency ω. If z_n includes some damping, the previous work is applicable in the extreme limit in question. If the damping vanishes, the contribution from modes having $k_n \simeq k_r$ must be separately considered even in the limit.

5. Approach to the limiting solution as the lateral dimension a increases at fixed depth $|y|$

It may be suggested that the asymptotic solution found for the limit as $a \to \infty$ and $|y| \to \infty$ with $y/a \to 0$ represents also the correction to lowest order in a^{-1} of the asymptotic solution applicable in the limit where y remains fixed as $a \to \infty$, i.e., in the case of circular geometry, that the term (63) or (71) may be added to the term (50) to give the solution to the lowest two orders in a^{-1}, namely to orders a^o and $a^{-3/2}$, as $a \to \infty$ at fixed y. That this result probably does not hold, however, can be seen by proceeding with the full sum on n for p(x,y) as given at (11), as we did previously with the truncated sum at (16). For definiteness, we consider the case of rectangular geometry, rigid walls, and excitation $p_o \cos kx$, but the same procedure applies directly in the other cases.

A corrected version of Eq. (20) again holds when $N = \infty$, provided we sum explicitly over an interval $n = s - n_o + 1$ to $n = s + n_o$ which includes the additional point $k_n = k$ at which $f(n)$ is not analytic, where s is the integer defined by

$$k_s \le k < k_{s+1}$$

and we let $n_o \to \infty$ as $a \to \infty$ but as usual with $n_o/ka \to 0$. The form of Eq. (20), thus modified, merely has added to the right side the quantity S_k, where

$$(80) \qquad S_k = \sum_{\substack{n=s-n_o+1 \\ \text{even}}}^{s-1+n_o} \left[f(n) - f(n+1)\right] - (1/2)f(s + \tfrac{1}{2} - n_o) + (1/2)f(s + \tfrac{1}{2} + n_o)$$

and we assume s to be odd for definiteness and choose n_o to be even. To form $p(x,y)$ in (11), however, the sum (20) is multiplied by $a^{-1}\sin ka$ (among other factors), and it is convenient to consider S_k together with this factor and write

$$(81) \qquad U_k \equiv a^{-1} \sin ka \; S_k = \sum_{n=s-n_o+1}^{s+n_o} (-)^n f(n) \cdot a^{-1} \sin ka$$

$$+ a^{-1} \sin ka \left[-(1/2)f(s + \tfrac{1}{2} - n_o) + (1/2)f(s + \tfrac{1}{2} + n_o)\right].$$

Inserting (12), replacing $\mp (1/2) \exp(ik_{2n}|y|)$ by $\hat{g}(k_n, k_{2n}, y)$ for generality in accord with Sec. 4, and noting that $(-)^n \sin ka = \sin(ka - n\pi)$, we may rewrite the summation term in (81) as

$$U_k' \equiv \sum_{n=s-n_o+1}^{s+n_o} (-)^n f(n) \cdot a^{-1} \sin ka$$

(82)

$$= \sum_n \frac{\sin(ka-n\pi)}{ka - n\pi} \frac{\cos k_n x}{k + k_n} \hat{g}(k_n, k_{2n}, y),$$

where the limits on the sum remain the same. As $a \to 0$, the range of k_n in the sum contracts to zero relative to k_n or k itself, since $n_o/ka \to 0$. In (82) let

(83) $$H(k_n) \equiv \frac{\cos k_n x}{k + k_n} \hat{g}(k_n, k_{2n}, y).$$

Since

(83.1) $$\sum_{n=-\infty}^{\infty} \frac{\sin(ka-n\pi)}{ka - n\pi} = 1,$$

the sum (82) can be rewritten as

(84) $$U_k' = H(k) - H(k)\left(\sum_{n=-\infty}^{s-n_o} + \sum_{n=s+n_o+1}^{\infty} \right) \frac{\sin(ka-n\pi)}{ka - n\pi}$$

$$+ \sum_{n=s-s_o+1}^{s+n_o} \frac{\sin(ka-n\pi)}{ka - n\pi} \left[H(k_n) - H(k) \right].$$

When $a \to \infty$ and $n_o \to a$ with $n_o/a \to 0$ and fixed y, all of the terms in (84) but the first vanish, yielding by (11) and (83) the appropriate result for this limit:

$$p(x,y) \to p_{\infty}(x,y) = p_o \cos kx\ \hat{g}(k,k_2,y).$$

We wish to consider the correction to this result to next higher order in a^{-1}, so far as contributed by the subject interval near $k_n = k$. This correction is due to the remaining terms of U_k' in (84), to the second part of the right member of (81), and to the residual terms in Eq. (20) that derive from values n just outside the interval $s - n_o + 1 \leq n \leq s + n_o$ which is summed over explicitly. We find readily (noting $(ka-n\pi)^{-1} \sim n_o^{-1}$) that the contribution of the second group of terms in (84) is of the order of $a^o n_o^{-1}$ relative to the zero-order contribution from the first term. The contribution of the last group in (84) we find to be of the higher order $a^{-2}n_o$. The pertinent residual terms of (20) (related to the third derivative of $(ka-n\pi)^{-1} = [(k-k_n)a]^{-1}$ with respect to k_n) yield a contribution of the order given by

$$(\pi/a)^3\ a^{-1}(a/\pi) \int_{k+n_o\pi/a}^{\infty} dk_n (k-k_n)^{-4} \propto a^o n_o^{-3}\ .$$

Also, the latter set of terms in (81) immediately yield a contribution of the order $a^o n_o^{-1}$. This contribution is found, however, to cancel that of the second group in (84) in this order. All remaining contributions are then at most of one of the following orders:

$$n_o^{-3}\ ,\quad a^{-2}n_o\ . \tag{85}$$

Unless the terms of one or the other of these orders cancel, the highest order in a^{-1} achievable for the lowest-order one of the pair (85) by choosing δ, where $n_o \propto a^{\delta}$, is obtained for $\delta = 1/2$ and is of the order $a^{-3/2}$. This, however, is the same order as the asymptotic forms obtained in the previous sections for the case $|y| \to \infty$ [e.g. (63) or (71)]. Though cancellation of the leading-order terms considered above has not been excluded, it appears unlikely. If there is no such cancellation, we thus see that such results as (63) and (71), though representing the leading terms as $a \to \infty$ when $|y| \to \infty$, represent only part of the correction to next-to-leading order in a^{-1} in the limit when y remains fixed.

6. Use of limiting forms or simplified sums as approximations

The approach in this appendix has been to consider results that hold rigorously in a certain limit without detailed regard to how this limit is approached or whether it is approached sufficiently that the limiting form constitutes a useful approximation in applications of concern. These questions are considered in the text. Some of the intermediate work is useful for approximation under conditions where limiting expressions are not. In particular, we can make the expansion (20) and may drop the residual terms without, however, passing to the limit $a \to \infty$, $n_2 \to \infty$ in (20.1) and without, therefore, using the limiting forms (24). In this way we merely use (20) to sum out the intervals where $f(n)$ changes slowly and which therefore contribute little to the sum, leaving explicit sums over intervals where the contribution is large. A similar procedure was applied in (52) and (53), and for isolating and summing explicitly over the interval near resonance in the more general case of Sec. 4. The point here is just that the intervals that tend to contribute most to the sum (20) are those where the third derivative of f(n) with respect to k_n is large or where f(n) is not an analytic function of k_n at all. Such approximations are expounded in the text.

APPENDIX 3

RADIATED POWER

We consider the power radiated by the dome or plate in the model considered in the test. First, for the laterally infinite case, the frequency spectrum of the time-average radiated power per unit area is given by

$$W_\infty(\omega) = \int_0^\infty dkkI_0(k,\omega)\left|\Gamma_1(k,\omega)\right|^2 A(k,\omega) \tag{1}$$

where A, one half the real part of the reciprocal outside radiation impedance, is simply

$$A = \begin{cases} k_2^+/2\rho^+\omega & \text{if } k < \omega/c^+ \\ 0 & \text{if } k > \omega/c^+ \end{cases}$$

(we recall $k_2^+ = [(\omega/c^+)^2 - k^2]^{1/2}$).

For the finite circular cylindrical dome or plate of radius a, the spectrum of the total time-average radiated power is given by

$$W(\omega) = \sum_m \sum_n A(k_{mn},\omega)\left|\Gamma_1(k_{mn},\omega)\right|^2 \int_0^\infty dkkI_0(k,\omega)\left|c_{mn}(k)\right|^2 \tag{2}$$

where $A(k,\omega)$ represents the same function as above and the sums run over all modes m, n. $c_{mn}(k)$ was given at Eq. (2-11). It is seen that the sums can be limited to propagated modes, i.e., to m, n such that $k_{mn} < \omega/c^+$. Each such mode, however, is excited even by driving components with $k \gg \omega/c^+$.

Expression (2) could be simplified on assumption of large a in a manner similar to that employed in Part 2 of the text, with a contribution given approximately by W_∞ of Eq. (1) above and a further a-dependent contribution deriving from low k_{mn}.

The radiated power is more often of interest for a rectangular rather than a circular plate. The present model could be altered accordingly and corresponding results obtained. The power radiated by rectangular plates has been considered previously by Kraichnan (Ref. 4), among others.* His assumption of no correlation among contributions from different orthogonal modes is exactly true in the present model, as seen from Eq. (2), having resulted here from the assumption of certain conjugate boundary conditions on an imagined cylindrical projection of the plate boundary normally into the outer fluid.

*Kraichnan computes results for an explicit driving turbulent pressure spectrum $P_0(\bar{k},\omega)$ obtained from a crude model.

APPENDIX 4

CONJUGATE BOUNDARY CONDITIONS FOR EXTENDED-WALL DOME MODEL

We consider here the simplified dome model introduced in the text, where the cylindrical lateral wall of the dome is imagined, so far as the acoustic fields are concerned, to be extended infinitely outward into the outer fluid with the same boundary condition applied (see Fig. A2-1).

Suppose first that the dome cover is a membrane, not a plate. Suppressing time dependence and following the notation of App. 1, we may expand the cover displacement and acoustic potentials as

$$\eta(\overline{R}) = (2\pi)^{-1/2}\sum_{m}\sum_{n}\eta_{mn}N^{o}_{mn}J_m(k^{o}_{mn}R)\,e^{im\theta}, \tag{1}$$

$$\phi^{+}(\overline{r}) = (2\pi)^{-1/2}\sum_{m}\sum_{n}\phi^{+}_{mn}N_{mn}J_m(k_{mn}R)\,e^{im\theta}\exp(ik^{+}_{2mn}y), \tag{2}$$

$$\phi^{-}(\overline{r}) = (2\pi)^{-1/2}\sum_{m}\sum_{n}\phi^{-}_{mn}N_{mn}J_m(k_{mn}R)\,e^{im\theta}\,\frac{\cos[k^{-}_{2mn}(y+L)+\alpha_{mn}]}{\cos(k^{-}_{2mn}L+\alpha_{mn})}, \tag{3}$$

where η_{mn} and $\phi^{\pm}_{mn}$ are expansion coefficients, N^{o}_{mn} and N_{mn} are known normalizing coefficients for the J_m, and

$$k^{\pm}_{2mn} = (\omega/c^{\pm})^2 - k^2_{mn} \qquad (\text{if } k_{mn} < \omega/c^{\pm}),$$

$$\tan\alpha_{mn} = i\omega\rho^{-}/k^{-}_{2mn}z_{Lmn};$$

the eigenvalues k_{mn} are to be fixed by the boundary conditions on the acoustic fields in the fluids at the lateral wall (R=a) and the eigenvalues k^{o}_{mn} by those on the membrane at its periphery (R=a).

If the boundary conditions on the membrane are so related to those on the fluids that $k^{o}_{mn} = k_{mn}$, the eigenfunctions in the expansions (1) and (2)-(3) for the membrane and fluids are the same, and the equations that determine the coefficients η_{mn} and $\phi^{\pm}_{mn}$, namely Eqs. (2), (5), and (6) of App. 1, relate η_{mn} and $\phi^{\pm}_{mn}$ independently for each m and n. For each mode m,n, the relations and response coefficients of App. 1 are again obtained, with $k = k_{mn}$ and the previous quantities indexed by m,n as required.

In particular, for a rigid wall on the fluid, the k_{mn} are fixed by

$$J_m'(k_{mn}a) = 0. \tag{4}$$

The conjugate condition on the membrane must likewise yield

$$J_m'(k_{mn}^o a) = 0,$$

and by Eq. (1) this condition is seen to be that

$$\partial\eta(\bar{R})/\partial R = 0 \text{ at } R=a. \tag{5}$$

This condition of a vanishing normal derivative of displacement of the membrane at its periphery corresponds to a vanishing force in the η direction, i.e., a freely sliding attachment, as in slots perpendicular to the equilibrium plane. Similarly, for a pressure-release fluid wall, such that

$$J_m(k_{mn}a) = 0, \tag{6}$$

the condition on the membrane yielding $k_{mn}^o = k_{mn}$ is

$$\eta(a) = 0. \tag{7}$$

This corresponds to an immovable attachment of the membrane at its periphery.

Now suppose the dome cover is a thin plate rather than a membrane. In this case Eq. (1) does not represent the most general expansion for arbitrary boundary conditions. In general, apart from the J_m, another type of Bessel function must be included, the expansion being one in eigenfunctions of the fourth-order equation $[\nabla^2-(k_{mn}^o)^2][\nabla^2+(k_{mn}^o)^2]\eta=0$ rather than the second-order equation $[\nabla^2+(k_{mn}^o)^2]\eta=0$. We may, nevertheless, achieve the simplifying identity of the plate and fluid eigenfunctions, as before, by prescribing boundary conditions on the periphery of the plate such that even the generalized expansion (1) reduces, in fact, to the given form of (1) containing only the functions J_m and such that we again have $k^o_{mn} = k_{mn}$. In particular, for a rigid wall on the fluid, so that Eq. (4) applies, we again require Eq. (5) to apply. We confine attention to the modes with m=0, which are the only ones that contribute to averages over a circular area concentric with the cylindrical fluid, dealt with in the text. For these modes, with the constraint that the radial functions include only the oscillatory type J_m of Eq. (1), the vanishing slope, [Eq. (5)] implies that the force on the plate (in the η direction) vanishes. The boundary condition on the plate thus corresponds to a freely sliding clamp, which fixes the slope at the periphery but offers no resistance to displacement normal to the plate. For a pressure-release wall on the fluid, on the other hand, so that Eq. (6) applies, we again require Eq. (7). For the m=0 modes, the vanishing displacement, Eq. (7), and the previous limitation of the radial functions to J_m, may be shown to imply a moment at the plate periphery which vanishes in the limit of large plate radius ($k^o_{mn}a \to \infty$). The boundary condition on the plate in this instance thus corresponds to a hinged periphery, which fixes the displacement but offers no resistance to rotation.*

In the text and App. 1, hysteretic and viscous damping are attributed to the dome material. In practice, depending on what the dome material or construction is, the main damping may occur rather at the joinings with the supporting structure, i.e., in the present model, at the periphery of the dome cover. Such an effect can be incorporated in the model by assumption of a resistive boundary condition on the plate (or membrane) displacement $\eta(\overline{R})$. In the case of the membrane, for example, the force per unit length in the normal (η) direction exerted by the membrane on its periphery at R=a is given in the usual notation (e.g., see App. 1) by $-T(1-i\zeta) \times (\partial\eta/\partial R)_{R=a}$ and the velocity there by $-i\omega\eta$. If the ratio of these two quantities, the mechanical impedance per unit length of boundary, is the same for each mode, say

$$z_p \equiv r_p + ix_p, \tag{8}$$

*That the condition corresponds to a hinge only in the limit of large radius accords with the fact that free motion on hinges is possible only for a straight-line boundary.

from Eq. (1) we obtain

$$k^o_{mn}\, a \frac{J'_m(k^o_{mn}a)}{J_m(k^o_{mn}a)} = \frac{i\omega}{T(1-i\zeta)}(r_p + ix_p)\,a. \qquad (9)$$

Previously we considered explicitly only the cases $J'_m(k^o_{mn}a) = 0$ or $J_m(k^o_{mn}a) = 0$, corresponding respectively to vanishing or infinite impedance z_p. In an intermediate case where the ratio (9) is finite, a resistive component of force, ie., damping in the boundary attachment, is present if $r_p \neq 0$. In order to achieve identity between dome-cover and fluid eigenvalues and eigenfunctions, as before, we must attribute to the fluid walls a certain damping related to that at the dome-cover periphery.

The eigenvalues k^o_{mn} from (9) will be complex unless it happens that $r_p/x_p = \zeta$. With boundary damping inserted, then, we have, in general, complex k^o_{mn}. The imaginary part of k_{mn} contributes an additional imaginary (damping) term in the denominator of the acoustic response coefficients $\Gamma(k_n,\omega,y)$ used in the text (see also App. 1). Correspondingly, in the consideration of resonances an additional contribution will be present in the wavenumber half width δk and similarly in the frequency half width $\delta\omega$ (see Part I). It would be needlessly detailed to consider these expressions explicitly. It is worthwhile, however, to recognize that boundary damping would appear in a way similar to other types of damping explicitly treated in the text, so that the more general discussions of resonances and their role in determining average pressure spectra may be regarded as applicable whatever the main sources of mechanical damping.

APPENDIX 5

AVERAGE-PRESSURE SPECTRA ON A CIRCULAR AREA FOR CERTAIN WAVE-NUMBER-FREQUENCY SPECTRA

A.5.1 Area Dependence vs Spectrum at Low Wave Numbers

We point out first the dependence on the radius R_o of the averaging area that results from a sample wavenumber frequency spectrum of pressure $I_o(k,\omega)$ in a turbulent boundary layer. I_o is defined to include an integral over direction θ of the surface wave vector $\bar{k}$: $I_o(k) = \int d\theta P_o(k,\theta)$. Let the dependence of $I_o(k)$ at low k correspond to a constant value up to $k=k_c$, followed by a power-law rise to $k=k_1$, followed by arbitrary dependence (see Fig.A-5-1).

$$I_o(k) = \begin{cases} B_n(k_c/k_1)^n & \text{for } 0 < k < k_c \\ B_n(k/k_1)^n & \text{for } k_c < k < k_1 \\ I_o(k) & \text{for } k_1 < k \end{cases} \tag{1}$$

where the frequency dependence of I_o and B_n are suppressed. Here B_n is independent of k; if the assumed form (1) is required to yield a fixed value, independent of n, for the spectrum deriving from $k < k_1$; i.e., for $P_-(\omega) = \int_o^{k_1} dkkI_o(k,\omega)$, then B_n is given by

$$B_n = (n+2)k_1^{-2}\left[1+n(k_c/k_1)^{n+2}\right]^{-1}P_-.$$

The spectrum of average pressure is given as usual by

$$Q_o(\omega) = \int_o^\infty dkk[2J_1(kR_o)/kR_o]^2 I_o(k,\omega). \tag{2}$$

Eq. (1) yields

$$Q_o = Q_- + Q_+, \tag{3}$$

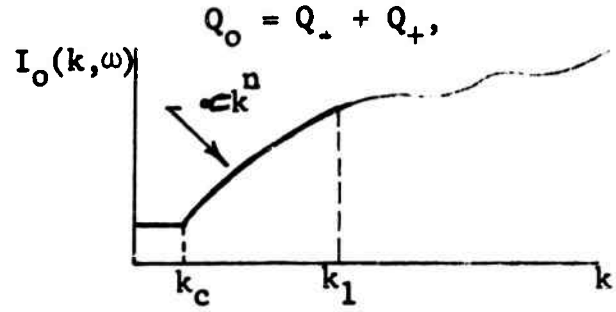

Figure A5-1. Hypothetical wavenumber spectrum of boundary-layer pressure fluctuations at low wave number.

where

$$Q_- = 4B_n R_o^{-2}\left[(k_c/k_1)^n \int_o^{k_cR_o} dx x^{-1} J_1^{\,2}(x) + (k_1R_o)^{-n}\int_{k_cR_o}^{k_1R_o} dx x^{n-1} J_1^{\,2}(x)\right], \quad (4)$$

$$Q_+ = \int_{k_1}^{\infty} dkk[2J_1(kR_o)/kR_o]^2 I_o(k). \quad (5)$$

We consider R_o such that $k_1R_o >> \pi$. Then, as at Eq.(2-65b) $Q_+ \propto R_o^{-3}$, and, assuming that $I_o(k)$ changes relatively little over an interval πR_o^{-1}, we have

$$Q_+ \simeq (4/\pi)R_o^{-3}\int_{k_1}^{\infty} dkk^{-2} I_o(k). \quad (5.1)$$

We consider R_o such that, however, $k_cR_o < 1$. Eq.(4) then rapidly reduces to

$$Q_- \simeq B_n\left\{[n/(n+2)]k_c^{\,2}(k_c/k_1)^n + 4R_o^{-2}(k_1R_o)^{-n}F_n(k_1R_o)\right\}, \quad (6)$$

where

$$F_n(z) \equiv \int_o^z dx x^{n-1} J_1^{\,2}(x).$$

For the assumed large k_1R_o, we have asymptotically

$$4F_n(k_1R_o) \rightarrow \begin{cases} f_n & \text{for } 0 \le n < 1 \\ h_n \ln k_1R_o & \text{for } n = 1 \\ g_n(k_1R_o)^{n-1} & \text{for } 1 < n, \end{cases}$$

where f_n, h_n, and g_n are numbers depending only on n. For $k_1R_o >> \pi$, we may thus write (6) as

$$Q_- \simeq B_n\left\{[n/(n+2)]k_c^{\,2}(k_c/k_1)^n + q_n(k_1,R_o)\right\} \quad (7)$$

with

$$q_n(k_1,R_o) = \begin{cases} f_n k_1^{-n} R_o^{-(2+n)} & \text{for } 0 \leq n < 1 \\ h_n k_1^{-1} R_o^{-3} \ell n k_1 R_o & \text{for } n = 1 \\ g_n k_1^{-1} R_o^{-3} & \text{for } 1 < n \end{cases} \tag{8}$$

The first term in the partial average-pressure spectrum (7) is independent of R_o (on account of the assumption $R_o \lesssim k_c^{-1}$). If $I_o(k)$ in the assumed intermediate, power-law range $k_c < k < k_1$ increases more rapidly than as k, the second term in (7), according to (8), depends on R_o as R_o^{-3}, just as the high-k contribution Q_+ of (5.1). If I_o increases less rapidly than as k, i.e., as k^n with $n < 1$, the second term depends on R_o rather as $R_o^{-(2+n)}$. Hence we see that an observed dependence of $Q_o(\omega)$ as R_o^{-m} with $m \simeq 2$ implies a rather rigid restriction on the rate of increase of $I_o(k)$ in the range where kR_o is not very large. There is, of course, a substantial uncertainty in the experimentally observed area dependence, on account of the measurement uncertainty, the limited number of areas used and the variability of extraneous factors.

The actual dependence of $I_o(k,\omega)$ at low k is naturally less simple than that of (1). In particular, when compressibility is considered there is reason to expect peculiarity in the behavior of I_o where $k \simeq \omega/c$, i.e., near the wave number of sound at the given frequency.* This range of k, like the initial range in (1), yields a contribution to Q_o that is independent of R_o for $R_o \lesssim c/\omega$. Theoretically, in a somewhat higher range, where $k \gtrsim \omega/c$, it appears that $I_o(k)$ should increase as k^2 (e.g., see Ref. 14).** If so, this range would contribute to $Q_o(\omega)$ a term varying as R_o^{-3} according to the third form of (8). It is suggested, therefore, that average-pressure spectra in some area range with $U_\infty/\omega << R_o \lesssim c/\omega$ may have an area dependence more as

*In particular, the view that the turbulent velocity field, which plays the role of a source in the equation for pressure fluctuation, may be regarded as unaffected by compressibility even if the latter is otherwise incorporated in that equation, i.e., the view that the turbulence drives the sound field without being appreciably affected by it, appears to fail where wave-numbers $k \simeq \omega/c$ are in question.

**Inhomogeneity in the direction of flow, however, represents another complicating effect of presently unknown consequence.

$$B + CR_o^{-3} \tag{9}$$

than as R_o^{-2}. Further consideration of the available measurements with this point in mind might be useful.

A.5.2 Results For Certain Wavenumber-Frequency Spectra

Let $W_o(\vec{\zeta},\tau)$ denote the space-time correlation function of pressure for separation vector $\vec{\zeta}$ in the boundary plane and time delay τ. Then the spectral density is

$$P_o(\vec{k},\omega) = 2(2\pi)^{-3}\int d^2\vec{\tau}\int d\tau e^{-i\vec{k}\cdot\vec{\zeta}+i\omega\tau} W_o(\vec{\zeta},\tau), \tag{10}$$

where P_o is so normalized that the mean squared pressure is

$$\langle p^2\rangle = \int_0^\infty d\omega\int d^2\vec{k}P_o(\vec{k},\omega). \tag{11}$$

We consider first the form sometimes used in previous work on boundary vibration:

$$W_o(\vec{\zeta},\tau) = \langle p^2\rangle\exp\left\{-\alpha\left[(\zeta_1-v\tau)^2 + \zeta_3^2\right]^{1/2} - \theta^{-1}|\tau|\right\}. \tag{12}$$

Here v represents a mean convection velocity, α^{-1} a correlation distance corresponding to a fixed time, and θ a decay time such that the correlation corresponding to a point fixed in the convected frame decreases by a factor e. By a gross fit to the measured $W_o(\vec{\zeta},\tau)$, Ref. 2 specifies the parameters as

$$\alpha \simeq 2\delta_*^{-1},\ \theta \simeq 30\delta_*/U_\infty,\ v \simeq 0.8U_\infty \tag{13}$$

From (10) we find

$$P_o(\vec{k},\omega) = 4(2\pi)^{-2}\langle p^2\rangle \frac{\alpha}{(k^2+\alpha^2)^{3/2}} \frac{\theta^{-1}}{(\omega-vk_1)^2+\theta^{-2}}. \tag{14}$$

Suppose $\omega/v\alpha >> 1$ and $\omega >> \theta^{-1}$, as will be true according to (13) if $\omega\delta_*/U_\infty >> 1$. Then the point spectrum is given by

$$P_o(\omega) = \int d^2\vec{k}P_o(\vec{k},\omega) \simeq 4(2\pi)^{-1}\langle p^2\rangle(1/\alpha v\theta+1)\alpha v\omega^{-2} \tag{15}$$

in agreement with Ref. 2 , Eq. (29), in this approximation. The first term derives from the region $k_1 \lesssim \alpha$ and the second from $k_1 \simeq \omega/v$. According to (13), the second is larger by a factor ~ 48.

The average-pressure spectrum on a large area, i.e., where $\omega R_o/v \gg \pi$, derived from (14) is found to be given approximately by

$$Q_o(\omega) \approx 4(2\pi)^{-1}\langle p^2\rangle \theta^{-1}\omega^{-2}\left[S+(3/4)\alpha v\theta(\omega R_o/v)^{-3}\right], \tag{16}$$

where

$$S = \begin{cases} 1 & \text{if } R_o \ll \alpha^{-1} \\ 2(\alpha R_o)^{-2} & \text{if } R_o \gg \alpha^{-1} \end{cases} \tag{17}$$

and it is again assumed that $\omega/\alpha v \gg 1$ and $\omega \gg \theta^{-1}$. The second term in (16) derives from $k_1 \simeq \omega/v$, and the first from $k_1 \lesssim \alpha$ if $R_o \ll \alpha^{-1}$ and from $k_1 \lesssim \pi R_o^{-1}$ if $R_o \gg \alpha^{-1}$. With the values (13), in the case $R_o \ll \alpha^{-1}$ the second term in (16) is smaller than the first roughly if $\omega R_o/v > 3$, which is true by assumption; in the case $R_o \gg \alpha^{-1}$ the second term is smaller if $(\alpha v/\omega)^3(v\theta/R_o) \lesssim 3$, i.e., according to (13), if $(\omega R_o/U_\infty)^{-1}(\omega\delta_*/U_\infty)^{-2} \lesssim 0.03$. The second term, it is noted, represents a high-k component that varies, as usual, as R_o^{-3}, and the first term a low-k component that in the case $R_o \gg \alpha^{-1}$ (corresponding to a nearly flat wavenumber spectrum where $k \lesssim \pi R_o^{-1}$), varies as R_o^{-2}.

We note further the results obtained when $\alpha \gg (\omega+\theta^{-1})/v$, which is implied, in particular, if one approximates by assuming a vanishing correlation scale at the outset, so that (12) becomes

$$W_o(\zeta,\tau) \propto \delta(\zeta_1 - v\tau)\delta(\zeta_3)\exp(-\theta^{-1}|\tau|) \tag{18}$$

(e.g., see Ref. 2). The point spectrum in this regime is flat, namely

$$P_o(\omega) = 2(2\pi)^{-1}\langle p^2\rangle(\alpha v)^{-1} \tag{19}$$

(cf. (15)). For $\alpha R_o/v \gg \pi$, i.e., a large area, we find

$$Q_o(\omega) \simeq 8(2\pi)^{-1}\langle p^2\rangle\theta^{-1}\omega^{-2}(\alpha R_o)^{-2}(1+2\pi^{-1}\theta v/R_o). \quad (20)$$

The contribution of the first term is the same as in the case of (16) (i.e., for $\omega/\alpha v \gg 1$) when, in reference to (17), $\alpha R_o \gg 1$. That of the second term, however, which derives from $k_1 \sim \omega/v$ and varies as R_o^{-3}, differs drastically from that of the second term in (16). This reflects the fact that (18) may be regarded as a valid approximation to (12) only so far as low wave numbers ($k \lesssim \alpha$) are concerned.

For comparison with the results based on (12), we now consider a time correlation, in place of $\exp(-\theta^{-1}|\tau|)$, whose frequency transform declines more rapidly with ω, namely

$$g(\tau) = (1+\theta^{-1}|\tau|)\exp(-\theta^{-1}|\tau|). \quad (21)$$

This form satisfies $g(0)=1$, as required, and, unlike the earlier form, has a continuous derivative at $\tau=0$: $g'(0)=0$. In place of (14) we find

$$P_o(k,\omega) = 8(2\pi)^{-2}\langle p^2\rangle \frac{\alpha}{(k^2+\alpha^2)^{3/2}} \frac{\theta^{-3}}{[(\omega-vk_1)^2+\theta^{-2}]^2} \quad (22)$$

If $\omega/v\alpha \gg 1$ and $\omega \gg \theta^{-1}$, the point spectrum is found to be given approximately by

$$P_o(\omega) \simeq 4(2\pi)^{-1}\langle p^2\rangle\alpha v\omega^{-2}[2(\alpha v\theta)^{-1}(\theta\omega)^{-2}+1]. \quad (23)$$

The contribution from high wave numbers ($k_1 \sim \omega/v$, second term) is the same as in (15); now, however, except at very low frequency it dwarfs the contribution from low wave numbers (first term). For a large area ($\omega R_o/v \gg \pi$) we find

$$Q_o(\omega) \simeq 8(2\pi)^{-1}\langle p^2\rangle\theta^{-3}\omega^{-4}[S + (3/8)\alpha v\theta(\omega R_o/v)^{-1}(v\theta/R_o)^2], \quad (24)$$

where S is again defined by (17). In this instance, unlike that of (16), the contribution from high wave numbers (second term), which varies as R_o^{-3} and is the same as in (16), exceeds that from low wave numbers (first term) typically up to rather high frequency.

The form (12) corresponds better to the gross observed time decay than the form based on (21), and the magnitudes and frequency dependence of the resulting $P_o(\omega)$ and $Q_o(\omega)$ accord generally better with observed spectra. Still, there is no apparent reason to consider that it has any detailed or unique validity.

We note now the area scale, or correlation area, corresponding to (12) and to the modification using (21). The correlation area A_c is properly defined, in the elementary statistical sense, by writing the force spectrum $A^2Q_o(\omega)$ on an area A, in the limit of large A, in terms of the point-pressure spectrum $P_o(\omega)$ as

$$A^2Q_o(\omega) = AA_cP_o(\omega).$$

Since in this limit we have

$$Q_o(\omega) \rightarrow (2\pi)^2A^{-1}P_o(0,\omega),$$

we may write A_c also as

$$A_c = (2\pi)^2[P_o(0,\omega)/P_o(\omega)]. \qquad (25)$$

Corresponding to (12), we then find in the approximation of (15)

$$A_c = 2\pi\alpha^{-2}/(1+\alpha v\theta).$$

With acceptance of (13), this becomes

$$A_c \simeq 2\pi(0.0051)\delta_*^2 \qquad (26)$$

Corresponding to (21) [i.e., (22)], on the other hand, in the approximation of (23) we find

$$A_c \simeq 2\pi\alpha^{-2}/[1+(1/2)\alpha v\theta(\theta\omega)^2].$$

With acceptance of (13) this becomes (except at very low frequency, where $\omega\delta_*/U_\infty \lesssim 10^{-4}$)

$$A_c \simeq 2\pi(1.2)(10^{-5})(U_\infty/\omega)^2. \qquad (27)$$

The magnitude of (27) is too small to accord with measured values of $Q_o(\omega)$; it does, however, depend on frequency and the flow parameters in a manner more consistent with observation (see Ref. 3) than the result (26).

The correlation field prescribed by (12), or (14), may be formally expressed as a superposition of contributions from a range of convection velocities u without decay. A fixed convection velocity is reflected in a factor $\delta(\omega-uk_1)$ in the wavenumber-frequency spectrum; for example, if $\theta^{-1}\to 0$ in (12), corresponding to pure convection (at velocity v) without decay, $P_o(\mathbf{k},\omega)$ of (14) contains a factor with limiting form given by

$$\frac{\theta^{-1}}{(\omega-vk_1)^2+\theta^{-2}} \to \pi\delta(\omega-vk_1). \qquad (\theta^{-1}\to 0) \tag{28}$$

In general, we write this factor in (14) as

$$\frac{\theta^{-1}}{(\omega-vk_1)^2+\theta^{-2}} = \pi \int_{-\infty}^{\infty} du\ f(\omega,k_1,u)\,\delta(\omega-uk_1), \tag{29}$$

whence

$$f(\omega,k_1,\omega/k_1) = \frac{\pi^{-1}\theta^{-1}/k_1}{(\omega/k_1-v)^2 + (\theta^{-1}/k_1)^2} .$$

This relation does not determine the weighting function $f(\omega,k_1,u)$ uniquely. If, however, we prescribe that f depend only on the velocity difference u-v, we obtain

$$f(\omega,k_1,u) = \frac{\pi^{-1}\theta^{-1}/k_1}{(u-v)^2 + (\theta^{-1}/k_1)^2} . \tag{30}$$

The weight f, we note, has a nonvanishing value for all convection velocities $-\infty < u < \infty$. The present formal superposition (29), however, though suggestive, is contrived and presumably not significant, since the point of the form (12) or (14) with $\theta^{-1}\neq 0$ is to embody a deviation from pure convection in the correlation function. That is, the superposition (29) does not imply that a correlation function of the character of (12) is applicable only if some portion of fluid is actually convected at each velocity u. If one actually had pure convection, but with a superposition over all convection velocities $0 < u(x_2) < U_\infty$ that are equal to the mean velocity at some depth x_2 in the boundary layer, the corresponding factor $\delta(\omega - u(x_2)k_1)$ contained by $P_o(\bar{k},\omega)$, after integration over x_2, would imply $P_o(\bar{k},\omega)=0$ for $k_1 < \omega/U_\infty$, in contrast to (14).

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13. ABSTRACT

Flow-induced noise generated by pressure fluctuations in a turbulent boundary layer is studied here theoretically with regard to its acoustic transmission through a dome to a transducer element or array. Particular attention is given to a unilayer dome in the form of a finite slab, described as a fluid but serving as a prototype for a realizable solid slab. A similar dome with a thin outer cover having nonvanishing impedance is also considered. Likewise, attention is given to acoustic modification of the flow noise on a flush element due to mobility of the surrounding baffle.

To demonstrate clearly what properties of the boundarylayer pressure fluctuations are essential, high- and low-wavenumber ranges are distinguished in the wavenumber spectrum of pressure at given frequency, the former range corresponding to possible generation by a convected eddy field. By various approximations and simplification of boundary conditions, the contributions of these respective ranges to the frequency spectrum of average pressure on a dome-shielded element are related to their respective contributions in the reference case of an element flush-mounted in a rigid plate. The high-wavenumber component in the flush case varies with element radius R_o as R_o^{-3}, whereas the low-wavenumber
(continued on attached page)

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component may vary more as R_o^{-2}. If the latter component predominates in the spectrum at given frequency for a large flush element, it predominates still more for a shielded element. The high-wavenumber component for a shielded element contains a part that is independent of lateral dome size (face area) when this is large, but this part is highly attenuated by the dome if $L \gg U_\infty/\omega$, where L is dome thickness, ω angular frequency, and U_∞ asymptotic flow velocity. The other part of this high-wavenumber component is reduced somewhat as though the pressure were averaged over the face area of the dome section. On assumption that the wavenumber spectrum changes little in the pertinent interval (in accord with area dependence as R_o^{-2} for a flush element), the low-wavenumber component is reduced rather as though averaged over an area of radius given roughly by the smaller of one-third the sound wave length or three times the dome thickness whenever this area is larger than the actual area of the element. With reference to the noise-to-signal ratio for an array of elements, though a thin unilayer dome can thus be very effective against the high-wavenumber component of flow noise, it can reduce the low-wavenumber component by no more than the array factor, or by not as much if $L \lesssim d/5$, where D is the element spacing.

In the case of a dome with cover having impedance, the effect of flexural resonances of the dome-fluid system is studied.

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SUPPLEMENTARY

INFORMATION

Errata

for Report TRG-011-TN-66-1, "Flow Noise Transmitted Through Domes or Acoustically Modified by Non-Rigid Boundaries," by David M. Chase, January, 1966.

p. 0-3, line (-5): Preceding "a range" insert "in."

p. 0-5, line (-2): For "borader, having" read "broader, leaving."

p. 0-25, Eq. (0-32): For "$(\omega a/c)^{1/2}$" read "$(\omega a/c)^{-1/2}$."

p. 0-25, line (-4): For "(0-4)" read "(0-24)."

p. 1-11, Eq. (1-31): Second form should read

$$"(4/\pi R_o^3) \int_{k_m}^{\infty} dk k^{-2} I_o(k,\omega)."$$

p. 2-36, Eq. (2-94): Following $Q_-(\omega,-L)$, for "-" read "$\simeq$."

p. 2-49, line 5: For "$A(D/\pi r_o^2)$" read "$A(D^2/\pi r_o^2)$."

p. 2-54, Eq. (2-103.8): For "$(\pi R_o^2)^{1/2}$" read "$(\pi R_o^2)^{-1/2}$."

p. 2-54, line (-2): For "k_2" read "k_ω."

p. 2-74, Eq. (2-143): For "$\hat{Q}_+(\omega)$" read "$\delta\hat{Q}_{\mp}(\omega)$."

p. A1-1, line (-2): For "p_o" read "ρ_o."

p. A1-4, Eq. (13c): Left member should be "$\Gamma(k,\omega,y)$."

P. A5-4, Eq. (10): For "$\int d^2\bar{\tau}$" read "$\int d^2\bar{\zeta}$."