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PROBABILITY OF DETECTION FOR SOME
ADDITIONAL FLUCTUATING
TARGET CASES

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**PROBABILITY OF DETECTION FOR SOME ADDITIONAL
FLUCTUATING TARGET CASES**

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The information in a Technical Operating Report is developed for a particular program and is therefore not necessarily of broader technical applicability.

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ABSTRACT

The chi-square family of signal fluctuation distributions is defined. Rules are given for embedding the Swerling cases, and other cases of interest, in this family. Probability of detection curves are presented for the chi-square family of fluctuations, including cases whose probability of detection curves cannot be bracketed by the Swerling cases and the non-fluctuating case. The work of Weinstock has indicated such cases to be of practical interest; the fluctuation loss, for probability of detection exceeding .50, can be much larger than that for Swerling Case I.

A discussion and partial analysis, accompanied by examples, is devoted to the question: when can the detection probability curves for signal fluctuations not belonging to the chi-square family be well approximated by curves resulting from chi-square fluctuations, and what methods can be used to choose adequately fitting chi-square fluctuation models when such a fit is possible?

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SECTION 1

INTRODUCTION

It is often assumed that the Swerling cases⁽¹⁾, together with the non-fluctuating case⁽¹⁾, bracket the behavior of fluctuating targets of practical interest. However, recent investigations of target cross section fluctuation statistics indicate that some targets may have probability of detection curves which lie considerably outside the range of cases which are satisfactorily bracketed by the Swerling cases. The curves of detection probability vs. average signal-to-noise ratio for such targets may be considerably flatter than that for Swerling Case I; put another way, the fluctuation loss for detection probabilities greater than .50, based on required average signal-to-noise ratio to achieve a given detection probability, may be considerably greater than for Swerling Case I. The recent work of Weinstock⁽²⁾ provides examples. It is of interest to compute probability of detection curves for cases of this type, of which those considered by Weinstock may be regarded as extrapolations of the family of Swerling cases, and also to interpolate to various cases intermediate between the Weinstock and Swerling cases, and the non-fluctuating target. Moreover, many examples can be given of fluctuation statistics, of potential practical interest, which lead to detection curves which cannot be well approximated by a fluctuation model belonging to any simple family of extrapolations or interpolations of the Swerling cases.

This report contains a collection of results on these topics, motivated by cases of possible practical interest which the author has encountered. Even though the results presented are an extension of previous calculations, they by no means provide good approximations for all cases which may arise in practice.

Section 2 begins by stating some general, well-known results concerning the properties of fluctuation distributions which determine the probability of detection curves. Next, the chi-square family of fluctuation distributions is defined, and rules are given for embedding the Weinstock cases, the Swerling cases, and the non-fluctuating case in the chi-square family.

Weinstock⁽²⁾ has investigated the fluctuation statistics likely to characterize certain target classes consisting of various relatively simple shapes, or combinations of shapes, similar to some types of earth satellites, with particular reference to the question of whether their fluctuation statistics can be well approximated by chi-square distributions. His distributions are derived by considering both the theoretical scattering patterns of such objects and their scattering patterns as measured from scale models.

He found that the chi-square family does not yield entirely satisfactory fits to the empirically-derived distributions; more important for present purposes, however, is his finding that the best chi-square fits to some of the distributions call for using chi-square distributions with less than two degrees of freedom, in fact, with degrees of freedom between one and two (Swerling Case I is chi-square with two degrees of freedom).

He estimates that the fluctuation loss for such cases, based on average required signal-to-noise ratio, is significantly greater than for Swerling Case I, the "worst" of the Swerling cases. However, Weinstock's estimate of fluctuation loss is not based on detection curves calculated from the probability distribution of signal plus noise at the output of the post-detector integrator, but rather, on a reasonable rule of thumb involving just the signal fluctuation statistics. In Section 3 of the present report, curves are given of probability of detection vs. average signal-to-noise ratio, based on the actual signal plus noise distribution, for the Weinstock case with one degree of freedom. The results show deviations from the detection curves for Swerling Case I which are even greater than those estimated by Weinstock.

In Section 4, a rapid, simple, and reasonably accurate method is given for interpolating to intermediate cases which still belong in some sense to the chi-square family. This method enables complete results to be derived for the family of cases considered, without recourse to further computation; the results are all derived by graphical interpolation utilizing only the graphical results for the Swerling cases and the non-fluctuating case presented in Ref. 1, and for the Weinstock case presented in Section 3 of this report. Also, in Section 4, a discussion is given of procedures for applying these results and of some of the limitations to which they are subject.

Section 5 presents a partial analysis of a class of fluctuation distributions of potential practical interest, which is characterized by the property that for one region of the parameters involved, these distributions can be well approximated by distributions belonging to the chi-square family, while for other values of the relevant parameters, no such good chi-square fit is possible.

SECTION 2

THE CHI-SQUARE FAMILY OF DISTRIBUTIONS

2.1 PRELIMINARY DISCUSSION

Suppose the receiver model is the familiar one usually considered^{(1), (3), (4)} with white receiver noise, independent from pulse to pulse; a normalized square-law envelope detector; and uniform integration of N detector outputs followed by a threshold. Denote by x_i , $i = 1, \dots, N$, the signal-to-noise ratios of the individual pulses, with N the number of pulses integrated; and let

$$X = \sum_{i=1}^N x_i \quad (1)$$

If the signal is fluctuating, x_i and X are random variables. As shown in Ref. 3, for the receiver model under consideration, the probability density function of the output of the post-detection integrator is determined by the probability distribution of X which in turn is determined by the joint probability distribution of x_1, \dots, x_N .

2.2 THE CHI-SQUARE FAMILY

Let v be any random variable, with mean \bar{v} . The random variable v will be said to have a chi-square distribution with $2k$ degrees of freedom if the probability density function of v is

$$w_k(v, \bar{v}) = \frac{1}{(k-1)!} \left(\frac{k}{\bar{v}}\right)^k \left(\frac{v}{\bar{v}}\right)^{k-1} \exp\left(-\frac{kv}{\bar{v}}\right) \quad v > 0 \quad (2)$$
$$= 0 \quad v \leq 0$$

(In conventional terminology of statistics, the chi-square distribution is obtained when $\bar{v} = 2k$ and $2k$ is an integer; however, a terminology will be used here according to which the chi-square family of distributions is the two-parameter family defined by eq. (2), and $2k$ is not restricted to be an integer.)

Now, denote by \bar{x}_i the ensemble average of x_i , and by \bar{X} the ensemble average of X . Then, from (1),

$$\bar{X} = \sum_{i=1}^N \bar{x}_i = N\bar{x} \quad (3)$$

where

$$\bar{x} = \frac{1}{N} \sum_{i=1}^N \bar{x}_i \quad (4)$$

The fluctuation distribution will be called strict-sense chi-square if X has a chi-square distribution. In such a case, the probability density function of X will be denoted by

$$w_K(X, \bar{X}) = \frac{1}{(K-1)!} \frac{K}{\bar{X}} \left(\frac{KX}{\bar{X}}\right)^{K-1} \exp\left(-\frac{KX}{\bar{X}}\right), \quad X > 0 \quad (5)$$

$$= 0, \quad X \leq 0$$

where $2K =$ number of degrees of freedom of X .

Ordinarily, it is more usual to specify fluctuation distributions in terms of the joint distributions of the single-pulse signal-to-noise ratios x_i (and hence of the target cross sections seen on each pulse, which are related to x_i by a

multiplicative constant, assuming other factors in the radar range equation do not vary). The fluctuations will be called wide-sense chi-square if the individual x_i have chi-square distributions. In this case, the probability density functions of the individual x_i will be denoted by

$$w_k(x_i, \bar{x}_i) = \frac{1}{(k-1)!} \frac{k}{\bar{x}_i} \left(\frac{kx_i}{\bar{x}_i} \right) \exp\left(-\frac{kx_i}{\bar{x}_i} \right), \quad x_i > 0 \quad (6)$$

$$= 0, \quad x_i \leq 0$$

where $2k$ = number of degrees of freedom of x_i .

In some special cases, as described immediately below, X as well as the individual x_i are chi-square distributed. In more general cases, the fact that the x_i are chi-square distributed does not imply that X is also, although in many such cases it may be true that the distribution of X can be closely approximated by a chi-square distribution.

2.3 RULES FOR EMBEDDING VARIOUS CASES IN THE CHI-SQUARE FAMILY

Embedding of the Weinstock, Swerling, and non-fluctuating cases in the chi-square family is governed by the following rules:

Rule 1: If the individual x_i have chi-square distributions all with the same mean \bar{x} and the same number of degrees of freedom $2k$, and if the fluctuations are pulse-to-pulse independent, then X has a chi-square distribution with mean $\bar{X} = N\bar{x}$ and degrees of freedom $2K = 2kN$.

Rule 2: If the individual x_i have chi-square distributions all with the same number of degrees of freedom $2k$ (but not necessarily with the same means), and if the fluctuations are scan-to-scan, then X has a chi-square distribution with mean $\bar{X} = N\bar{x}$, \bar{x} being given by eq. (4), and degrees of freedom $2k = 2K$. (If the individual means \bar{x}_i are different, scan-to-scan fluctuation means that the values of x_i/\bar{x}_i are all equal for the N pulses on a single scan.)

Rule 3: Suppose the individual x_i have chi-square distributions all with the same mean \bar{x} and the same number of degrees of freedom $2k$. Also suppose the N pulses can be divided into f groups, each containing N/f pulses, such that the fluctuations are completely correlated (i. e., the values of x_i are equal) for all pulses within a group, but the values of x_i are statistically independent from group to group. Then, X has a chi-square distribution with mean $\bar{X} = N\bar{x}$ and degrees of freedom $2K = 2kf$.

Fluctuations of the type described in Rule 3 might arise, for example, as the result of frequency hopping, where the frequency increments are large enough to produce independent samples for the target in question, and where f different frequencies are used. Fluctuations resulting from target motion would generally not be of the type described in Rule 3.

It is also of interest to consider the case where the fluctuations are of the kind described in Rule 3, with the single exception that the f groups do not all contain the same number of pulses. Such a case might arise, for example, if fluctuations were produced by f different frequencies but if N were not an integral multiple of f . In this case, X is not chi-square distributed. However, it can be shown, by methods to be outlined in Section 4, that the distribution of X in most cases of this type, if the groups are not too disparate in size, is closely approximated by a chi-square distribution with mean $\bar{X} = N\bar{x}$ and degrees of freedom $2kf$.

The rules stated above imply the following table showing the relation of various standard cases to the chi-square family.

Discussion of procedures to apply when X is not rigorously chi-square (such as, for more general types of fluctuation correlations, or when x_i are not chi-square), and the limitations of such procedures, will be deferred to later sections.

Table 1. Relation of Various Fluctuating Target Cases to Chi-Square Family

Fluctuating Target Model	Value of k	Value of K
Swerling Case I	1	1
Swerling Case II	1	N
Swerling Case III	2	2
Swerling Case IV	2	2N
Weinstock Case, Scan-To-Scan	1/2	1/2
Weinstock Case, Pulse-To-Pulse	1/2	N/2
Non-Fluctuating Case	∞	∞

SECTION 3

DETECTION CURVES FOR WEINSTOCK CASE $k = K = \frac{1}{2}$

Figures 1, 2, and 3 show probability of detection curves vs. \bar{x} for the case $k = K = \frac{1}{2}$ (i. e., the Weinstock case with scan-to-scan fluctuations), for $N = 1, 10, 100$ respectively, and for false alarm number $\alpha = 10^{-10}$ (defined as in Marcum⁽¹⁾).

As can be seen, the curves lie significantly outside the range of cases bracketed by the Swerling cases. It is unnecessary to compute separate curves for the Weinstock case for pulse-to-pulse fluctuations, since, by Rule 1, such curves are identical to those for scan-to-scan fluctuations with higher values of k .

It is, of course, of interest to extend these results to all values of k between one-half and infinity, as well as to different values of n and N . This could be done by repeating the computations using the general formulas presented in Section 3.1, a simpler method, without requiring recourse to additional computations, is given in Section 4.

3.1 DERIVATION OF FIGURES 1, 2, 3

Suppose the random variables x_i are chi-square distributed with common means \bar{x} and degrees of freedom $2k$, i. e., with density function given by eq. (6). According to the method described in Ref. 3, the characteristic function of the signal plus noise output y of the post-detector integrator, assuming N completely correlated pulses, is (using the same notation and definition of characteristic function as in Refs. 1, 3, and 4)

$$\begin{aligned}
 C(p) = E[e^{-py}] &= (1+p)^{-N} \left[1 + \frac{N\bar{x}p}{k(1+p)} \right]^{-k} \\
 &= \left(1 + \frac{N\bar{x}}{k} \right)^{-k} (1+p)^{-(N-k)} \left[p + \frac{1}{1 + \frac{N\bar{x}}{k}} \right]^{-k}
 \end{aligned} \tag{7}$$

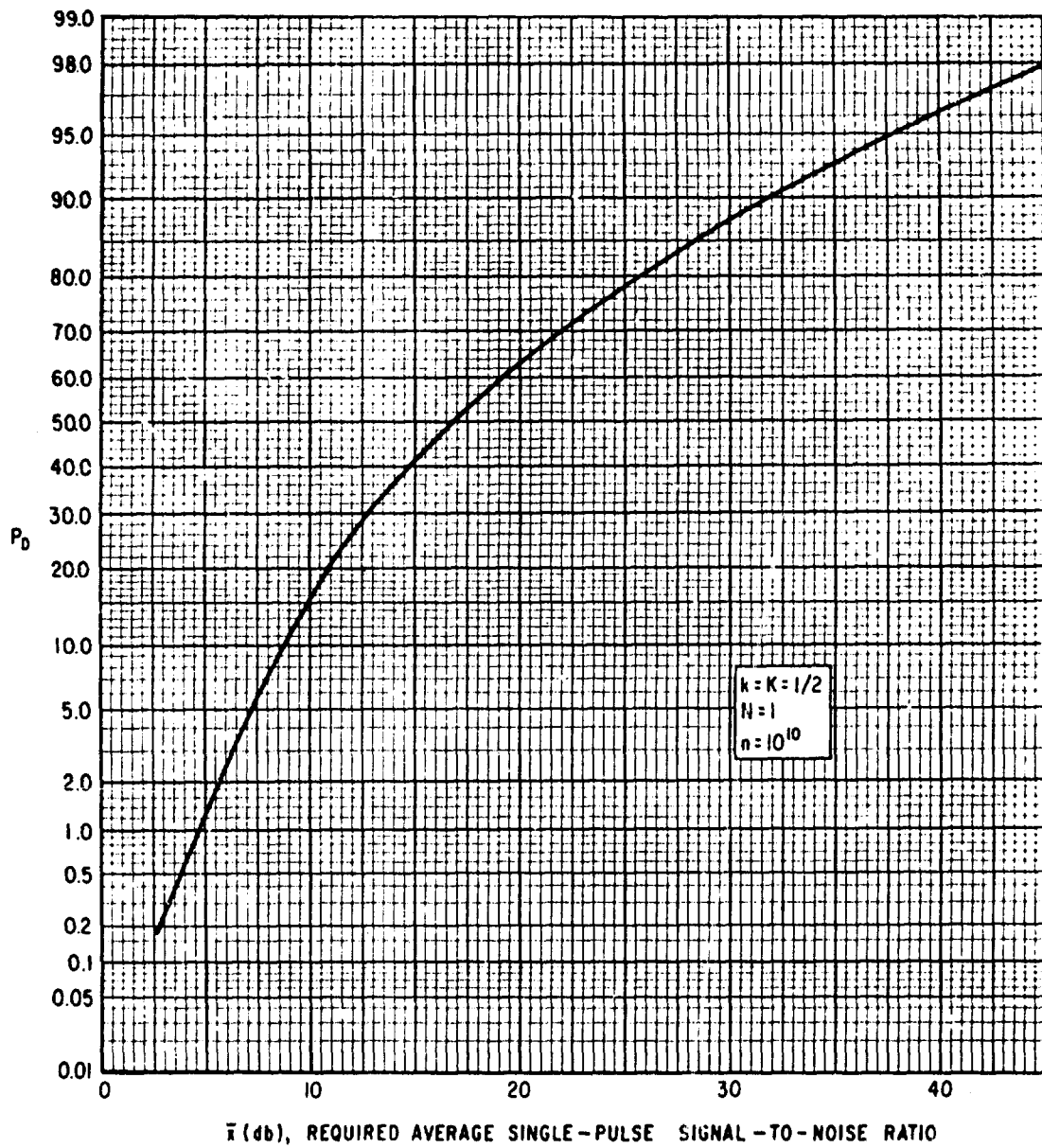


Figure 1. P_D vs. \bar{x} for $N = 1$

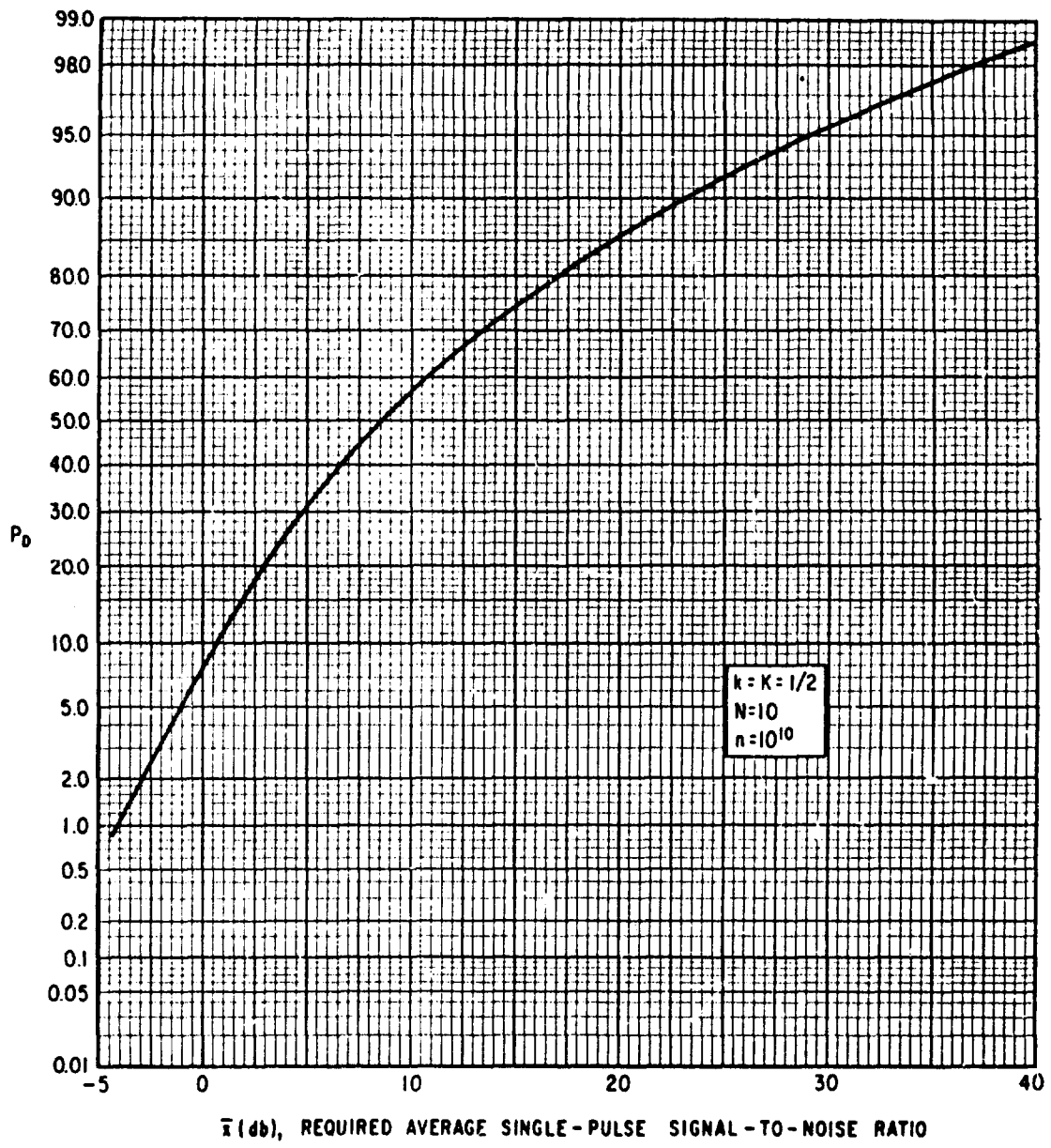


Figure 2. P_D vs. \bar{x} for $N = 10$

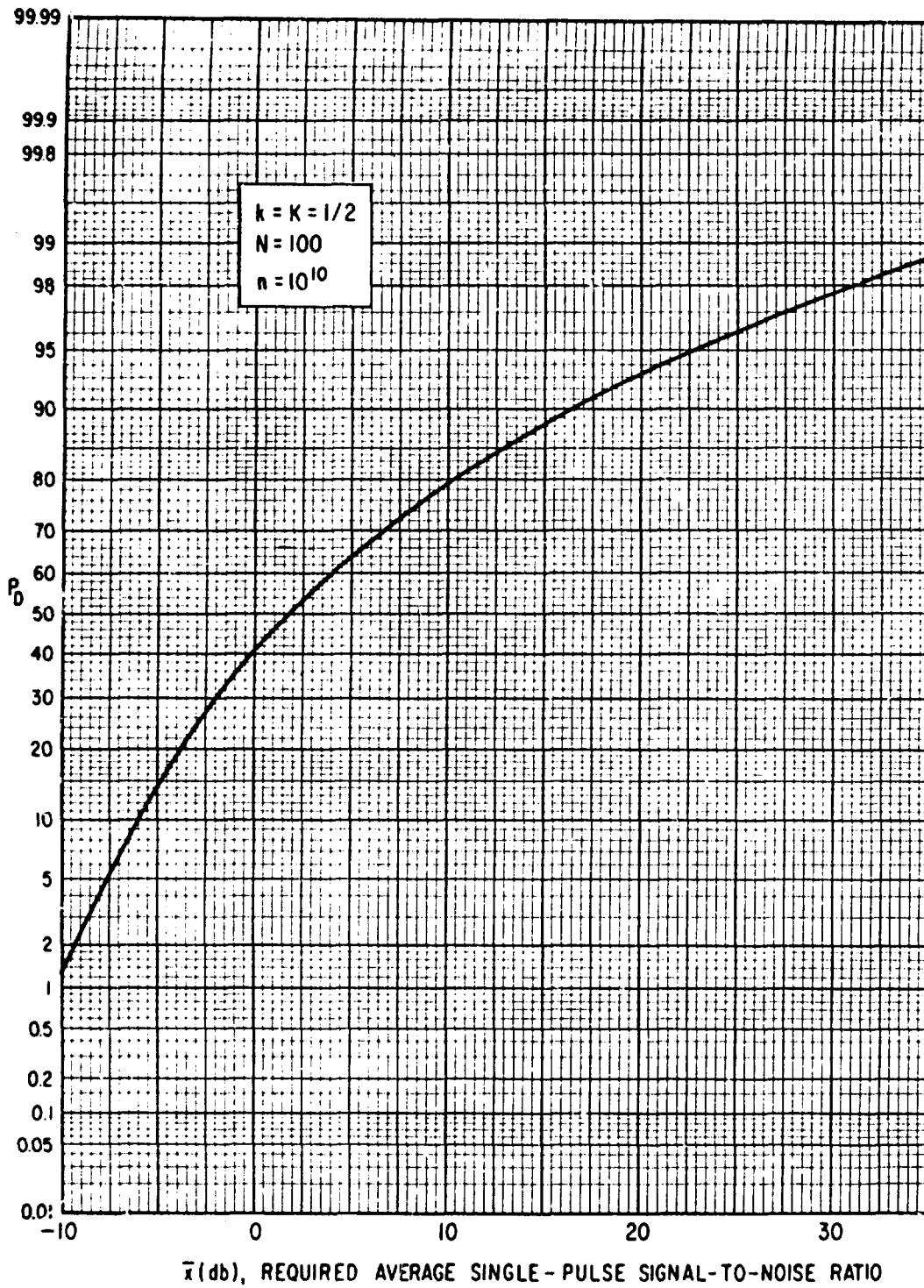


Figure 3. P_D vs. \bar{x} for $N = 100$

Utilizing pair no. 581.1 of Ref. 5, with appropriate changes of notation, this means that the probability density function of y is

$$dP(y, N, k) = \frac{\left[1 + \frac{N\bar{x}}{k}\right]^{-k}}{(N-1)!} y^{N-1} e^{-y} {}_1F_1\left[k, N, \frac{y}{1 + \frac{k}{N\bar{x}}}\right] dy, \quad y \geq 0 \quad (8)$$

$$= 0, \quad y < 0$$

where ${}_1F_1$ is the confluent hypergeometric function^{(1), (6)}.

The probability that y exceeds a threshold Y_b is

$$P_D = \int_{Y_b}^{\infty} dP(y, N, k) \quad (9)$$

The detection probability P_D can be desk-computed by utilizing the asymptotic expansion of the confluent hypergeometric function. Using the derivation of Copson⁽⁶⁾, pp. 260-265, the first three terms of this asymptotic expansion are

$${}_1F_1(k, N, z) = \frac{\Gamma(N)}{\Gamma(k)} \frac{e^z}{z^{N-k}} \left\{ 1 + \frac{(N-k)(1-k)}{z} + \frac{(N-k)(N+1-k)(1-k)(2-k)}{2z^2} + \dots \right\} \quad (10)$$

where Γ is the gamma function.

The variable $z = y(1 + k/N\bar{x})^{-1}$ is all in cases of interest sufficiently large for the asymptotic expansion to yield good results*.

* Refs. 7 and 8, in stating the asymptotic expansion of ${}_1F_1$, give a series of terms additional to those which appear from Copson's formula⁽⁶⁾. These terms are much smaller than those arising from Copson's formula. Moreover, it is difficult to see how these additional terms could be applicable in the case at hand, since for $k = 1/2$, z real and positive, they would be pure imaginary while ${}_1F_1$ is purely real.

Finally, a formula for P_D , suitable for desk computation, is obtained by inserting (10) into (8) and (9) and performing the integration. In the cases at hand, after various obvious integrations by parts, the following result is obtained. Let

$$Z_b = \frac{Y_b}{\left[1 + \frac{N\bar{x}}{k}\right]} \quad (11)$$

Then, for $k \neq \text{integer}$,

$$P_D = A_1 + A_2 + A_3 \quad (12)$$

In the latter expression, the terms A_1 , A_2 , and A_3 are given by:

$$A_1 = \left[1 + \frac{k}{N\bar{x}}\right]^{N-k} \left[1 - I\left(\frac{Z_b}{\sqrt{k}}, k - 1\right)\right] \quad (13)$$

where I is the incomplete gamma function⁽¹⁾:

$$I(u, p) = \int_0^{u\sqrt{1+p}} \frac{e^{-v} v^p dv}{p!}$$

$$A_2 = \frac{\left[1 + \frac{k}{N\bar{x}}\right]^{N-k+1}}{\left[1 + \frac{N\bar{x}}{k}\right]} (N - k) \left\{ \frac{e^{-Z_b} Z_b^{k-1}}{\Gamma(k)} - (1 - I) \right\} \quad (14)$$

and

$$A_3 = \frac{\left[1 + \frac{k}{N\bar{x}}\right]^{N-k+2} (N-k)(N-k+i)}{2\left[1 + \frac{N\bar{x}}{k}\right]^2} \left\{ (1-I) - \frac{e^{-Z_b} Z_b^{k-2}}{\Gamma(k-1)} - \frac{e^{-Z_b} Z_b^{k-1}}{\Gamma(k)} \right\} \quad (15)$$

In (14) and (15), I is the incomplete gamma function with the same arguments as in (13).

Somewhat simpler appearing formulas can be obtained by collecting terms; however, it is computationally convenient to segregate the terms A_1 , A_2 , and A_3 arising from the first, second, and third terms of the asymptotic expansion of ${}_1F_1$.

The actual computations of Figs. 1, 2, 3 were done using Table 26.7 of Ref. 7, which gives values of the incomplete gamma function. The values of Y_b used were 23.4, 42.3, and 168 respectively for $n = 10^{10}$, $N = 1, 10, 100$; these were taken from Figs. 27 and 57a, pp. 141-142 of Ref. 1 (actually, from the author's records of the original calculations of the curves of Ref. 1). For many points, the first two terms of the asymptotic expansion suffice.

SECTION 4

INTERPOLATION TO OTHER VALUES OF K

Figures 4, 5, and 6 give results for essentially all cases where X is chi-square distributed, for $N = 1, 10, \text{ and } 100$ and $n = 10^{10}$, and for $1/2 \leq K \leq \infty$ (the case $K = 1000$ is, within a tenth of a db. or less, equivalent to $K = \infty$). The results are plotted, in conformity with usual practice, against \bar{x} , the required average signal-to-noise ratio for individual pulses; however, the vertical axis is K, half the number of degrees of freedom for X. Methods of extending these results to other values of n and N will be stated shortly.

The remainder of this section comprises: a description of the interpolation method used to obtain these results, with an estimate of the accuracy achieved; rules for extending the results to other values of n and N; rules for applying the results to various cases; and an outline of a possible accuracy investigation when applying the results to cases where X is not rigorously chi-square distributed but may be approximately so distributed.

4.1 METHOD OF DERIVATION OF FIGURES 4, 5, AND 6

The method of obtaining Figs. 4, 5, and 6 is based on Table I, Section 2.3. This table enables many of the points in Figs. 4, 5, and 6 to be read directly from existing graphical results, either in Ref. 1 or in Section 3 of this report. In this manner, the following points can be directly obtained: For $N = 1, K = 1/2, 1, 2, 1000$; for $N = 10, K = 1/2, 1, 2, 10, 20, 1000$; for $N = 100, K = 1/2, 1, 2, 100, 200, 1000$. ($K = 1000$ is equivalent within less than .1 db, to the non-fluctuating case.) These points alone are almost sufficient for a graphical interpolation. However, to get a good interpolation, they should be supplemented by a few intermediate points. This can be conveniently done as follows.

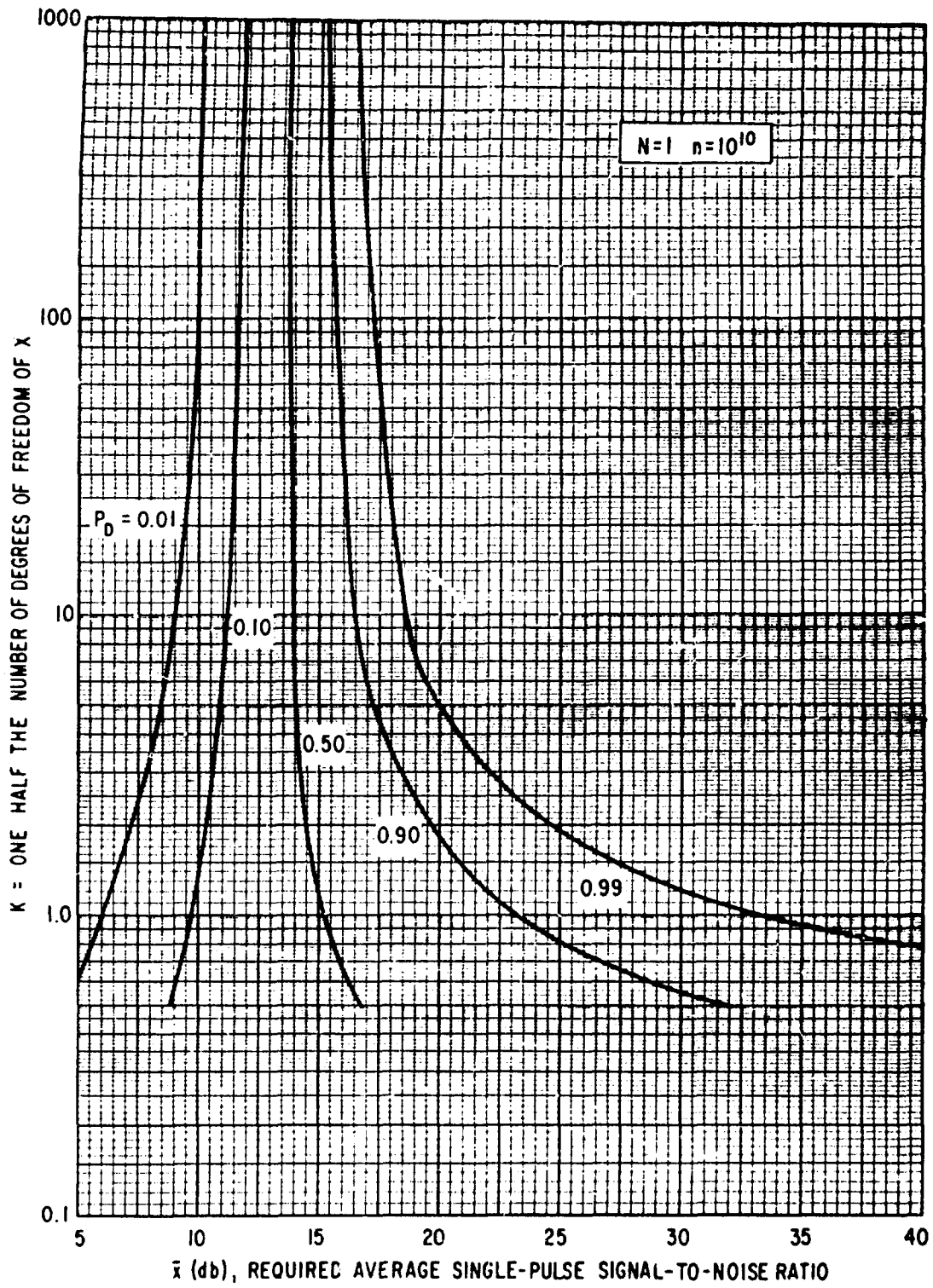


Figure 4. \bar{x} vs. K for Various P_D and $N = 1$

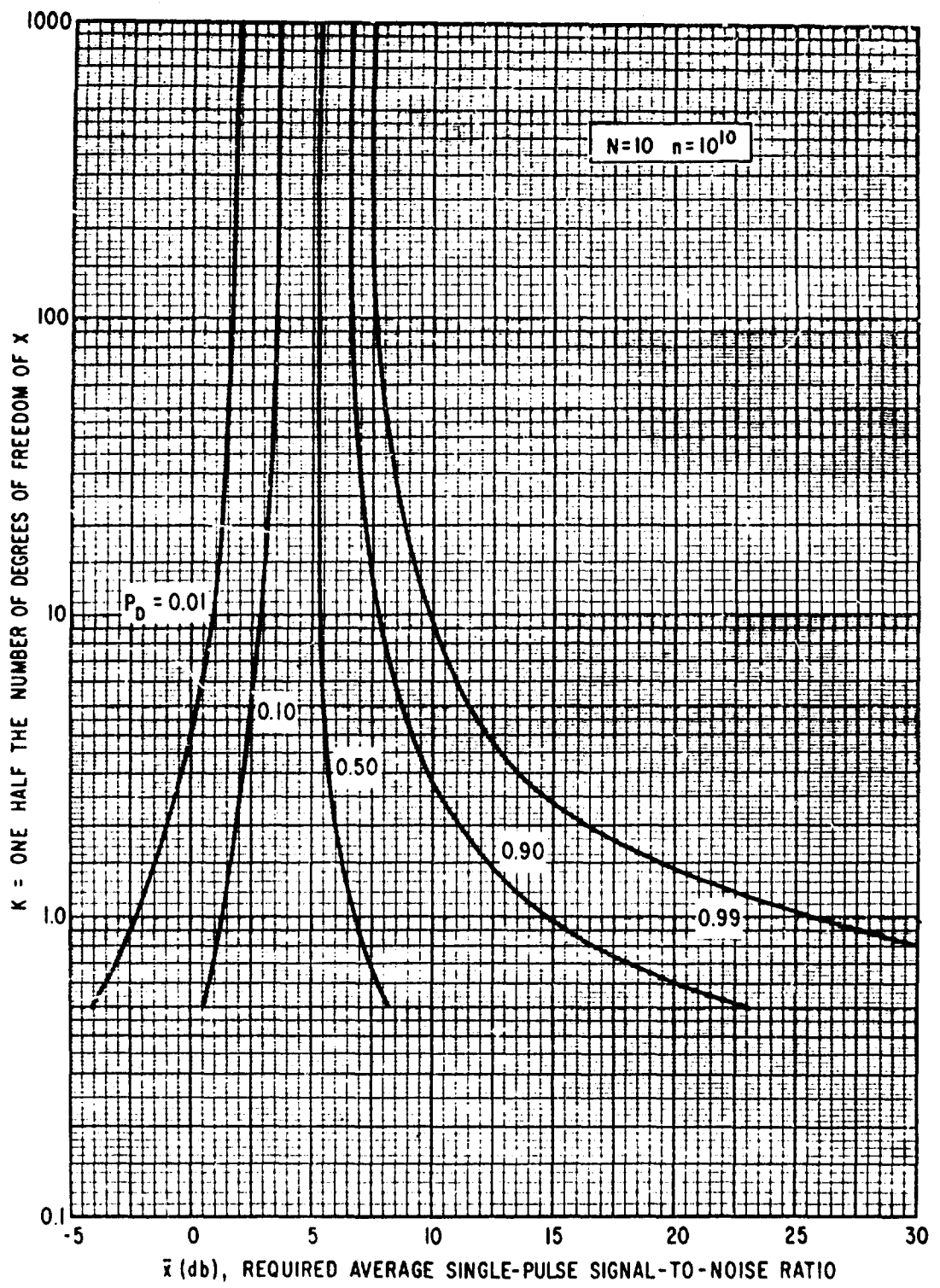


Figure 5. \bar{x} vs. K for Various P_D and $N = 10$

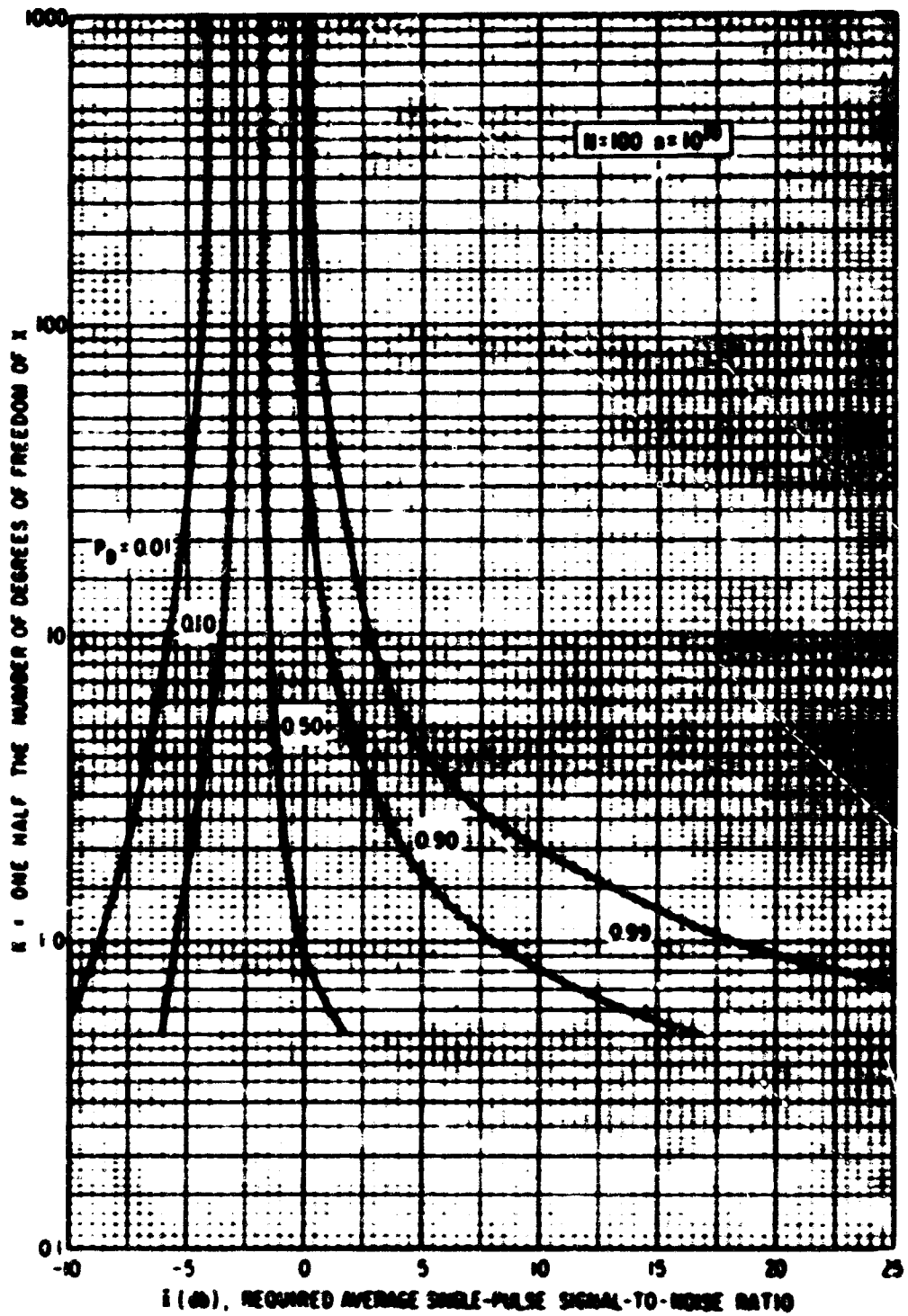


Figure 6. \bar{x} vs. K for Various P_D and $N = 100$

For any fixed value of K , let

$$\bar{x}(K, N, P_D) = \text{required value of } \bar{x} \text{ to get } P_D \text{ for } N \text{ integrated pulses} \quad (16)$$

Then approximately,

$$10 \log_{10} \{\bar{x}(K, N', P_D)\} = 10 \log_{10} \{\bar{x}(K, N, P_D)\} + 10 \log_{10} \left(\frac{N}{N'} \right) + L_{INT}(K, P_D, N') - L_{INT}(K, P_D, N) \quad (17)$$

where L_{INT} is the integration loss expressed in db. However, the integration loss L_{INT} is, within a few tenths of a db, independent of K . One can verify from Swerling's curves⁽¹⁾ that for Swerling Case 1, with $K = 1$, the integration loss is within half a db or less of that for $K = \infty$, for the range of parameter values of interest. Moreover, in order to get intermediate points for Figs. 4, 5, and 6, it is necessary only to apply eq. (17) for higher values of K , e.g., $K \geq 4$. For such values, the integration loss is within two or three tenths of a db of that for $K = \infty$. Thus, the values of L_{INT} for the non-fluctuating case can be used.

In this way, points on the curves of Figs. 4, 5, and 6 can be obtained for intermediate values of K . As an illustration, suppose one wishes to obtain the points $K = 10$ and $K = 20$ for Fig. 6, i.e., for $N = 100$. The procedure is as follows. First determine $\bar{x}(10, 10, P_D)$ and $\bar{x}(20, 10, P_D)$ directly from the curves for $N = 10$, $n = 10^{10}$, Swerling Cases II and IV, respectively. Then determine $\bar{x}(10, 100, P_D)$ and $\bar{x}(20, 100, P_D)$ from eq. (17).

The plotting of curves such as those in Figs. 4, 5, and 6 can proceed very rapidly; e.g., about 15 to 30 minutes suffices for an entire set of curves for a given value of N .

The major sources of error in the resulting curves are:

- (1) errors resulting from reading values of \bar{x} off the graphs in Ref. 1;
- (2) errors in applying integration loss for intermediate points;
- (3) errors in fitting a curve through the resulting points.

Of these, probably (1) is the major source of error. It is estimated that, for all points on the curves, the resultant of such errors is bounded by about .5 db; and it may, for many points, be of the order of two or three tenths of a db.

4.2 EXTENSION TO OTHER VALUES OF N AND n

The extension to other values of N may be accomplished by plotting the points $N = 1, 10, 100$, together with other points obtained from eq. (17), and fitting a curve to the resulting points. The error in applying non-fluctuating integration loss can be expected to be small, even for small K, since one will only need to apply the difference between the integration losses for relatively close values of N and N' (such as 10 and 30), and these differences will to high accuracy be independent of K.

Extension to other values of n can also most conveniently be accomplished by applying a small correction to \bar{x} , which to high accuracy can be taken equal to the correction that would apply for non-fluctuating targets. (This correction is roughly .35 db per order of magnitude change in n for n near 10^{10} , and increases to about .7 db per order of magnitude for n near 10^6 .)

It is estimated that, for the parameter regions of interest, these methods of extending to other n and N could introduce resultant errors of the order of .2 to .3 db; the resultant of these with the previously mentioned errors in Figs. 4, 5, and 6 is estimated to be within about .6 db throughout the parameter region of interest.

The results presented here for $K \geq 1$ should agree with those of Refs. 9 and 10 for the cases labelled $V \leq 1$ in those references with $V = 1/K$. (The cases labelled $V > 1$ in Refs. 9 and 10 have however no relation to our cases $K < 1$.) The results in Refs. 9 and 10 were obtained by digital computation. Comparison shows agreement, within the expected accuracy, in those cases where there should be agreement. Great caution must be exercised in utilizing the material in Refs. 9 and 10, which state some highly misleading assumptions and conclusions as to the manner in which the results can be applied and interpreted.

4.3 RULES FOR APPLICATION OF FIGS. 4, 5, AND 6

As previously mentioned, fluctuation distributions are more commonly specified in terms of the joint distribution of (x_i) than in terms directly of the distributions of X . Thus, the procedure for applying Figs. 4, 5, 6, or their extensions to other values of n and N consists of two steps:

- (1) Given the joint distribution of (x_i) , determine the number of degrees of freedom in the chi-square distribution of X , for cases where X has a chi-square distribution; or more generally, if the distribution of X can be well approximated by a chi-square distribution, determine the parameters \bar{X} and K of a chi-square distribution giving an adequate fit.
- (2) Apply Figs. 4, 5, 6, or similar results. (If the "best fitting" chi-square has mean different from the true mean of X , as can sometimes happen, then \bar{x} in Figs. 4, 5, 6 is simply \bar{X}/N , where \bar{X} is the mean of the best-fitting chi-square rather than the true mean of X . In such cases, one must specify the relation of \bar{X} to some parameter of the true distribution of X ; see Section 5 for an example.)

In general, there is no difficulty in the second step; the difficult step is the first. Various possible situations can be delineated:

- (a) Suppose the joint distribution of (x_i) is such that X has a chi-square distribution, such as the cases described in Rules 1, 2, 3 of Section 2.3. Then K is found by a direct application of these rules stated in Section 2.3.

- (b) Suppose the individual x_i have chi-square distributions, but for one reason or another their joint distribution does not fall within the compass of the special cases in which X has also a chi-square distribution. This may be the case, for example, for more general types of fluctuation correlations; or if x_i do not all have the same number of degrees of freedom; or if x_i have different mean values.

In many such cases, it is probable that the distribution of X can be well approximated by a chi-square distribution. One indicated method of obtaining the approximating chi-square distribution is to use the chi-square with the same mean and second moment as X , which is to say, to use the leading term in the Laguerre series expansion (11) of the distribution of X . The second moment of X can be obtained, given the first and second moments of (x_i) , including cross moments. It is not certain that this always gives the best fit, for purposes of approximating the detection probability curves for some given interval of detection probabilities. In any case, such a procedure should be accompanied by an estimate of the accuracy achieved. So far as the author is aware, no analysis has been made of the interesting question of what accuracy is achievable by applying this procedure when the individual x_i have chi-square distributions but X does not. An outline of possible approaches to such an investigation is given in Section 4.4 below.

- (c) If the individual x_i themselves are not restricted to have chi-square distributions, then the probability of detection curves may or may not be obtainable, to an adequate degree of approximation, by the procedure of finding a chi-square fit to the distribution of X . Examples can be given to illustrate both situations (see Section 5, for instance). Each situation must be analyzed on its own merits. Methodology for doing this is illustrated in Section 5.

One point which should be emphasized is that two fluctuation distributions may have identical first and second moments, and still yield completely different detection probabilities; or, the ratio of mean to standard deviation may be the same, but the curves of detection probability vs. \bar{x} may be widely different. Examples of this are easy to give. Conversely, two fluctuation distributions may have very different ratios of mean to standard deviation, and still have very similar detection probability curves; examples of this situation are also easy to give. Thus, even if a good chi-square fit is possible, it cannot necessarily be obtained by matching the first two moments of the distribution of X. Section 5 illustrates this also.

Suppose one wishes to find a chi-square fit to the fluctuation distribution, in such a way as to yield a good approximation to the probability of detection curves over some interval of detection probabilities, say $P_1 \leq F_D \leq P_2$. A good rule of thumb is that the cumulative distribution function of X must also be approximated, to about the degree of accuracy desired, over the same region of probabilities of the fluctuation distribution (i. e., between the P_1 and P_2 percentiles of the fluctuation distribution). It is assumed here that a "good fit" requires that the value of K must be constant, and that \bar{X} must have a fixed ratio to the true mean, for all probabilities of interest. Accuracy of approximation can be defined in terms of the difference, in db, between the values of X at a given probability.

4.4 OUTLINE OF ACCURACY INVESTIGATION FOR CHI-SQUARE FITS

It was mentioned that it would be of interest to investigate the accuracy obtainable by applying the procedure stated in Section 4.3. to cases where the individual x_i have chi-square distributions but X does not.

One approach would be to expand, in specific instances, the exact distribution of X in a Laguerre series, and determine the magnitude of the second or higher terms. This requires the third or higher moments of X to be computed, which in turn can be done⁽³⁾ if one knows the third or higher moments.

including cross moments, of $\{x_i\}$. However, such an approach leaves something to be desired, since it would be preferable to obtain accuracy estimates by direct comparison of the exact and approximate probability of detection curves themselves, rather than of the distributions of X ; also, it would be desirable to obtain generally applicable bounds on the error in addition to error estimates for individual cases.

The analytical tools for comparing the detection probabilities themselves exist, and these tools may also make possible the derivation of general error bounds. The starting point is the fact that the characteristic function of the exact probability distribution of the integrator output has been derived⁽⁴⁾.

For example, consider the case where the x_i have common means \bar{x} and number of degrees of freedom $2k$, but the fluctuation correlations are arbitrary. Then, the exact characteristic function of the integrator output is

$$C(p) = (1+p)^{N(k-1)} \prod_{i=1}^N \left\{ 1 + p \left[1 + \frac{\mu_i \bar{x}}{k} \right] \right\}^{-k} \quad (18)$$

where μ_i are non-negative constants related to the fluctuation correlations, they are the eigenvalues of a certain matrix⁽⁴⁾.

On the other hand, the characteristic function of the integrator output, assuming that X has an approximate chi-square distribution with mean $N\bar{x}$ and degrees of freedom $2K$, is

$$C(p) = (1+p)^{-N} \left[1 + \frac{N\bar{x}p}{K(1+p)} \right]^{-K} \quad (19)$$

The exact cumulants (and hence the exact moments) of the integrator output can be determined in terms of the quantities $\sum_{i=1}^N \left(1 + \frac{\mu_i \bar{x}}{k}\right)^m$ for

$m = 1, 2, \dots$. Thus, one might proceed as follows.

Suppose the type of fluctuations is simply specified by k and by the set (μ_i) , $i = 1, \dots, N$. One can then determine bounds on the quantities

$$\sum_{i=1}^N \left(1 + \frac{\mu_i \bar{x}}{k}\right)^m \quad \text{for } m = 3, 4, \dots \text{ given that } \mu_i \geq 0 \text{ and}$$

$$\sum_{i=1}^N \left(1 + \frac{\mu_i \bar{x}}{k}\right) = C_1 \quad (20)$$

$$\sum_{i=1}^N \left(1 + \frac{\mu_i \bar{x}}{k}\right)^2 = C_2 \quad (21)$$

where C_1 and C_2 are given constants related to the first and second moments of the integrator output.

Similarly, the moments of the approximate integrator output distribution determined from eq. (19) can be calculated. This could be done on the basis that $N\bar{x}$ in eq. (19) is the true mean value of X and \bar{X}^2/K is equal to the true variance of X . This amounts to applying the procedure stated in Section 4.3, using a chi-square fit to X with the same mean and second moment as X actually has. Alternatively, one could assume $N\bar{x}$ and K in (19) to have slightly different values, provided these are related in a specified way to the true distribution of X , since in some cases better fits could be obtained in this way.

Finally, one could expand both the true and approximate distributions of the integrator output in say a Laguerre series. The results obtained as just described would then enable general bounds to be placed on the differences in successive terms of these two Laguerre series.

Such an investigation would probably show that for sufficiently small values of N , say $N \leq 5$, good approximations are always obtained by using a chi-square approximation to the distribution of X , having first and second moments equal to the true first and second moments of X . On the other hand, as N increases, the possible variations in the distribution of X , subject to its having given first and second moments, get larger, so that for sufficiently large N (possibly 30 or more), it is quite likely that additional restrictions would have to be placed on the fluctuation correlations in order to ensure good approximation by this method.

SECTION 5

SOME RESULTS ON LOG-NORMAL DISTRIBUTIONS

Some evidence resulting from the analysis of cross section measurements of ships and missiles⁽¹¹⁾ has indicated that the radar cross section distributions of some targets of these types may be log-normal (to be more precisely defined below). It is thus of interest to investigate whether and when the log-normal distributions can be well fitted by chi-square distributions, how good the fit is, and what are the parameters of the best-fitting chi-square distribution for given parameters of the log-normal distribution.

If the radar cross section is log-normally distributed, so will be x , the single-pulse signal-to-noise ratio (assuming other factors in the radar range equations do not vary). If the fluctuations are scan-to-scan, X will also be log-normally distributed. It is thus of interest to investigate how well a log-normal distribution of x can be fitted by a chi-square distribution. For more general types of fluctuation correlations, X will not be log-normally distributed, but the fitting of the distribution of x by a chi-square will still be of interest, since the existence of a good chi-square fit for the distribution of x will in general imply the existence of a good chi-square fit for X .

The procedure for using the following results to calculate detection probability when cross section is log-normally distributed, would then be:

- (1) See whether the log-normal parameters are such that a good chi-square fit to the distribution of x is possible; if it is, determine the parameters \bar{x} and k of the best fitting chi-square. As will be seen, \bar{x} is no longer necessarily the true mean of x nor is the variance of the best-fitting chi-square distribution necessarily equal to the true variance of x ; however, \bar{x} and k can be related in a definite way to the parameters of the log-normal distribution of x .

(2) Next, using the procedures outlined in Section 4, obtain the best chi-square fit to the distribution of X, taking into account the type of fluctuation correlations.

(3) Then, apply Figs. 4, 5, 6, or their extensions.

The analysis to be presented here will not be concerned with signal plus noise distributions, but just with the question of fitting given log-normal distributions by chi-square distributions. According to the rule of thumb cited in Section 4.3, the results should be strongly indicative of when the detection curves resulting from log-normal distributions of x can be well approximated by the procedure just stated, and when not. However, such conclusions ought finally to be verified by calculation of detection probability curves based on signal plus noise distributions, explicitly based on log-normal distributions for x. The problem of obtaining such detection curves is a useful one to investigate (the author understands that this problem is under active investigation⁽¹¹⁾).

Precisely what is meant by a good fit of a log-normal distribution by a chi-square distribution will become clear in the course of presenting the results.

5.1 LOG-NORMAL DISTRIBUTIONS AND CHI-SQUARE FITS

A random variable x is said to have a log-normal distribution if the logarithm of x is normally distributed. The probability density function for x itself is

$$w(x, \sigma, a) = \begin{cases} \frac{1}{\sigma x \sqrt{2\pi}} \exp\left\{-\frac{1}{2\sigma^2} \left[\ln\left(\frac{x}{a}\right)\right]^2\right\} & (x > 0) \\ 0 & x \leq 0 \end{cases} \quad (22)$$

Here, σ is the standard deviation of the natural logarithm of x, and a is the median of x.

Fig. 7 plots some log-normal distributions vs. $\ln(x/a)$. Figs. 8 and 9 show plots of chi-square distributions for various values of k = one half the number of degrees of freedom, vs. $\ln(x/\bar{x})$, \bar{x} being the mean of the chi-square distribution.

Goodness of fit will be defined with respect to some given interval of probabilities. For the results to be presented, this interval will be chosen to be $.01 \leq P \leq .99$. The goodness of fit will be determined by how well a curve such as those shown in Figs. 8 and 9 can fit a given curve of the kind shown in Fig. 7, over the whole interval $.01 \leq P \leq .99$. In fact, let \bar{x} be the mean value of a given chi-square distribution, and let \bar{x}' be the mean value of another chi-square distribution which intersects a given log-normal distribution at a certain value of P . Then, the deviation of the chi-square distribution with mean \bar{x} , from the given log-normal distribution, at probability P , will be defined (in db) as

$$\left| 10 \log_{10} \left(\frac{\bar{x}'}{\bar{x}} \right) \right|.$$

The goodness of fit of a given chi-square to a given log-normal distribution is defined as the maximum deviation for all P in the interval $.01 \leq P \leq .99$.

It is also of interest to define chi-square distributions which bracket any given log-normal distribution over a given range of probabilities. Such bracketing is then a strong indication that the true detection probability will be bracketed by the values calculated by assuming x to have the chi-square distributions which bracket the true log-normal distribution of x , provided the true detection probability lies in about the same range as that over which the log-normal distribution is bracketed. (By distribution in the above, reference is had to the cumulative probability distribution, not the density function.)

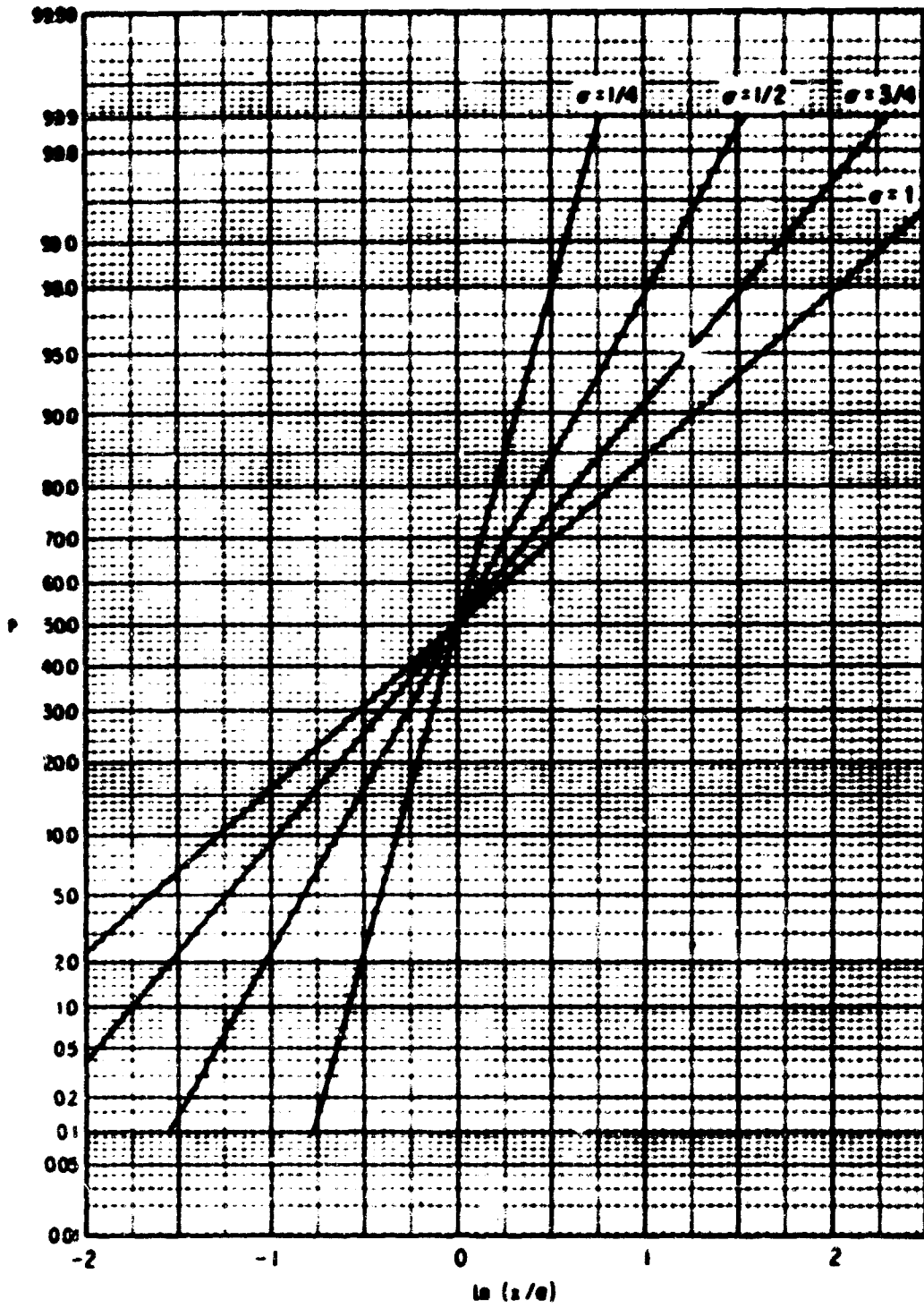


Figure 7. Log-Normal Distribution

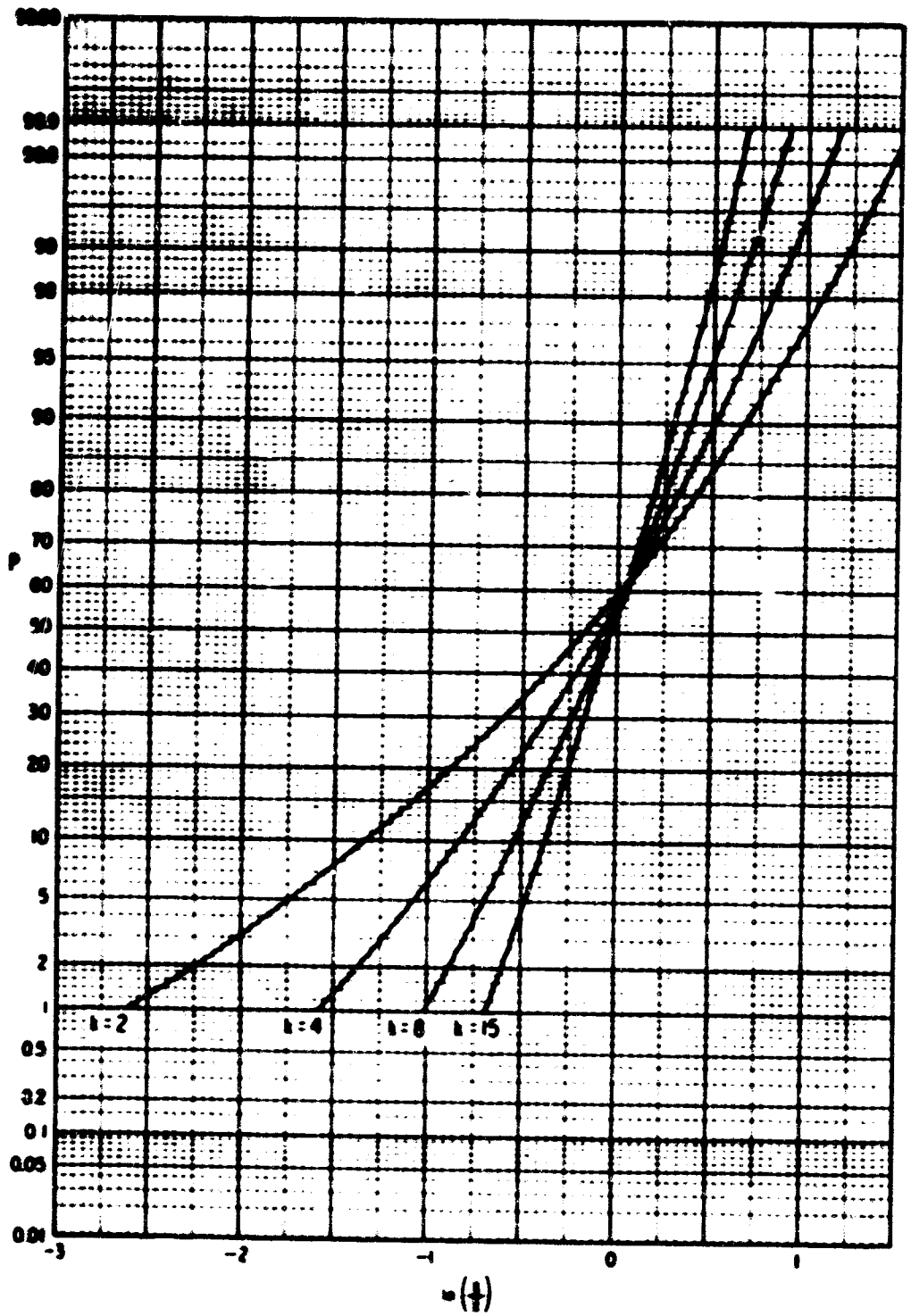


Figure 8. Chi-Square Distribution for $k = 2, 4, 8, 15$

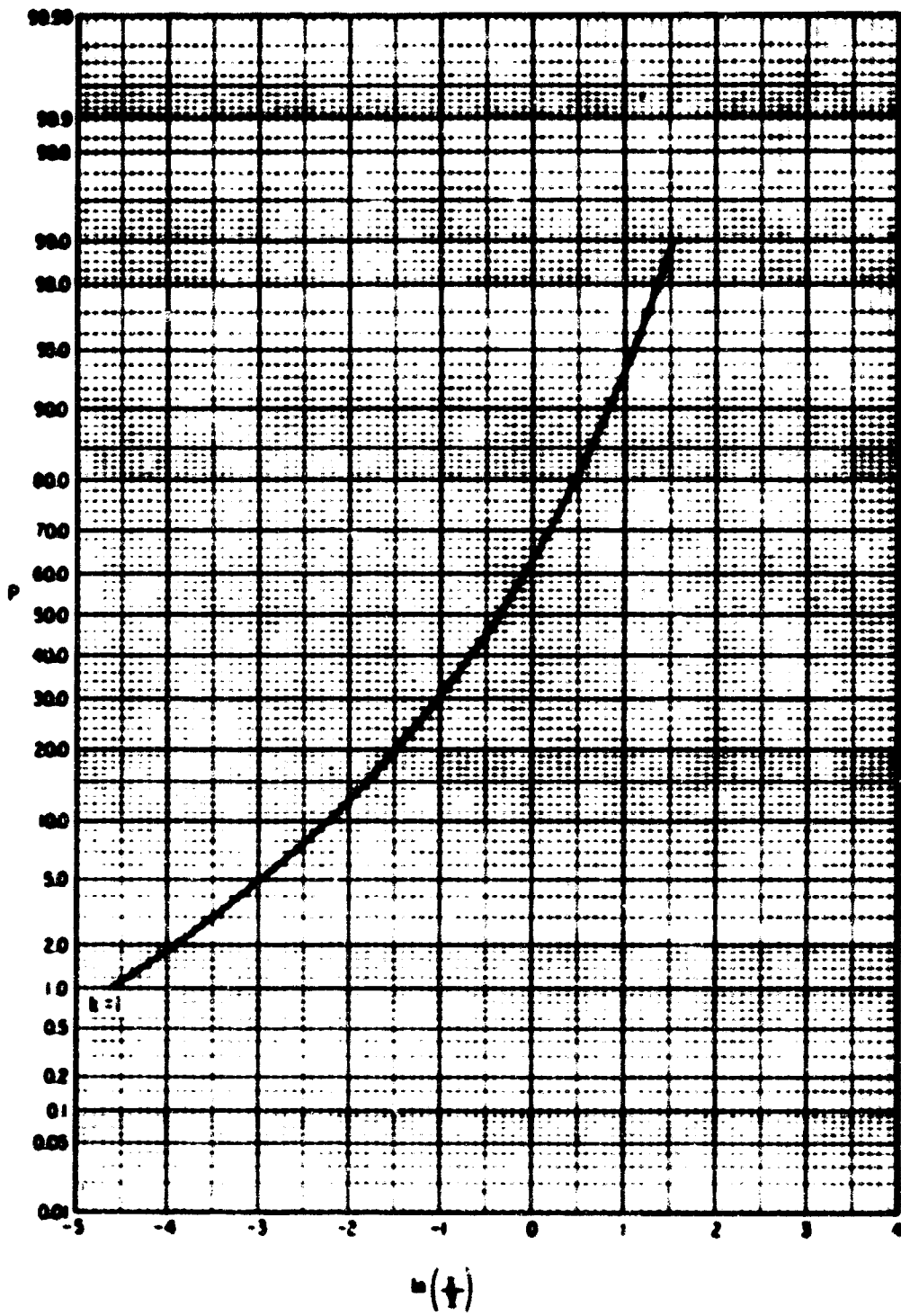


Figure 9. Chi-Square Distribution for $k = 1$

Now, a glance at Figs. 7, 8, and 9 shows that chi-square distributions which fit or bracket given log-normal distributions can be obtained in the following manner, and moreover that for all practical purposes the best fits or the closest brackets are thus obtained:

Consider a given log-normal family characterized by a fixed value of σ , as given by a curve such as those in Fig. 7. Find the three chi-square distributions, all having a fixed value of k , and means \bar{x}_0 , \bar{x}_1 , \bar{x}_2 , such that:

- (a) The curve with mean \bar{x}_1 intersects the given log-normal curve at $P = .01$ and $P = .99$ (this determines both k and \bar{x}_1).
- (b) The curve with mean \bar{x}_2 and the same value of k is tangent to the given log-normal curve.
- (c) The curve with mean \bar{x}_0 has the same k and

$$\bar{x}_0 = (\bar{x}_1 \bar{x}_2)^{1/2} \quad (23)$$

Then, the (k, \bar{x}_0) curve gives the best fit to the given log-normal curve, and the (k, \bar{x}_1) and (k, \bar{x}_2) curves give the closest bracket of the given log-normal curve (all with respect to $.01 \leq P \leq .99$). Best here means with respect to minimizing the maximum deviation of fit, or minimizing the ratio of \bar{x}_2 to \bar{x}_1 for the bracket; it is not certain that this gives the best fit or closest bracket for the probability of detection curves over the given probability interval.

Define

$$\begin{aligned} \text{goodness of fit} &= \left| 10 \log_{10} \left(\frac{\bar{x}_1}{\bar{x}_0} \right) \right| \\ &= \left| 10 \log_{10} \left(\frac{\bar{x}_2}{\bar{x}_0} \right) \right| \end{aligned} \quad (24)$$

= Maximum deviation of fit over
range $.01 \leq P \leq .99$

$$\begin{aligned} \text{closeness of bracket} &= \left| 10 \log_{10} \left(\frac{\bar{x}_2}{\bar{x}_1} \right) \right| \\ &= 2 \times (\text{goodness of fit}) \end{aligned} \quad (25)$$

It is also convenient to define

$$\Delta x = \frac{\bar{x}_0}{\bar{x}_1} - \frac{\bar{x}_2}{\bar{x}_0} \quad (26)$$

Thus, Δx is a ratio rather than an additive increment.

Figs. 10-15 show the results of such an analysis for a parameter range of interest. Fig. 10 gives the values of k , for the best-fitting chi-square distribution, for given values of σ (this is independent of a). Fig. 11 shows the value of $10 \log_{10} (\bar{x}_0/a)$, i. e., the ratio of the mean of the best-fitting chi-square curve to the median of the log-normal distribution to which the fit is being made, plotted against σ ; Fig. 12 shows the same plotted against k .

Fig. 13 shows the goodness of fit $10 \log_{10} \Delta x$ vs. σ (it is independent of a), and Fig. 14 plots the same information against k . The closeness of bracketing is, of course, equal to $20 \log_{10} \Delta x$.

It is also of interest to relate \bar{x}_0 , the mean of the best-fitting chi-square distribution, to the true mean $E(x)$ of the given log-normal distribution. This can be done by noting that, for log-normal distributions as given by eq. (22),

$$\frac{E(x)}{a} = \exp\left[\frac{1}{2} \sigma^2\right] \quad (27)$$

Fig. 15 plots $E(x)/a$ vs. σ ; from this and Figs. 11 or 12, one can relate \bar{x}_0 to $E(x)$.

From Fig. 13, it can be concluded that good chi-square fits to log-normal distribution can be obtained for $\sigma < .75$, approximately, while good chi-square fits cannot be obtained for $\sigma > 1.1$, approximately. Of course, this depends on ones definition of "good". The range $.75 \leq \sigma \leq 1.1$ is a transitional region. It is also to be noted that for a significant part of the region where good chi-square fits can be obtained, they cannot be obtained by matching the first two moments of the given log-normal distribution.

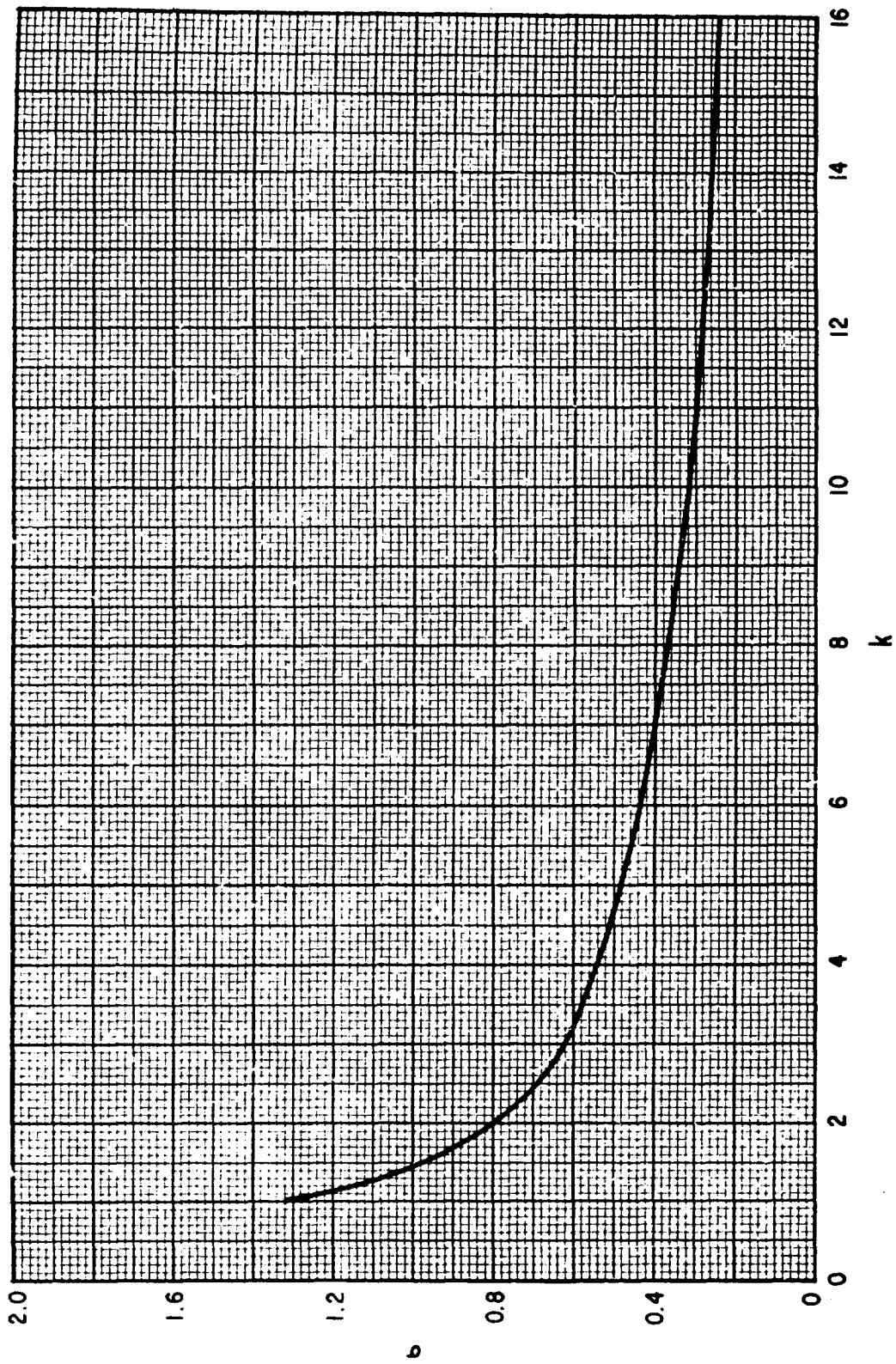


Figure 10. σ vs. k

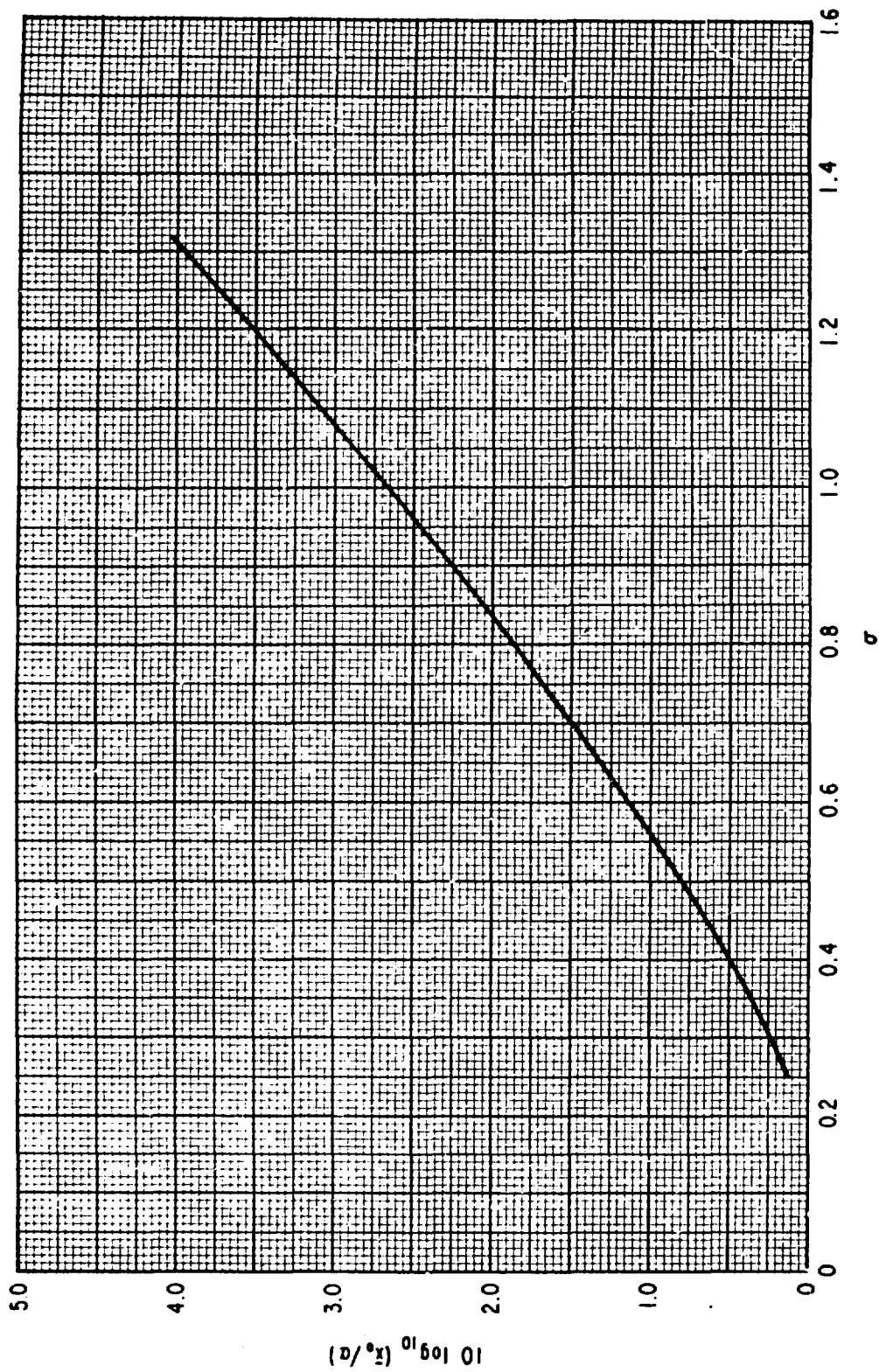


Figure 11. $10 \log_{10} (\bar{x}_0/a)$ vs. σ

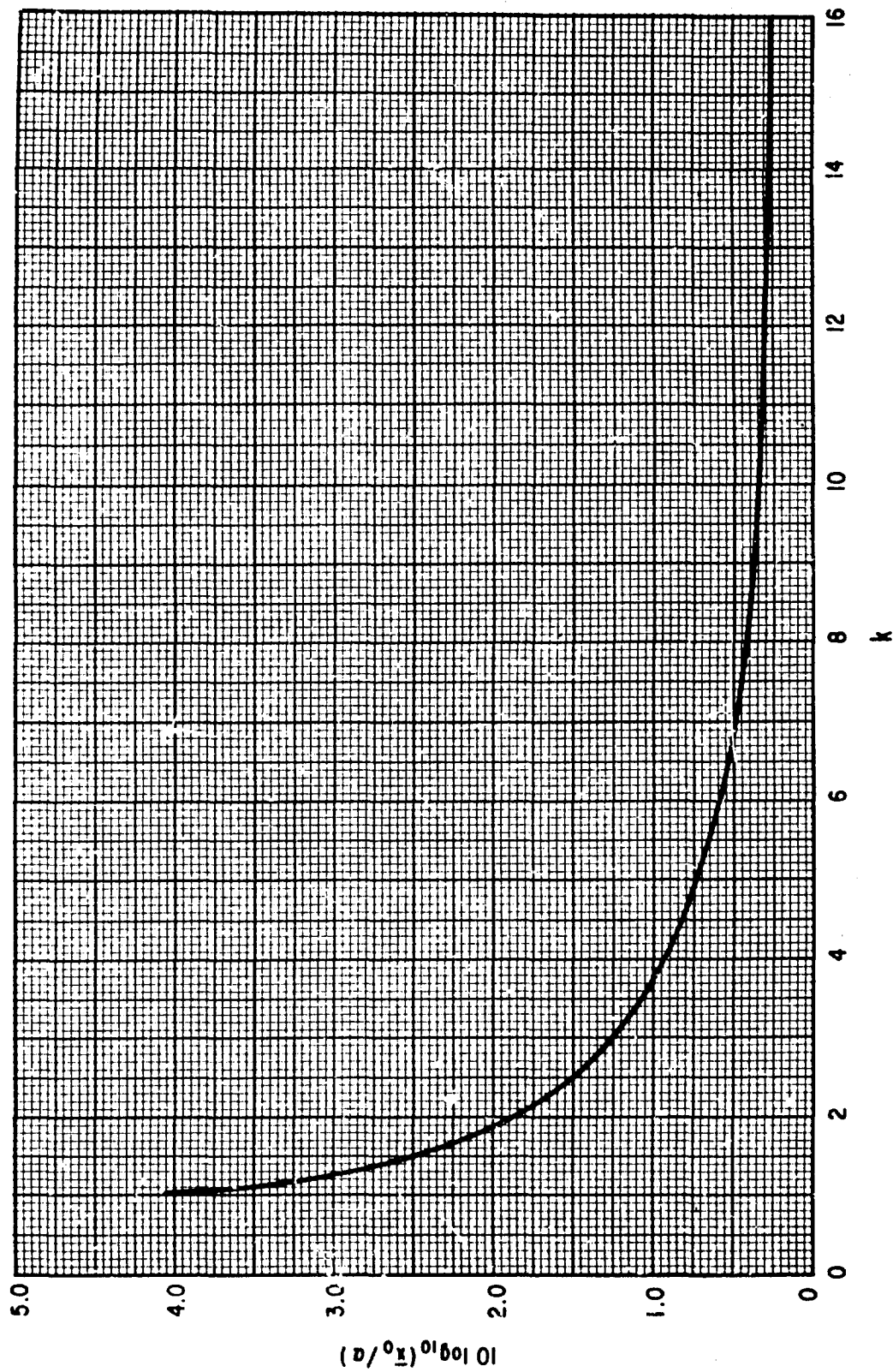


Figure 12. $10 \log_{10} (\bar{x}_0/a)$ vs. k

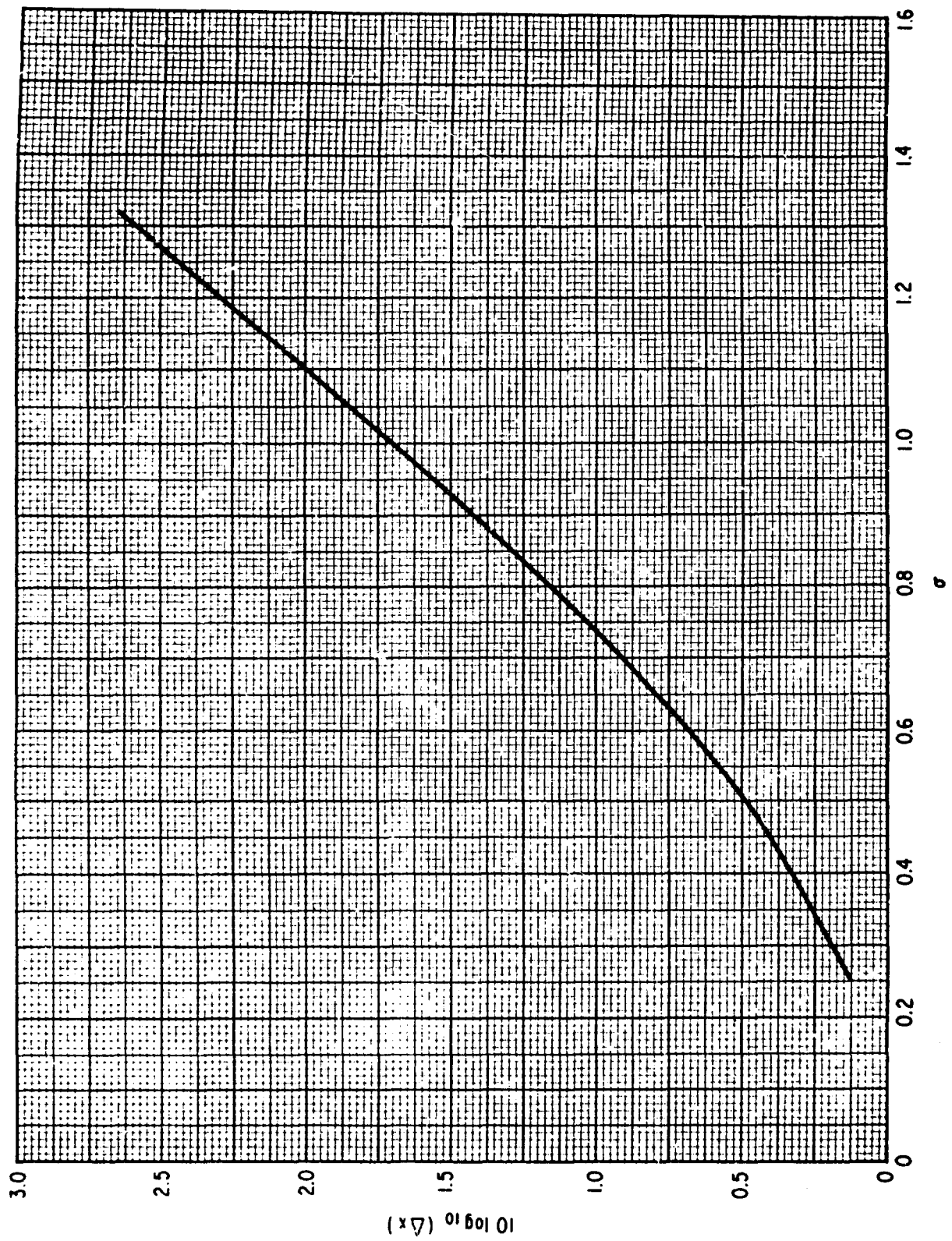


Figure 13. $10 \log_{10} (\Delta x)$ vs. σ

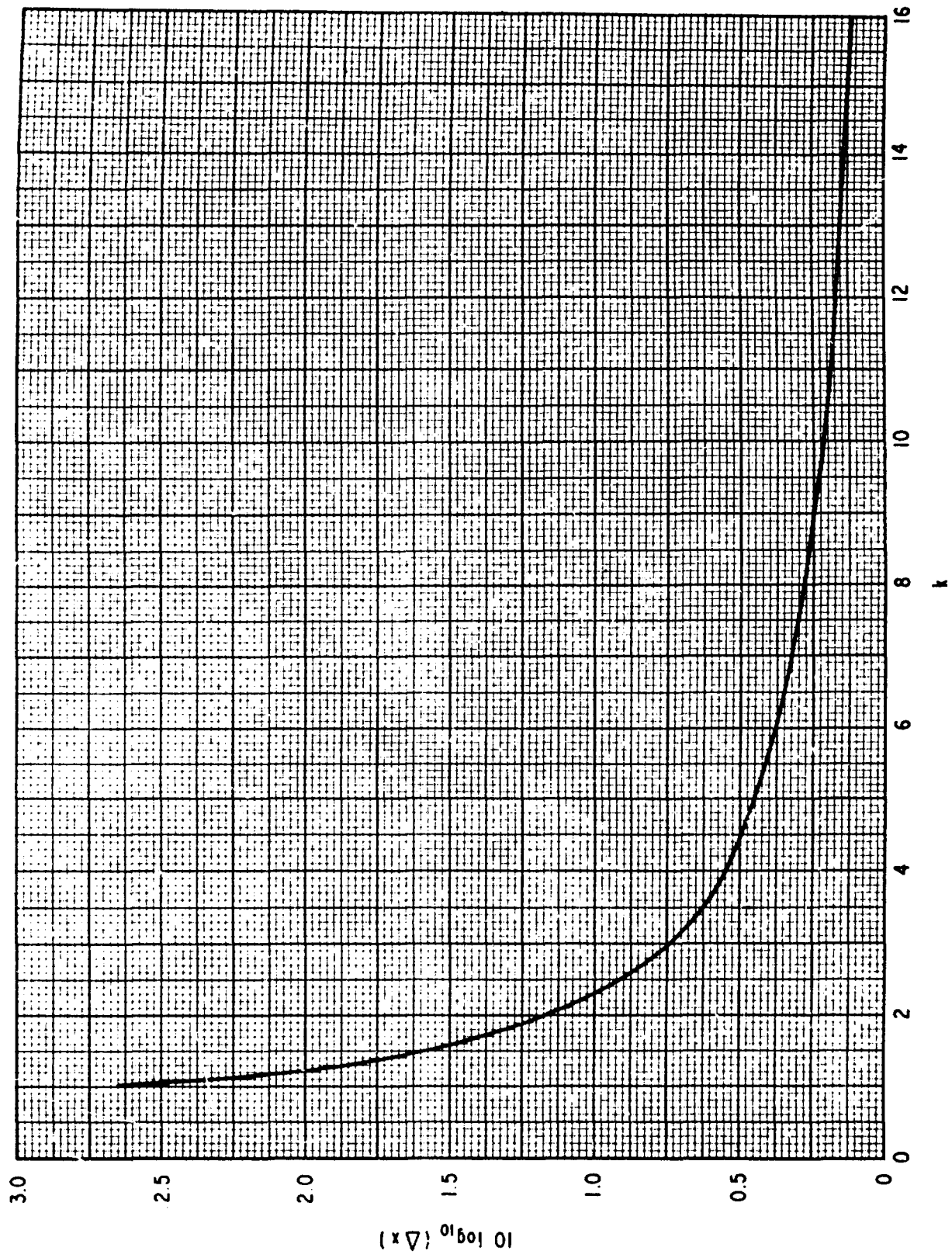


Figure 14. $10 \log_{10}(\Delta x)$ vs. k

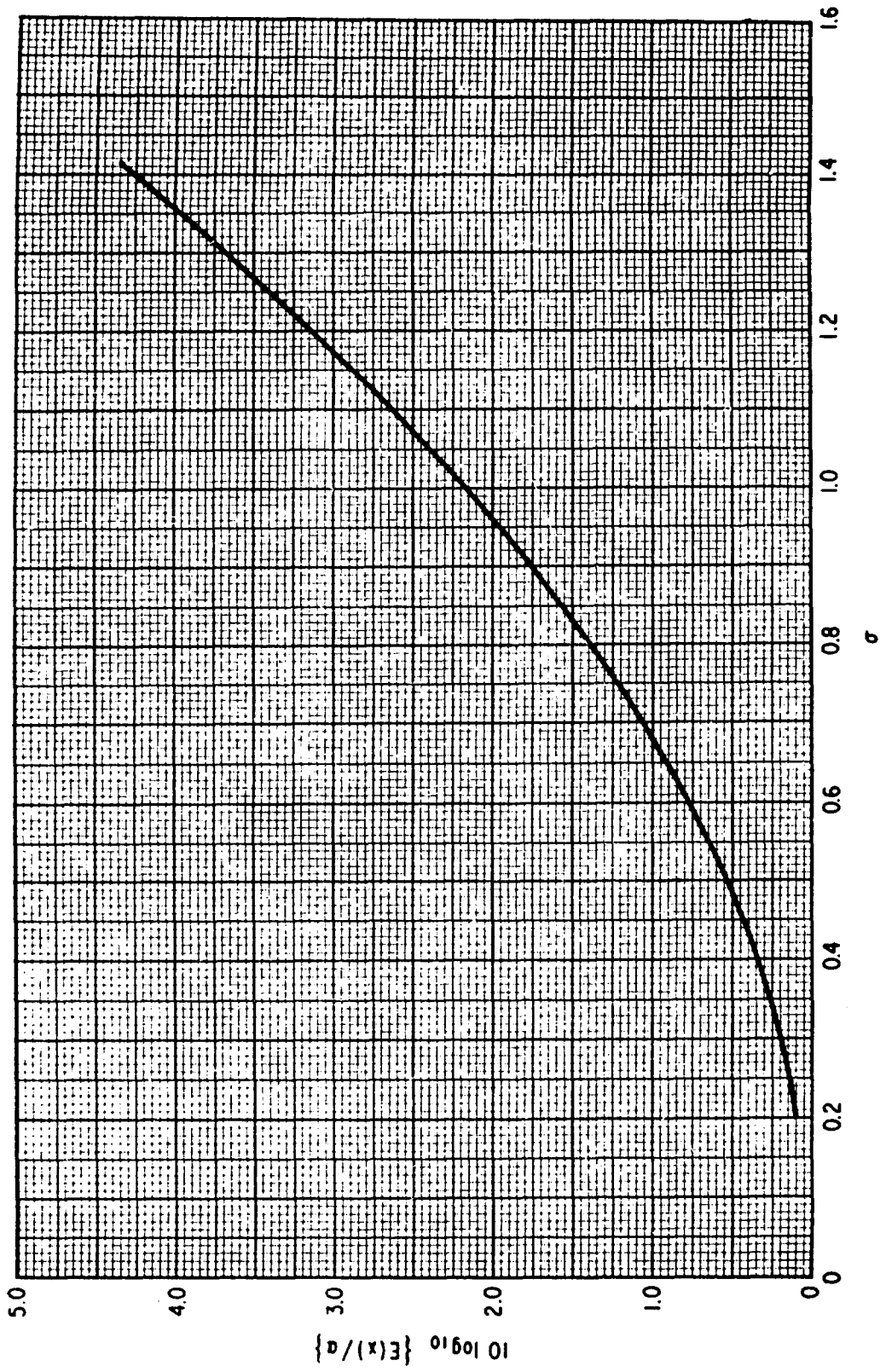


Figure 15. $10 \log_{10} \{E(x)/a\}$ vs. σ

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13. ABSTRACT The chi-square family of signal fluctuation distributions is defined. Rules are given for embedding the Swerling cases, and other cases of interest, in this family. Probability of detection curves are presented for the chi-square family of fluctuations, including cases whose probability of detection curves cannot be bracketed by the Swerling cases and the non-fluctuating case. The work of Weinstock has indicated such cases to be of practical interest; the fluctuation loss, for probability of detection exceeding .50, can be much larger than that for Swerling Case I. A discussion and partial analysis, accompanied by examples, is devoted to the question: when can the detection probability curves for signal fluctuations not belonging to the chi-square family be well approximated by curves result- ing from chi-square fluctuations, and what methods can be used to choose adequately fitting chi-square fluctuation models when such a fit is possible?		

Radar
Detection Probability
Fluctuating Targets
Scintillation
Radar Cross Section
Chi-Square Distribution
Log-Normal Distribution

Abstract (Continued)