AD NUMBER

AD483249

NEW LIMITATION CHANGE

TO

Approved for public release, distribution unlimited

FROM

Distribution authorized to U.S. Gov't. agencies and their contractors; Administrative and Operational Use, Export Control; May 1966. Other requests shall be referred to the Air Force Operational Requirements and Development Plans, Attn: DCS/Research and Development, Washington,

AUTHORITY

OSD, per ltr, dtd 18 Mar 1968

AR 66-2

PREFERENTIAL STRATEGIES

J. D. MATHESON

May 1966

ANALYTIC SERVICES INC.

ANSER REPORT

AR 66-2

PREFERENTIAL STRATEGIFS

J. D. Matheson

May 1966

This document presents results of work sponsored by the Director of Operational Requirements and Development Plans, DCS/Research and Development, Headquarters United States Air Force, under Contract AF 49(638)-1259. These results do not necessarily represent Air Force official opinion or policy.

Analytic Services Inc.
5613 Leesburg Pike, Falls Church, Virginia 22041

ABSTRACT

This report discusses targeting strategies for attack and defense. Weapons with unit probability of kill are pre-allocated by each combatant, who knows the enemy's total force size but not his weapon allocation. Canonical minimax solutions are formulated for all integral numbers of targets and weapons. In general, multilevel strategies are preferable to uniform strategies.

TABLE OF CONTENTS

TABLE OF CONTENTS—Continued

LIST OF FIGURES

LIST OF TABLES

PREFERENTIAL STRATEGIES

I. INTRODUCTION

Multilevel strategies of attack and defense are preferable to uniform strategies under certain assumptions about weapon capabilities and mode of employment and the information available to attacker and defender. This report states the assumptions and the problem, illustrates the principles of solution, and formulates canonical solutions for all integral numbers of targets and weapons.

II. THE PROBLEM

Suppose there are T targets of unit value. An attacker has A weapons, each of which, if not nullified by the defense, will destroy the target at which it is fired. These are capable of attacking any target. The defender has D weapons, each of which will nullify an attacking weapon at which it is fired. These are capable of defending any target. The numbers represented by T, A, and D are known to both attacker and defender.

Before the attack, the opponents allocate their weapons to the attack and defense of specific targets. Neither knows the other's allocation, and neither changes his own allocation during the attack.

Let the attack allocation, or strategy, be represented by $R \equiv (R_0, R_1, \cdots, R_\alpha, \cdots)$, where R_α is the number of targets each attacked by α weapons. Similarly, let the defense strategy be represented by $S \equiv (S_0, S_1, \cdots, S_\delta \cdots)$, where S_δ is the number of targets each defended by δ weapons. The strategies satisfy the following equations of constraint:

$$T = R_0 + R_1 + R_2 + \cdots = \sum_\alpha R_\alpha ,$$

$$A = R_1 + 2R_2 + 3R_3 + \cdots = \sum_\alpha \alpha R_\alpha ,$$

$$T = S_0 + S_1 + S_2 + \cdots = \sum_\delta S_\delta ,$$

$$D = S_1 + 2S_2 + 3S_3 + \cdots = \sum_\delta \delta S_\delta .$$

A target is destroyed if, and only if, the number of weapons attacking it is greater than the number defending it. The probability that a target is destroyed when it is defended at the δ level is $\frac{1}{T}(R_{\delta+1} + R_{\delta+2} + \cdots) = \frac{1}{T}\sum_{\alpha>\delta} R_\alpha$. The expected number of targets destroyed is the sum of these probabilities for all targets; that is,

$$E = \frac{S_0}{T}(R_1 + R_2 + R_3 + \cdots)$$

$$+ \frac{S_1}{T}(R_2 + R_3 + R_4 + \cdots)$$

$$+ \cdots$$

$$= \frac{1}{T}\sum_\delta S_\delta \sum_{\alpha>\delta} R_\alpha ,$$

which can also be written

$$E = \frac{1}{T}\sum_\alpha R_\alpha \sum_{\delta<\alpha} S_\delta .$$

The attacker chooses R to maximize E; the defender chooses S to minimize it. The problem is to determine, for every set (T, A, D), the minimax value of E, which is denoted by V and called the value of the game. In addition, attack strategies that guarantee at least this expectation and defense strategies that guarantee no more than this expectation are to be determined.

III. EXAMPLES

A numerical example clarifies the ideas and illustrates the fundamental principles of solution. Consider (T, A, D) = (4, 5, 5) ; that is, 4 targets, 5 attack weapons, 5 defense weapons. The attacker has six playable strategies available. One of these is $R_0 = 3$, $R_1 = R_2 = R_3 = R_4 = 0$, $R_5 = 1$, $R_{\alpha>5} = 0$. For convenience, this is written (3,0,0,0,0,1). Other attack allocations are (2,1,0,0,1), (2,0,1,1), (1,2,0,1), (1,1,2) and (0,3,1). The defender has the same six strategies available. If the attacker should choose (2,0,1,1) and the defender should choose (0,3,1), then the expected number of targets destroyed would be

$$E = \frac{3}{4}\,(1 + 1) + \frac{1}{4}\,(1) = \frac{7}{4}\ ,$$

or

$$TE = 4E = 7\ .$$

Every playable attack strategy is compared with every playable defense strategy, and the resulting values of TE are shown as a matrix in Table 1.

TABLE 1

MATRIX OF EXPECTED VALUES OF TE FOR (T, A, D) = (4, 5, 5)

Attack \ Defense	3,0,0,0,0,1	2,1,0,0,1	2,0,1,1	1,2,0,1	1,1,2	0,3,1	
3,0,0,0,0,1	3	4	4	4	4	4	
2,1,0,0,1	6	5	6	5	5	4	
2,0,1,1	6	6	5	6	6	7	f_1
1,2,0,1	9	7	7	5	6	4	f_2
1,1,2	9	8	6	7	5	6	f_3
0,3,1	12	9	8	6	5	3	f_4
			g_1	g_2	g_3	g_4	

Examination of this matrix shows that every element of the first row is less than or equal to the corresponding element of the second row. Since the goal of the attacker is to maximize expected destruction, he rejects the (3,0,0,0,0,1) strategy of the first row as a preferred strategy. In like fashion, he rejects the (2,1,0,0,1) strategy because every element of the second row is less than or equal to every element of the fourth row. With these two rows eliminated, the remaining elements of each column are compared with those of every other column. Since the goal of the defender is to minimize destruction, he rejects the (3,0,0,0,0,1) and (2,1,0,0,1) strategies because the remaining elements in the first two columns are greater than or equal to those in the third column.

The resulting 4-by-4 kernel matrix is enclosed by double lines in Table 1. No other rows or columns may be eliminated by inspection. No single row guarantees the attacker more than 5; no single column guarantees the defender less than 6. Both can do better from their respective points of view with mixed strategies—the attacker by playing the four rows of the kernel with frequencies (or probabilities) f_1, f_2, f_3, and f_4, and the defender by playing the four columns of the kernel with frequencies g_1, g_2, g_3, and g_4, as shown in the margin of the table.

To determine the proper frequencies of attack, the following conditions must be satisfied:

$$f_1, \ f_2, \ f_3, \ f_4 \geq 0 \ ,$$

$$f_1 + f_2 + f_3 + f_4 = 1 \ ,$$

$$5f_1 + 7f_2 + 6f_3 + 8f_4 \geq TV \ ,$$

$$6f_1 + 5f_2 + 7f_3 + 6f_4 \geq TV \ ,$$

$$6f_1 + 6f_2 + 5f_3 + 5f_4 \geq TV \ ,$$

$$7f_1 + 4f_2 + 6f_3 + 3f_4 \geq TV \ ,$$

where V is the unknown value of the game, to be maximized by the attacker. The largest value satisfying the conditions is: $V = 35/24$. Two mixed solution strategies yielding this value are $(f_1, f_2, f_3, f_4) = (4/6, 1/6, 0, 1/6)$ and $(f_1, f_2, f_3, f_4) = (3/6, 2/6, 1/6, 0)$. Any linear combination of these strategies with positive multipliers summing to unity is also a solution, for example:

$$\begin{aligned}(f_1, f_2, f_3, f_4) &= 1/2\ (4/6, 1/6, 0, 1/6) \\ &+ 1/2\ (3/6, 2/6, 1/6, 0) \\ &= (7/12, 3/12, 1/12, 1/12) \ .\end{aligned}$$

Thus, there are an infinite number of minimax strategies for the attacker, involving playing the four kernel strategies with proper frequencies. However, these are all equivalent to a single strategy (or allocation) which is called the canonical strategy.

The canonical strategy is the vector sum of the products of the component strategies and their frequencies in a mixed solution strategy. If the solution frequencies are $(f_1, f_2, f_3, f_4) = (4/6, 1/6, 0, 1/6)$, and the components are (2,0,1,1), (1,2,0,1), (1,1,2) and (0,3,1), respectively, then the canonical strategy is:

$$\begin{aligned}(R_0, R_1, R_2, R_3) &= 4/6\ (2,0,1,1) \\ &+ 1/6\ (1,2,0,1) \\ &+ 1/6\ (0,3,1) \\ &= (9/6,\ 5/6,\ 5/6,\ 5/6)\ .\end{aligned}$$

The other mixed strategy with frequencies (3/6, 2/6, 1/6, 0) yields the same canonical strategy. It follows that any linear combination, such as (7/12, 3/12, 1/12, 1/12), does the same.

A similar result is found for the defense. The conditions to be satisfied are:

$$g_1,\ g_2,\ g_3,\ g_4 \geq 0\ ,$$

$$g_1 + g_2 + g_3 + g_4 = 1\ ,$$

$$5g_1 + 6g_2 + 6g_3 + 7g_4 \leq TV\ ,$$

$$7g_1 + 5g_2 + 6g_3 + 4g_4 \leq TV\ ,$$

$$6g_1 + 7g_2 + 5g_3 + 6g_4 \leq TV\ ,$$

$$8g_1 + 6g_2 + 5g_3 + 3g_4 \leq TV\ ,$$

where the defense is to minimize TV. The solution value is $V = 35/24$, as before. Two mixed strategies are $(g_1, g_2, g_3, g_4) = (3/6, 0, 1/6, 2/6)$ and $(g_1, g_2, g_3, g_4) = (1/6, 2/6, 3/6, 0)$. Any linear combination of these with positive multipliers summing to unity is also a solution; for example: (2/6, 1/6, 2/6, 1/6) . The canonical strategy for the defense is found to be $(S_0, S_1, S_2, S_3) = (7/6, 7/6, 7/6, 3/6)$.

The canonical strategies are shown graphically in Figure 1. In this case, where (T, A, D) = (4, 5, 5) , the defense strategy is characterized by steps of equal width beginning at the bottom, $S_0 = S_1 = S_2 = 7/6$. In effect, this strategy leaves 7/6 of a target at risk to a one-weapon

V = 35/24

V/T = 35/96

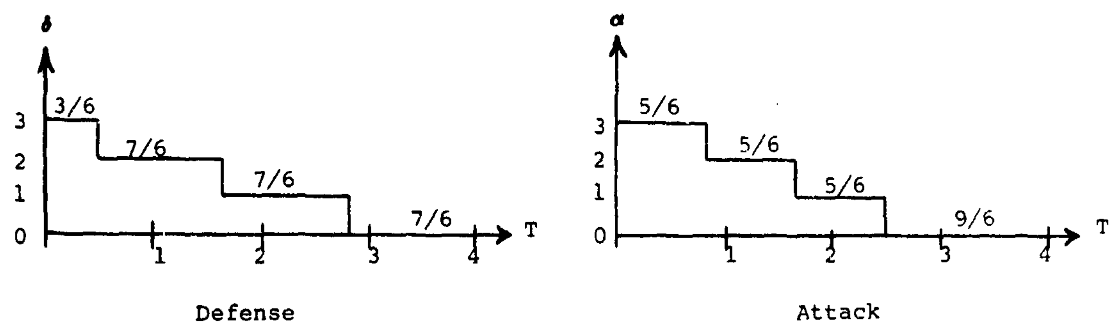

Figure 1. CANONICAL STRATEGIES FOR (T, A, D) = (4, 5, 5)

attack, 14/6 at risk to a two-weapon attack, 21/6 at risk to a three-weapon attack, and 24/6 at risk to attack at higher levels. Against this defense, the expectation per attacking weapon cannot exceed 7/24. As there are 5 attacking weapons, $V = 35/24$.

The attack strategy is characterized by steps of equal width beginning at the top, $R_3 = R_2 = R_1 = 5/6$. In effect, this strategy equalizes the per-weapon effectiveness of the defense, since a target defended at the 3 level is protected against the entire attack, a target at the 2 level against 2/3 of the attack, and a target at the 1 level against 1/3 of the attack. Thus, each defense weapon cannot be expected to protect one target against more than 1/3 of the attack. As there are 4 targets and 5 defense weapons, essentially $(4 - 5 \cdot 1/3) = 7/3$ targets are unprotected. As 15/24 of the target system is under attack, the value of the game is, again, $V = 7/3 \cdot 15/24 = 35/24$.

It is instructive to observe how the canonical strategies and the value of the game change as the number of defense weapons increases. These variations are illustrated in Figure 2. At (4, 5, 6) the attack strategy is unchanged, but the added defense weapon is divided, as shown by the

Figure 2. CANONICAL STRATEGIES FOR (T, A, D) = (4, 5, D)

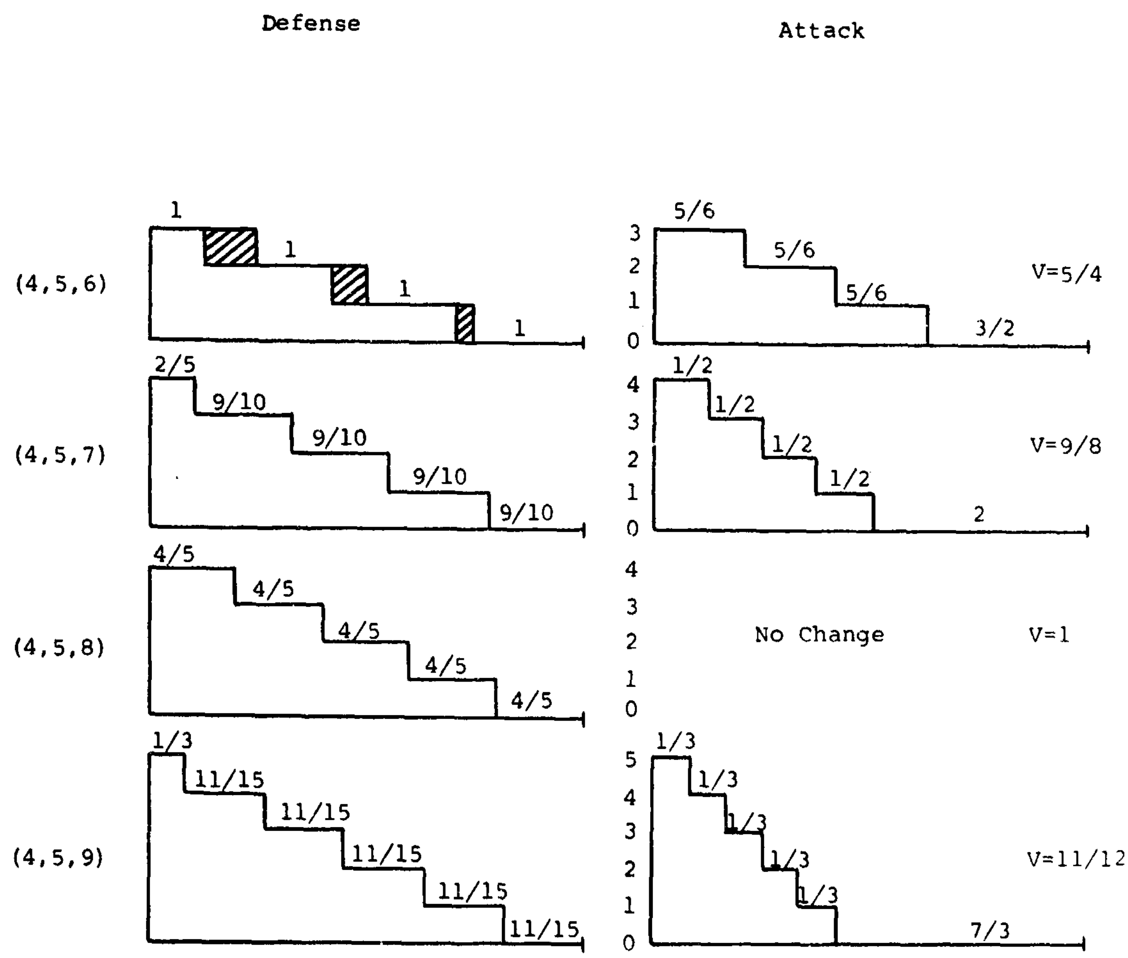

Figure 2. CANONICAL STRATEGIES FOR (T, A, D) = (4, 5, D)

(Continued)

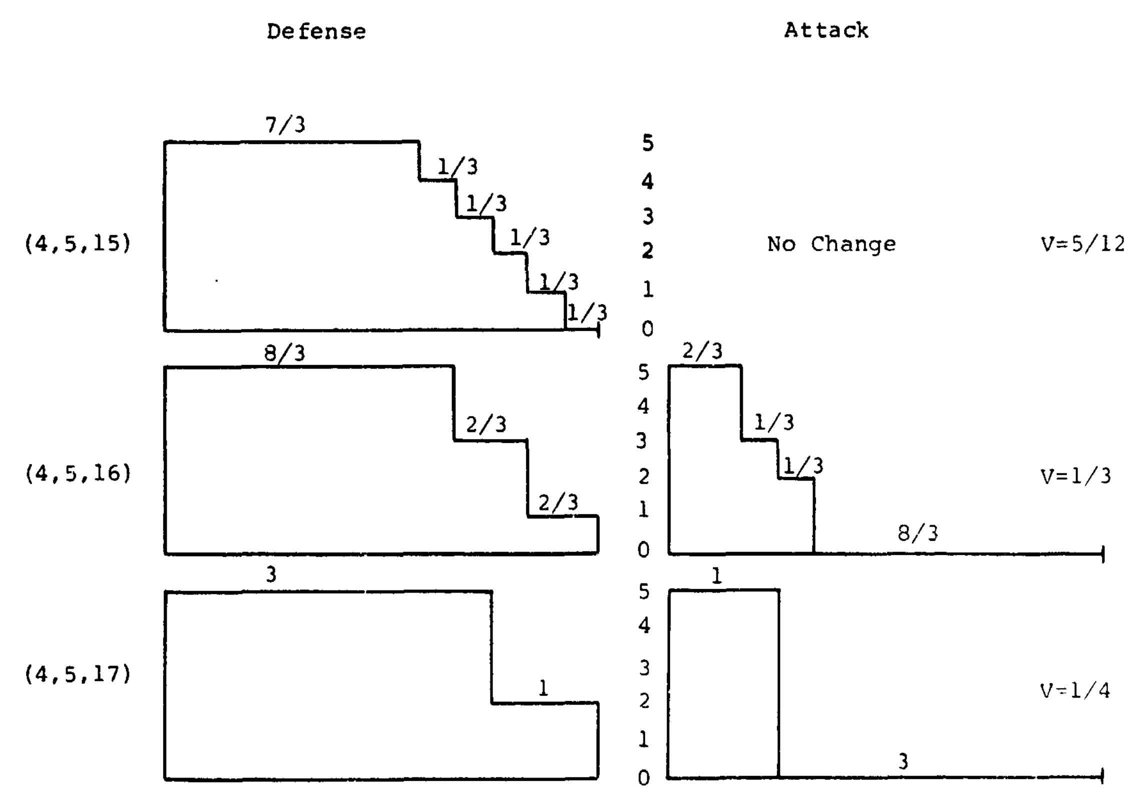

Figure 2. CANONICAL STRATEGIES FOR (T, A, D) = (4, 5, D)—Continued

shaded areas, in order to obtain equal steps. At (4, 5, 7), both strategies add another level, but maintain their equal step characters. At (4, 5, 8), again the attack is unchanged and the top defense step is the same as the others. At (4, 5, 9) both strategies add another level. Throughout this sequence, it is apparent that the upper level of canonical strategy is determined by the magnitude of the defense, with the attack being adjusted to match. On the other hand, the attack is unchanged except when the defense adds another level. For obvious reasons, these particular strategies are called "defense dominated."

At the 5 level, another effect begins to be felt. Since the total number of attack weapons is only 5, the defense need never protect any target above this level. Accordingly, from (4, 5, 9) to (4, 5, 15), the attack strategy is unchanged, while the defense plays an equal step strategy with an extra-long upper step.

At (4, 5, 16), the defense can no longer play according to the previous canon because, with $D > A (T - 1)$, none of his allowable plays includes undefended targets, unless other targets are wastefully defended above the 5 level.

Here begins a transition region, where allowable plays are increasingly constrained. This region ends at (4, 5, 20), where D = AT and V = 0 .

Changes in canonical strategies and the value of the game as the number of attack weapons is increased from the base case (4, 5, 5) are illustrated in Figure 3. At (4, 6, 5), there is no change in the defense strategy, but the added attack weapon has been divided, as shown by the shaded areas to obtain equal steps from the top (and coincidentally the bottom). A similar process is carried out for the next two added attack weapons, so that at (4, 8, 5) all targets are attacked. This case marks the change to an attack-dominated strategy which appears in (4, 9, 5), where all targets are attacked, where the canonical attack strategy is characterized by equal steps from the bottom, where the attacking force determines the upper level of attack, and where the canonical defense strategy is characterized by a top step one level below the top attack step and by equal steps from there down.

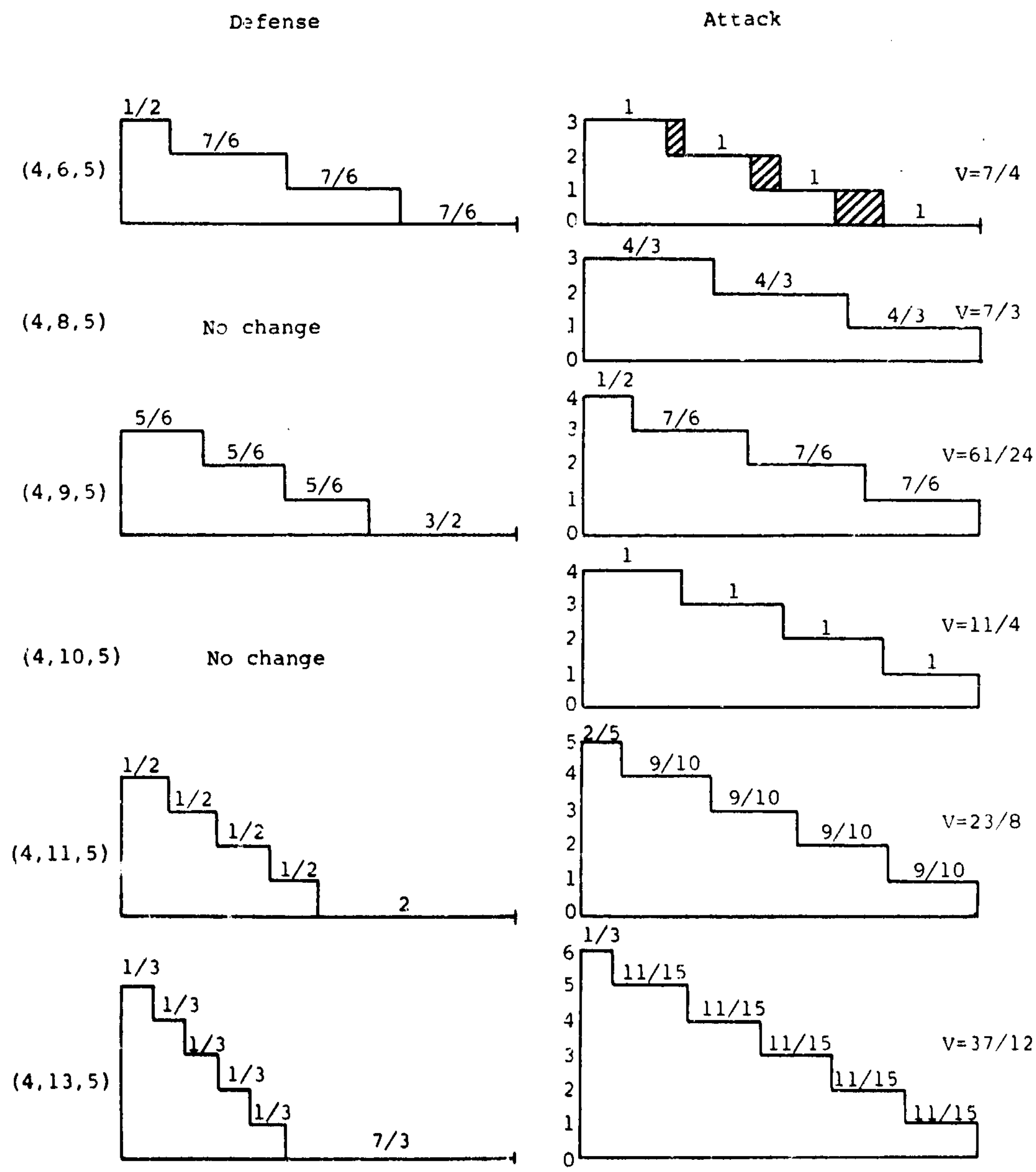

Figure 3. CANONICAL STRATEGIES FOR (T, A, D) = (4, A, 5)

(Continued)

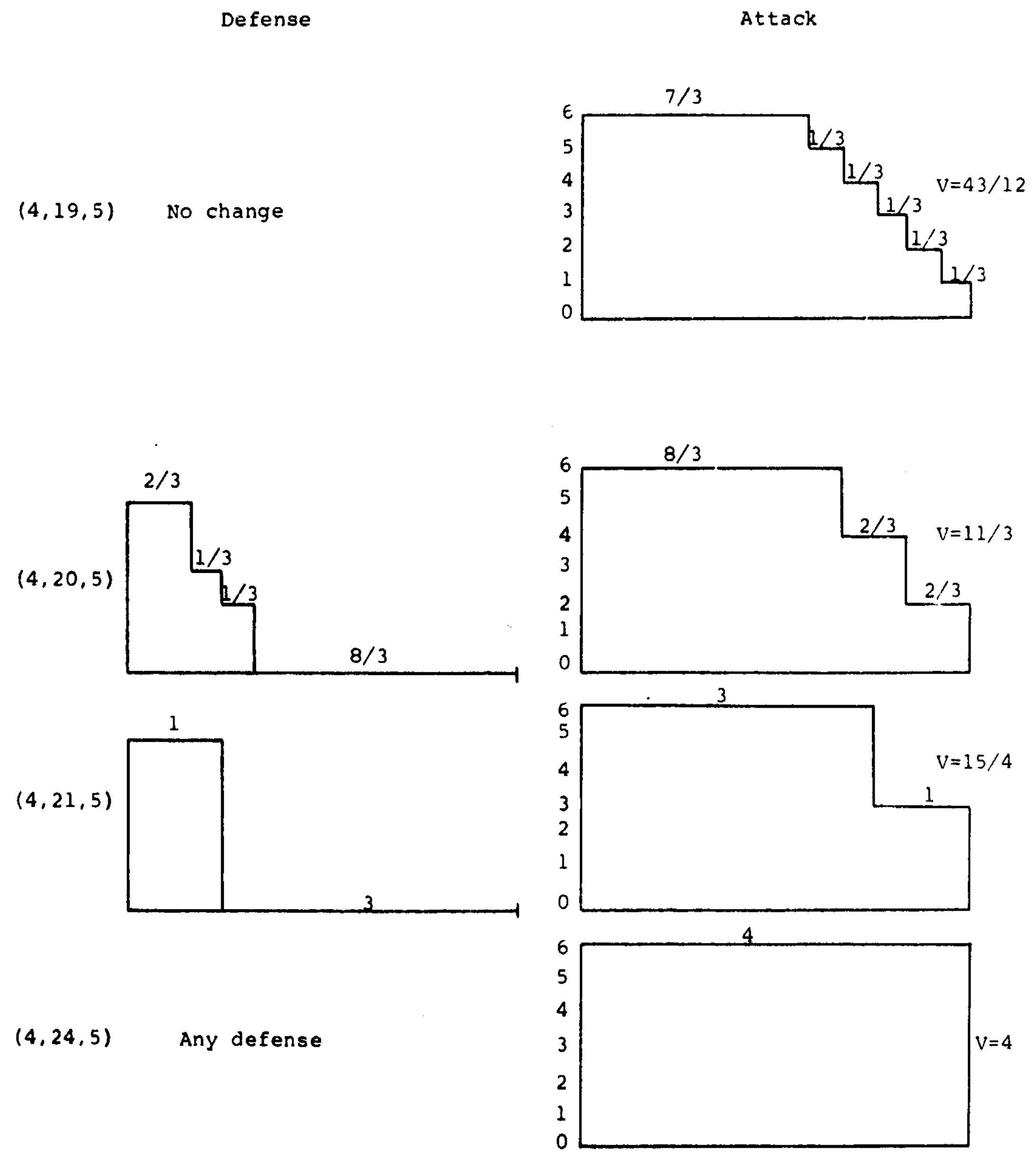

Figure 3. CANONICAL STRATEGIES FOR (T, A, D) = (4, A, 5)—Continued

In the attack-dominated cases, the attacker never plays above the $D + 1 = 6$ level. Also there is a transition region beginning at (4, 20, 5) where the attack cannot play according to the regular canon because $A > (D + 1)(T - 1) + 1$.

The principles and rules exemplified above have been found to apply for all sets of (T, A, D) examined. It has also been observed that the density of attack and defense, rather than the absolute numbers, are the factors determining strategy in the canonical region. S if (T, A, D) = (480, 600, 600) instead of (4, 5, 5), then the canonical defense strategy would be to leave 140 targets undefended, to defend 140 at the 1 level, 140 at the 2 level, and 60 at the 3 level. The canonical attack strategy would be to attack 100 targets at the 3 level, 100 at the 2 level, 100 at the 1 level, and 180 not at all. The expected number of targets destroyed would be 175, and the probability of destroying any target would be $V/T = 35/96$, as in the (4, 5, 5) case.

IV. GENERAL SOLUTIONS

The illustrative examples are a guide to the general solutions.

The first question to settle is that of dominance, because the dominant player uses what might be called his natural strategy across the entire target system, whereas the dominated player has to play a step-matching strategy over a smaller part of the target system.

The natural strategy for the defense is characterized by a top step at what is called the $\bar{\delta}$ level and by the relationships:

$$S_0 = S_1 = \cdots = S_{\bar{\delta}-1} \geq S_{\bar{\delta}} > 0 .$$

These are combined with the basic constraints

$$\sum_{0}^{\bar{\delta}} S_\delta = T \quad \text{and} \quad \sum_{0}^{\bar{\delta}} \delta \, S_\delta = D$$

and are solved. The equations reduce to the pair

$$\begin{cases} S_0 \, \bar{\delta} + S_{\bar{\delta}} = T \\ S_0 \, (\bar{\delta} - 1) \, \bar{\delta}/2 + S_{\bar{\delta}} \, \bar{\delta} = D . \end{cases}$$

From these:

$$\begin{cases} S_0 = S_1 = \cdots = S_{\bar{\delta}-1} = \dfrac{2\,(\bar{\delta}\,T - D)}{\bar{\delta}\,(\bar{\delta} + 1)} \\ S_{\bar{\delta}} = \dfrac{2D - (\bar{\delta} - 1)\,T}{\bar{\delta} + 1} . \end{cases}$$

The inequalities require that

$$\frac{2\,(\bar{\delta}\,T - D)}{\bar{\delta}\,(\bar{\delta} + 1)} \geq \frac{2D - (\bar{\delta} - 1)\,T}{\bar{\delta} + 1} > 0 ,$$

and imply

$$\frac{2D}{T} + 1 > \bar{\delta} \geq \frac{2D}{T} .$$

$\bar{\delta}$ must be integral. Hence, except for integral values of $\frac{2D}{T}$, there is a unique solution.

$$\bar{\delta} = 1 + \left[\frac{2D}{T}\right] ,$$

where the square bracket symbol $[x]$ is to be read "the greatest integer contained in x." The fact that integral values of $\frac{2D}{T}$ make $\bar{\delta}$ seem one step too high is not a real problem, since $S_{\bar{\delta}} = 0$ in these cases, and the actual strategy is not affected. Of course, integral values of $\frac{2D}{T}$

are those where all steps are equal, including the top step. With $\bar{\delta}$ thus determined, the natural strategies for the defense, which have already been expressed as functions of $\bar{\delta}$, are also determined.

The natural strategy for the attack is characterized by a top step at what is called the $\bar{\alpha}$ level, and by the relationships:

$$\begin{cases} R_0 = 0 \\ R_1 = R_2 = \cdots = R_{\bar{\alpha}-1} \geq R_{\bar{\alpha}} > 0 \ . \end{cases}$$

These are combined with the basic constraints

$$\sum_{0}^{\bar{\alpha}} R_\alpha = T \quad \text{and} \quad \sum_{0}^{\bar{\alpha}} \alpha \, R_\alpha = A$$

and are solved. The solutions of the equations are:

$$\begin{cases} R_1 = R_2 = \cdots = R_{\bar{\alpha}-1} = \dfrac{2\,(\bar{\alpha}\,T - A)}{(\bar{\alpha} - 1)\,\bar{\alpha}} \\ R_{\bar{\alpha}} = \dfrac{2A}{\bar{\alpha}} - T \ . \end{cases}$$

The inequalities require that

$$\frac{2\,(\bar{\alpha}\,T - A)}{(\bar{\alpha} - 1)\,\bar{\alpha}} \geq \frac{2A}{\bar{\alpha}} - T > 0 \ ,$$

and imply

$$\frac{2A}{T} > \bar{\alpha} \geq \frac{2A}{T} - 1 .$$

Since $\bar{\alpha}$ must be integral, there is the unique solution

$$\bar{\alpha} = \left[\frac{2A}{T}\right],$$

which is valid except for integral values of $\frac{2A}{T}$, and is actually correct for all values by an argument similar to that given for the natural defense strategy. With $\bar{\alpha}$ thus determined, the natural attack strategies are also determined.

The matter of dominance can now be stated explicitly. If $\bar{\alpha} > \bar{\delta}$, the attack is dominant; if $\bar{\delta} \geq \bar{\alpha}$ the defense is dominant, at least from the point of view of determining canonical strategies. Regions of dominance are shown in Figure 4. On the boundaries, both alternative solutions are valid.

A. Defense-Dominated Cases

When the defense is dominant, the defender's canonical strategy is the same as his natural strategy. The canonical strategy for the attack is a $\bar{\delta}$-strategy, characterized by the following relationships:

$$R_{\bar{\delta}} = R_{\bar{\delta}-1} = \cdots = R_1 .$$

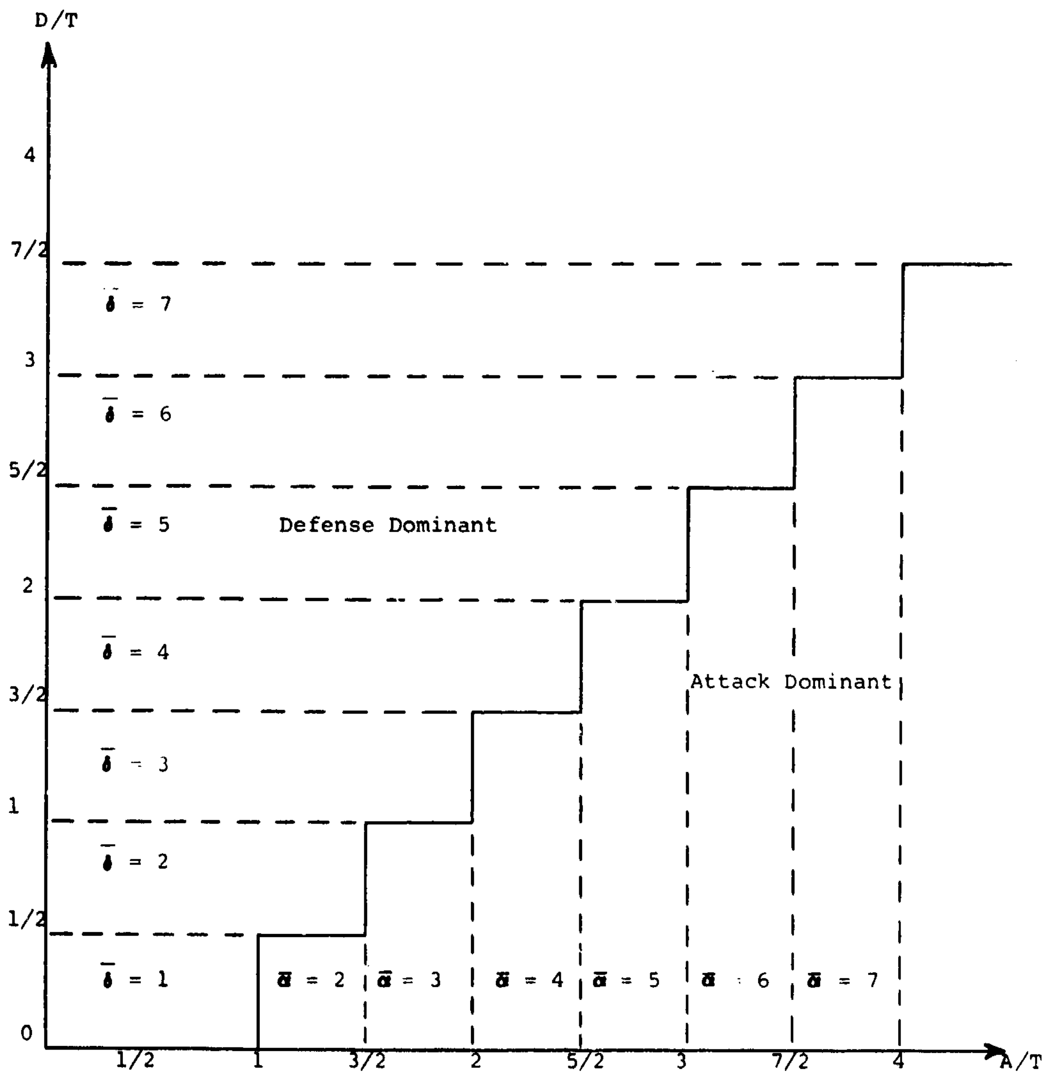

Figure 4. REGIONS OF DOMINANCE

Solving these with the basic constraint equations, it is found that

$$R_{\bar{\delta}} = \cdots = R_1 = \frac{2A}{\bar{\delta}\,(\bar{\delta} + 1)} ,$$

and

$$R_0 = T - \frac{2A}{\bar{\delta} + 1} .$$

The value of the game in this situation is determined by substituting the canonical strategies for attack and defense into the expectation formula,

$$E = \frac{1}{T} \sum_{\alpha} R_\alpha \sum_{\delta<\alpha} S_\delta .$$

In extended form, this becomes

$$TV = \frac{2A}{\bar{\delta}\,(\bar{\delta} + 1)} \left\{1 \cdot \frac{2\,(\bar{\delta}\,T - D)}{\bar{\delta}\,(\bar{\delta} + 1)}\right\}$$

$$+ \frac{2A}{\bar{\delta}\,(\bar{\delta} + 1)} \left\{2 \cdot \frac{2\,(\bar{\delta}\,T - D)}{\bar{\delta}\,(\bar{\delta} + 1)}\right\}$$

$$+ \cdots + \frac{2A}{\bar{\delta}\,(\bar{\delta} + 1)} \left\{\bar{\delta} \cdot \frac{2\,(\bar{\delta}\,T - D)}{\bar{\delta}\,(\bar{\delta} + 1)}\right\} .$$

So,

$$TV = \frac{2A}{\bar{\delta}\,(\bar{\delta} + 1)} (\bar{\delta}\,T - D) .$$

This can be rewritten as

$$\frac{V}{T} = \frac{A/T}{\sum_{1}^{\bar{\delta}} \delta} \left(\bar{\delta} - \frac{D}{T} \right),$$

a form that emphasizes the fact that the probability of target destruction in this case depends upon the densities of attack and defense rather than their absolute values.

It can be proved that this value of the game is a minimax value by showing that if either side plays canonically, the other side cannot improve the result from his point of view. So, if the attack plays canonically and the defense plays arbitrarily, the expectation in extended form is

$$\begin{aligned} TE &= \frac{2A}{\bar{\delta}\,(\bar{\delta} + 1)} \left\{ S_0 \right\} \\ &+ \frac{2A}{\bar{\delta}\,(\bar{\delta} + 1)} \left\{ S_0 + S_1 \right\} \\ &+ \cdots \\ &+ \frac{2A}{\bar{\delta}\,(\bar{\delta} + 1)} \left\{ S_0 + S_1 + \cdots + S_{\bar{\delta}-1} \right\} \\ &= \frac{2A}{\bar{\delta}\,(\bar{\delta} + 1)} \left\{ \bar{\delta}\, S_0 + (\bar{\delta} - 1)\, S_1 + \cdots + S_{\bar{\delta}-1} \right\} \\ &= \frac{2A}{\bar{\delta}\,(\bar{\delta} + 1)} \left\{ \bar{\delta} \sum_{0}^{\bar{\delta}-1} S_\delta - \sum_{1}^{\bar{\delta}-1} \delta\, S_\delta \right\} \end{aligned}$$

$$= \frac{2A}{\bar{\delta}\,(\bar{\delta} + 1)} \left\{ \bar{\delta}\,(T - S_{\bar{\delta}} - S_{\bar{\delta}+1} - \cdots) - (D - \bar{\delta}\, S_{\bar{\delta}} - (\bar{\delta} + 1)\, S_{\bar{\delta}+1} - \cdots) \right\}$$

$$= TV + \frac{2A}{\bar{\delta}\,(\bar{\delta} + 1)} \left\{ S_{\bar{\delta}+1} + 2\, S_{\bar{\delta}+2} + \cdots \right\} \geq TV \,.$$

Therefore, the defense cannot improve on the value. On the other hand, if the defense plays canonically and the attack plays arbitrarily, the expectation is

$$TE = \frac{2\,(\bar{\delta}\,T - D)}{\bar{\delta}\,(\bar{\delta} + 1)} \left\{ R_1 + R_2 + \cdots \right\}$$

$$+ \frac{2\,(\bar{\delta}\,T - D)}{\bar{\delta}\,(\bar{\delta} + 1)} \left\{ R_2 + R_3 + \cdots \right\}$$

$$+ \cdots$$

$$+ \frac{2\,(\bar{\delta}\,T - D)}{\bar{\delta}\,(\bar{\delta} + 1)} \left\{ R_{\bar{\delta}} + R_{\bar{\delta}+1} + \cdots \right\}$$

$$+ \frac{2D - (\bar{\delta} - 1)\,T}{\bar{\delta} + 1} \left\{ R_{\bar{\delta}+1} + R_{\bar{\delta}+2} + \cdots \right\}$$

$$= \frac{2\,(\bar{\delta}\,T - D)}{\bar{\delta}\,(\bar{\delta} + 1)} \left\{ R_1 + 2R_2 + \cdots + \bar{\delta}\,(R_{\bar{\delta}} + R_{\bar{\delta}+1} + \cdots) \right\}$$

$$+ \frac{2D - (\bar{\delta} - 1)\,T}{\bar{\delta} + 1} \left\{ R_{\bar{\delta}+1} + R_{\bar{\delta}+2} + \cdots \right\}$$

$$= \frac{2\,(\bar{\delta}\,T - D)}{\bar{\delta}\,(\bar{\delta} + 1)} \left\{ A - R_{\bar{\delta}+1} - 2\,R_{\bar{\delta}+2} - \cdots \right\}$$

$$+ \frac{2D - (\bar{\delta} - 1)\,T}{\bar{\delta} + 1} \left\{ R_{\bar{\delta}+1} + R_{\bar{\delta}+2} + \cdots \right\}$$

$$= TV - \left\{ \frac{2\,(\bar{\delta}\,T - D)}{\bar{\delta}\,(\bar{\delta} + 1)} \sum_{j=1}^{\infty} jR_{\bar{\delta}+j} - \frac{2D - (\bar{\delta} - 1)\,T}{\bar{\delta} + 1} \sum_{j=1}^{\infty} R_{\bar{\delta}+j} \right\} .$$

The expression in braces is ≥ 0 , because $\bar{\delta}$ was so chosen that

$$\frac{2\,(\bar{\delta}\,T - D)}{\bar{\delta}\,(\bar{\delta} + 1)} \geq \frac{2D - (\bar{\delta} - 1)\,T}{\bar{\delta} + 1}$$

and, clearly,

$$\sum_{j=1}^{\infty} jR_{\bar{\delta}+j} \geq \sum_{j=1}^{\infty} R_{\bar{\delta}+j} .$$

Hence, $TE = TV - \left\{\cdots\right\} \leq TV$. This completes the proof that the defense-dominant canonical strategies, as formulated, are minimax strategies.

Their playability is still an open question. No proof has yet been found of the existence everywhere in the canonical region of integral-valued mixed strategies equivalent to the canonical strategies that have fractional values of R_α and S_δ , but the evidence strongly supports the conjecture that they do exist.

Of course, the derived formulas are not valid if $\bar{\delta} > A$, because the attack of a single target cannot exceed the

A level, and the defense would not profit by allocating more than A weapons to defend any target. The defense in this case is characterized by the relationships

$$S_0 = S_1 = \cdots = S_{A-1} \leq S_A ,$$

and the attack by $R_A = \cdots = R_1$, and $R_\alpha = 0$ for $\alpha > A$. These lead to the canonical formulations

$$S_0 = \cdots = S_{A-1} = \frac{2\ (AT - D)}{A\ (A + 1)} ,$$

$$S_A = \frac{2D - (A - 1)\ T}{A + 1} ,$$

$$R_A = \cdots = R_1 = \frac{2}{A + 1} ,$$

$$R_0 = T - \frac{2A}{A + 1} ,$$

which are precisely the same as those obtained before except that A is substituted for $\bar{\delta}$. In this case,

$$\frac{V}{T} = \frac{A/T}{\sum_1^A \delta} \left(A - \frac{D}{T} \right) ,$$

which can be proved minimax by a procedure similar to that used in the $\bar{\delta}$ case. Again, the playability of these strategies has not been proved, but is conjectured for values of $D \leq A\ (T - 1)$.

If $D > A\ (T - 1)$, the defense cannot reasonably leave any target completely undefended, as explained in the illustrative examples. For this constrained situation, the canonical strategies are determined by the parameter $\gamma \equiv D - A\ (T - 1)$, which is the excess of D over $A\ (T - 1)$, and they are characterized by steps of rise $\gamma + 1$.

A precise formulation of the canonical attack strategy shows $R_\alpha = 0$ for all α except $\alpha = 0\ ,\ \gamma + 1\ ,\ 2\ (\gamma + 1)\ ,$ $\ldots\ ,\ \left(\left[\frac{A/2}{\gamma + 1}\right] - 1\right)(\gamma + 1)\ ,\ \left[\frac{A/2}{\gamma + 1}\right](\gamma + 1)\ ,$ $A - \left[\frac{A/2}{\gamma + 1}\right](\gamma + 1)\ ,\ A - \left(\left[\frac{A/2}{\gamma + 1}\right] - 1\right)(\gamma + 1)\ ,\ \ldots\ ,$ $A - (\gamma + 1)\ ,\ A$.

The values of R_α are

$$R_0 = (T - 2) + R_0' \ ,$$

where

$$R_0' \equiv R_{\gamma+1} = \cdots = R_{\left(\left[\frac{A/2}{\gamma+1}\right]-1\right)(\gamma+1)}$$

$$= R_A = R_{A-(\gamma+1)} = \cdots = R_{A-\left(\left[\frac{A/2}{\gamma+1}\right]-1\right)(\gamma+1)} \ .$$

A special condition determines the values of R_α for the two middle values of α. If

$$A - 2\left[\frac{A/2}{\gamma + 1}\right](\gamma + 1) \geq (\gamma + 1) ,$$

then

$$R_{\left[\frac{A/2}{\gamma+1}\right](\gamma+1)} = R_{A-\left[\frac{A/2}{\gamma+1}\right](\gamma+1)} = R_0' .$$

If, however,

$$A - 2\left[\frac{A/2}{\gamma + 1}\right](\gamma + 1) < (\gamma + 1) ,$$

then

$$R_{\left[\frac{A/2}{\gamma+1}\right](\gamma+1)} = R_{A-\left[\frac{A/2}{\gamma+1}\right](\gamma+1)} = 1/2\ R_0' .$$

In the former case, application of the basic constraint equations yields

$$R_0' = R_{\gamma+1} = \cdots = R_A = \frac{1}{1 + \left[\frac{A/2}{\gamma + 1}\right]} .$$

In the latter case,

$$R'_0 = R_{\gamma+1} = \cdots = R_{\left(\left[\frac{A/2}{\gamma+1}\right]-1\right)(\gamma+1)}$$

$$= R_A = R_{A-(\gamma+1)} = \cdots = R_{A-\left(\left[\frac{A/2}{\gamma+1}\right]-1\right)(\gamma+1)}$$

$$= \frac{2}{1 + 2\left[\frac{A/2}{\gamma + 1}\right]} ,$$

and

$$R_{\left[\frac{A/2}{\gamma+1}\right](\gamma+1)} = R_{A-\left[\frac{A/2}{\gamma+1}\right](\gamma+1)} = \frac{1}{1 + 2\left[\frac{A/2}{\gamma + 1}\right]} .$$

In the event that $\frac{A/2}{\gamma + 1}$ is an integer, then

$$\left[\frac{A/2}{\gamma + 1}\right](\gamma + 1) = A - \left[\frac{A/2}{\gamma + 1}\right](\gamma + 1) = \frac{A}{2} ,$$

the condition for the second case is satisfied, and the two middle steps, which are actually at the same level, combine to give

$$R_{A/2} = \frac{2}{1 + \frac{A}{\gamma + 1}} = R'_0 = \cdots .$$

The canonical defense strategies in these cases are $S_\delta = 0$ for all δ except $\delta = \gamma$, $\gamma + (\gamma + 1)$, $\cdots$,

$$\gamma + \left[\frac{A/2}{\gamma + 1}\right] (\gamma + 1) \ , \ A - \left[\frac{A/2}{\gamma + 1}\right] (\gamma + 1) \ , \ \ldots \ , \ A - (\gamma + 1) \ ,$$

A, and the actual values of S_δ are the same as those of corresponding R_α , as follows:

$$S_A = R_0 = (T - 2) + R_0' \ ,$$

$$S_{A-(\gamma+1)} = R_{\gamma+1} \ , \ \ldots \ ,$$

$$S_{A-\left[\frac{A/2}{\gamma+1}\right](\gamma+1)} = R_{\left[\frac{A/2}{\gamma+1}\right](\gamma+1)} \ ,$$

$$S_{\gamma+\left[\frac{A/2}{\gamma+1}\right](\gamma+1)} = R_{A-\left[\frac{A/2}{\gamma+1}\right](\gamma+1)} \ , \ \ldots \ ,$$

$$S_\gamma = R_A \ .$$

If the two middle steps coincide; that is, if

$$A - \left[\frac{A/2}{\gamma + 1}\right] (\gamma + 1) = \gamma + \left[\frac{A/2}{\gamma + 1}\right] (\gamma + 1) \ ;$$

then they combine to yield a single step, as illustrated in Figure 2 for the case (4, 5, 16),

where

$$\gamma = 1 \ , \ \gamma + 1 = 2 \ , \ \left[\frac{A/2}{\gamma + 1}\right] = 1 \ ,$$

$$A - \left[\frac{A/2}{\gamma + 1}\right] (\gamma + 1) = 3 = \gamma + \left[\frac{A/2}{\gamma + 1}\right] (\gamma + 1) \ ,$$

and the combined

$$S_3 = 1/3 + 1/3 = 2/3 \ .$$

The expectation computed for the canonical strategies is

$$V = \frac{2}{T} \Big/ \left\{1 + 1\Big/\left[\frac{A}{\gamma + 1}\right]\right\}$$

in both cases. It can be proved that this is a minimax value and that the canonical strategies are always playable. The formula confirms the result $V = 0$, if $D \geq AT$, where all targets can be completely protected against any attack.

B. <u>Attack-Dominated Cases</u>

In cases dominated by the attack, where $\bar{\alpha} > \bar{\delta}$, methods like those above can be used to obtain similar results with the same degree of assurance of proof or conjecture. General results are presented in Table 2.

It is interesting to observe the duality between attack-dominated cases and defense-dominated cases. In attack-dominated cases the roles of attacker and defender can be thought of as being reversed. In effect, T attack weapons (1 per target) become targets for the D defense weapons, and they are defended by the A-T remaining attack weapons. The

TABLE 2

GENERAL RESULTS FOR ARBITRARY (T, A, D)

	Conditions	Strategies	V/T
$\bar{\delta} \geq \bar{\alpha}$	$D \leq A(T-1)$	$L \equiv \text{Min}\ (A, \bar{\delta})$ $R_L = \cdots = R_1 = \frac{A}{\sum_1^L j}$ $R_0 = T - \frac{2A}{L+1}$ $S_0 = \cdots = S_{L-1} = \frac{LT-D}{\sum_1^L j}$ $S_L = \frac{2D-T\ (L-1)}{L+1}$	$\frac{A}{T\sum_1^L j}\ (L - D/T)$
	$D > A(T-1)$	See text, pp. 33 to 36.	$\frac{2/T^2}{1 + 1/\left[\frac{A}{D+1-A(T-1)}\right]}$
$\bar{\alpha} > \bar{\delta}$	$A-T \leq D\ (T-1)$	$L \equiv \text{Min}\ (D+1, \bar{\alpha})$ $R_0 = 0$ $R_1 = \cdots = R_{L-1} = \frac{LT-A}{\sum_1^{L-1} j}$ $R_L = \frac{2A}{L} - T$ $S_{L-1} = \cdots = S_1 = \frac{D}{\sum_1^{L-1} j}$ $S_0 = T - \frac{2D}{L}$	$1 - \frac{D}{T\sum_1^{L-1} j}\ (L-A/T)$
	$A-T > D\ (T-1)$	See text, pp. 33 to 36, and apply duality principles, p. 37.	$1 - \frac{2/T^2}{1+1/\left[\frac{D}{A-(D+1)(T-1)}\right]}$

expected number destroyed of the T attack weapons acting as virtual targets can be computed from the formulas for the defense-dominated cases simply by substituting D for A and A-T for D in these formulas. The expected number of real targets destroyed is then T minus the expected number of these virtual targets destroyed.

C. Singular Strategies

Although the canonical strategies are always minimax strategies, there are some constrained cases in which singular strategies exist that are also minimax. These singular strategies guarantee the minimax value if the opponent plays properly, but may yield greater benefits if the opponent misplays. For example, in the case $(T, A, D) = (3, 6, 3)$, the canonical strategies are $(R_0, R_1, R_2, R_3) = (0,1,1,1)$ and $(S_0, S_1, S_2) = (1,1,1)$, and the minimax value is $V/T = 2/3$, but the singular attack strategy (0,0,3,0) is preferable because it penalizes the (0,3,0) defensive misplay, and the canonical attack strategy does not.

DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Analytic Services Inc. Falls Church, Virginia	2a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED 2b GROUP --

3. REPORT TITLE

PREFERENTIAL STRATEGIES

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

ANSER Report

5. AUTHOR(S) (Last name, first name, initial)

Matheson, J.D.

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
May 66	50	0

8a. CONTRACT OR GRANT NO. AF 49(638)-1259 b. PROJECT NO. c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) AR 66-2 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report) --

10. AVAILABILITY/LIMITATION NOTICES

This document is subject to special export controls and each transmittal to foreign governments or foreign nationals may be made only with prior approval of AFRDQAN, Headquarters U.S. Air Force.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
--	Directorate of Operational Requirements and Development Plans Headquarters USAF

13. ABSTRACT

This report discusses targeting strategies for attack and defense. Weapons with unit probability of kill are pre-allocated by each combatant, who knows the enemy's total force size but not his weapon allocation. Canonical minimax solutions are formulated for all integral numbers of targets and weapons. In general, multilevel strategies are preferable to uniform strategies.(U) (Author)

Unclassified

Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
12/01 Game Theory 12/02 Theoretical Operations Research 15/07 Strategic Warfare						

INSTRUCTIONS

1. ORIGINATING ACTIVITY: Enter the name and address of the contractor, subcontractor, grantee, Department of Defense activity or other organization (*corporate author*) issuing the report.

2*a*. REPORT SECURITY CLASSIFICATION: Enter the overall security classification of the report. Indicate whether "Restricted Data" is included. Marking is to be in accordance with appropriate security regulations.

2*b*. GROUP: Automatic downgrading is specified in DoD Directive 5200.10 and Armed Forces Industrial Manual. Enter the group number. Also, when applicable, show that optional markings have been used for Group 3 and Group 4 as authorized.

3. REPORT TITLE: Enter the complete report title in all capital letters. Titles in all cases should be unclassified. If a meaningful title cannot be selected without classification, show title classification in all capitals in parenthesis immediately following the title.

4. DESCRIPTIVE NOTES: If appropriate, enter the type of report, e.g., interim, progress, summary, annual, or final. Give the inclusive dates when a specific reporting period is covered.

5. AUTHOR(S): Enter the name(s) of author(s) as shown on or in the report. Enter last name, first name, middle initial. If military, show rank and branch of service. The name of the principal author is an absolute minimum requirement.

6. REPORT DATE: Enter the date of the report as day, month, year; or month, year. If more than one date appears on the report, use date of publication.

7*a*. TOTAL NUMBER OF PAGES: The total page count should follow normal pagination procedures, i.e., enter the number of pages containing information.

7*b*. NUMBER OF REFERENCES: Enter the total number of references cited in the report.

8*a*. CONTRACT OR GRANT NUMBER: If appropriate, enter the applicable number of the contract or grant under which the report was written.

8*b*, 8*c*, & 8*d*. PROJECT NUMBER: Enter the appropriate military department identification, such as project number, subproject number, system numbers, task number, etc.

9*a*. ORIGINATOR'S REPORT NUMBER(S): Enter the official report number by which the document will be identified and controlled by the originating activity. This number must be unique to this report.

9*b*. OTHER REPORT NUMBER(S): If the report has been assigned any other report numbers (*either by the originator or by the sponsor*), also enter this number(s).

10. AVAILABILITY/LIMITATION NOTICES: Enter any limitations on further dissemination of the report, other than those imposed by security classification, using standard statements such as:

(1) "Qualified requesters may obtain copies of this report from DDC."

(2) "Foreign announcement and dissemination of this report by DDC is not authorized."

(3) "U. S. Government agencies may obtain copies of this report directly from DDC. Other qualified DDC users shall request through ________________."

(4) "U. S. military agencies may obtain copies of this report directly from DDC. Other qualified users shall request through ________________."

(5) "All distribution of this report is controlled. Qualified DDC users shall request through ________________."

If the report has been furnished to the Office of Technical Services, Department of Commerce, for sale to the public, indicate this fact and enter the price, if known.

11. SUPPLEMENTARY NOTES: Use for additional explanatory notes.

12. SPONSORING MILITARY ACTIVITY: Enter the name of the departmental project office or laboratory sponsoring (*paying for*) the research and development. Include address.

13. ABSTRACT: Enter an abstract giving a brief and factual summary of the document indicative of the report, even though it may also appear elsewhere in the body of the technical report. If additional space is required, a continuation sheet shall be attached.

It is highly desirable that the abstract of classified reports be unclassified. Each paragraph of the abstract shall end with an indication of the military security classification of the information in the paragraph, represented as (*TS*), (*S*), (*C*), or (*U*).

There is no limitation on the length of the abstract. However, the suggested length is from 150 to 225 words.

14. KEY WORDS: Key words are technically meaningful terms or short phrases that characterize a report and may be used as index entries for cataloging the report. Key words must be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location, may be used as key words but will be followed by an indication of technical context. The assignment of links, rules, and weights is optional.

GPO 886-551

Unclassified

Security Classification