THERMAL STRESS AND BOWING OF A TUBE SUBJECTED TO A TEMPERATURE DISTRIBUTION INDEPENDENT OF TUBE LENGTH

WILLLIAM PERRY HAWORTH

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by

William Perry Haworth
Lieutenant, United States Naval Reserve

Submitted in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE
IN
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United States Naval Postgraduate School
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ABSTRACT

The theorem of least work was used to investigate the stress pattern and bowing, i.e. uniform curvature, of an initially straight thin tube subjected to a steady temperature distribution independent of the longitudinal coordinate of the tube. Simple equations were developed for stress, strain, and bowing. They show that if the thermal strain is expressed as a Fourier series, the tangential stress is small and is a function of the first harmonic only, the axial stress is a function only of the second and higher harmonics, and the bowing is a function only of the first harmonic.

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ACKNOWLEDGEMENT

The author is grateful to Professor John E. Brock for his advice and encouragement during the preparation of this thesis.

1. Nomenclature.

A = cross section area of tube wall

a, b = coefficients used to specify axial strain (See Equation 4)

$\mathcal{E}(\theta)$ = local thermal strain, presumed a function of θ only, not of r or z.

$\mathcal{E}_n, \bar{\mathcal{E}}_n$ = coefficients in Fourier expansion of $\mathcal{E}(\theta)$

E = Young's modulus of elasticity

n = an integer denoting the nth term of a Fourier series

r = radial coordinate, measured positive outward from mid-thickness of tube

r_0 = tube radius to mid-thickness of wall

t = tube wall thickness

$T(\theta)$ = local temperature

T_B = base temperature from which thermal strain is measured

U = strain energy per unit length of tube

$u(\theta)$ = local radial displacement of mid-thickness curve of tube

$u_n, \bar{u}_n$ = coefficients in Fourier expansion of $u(\theta)$

$v(\theta)$ = local tangential displacement of mid-thickness curve of tube

$v_n, \bar{v}_n$ = coefficients of Fourier expansion of $v(\theta)$

x, y, z = tube coordinates (see Figure 1.)

α = coefficient of linear thermal expansion

$\beta = 1 + \frac{t^2}{12 r_0^2}$

γ_θ, γ_z = elastic strain components in tangential and longitudinal directions, respectively

$$\delta = 1 + \frac{t^2}{4r_o^2}$$

$\epsilon_\theta, \epsilon_z$ = total strain (elastic plus thermal) in the tangential and longitudinal directions respectively

$$\zeta_n = 1 + \frac{t^2}{12r_o^2}(n^4-1)$$

θ = angular coordinate measured clockwise from the positive z-axis, (see Figure 1).

θ_R = angular position of the plane of bowing

K = curvature of bowed tube

ν = Poisson's ratio

$$\xi = 1 - \frac{t^2}{12r_o^2}$$

$$\rho_n = 1 + \frac{t^2}{12r_o^2}(n^2-1)$$

$\sigma_r, \sigma_\theta, \sigma_z$ = direct stress in the radial, tangential, and longitudinal directions, respectively

$\tau_{r\theta}$ = shear stress in the $r\theta$-plane

$$\Phi_o = 1+\nu(1-\xi\psi_o)$$

$$\Phi_1 = \frac{\beta}{\delta}\left[1+\nu(1-\psi_1)\right]$$

$$\chi = \frac{2(1-\nu^2)}{\pi r_o t E}U$$

$$\psi_o = \frac{1-\nu^2}{1-\nu^2\xi}$$

$$\psi_1 = \frac{(1+\nu)\left(1-\frac{\nu\beta^2}{\delta}\right)}{1-\frac{\nu^2\beta^2}{\delta}}$$

$\dot{\omega} = \dfrac{d\omega}{d\theta}$ where ω is any representative variable

2. Introduction

The problem of stress distribution in a tube with temperature variation independent of the axial coordinate has been solved under the following two sets of circumstances: (1) an exact solution [2]* in the sense of the theory of elasticity for steady-state temperature configuration, and (2) a very elementary solution [5] in the framework of elementary strength of materials. Solution (1) is of limited usefulness, not only because of its limitation to the steady-state configuration but also because of its mathematical complexity. Solution (2) was developed to investigate technically important cases of the bowing of pipes partially filled with a cryogenic fluid, such as liquid oxygen. This solution assumes that only longitudinal strains and stresses in the pipe are an important cause of bowing. This in effect is taking Poisson's ratio equal to zero. Presuming that Poisson's ratio is zero, while actually it is in the neighborhood of 0.3 for most piping materials, led to a simple and useful solution for bowing radius and axial stress but said nothing about tangential stress. The degree of approximation involved was quite unknown; therefore, it was the purpose of the present investigation to obtain a more accurate solution than that of Flieder, Loria, and Smith [5] without involving the mathematical com-

*Numbers in brackets refer to works listed in the bibliography, page 45.

Then strain energy is differentiated partially with respect to each such coefficient, each such partial derivative is equated to zero, and finally the resulting set of equations is solved for the corresponding values of the coefficients.

This procedure, which was used for the present investigation, gives the displacement which minimizes the strain energy (subject to the implied constraints of the assumed displacement) and thus corresponds to equilibrium.

That the theorem of least work can be applied to the problem at hand, where temperature changes are involved, is not obvious. Argyris[1] and Boley and Weiner[2] have discussed, in general terms, the energy theorems and thermal stresses where temperature changes and non-linear elasticity apply. Even so it seems appropriate to establish, by using the following physical considerations, that the theory of least work can be applied to the present problem.

Consider the structure cut into a number of elemental volumes such that the temperature change is uniform in each element. A change in temperature causes thermal strains but not stresses; however, the elements no longer fit together continuously. In order to fit the elements together into a continuous structure having an assumed displacement pattern from the original shape, stresses must be applied which cause elastic strains in accordance with the theory of elasticity. Since the stresses are applied holding temperature constant, the strain energy can be calculated using the equation previously cited, $\int \sigma \, d\gamma$. The values of the elastic strains are determined by the difference between the total strain, determined from geometric considerations of the assumed displacement pattern, and the thermal strains resulting from known temperature changes. The theory of least work now applies and states that the strain energy is a minimum when the assumed displacement pattern is that corresponding to equilibrium.

3. Analysis of Stress and Strain.

In order to proceed with the analysis, certain conditions must be met and assumptions made. The tube must be constructed of an isotropic elastic material and the temperature distribution must be a known function of the angular coordinate, θ , and independent of time and the longitudinal coordinate, z. This is the resulting temperature distribution when a tube is partially filled with a cold liquid. Also no external forces or moments may be applied to the tube. The necessary assumptions are as follows:

1. The tube is "thin" and thus σ_r and $\tau_{r\theta}$ are small and can safely be neglected in the analysis.*

2. The temperature is constant across the tube wall thickness.

3. Young's modulus of elasticity and Poisson's ratio are constant. This is a reasonable assumption as shown by values for 302 stainless steel in Figures 1 and 2. Stainless steel is typical of most metals.

4. Plane normal cross sections remain plane.

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*By neglecting σ_r and $\tau_{r\theta}$, equilibrium cannot be satisfied and any results are necessarily approximate. A similar assumption has been made by others under similar circumstances and the solutions obtained were acceptable when compared with the exact solutions according to the theory of elasticity. See DenHartog[4] pages 221 to 223 and page 240. The latter portion of Appendix III shows the relative magnitudes which may be expected of σ_r and $\tau_{r\theta}$.

5. In accordance with Saint Venant's principle the stress pattern exists uniformly along the tube except at short distances from the ends.

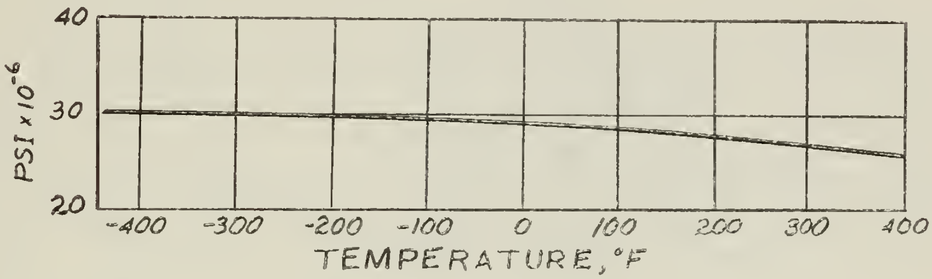

Figure 1. Modulus of Elasticity of AISI 302 Stainless Steel

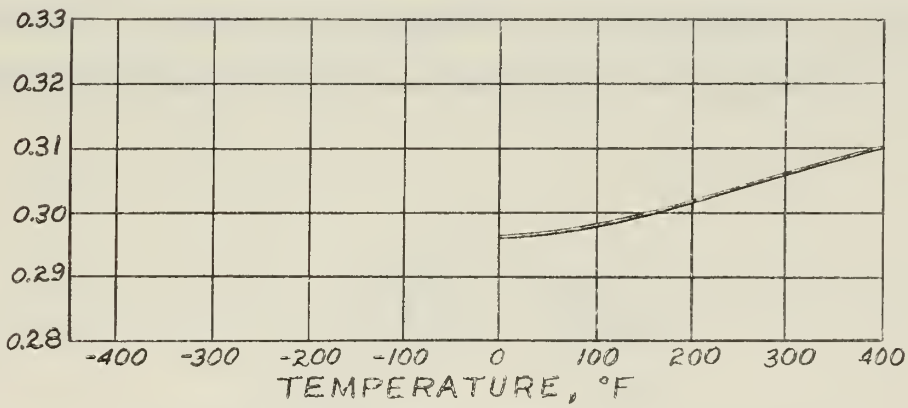

Figure 2. Poisson's Ratio for AISI 302 Stainless Steel

From T. F. Durham, R. M. McClintock, and R. P. Reed, "Cryogenic Materials Data Handbook", National Bureau of Standards, PB Report 171809, 1961.

A. Symmetric Case.

Temporarily the temperature distribution will be limited to one symmetric about the y-axis. The coordinate system used is shown in Figure 3.

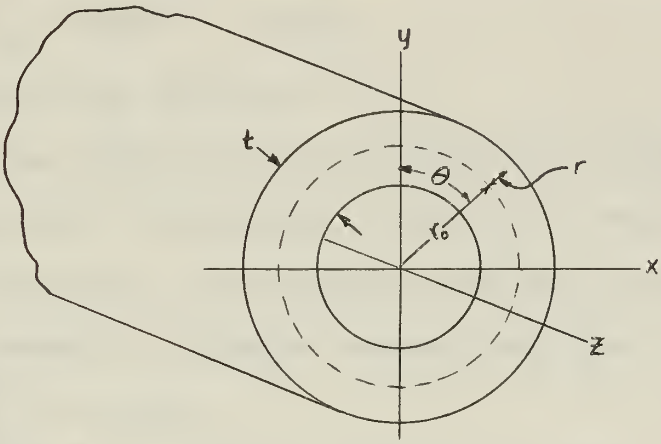

Figure 3. Typical Section of Tube

When there is a change in temperature of the tube, a point P located on the mid-thickness curve will, in general, be displaced to a new location P'. The radial displacement, u, and the tangential displacement, v, are positive in the directions indicated in Figure 4.

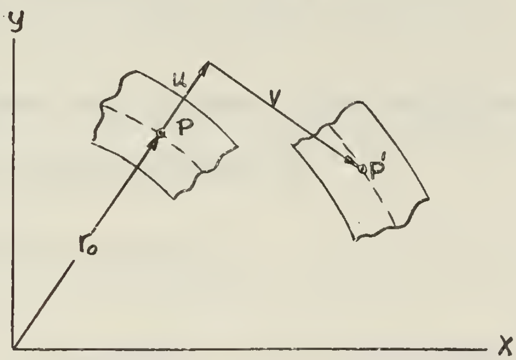

Figure 4. Displacement of Mid-thickness Curve

The tangential strain at any point can be expressed as the strain of the mid-thickness curve plus the strain due to bending of the tube wall, which in turn can be expressed in terms of the radial and tangential displacements as follows

$$\epsilon_\theta = \frac{u + \dot{v}}{r_0} - \frac{u + \ddot{u}}{r_0^2} r \quad . \tag{1}$$

For derivation see Appendix I.

Since the temperature distribution is symmetric about $\theta = 0$, the radial displacement will be an even function of θ and the tangential displacement will be an odd function of θ. Expressed as Fourier series they are

$$u = \sum_{n=0}^{\infty} u_n \cos(n\theta) \tag{2a}$$

and $$v = \sum_{n=1}^{\infty} v_n \sin(n\theta) .$$

The necessary derivatives are

$$\ddot{u} = -\sum_{n=1}^{\infty} n^2 u_n \cos(n\theta)$$
$$\dot{v} = \sum_{n=1}^{\infty} n v_n \cos(n\theta) . \tag{2b}$$

Substituting Equations 2 into Equation 1 we obtain

$$\epsilon_\theta = \frac{1}{r_0} \sum_{0}^{\infty} \left\{ \left[1 + \frac{r}{r_0}(n^2-1) \right] u_n + n v_n \right\} \cos(n\theta) . \tag{3}$$

The assumption that plane cross sections remain plane requires that the axial strain be

$$\epsilon_z = a + b(r_0 + r)\cos(\theta) , \tag{4}$$

where a and b are constants to be determined.

Because there are no external forces or moments applied to the tube $\int_A \sigma_z \, dA = 0$ (5)

$$\int_A y \sigma_z \, dA = 0 \qquad (6)$$

$$\int_A x \sigma_z \, dA = 0 \qquad (7)$$

where $y = (r_0 + r)\cos\theta$ and $dA = (r_0 + r)\,dr\,d\theta$. (8)

Equation 7 is automatically satisfied as a consequence of symmetry.

With the assumption that radial stress is zero, the stress condition in the tube is that of plane stress. The equations of plane stress are

$$\sigma_z = \frac{E}{1-\nu^2}\left[\epsilon_z + \nu\epsilon_\theta - (1+\nu)\mathcal{E}(\theta)\right] \qquad (9)$$

$$\sigma_\theta = \frac{E}{1-\nu^2}\left[\epsilon_\theta + \nu\epsilon_z - (1+\nu)\mathcal{E}(\theta)\right], \qquad (10)$$

where $\mathcal{E}(\theta)$ is the thermal strain. Expressed as a Fourier series $\mathcal{E}(\theta) = \mathcal{E}_0 + \sum_1^\infty \mathcal{E}_n \cos(n\theta)$, (11)

where $\mathcal{E}_0 = \frac{1}{2\pi}\int_0^{2\pi}\int_{T_B}^{T} \alpha(T)\,dT\,d\theta$

and $\mathcal{E}_n = \frac{1}{\pi}\int_0^{2\pi}\int_{T_B}^{T} \alpha(T)\,dT \cos(n\theta)\,d\theta \quad ; \; n = 1, 2, 3, \cdots$

The thermal strain is an even function as a consequence of the temperature being symmetric about the y-axis. In general $\mathcal{E}(\theta) = \int_{T_B}^{T(\theta)} \alpha(T)\,dT$. Using the above expression for axial stress and integrating Equations 5 and 6 permits determining a and b in terms of displacement:

$$a = (1+\nu)\mathcal{E}_0 - \frac{\nu \xi u_0}{r_0} \tag{12}$$

$$b = \frac{\beta}{\delta r_0}\left[(1+\nu)\mathcal{E}_1 - \frac{\nu(u_1+V_1)}{r_0}\right] \tag{13}$$

where $\xi = 1 - \frac{t^2}{12 r_0^2}$, $\beta = 1 + \frac{t^2}{12 r_0^2}$, and $\delta = 1 + \frac{t^2}{4 r_0^2}$.

The integration is performed in Appendix II.

The elastic strain energy for the case of plane stress as discussed on page 4 is

$$U = \frac{1}{2}\int_A \left\{[\epsilon_z - \mathcal{E}(\theta)]\sigma_z + [\epsilon_\theta - \mathcal{E}(\theta)]\sigma_\theta\right\} dA \tag{14}$$

per unit length of tube.

Substituting Equations 9 and 10 for stress into Equation 4, multiplying and collecting terms, we get

$$\frac{2(1-\nu^2)}{E}U = \int_A \left[\epsilon_z^2 + \epsilon_\theta^2 + 2\nu\epsilon_z\epsilon_\theta - 2(1+\nu)\mathcal{E}(\theta)[\epsilon_z+\epsilon_\theta] + 2(1+\nu)\mathcal{E}(\theta)^2\right] dA. \tag{15}$$

Integrating Equation 15, we get the following expression:

$$\frac{2(1-\nu^2)}{\pi r_0 t E}U = X = 2a^2 + r_0\delta b^2 + \frac{1}{r_0^2}\left[2\xi u_0^2 + \sum_1^\infty(\xi_n u_n^2 + 2n\rho_n u_n V_n + n^2 V_n^2)\right]$$

$$+ \frac{2\nu}{r_0}\left[2\xi a u_0 + \beta b(u_1+V_1)\right] - 2(1+\nu)\left[2a\mathcal{E}_0 + \beta r_0 b\mathcal{E}_1\right]$$

$$- \frac{2(1+\nu)}{r_0}\left[2\xi u_0\mathcal{E}_0 + \sum_1^\infty(\rho_n u_n + nV_n)\mathcal{E}_n\right] + 2(1+\nu)\left[2\mathcal{E}_0^2 + \sum_1^\infty \mathcal{E}_n^2\right]. \tag{16}$$

See Appendix II for details of the integration.

Applying the theorem of least work as discussed previously, we are now in a position to evaluate the coefficients u_n and v_n by taking the partial derivative of χ with respect to each coefficient and equating the result to zero.

For the coefficient u_o,

$$\frac{\partial \chi}{\partial u_o} = 4a\left(\frac{-\nu\xi}{r_o}\right) + \frac{\xi u_o}{r_o^2} + \frac{4\nu a\xi}{r_o} + \frac{4\nu u_o\xi}{r_o}\left(-\frac{\nu\xi}{r_o}\right)$$
$$-4(1+\nu)\mathcal{E}_o\left(\frac{-\nu\xi}{r_o}\right) - \frac{4(1+\nu)\xi\mathcal{E}_o}{r_o} = 0 . \qquad (17)$$

Therefore, $u_o = \psi_o r_o \mathcal{E}_o$ where $\psi_o = \dfrac{1-\nu^2}{1-\nu^2\xi}$. (18)

For coefficients u_1 and v_1 ,

$$\chi = r_o\delta b^2 + \frac{1}{r_o^2}(u_1+v_1)^2 + \frac{2\nu}{r_o}\beta b(u_1+v_1) - 2(1+\nu)\beta r_o b\mathcal{E}_1$$
$$-\frac{2(1+\nu)}{r_o}(u_1+v_1)\mathcal{E}_1 \qquad (19)$$

where only those terms that contain u_1 or v_1 have been written. Note that b contains the sum (u_1 +v_1). It is not possible to evaluate u_1 and v_1 separately because $\frac{\partial \chi}{\partial u_1}$ equals $\frac{\partial \chi}{\partial v_1}$. However, it is sufficient to obtain their sum because that is the only form in which u_1 and v_1 appear as can be seen from Equation 19. Taking $\frac{\partial \chi}{\partial(u_1+v_1)}$ and equating to zero:

$$\frac{\partial \chi}{\partial(u_1+v_1)} = -\frac{2\nu\beta b}{r_o} + \frac{2\nu\beta b}{r_o} + \frac{2(u_1+v_1)}{r_o^2}\left[1-\frac{\nu^2\beta^2}{\delta}\right] - \frac{2(1+\nu)\mathcal{E}_1}{r_o}\left[1-\frac{\nu\beta^2}{\delta}\right] = 0 . \qquad (20)$$

Solving for (u_1+v_1) we get

$$(u_1+v_1) = \psi_1 r_0 \mathcal{E}_1 \quad \text{where} \quad \psi_1 = \frac{(1+\nu)\left[1-\frac{\nu\beta^2}{\delta}\right]}{1-\frac{\nu^2\beta^2}{\delta}}. \tag{21}$$

For all other coefficients, u_n and v_n, $n>1$:

$$\frac{\partial \chi}{\partial u_n} = \varsigma_n u_n + n V_n \rho_n - (1+\nu)\rho_n r_0 \mathcal{E}_n = 0\,; \tag{22}$$

and $$\frac{\partial \chi}{\partial V_n} = \rho_n u_n + n V_n - (1+\nu) r_0 \mathcal{E}_n = 0 \tag{23}$$

yields $$n V_n = (1+\nu) r_0 \mathcal{E}_n - \rho_n u_n . \tag{24}$$

Substituting Equation 24 into 22 yields $(\varsigma_n - \rho_n^2) u_n = 0$.

But $(\varsigma_n - \rho_n^2) \neq 0$, therefore,

$$u_n = 0 \quad ; \quad V_n = \frac{(1+\nu) r_0 \mathcal{E}_n}{n} . \tag{25a,b}$$

The radial and tangential displacements are now known as functions of the thermal strain, and thus temperature. Substituting Equations 18 and 21 into Equations 12 and 13 respectively, yields:

$$a = \Phi_0 \mathcal{E}_0 \qquad \text{where} \quad \Phi_0 = 1 + \nu(1 - \xi \psi_0) \tag{26}$$

and $$b = \frac{\Phi_1 \mathcal{E}_1}{r_0} \qquad \text{where} \quad \Phi_1 = \frac{\beta}{\delta}\left[1 + \nu(1-\psi_1)\right]. \tag{27}$$

Thus the axial strain is

$$\epsilon_z = \Phi_0 \mathcal{E}_0 + (1 + \tfrac{r}{r_0}) \Phi_1 \mathcal{E}_1 \cos(\theta) . \tag{28}$$

Substituting Equations 18, 21, and 25 into 3 gives the tangential strain,

$$\epsilon_\theta = \psi_0 \mathcal{E}_0 + \psi_1 \mathcal{E}_1 \cos(\theta) + (1+\nu)\sum_{n=2}^{\infty} \mathcal{E}_n \cos(n\theta) \quad . \tag{29}$$

Remembering that $\mathcal{E}(\theta) = \mathcal{E}_0 + \mathcal{E}_1 \cos(\theta) + \sum_{n=2}^{\infty} \mathcal{E}_n \cos(n\theta)$

then $\sum_{n=2}^{\infty} \mathcal{E}_n \cos(n\theta) = \mathcal{E}(\theta) - \mathcal{E}_0 - \mathcal{E}_1 \cos\theta$ and,

$$\epsilon_\theta = [(\psi_0 - 1) - \nu]\mathcal{E}_0 + [(\psi_1 - 1) - \nu]\mathcal{E}_1 \cos\theta + (1+\nu)\mathcal{E}(\theta) . \tag{30}$$

Substituting these expressionsfor strain into the stress formulas, Equations 9 and 10, determines the stress:

$$\sigma_z = -E\mathcal{E}(\theta) + \frac{E}{1-\nu^2}\left\{\left[(\Phi_0 - 1) + \nu(\psi_0 - 1) + (1-\nu^2)\right]\mathcal{E}_0 + \left[(\Phi_1 - 1) + \nu(\psi_1 - 1) + (1-\nu^2) + \Phi_1 \frac{r}{r_0}\right]\mathcal{E}_1 \cos\theta\right\} \tag{31}$$

and $$\sigma_\theta = \frac{E}{1-\nu^2}\left\{\left[(\psi_0 - 1) + \nu(\Phi_0 - 1)\right]\mathcal{E}_0 + \left[(\psi_1 - 1) + \nu(\Phi_1 - 1) + \nu\Phi_1 \frac{r}{r_0}\right]\mathcal{E}_1 \cos\theta\right\} . \tag{32}$$

This concludes the stress analysis for the symmetric case.

B. Anti-Symmetric Case.

If the temperature distribution is anti-symmetric with respect to $\theta = 0$, the thermal strain is anti-symmetric. Expressed as a Fourier series

$$\mathcal{E}(\theta) = \sum_{n=1}^{\infty} \bar{\mathcal{E}}_n \sin(n\theta) \quad , \tag{33}$$

where $\bar{\mathcal{E}}_n = \frac{1}{\pi}\int_0^{2\pi}\int_{T_B}^{T(\theta)} \alpha(T)dT \sin(n\theta)d\theta$.

Note that there is no constant term in the Fourier series.

It is seen that this is similar to the symmetric case rotated 90 degrees and we can apply some of the results of the previous case.

The first harmonic is the only one contributing to axial strain, so the axis of rotation is the y-axis and the longitudinal strain can be written as

$$\epsilon_z = a_1 + b_1(r_0+r)\sin\theta, \tag{34}$$

where a_1 is automatically zero because of anti-symmetry. The radial displacement will be an odd function and the tangential displacement an even function of the angular coordinate. Expressed as Fourier series they are

$$u(\theta) = \sum_1^\infty \bar{u}_n \sin(n\theta) \tag{35}$$

and $$v(\theta) = \sum_1^\infty \bar{v}_n \cos(n\theta). \tag{36}$$

Note that v contains no constant term. A constant term would imply a rotation of the tube.

Substituting Equations 35 and 26 into 3, we obtain

$$\epsilon_\theta = \frac{1}{r_0}\sum_1^\infty \left\{\left[1+\frac{r}{r_0}(n^2-1)\right]\bar{u}_n - n\bar{v}_n\right\}\sin(n\theta). \tag{37}$$

Letting $-\bar{V}_n = \lambda_n$ yields

$$\epsilon_\theta = \frac{1}{r_0}\sum_1^\infty \left\{\left[1+\frac{r}{r_0}(n^2-1)\right]\bar{u}_n + n\lambda_n\right\}\sin(n\theta)\,. \tag{38}$$

The expressions for ϵ_z and ϵ_θ are of the same form as in the symmetric case except that $\sin(n\theta)$ is substituted for $\cos(n\theta)$. The only integrals that do not vanish when integrated over the interval 0 to 2π are $\int \sin^2(n\theta)d\theta$. These integrals have the same values as $\int \cos^2(n\theta)d\theta$ for the interval. Thus by making the substitution $V_n = -\bar{V}_n$ and writing $\sin(n\theta)$ for $\cos(n\theta)$, the solution of the symmetric case is applicable and the results are as follows:

$$\epsilon_z = (1+\frac{r}{r_0})\Phi_1\bar{\mathcal{E}}_1 \sin\theta \tag{39}$$

$$\epsilon_\theta = [(\psi_1 - 1) - \nu]\bar{\mathcal{E}}_1 \sin\theta + (1+\nu)\mathcal{E}(\theta) \tag{40}$$

$$\sigma_z = -E\mathcal{E}(\theta) + \frac{E}{1-\nu^2}\left[(\Phi_1 - 1) + \nu(\psi_1 - 1) + (1-\nu^2) + \Phi_1\frac{r}{r_0}\right]\bar{\mathcal{E}}_1 \sin\theta \tag{41}$$

$$\sigma_\theta = \frac{E}{1-\nu^2}\left[(\psi_1 - 1) + \nu(\Phi_1 - 1) + \nu\Phi_1\frac{r}{r_0}\right]\bar{\mathcal{E}}_1 \sin\theta\,. \tag{42}$$

The stress analysis is now complete for the anti-symmetric case.

C. General Case.

For a general temperature distribution the thermal strain can be expressed as the sum of the symmetric and anti-symmetric cases, an even and an odd function.

Thus $$\mathcal{E}(\theta) = \mathcal{E}_o + \sum_1^\infty \mathcal{E}_n \cos(n\theta) + \sum_1^\infty \bar{\mathcal{E}}_n \sin(n\theta) \,, \tag{43}$$

a complete Fourier series. The general solution is as follows: $$\epsilon_z = \Phi_o \mathcal{E}_o + (1+\tfrac{r}{r_o})\Phi_1 [\mathcal{E}_1 \cos\theta + \bar{\mathcal{E}}_1 \sin\theta] \tag{44}$$

$$\epsilon_\theta = [(\psi_o - 1) - \nu]\mathcal{E}_o + [(\psi_1 - 1) - \nu][\mathcal{E}_1 \cos\theta + \bar{\mathcal{E}}_1 \sin\theta] + (1+\nu)\mathcal{E}(\theta) \tag{45}$$

$$\begin{aligned}\sigma_z = -E\,\mathcal{E}(\theta) + \frac{E}{1-\nu^2}\Big\{&[(\Phi_o - 1) + \nu(\psi_o - 1) + (1-\nu^2)]\mathcal{E}_o \\ &+ [(\Phi_1 - 1) + \nu(\psi_1 - 1) + (1-\nu^2) + \Phi_1 \tfrac{r}{r_o}][\mathcal{E}_1 \cos\theta + \bar{\mathcal{E}}_1 \sin\theta]\Big\}\end{aligned} \tag{46}$$

$$\sigma_\theta = \frac{E}{1-\nu^2}\Big\{[(\psi_o - 1) + \nu(\Phi_o - 1)]\mathcal{E}_o + [(\psi_1 - 1) + \nu(\Phi_1 - 1) + \nu\Phi_1 \tfrac{r}{r_o}][\mathcal{E}_1 \cos\theta + \bar{\mathcal{E}}_1 \sin\theta]\Big\} \tag{47}$$

The stress and strain resulting from a general temperature distribution is now known.

D. Simplified Equations for Stress and Strain.

It is instructive to consider the magnitude of Φ_o, ψ_o, Φ_1, and ψ_1 in order to simplify the equations of stress and strain. The stress and strain equations (Equations 45, 46 and 47) contain the terms $(\Phi_o - 1)$, $(\Phi_1 - 1)$, $(\psi_o - 1)$, and $(\psi_1 - 1)$ which vanish if the Φ's and ψ's are unity. Table I shows that for tubes of practical wall thickness this is very nearly so.

TABLE I

The Effect of Wall Thickness on Stress and Strain

t/r_o	0.1	0.05	0.01
ψ_o	0.99992	0.99996	1.00000
ψ_1	1.00027	1.00012	1.00005
Φ_o	1.00027	1.00014	1.00001
Φ_1	0.99825	0.99913	0.99972

for $\nu = 0.296$

Substituting unity for Φ_o, ψ_o, Φ_1, and ψ_1, Equations 44, 45, 46, and 47 become:

$$\epsilon_z = \mathcal{E}_o + \left(1+\frac{r}{r_o}\right)\left(\mathcal{E}_1 \cos\theta + \bar{\mathcal{E}}_1 \sin\theta\right) \tag{48}$$

$$\epsilon_\theta = (1+\nu)\mathcal{E}(\theta) - \nu\left(\mathcal{E}_o + \mathcal{E}_1 \cos\theta + \bar{\mathcal{E}}_1 \sin\theta\right) \tag{49}$$

$$\sigma_z = -E\left\{\mathcal{E}(\theta) - \left[\mathcal{E}_o + \left(1+\frac{r}{(1-\nu^2)r_o}\right)\left(\mathcal{E}_1 \cos\theta + \bar{\mathcal{E}}_1 \sin\theta\right)\right]\right\} \tag{50}$$

$$\sigma_\theta = \frac{\nu E r}{(1-\nu^2)r_o}\left(\mathcal{E}_1 \cos\theta + \bar{\mathcal{E}}_1 \sin\theta\right). \tag{51}$$

These equations should be sufficiently accurate for most design applications.

4. Bowing of Tube.

A tube subjected to the assumed temperature distribution will bow into the arc of a circle, or if restrained develop stresses equal to those required to straighten the bowed tube. This is of practical concern in pipe design and has been discussed by Flieder, Smith, and Wetmore [6]. From the analysis of the bending of an initially straight beam [7] it is known that the curvature, K, is $\frac{\epsilon}{c}$; where c is the distance from the neutral axis and ϵ is the strain resulting from bending. The axial strain on the mid-thickness curve of the tube is $\epsilon_z = \Phi_0 \mathcal{E}_0 + \Phi_1(\mathcal{E}_1 \cos\theta + \bar{\mathcal{E}}_1 \sin\theta)$, obtained by taking r=0 in Equation 44. The $\Phi_0\mathcal{E}_0$ term denotes a general elongation of the tube and does not contribute to bowing. The $\Phi_1\mathcal{E}_1\cos\theta$ term results in bowing in the yz-plane. The strain, ϵ, at $\theta=0$ is $\Phi_1\mathcal{E}_1$, at a distance r_0 from the neutral axis. Thus, $K_{yz} = \frac{\Phi_1\mathcal{E}_1}{r_0}$ is the curvature in the yz-plane. The $\Phi_1\mathcal{E}_1\sin\theta$ term results in bowing in the xz-plane. The strain at $\theta=\frac{\pi}{2}$ is $\Phi_1\bar{\mathcal{E}}_1$ and as before the distance to the neutral axis is r_0. Therefore, $K_{xz} = \frac{\Phi_1\bar{\mathcal{E}}_1}{r_0}$ is the curvature in the xz-plane. The curvature in general,

$$K = \frac{\Phi_1\sqrt{\mathcal{E}_1^2 + \bar{\mathcal{E}}_1^2}}{r_0} \tag{52}$$

is the vector sum of K_{xz} and K_{yz}, and the plane of bowing

is given by $\theta_R = \tan^{-1}\frac{\bar{\mathcal{E}}_1}{\mathcal{E}_1}$, (53) measured clockwise from the y-axis. Again it is possible to set $\Phi_1 = 1$ without significant error so that

$$K = \frac{\sqrt{\mathcal{E}_1^2 + \bar{\mathcal{E}}_1^2}}{r_0} \quad . \tag{54}$$

5. Results, Design Equations.

The analysis leads to the following simplified equations which can be used to design thin tubing systems subjected to steady temperature distributions that are independent of tube length.

1. Axial strain

$$\epsilon_z = \mathcal{E}_o + \left(1 + \frac{r}{r_o}\right)\left(\mathcal{E}_1 \cos\theta + \bar{\mathcal{E}}_1 \sin\theta\right) \tag{48}$$

2. Tangential strain

$$\epsilon_\theta = (1+\nu)\mathcal{E}(\theta) - \nu\left(\mathcal{E}_o + \mathcal{E}_1 \cos\theta + \bar{\mathcal{E}}_1 \sin\theta\right) \tag{49}$$

3. Axial stress

$$\sigma_z = -E\left\{\mathcal{E}(\theta) - \left[\mathcal{E}_o + \left(1 + \frac{r}{(1-\nu^2)r_o}\right)\left(\mathcal{E}_1 \cos\theta + \bar{\mathcal{E}}_1 \sin\theta\right)\right]\right\} \tag{50}$$

4. Tangential stress

$$\sigma_\theta = \frac{\nu E r}{(1-\nu^2)r_o}\left(\mathcal{E}_1 \cos\theta + \bar{\mathcal{E}}_1 \sin\theta\right) \tag{51}$$

5. Bowing curvature

$$K = \frac{\sqrt{\mathcal{E}_1^2 + \bar{\mathcal{E}}_1^2}}{r_o} \quad \text{at} \quad \theta_R = \tan^{-1}\frac{\bar{\mathcal{E}}_1}{\mathcal{E}_1} \qquad (54)\&(53)$$

Where $$\mathcal{E}(\theta) = \int_{T_{BASE}}^{T(\theta)} \alpha(T)\, dT = \mathcal{E}_o + \sum_{n=1}^{\infty}\left[\mathcal{E}_n \cos(n\theta) + \bar{\mathcal{E}}_n \sin(n\theta)\right]$$

$$\mathcal{E}_o = \frac{1}{\pi}\int_0^{2\pi}\int_{T_B}^{T(\theta)} \alpha(T)\, dT\, d\theta$$

$$\varepsilon_n = \frac{1}{2\pi}\int_0^{2\pi}\int_{T_B}^{T(\theta)} \alpha(T)\,dT\,\cos(n\theta)\,d\theta$$

$$\bar{\varepsilon}_n = \frac{1}{2\pi}\int_0^{2\pi}\int_{T_B}^{T(\theta)} \alpha(T)\,dT\,\sin(n\theta)\,d\theta\,.$$

6. Discussion of Results.

When thermal strain is expanded in a Fourier series, the tangential stress and bowing as given by Equations 51 and 54 are functions of the first harmonic only, and the axial stress as given by Equation 50 is a function of the second and higher harmonics. To interpret the physical meaning of this, consider the case where $\varepsilon(\theta) = \varepsilon_1 \cos\theta + \varepsilon_2 \cos 2\theta$. When the thermal strain is projected onto a yz-plane, the first harmonic transforms to a linear function and the second harmonic transforms to an even function. It is known [8] that linear thermal strain does not cause stress in a free beam, but causes bowing, and that an even thermal strain function causes stress but not bowing. Thus, the results are those expected from superimposing elementary solutions for simple beams.

Equations 44 thru 47 include a first approximation for the effect of wall thickness; however, the basic assumptions of this analysis, particularly that the temperature does not vary through the wall, suggest that the results be confined to thin walled tubes. Accordingly, Equations 48 thru 51 and 54 are suggested as being the appropriate ones to use for actual design and analysis. These results afford a slight extension of those obtained by Flieder, Loria, and Smith in the following respects: (1) they give a variation of axial stress through the wall thickness and

(2) they give a value for tangential stress.

The present results and those of Flieder, Loria, and Smith are comparable for the case where E, ν, and α are constant. When these conditions are imposed, Equation 50 is identical at the wall mid-thickness to the result developed by Flieder, Loria, and Smith. Note that σ_z is independent of Poisson's ratio. Equation 50 gives a minor variation of σ_z through the wall; they did not consider any radial variation of σ_z. Flieder, Loria, and Smith do not give a result for tangential stress. Equation 51 shows that tangential stress is zero at the tube wall mid-thickness and that it is small for tubes with slight bowing. Equation 51 also shows that tangential stress vanishes if Poisson's ratio is zero.

Comparison of the results for bowing show that there is a very slight effect due to finite wall thickness and non-zero Poisson's ratio as given by Equation 52. However, the simplified equation, Equation 54, which is here recommended for purposes of design and analysis, is essentially identical to that given by Flieder, Loria, and Smith.

Essentially, we have shown that influences neglected in the analysis of Flieder, Loria, and Smith have only the very slightest effect on the final result. However, the expression of the thermal strain as a Fourier series has simplified the final form of the equations and aids in the visualization of the effect that a given temperature distribution may be expected to have on a tube.

simplified the final form of the equations and aids in the visualization of the effect that a given temperature distribution may be expected to have on a tube.

APPENDIX I

Derivation of Expression for Tangential Strain

The temperature change from point to point around the circumference of the tube results in thermal strains. There is assumed to be no temperature gradient across the wall thickness of the tube, thus the thermal strain is independent of radius. Superimposed upon this strain is another strain which varies with radius and is caused by the change in curvature of the tube wall.

To derive an expression for that portion of the strain independent of radius, consider two points A and P, close together, on the mid-thickness curve. When the temperature changes, the points will move to new locations A_1 and P_1 as shown in Figure 5.

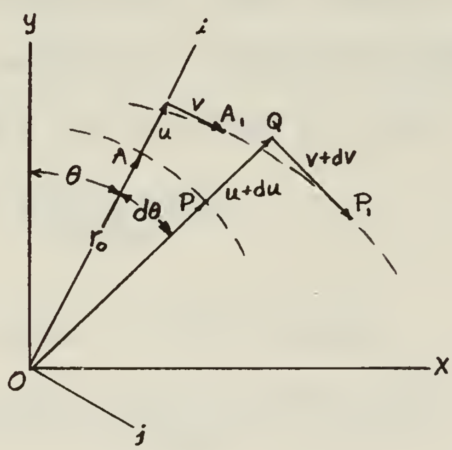

Figure 5. Tangential Strain at Mid-thickness.

Let $\bar{i}$ and $\bar{j}$ be unit vectors in the i and j directions respectively. The bar denotes a vector.

Then $\overline{OA_1} = v\bar{i} + (r_0+u)\bar{j}$, (1a-1)

$$\overline{OQ} = (r_0+u+du)[\sin(d\theta)\bar{i} + \cos(d\theta)\bar{j}] , \qquad (1a-2)$$

$$\overline{QP_1} = (v+dv)[\cos(d\theta)\bar{i} + \sin(d\theta)j] , \qquad (1a-3)$$

and $\overline{A_1P_1} = \overline{OQ} + \overline{QP_1} - \overline{OA_1}$. (1a-4)

Because $d\theta$ is a small angle, $\sin(d\theta) = d\theta$ and $\cos(d\theta) = 1$.

Then, $\overline{A_1P_1} = [(r_0+u)d\theta + du\,d\theta + dv]\bar{i} + [du - v\,d\theta - dv\,d\theta]\bar{j}$. (1a-5)

Neglecting infinitesimals of higher order,

$$\overline{A_1P_1} = [(r_0+u)d\theta + dv]\bar{i} + [du + v\,d\theta]\bar{j} \qquad (1a-6)$$

and $|A_1P_1| = [(r_0+u)^2 d\theta^2 + 2(r_0+u)\dot{v}\,d\theta^2 + \dot{u}^2 d\theta^2 - 2\dot{u}v\,d\theta^2 + v^2 d\theta^2]^{\frac{1}{2}}$. (1a-7)

The strain, $\epsilon_{\theta,1}$, on the mid-thickness curve is defined as the change in length per unit initial length of the curve as the tube wall deflects. For small angles the arc length and the cord length can be taken as equal. Therefore, the strain is

$$\epsilon_{\theta,1} = \frac{|A_1P_1| - r_0\,d\theta}{r_0\,d\theta} \, . \qquad (1a-8)$$

Substituting the value of $|A_1P_1|$ yields

$$\epsilon_{\theta,1} = \frac{[(r_0+u)^2 d\theta^2 + 2(r_0+u)\dot{v}\,d\theta^2 + \dot{u}^2 d\theta^2 - 2\dot{u}v\,d\theta^2 + v^2 d\theta^2]^{\frac{1}{2}} - r_0\,d\theta}{r_0\,d\theta} \, . \qquad (1a-9)$$

Again neglecting infinitesimals of higher order, the strain becomes $$\epsilon_{\theta,1} = \frac{r_o\left[1+\frac{2}{r_o}(u+\dot{v})\right]^{\frac{1}{2}} - r_o}{r_o} \quad . \tag{1a-10}$$

Since $\frac{2}{r_o}(u+\dot{v}) \ll 1$, $\left[1+\frac{2}{r_o}(u+\dot{v})\right]^{\frac{1}{2}} \cong \left[1+\frac{u+\dot{v}}{r_o}\right]$ (1a-11)

Therefore, $$\epsilon_{\theta,1} = \frac{u+\dot{v}}{r_o} \quad . \tag{1a-12}$$

on the mid-thickness curve.

The second component of strain is caused by the change in curvature of the tube wall. Because the tube is "thin" the theory for bending of straight beams can be used. The strain due to bending is $$\epsilon_{\theta,2} = r\,\Delta K \, , \tag{1a-13}$$

where r = distance from mid-thickness and

ΔK = change in curvature of tube wall mid-thickness curve.

To find the change in curvature consider a point P which after a change in temperature is at a new location P_1 as shown in Figure 6.

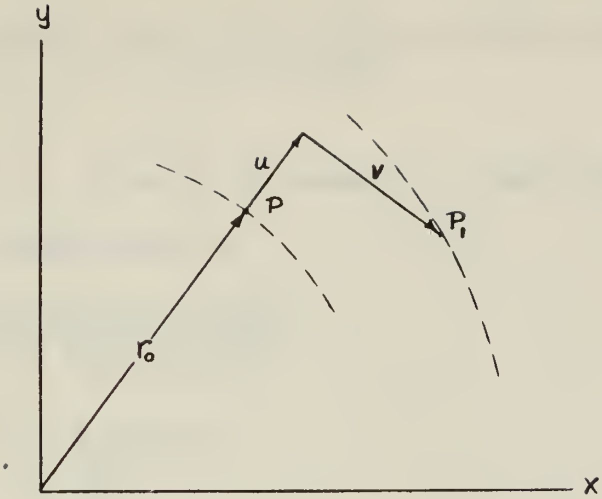

Figure 6. Displacement of a Point on Mid-thickness Curve.

The point is located by the vector $\bar{R} = (r_0 + u)\bar{e}_r + v\bar{e}_\theta$ (1a-14)

where $\bar{e}_r$ and $\bar{e}_\theta$ are unit vectors in the radial and tangential directions respectively. Taking derivatives;

$$\dot{\bar{R}} = \dot{u}\,\bar{e}_r + (r_0+u)\,\bar{e}_\theta + \dot{v}\,\bar{e}_\theta - v\,\bar{e}_r$$

$$= (\dot{u} - v)\bar{e}_r + (r_0 + u + v)\,\bar{e}_\theta \qquad (1a\text{-}15)$$

and $\ddot{\bar{R}} = (\ddot{u} - \dot{v})\bar{e}_r + (\dot{u} - \dot{v})\bar{e}_\theta + (\dot{u} + \ddot{v})e_\theta - (r_0 + u + \dot{v})\,\bar{e}_r$

$$= (\ddot{u} - 2\dot{v} - u - r_0)\bar{e}_r + (\ddot{v} + 2\dot{u} - v)\,\bar{e}_\theta \,. \qquad (1a\text{-}16)$$

The curvature after deflection is[3]

$$K_1 = \frac{(\dot{u}-v)(\ddot{u}+2\dot{u}-v)-(\ddot{u}-2\dot{v}-u-r_0)(r_0+u+\dot{v})}{\left[(\dot{u}-v)^2+(r_0+u+\dot{v})^2\right]^{3/2}} . \tag{1a-17}$$

Noting that r_0 greatly exceeds u, v and their derivatives, we get the approximation

$$K_1 = \frac{r_0^2+r_0(2u+3\dot{v}-\ddot{u})}{r_0^3\left[1+\frac{2}{r_0}(u+\dot{v})\right]^{3/2}}$$

$$= \frac{r_0^2-(u+\ddot{u})r_0}{r_0^3} . \tag{1a-18}$$

The change in curvature is the final curvature minus the initial curvature. The tube is initially round so the initial curvature is $1/r_0$. Thus,

$$\Delta K = \frac{r_0^2-(u+\ddot{u})}{r_0^3} - \frac{1}{r_0} = -\frac{u+\ddot{u}}{r_0^2} . \tag{1a-19}$$

The strain due to bending is then $\epsilon_{\theta,2} = -\frac{(u+\ddot{u})}{r_0^2} r$. (1a-20)

The tangential strain is the sum of the two components developed above:

$$\epsilon_\theta = \frac{u+\dot{v}}{r_0} - \frac{(u+\ddot{u})r}{r_0^2} . \tag{1}$$

Solving Equation 2a-4 for a, we have

$$a = (1+\nu)\varepsilon_0 - \left[1 - \frac{t^2}{12r_0^2}\right]\frac{\nu u_0}{r_0} \qquad (2a-5)$$

and $a = (1+\nu)\varepsilon_0 - \dfrac{\nu \xi u_0}{r_0}$

where $\xi = 1 - \dfrac{t^2}{12r_0^2}$. (12)

Equation 6.

$$\int_A \sigma_z \, y \, dA = 0 \qquad (6)$$

$$y = (r_0 + r)\cos\theta \quad \text{and} \quad dA = (r_0 + r)\,dr\,d\theta \qquad (8)$$

Substituting Equations 8 and 2a-1 into 6 and integrating over the same interval as before:

$$\frac{E}{1-\nu^2}\int_{-t/2}^{t/2}(r_0+r)^2 dr \int_0^{2\pi}\Big[a\cos\theta + b(r_0+r)\cos^2\theta + \nu\sum_{n=0}^{\infty}\frac{1}{r_0}\Big\{\Big[1+\frac{r}{r_0}(n^2-1)\Big]u_n + nV_n\Big\}\cos(n\theta)\cos\theta$$
$$-(1+\nu)\sum_{n=0}^{\infty}\varepsilon_n\cos(n\theta)\cos\theta\Big]d\theta = 0. \qquad (2a-6)$$

Note that $\displaystyle\int_0^{2\pi}\cos(m\theta)\cos(n\theta)\,d\theta = \begin{cases}\pi; & m=n\\ 0; & m\neq n\end{cases}$. (2a-7)

Henceforth in the analysis, terms which vanish in this way will not be written. Continuing the integration,

$$\frac{E}{1-\nu^2}\int_{-t/2}^{t/2}\Big[b(r_0+r)^3 + \nu(u_1+V_1)\frac{(r_0+r)^2}{r_0} - (1+\nu)\varepsilon_1(r_0+r)^2\Big]dr = 0 \qquad (2a-8)$$

$$\frac{2\pi E}{1-\nu^2}\Big\{r_0^3 t\Big[1+\frac{t^2}{4r_0^2}\Big]b + \nu r_0 t\Big[1+\frac{t^2}{12r_0^2}\Big](u_1+V_1) - (1+\nu)r_0^2 t\varepsilon_1\Big[1+\frac{t^2}{12r_0^2}\Big]\Big\} = 0. \qquad (2a-9)$$

Let $\delta = 1 + \frac{t^2}{4 r_0^2}$ and $\beta = 1 + \frac{t^2}{12 r_0^2}$,

then $b = \frac{\beta}{\delta r_0}\left[(1+\nu)\mathcal{E}_1 - \frac{\nu(u_1+v_1)}{r_0}\right].$ (13)

Equation 15.

$$\frac{2(1-\nu^2)U}{E} = \int_A \left[\epsilon_z^2 + \epsilon_\theta^2 + 2\nu\epsilon_z\epsilon_\theta - 2(1+\nu)\mathcal{E}(\theta)\epsilon_z \right. \tag{15}$$

$$\left. -2(1+\nu)\mathcal{E}(\theta)\epsilon_\theta + 2(1+\nu)\mathcal{E}(\theta)^2\right] dA$$

The integral is the sum of the following terms:

$$\text{I} \quad \int_A \epsilon_z^2\, dA = \int_{-t/2}^{t/2}(r_0+r)\,dr \int_0^{2\pi}\left[a^2 + b^2(r_0+r)^2\cos^2\theta\right] d\theta$$

$$= \int_{-t/2}^{t/2}(r_0+r)\left[2\pi a^2 + \pi b^2(r_0+r)^2\right] dr$$

$$= 2\pi a^2 r_0 t + \pi b^2 r_0^3 t\left[1 + \frac{t^2}{12 r_0^2}\right]$$

$$= 2\pi a^2 r_0 t + \pi r_0^3 t b^2 \delta$$

$$\text{II} \quad \int_A \epsilon_\theta^2\, dA = \int_{-t/2}^{t/2}(r_0+r)\,dr \int_0^{2\pi}\frac{1}{r_0^2}\sum_0^\infty\left\{\left[1+\frac{r}{r_0}(n^2-1)\right]^2 u_n^2 + 2n\left[1+\frac{r}{r_0}(n^2-1)\right]u_n V_n + n^2 V_n^2\right\}\cos(n\theta)\,d\theta$$

$$= \int_{-t/2}^{t/2}(r_0+r)\frac{\pi}{r_0^2}\left[2\left(1-\frac{r}{r_0}\right)u_0^2 + \sum_1^\infty\left\{\left[1+\frac{r}{r_0}(n^2-1)\right]^2 u_n^2 + 2n\left[1+\frac{r}{r_0}(n^2-1)\right]u_n V_n + n^2 V_n^2\right\}\right]$$

$$= \frac{\pi r_0 t}{r_0^2}\left[2\xi u_0^2 + \sum_1^\infty\left(\xi_n u_n^2 + 2n\rho_n u_n V_n + n^2 V_n^2\right)\right]$$

where $\xi_n = 1 + \frac{t^2}{12 r_0^2}(n^4-1)$ and $\rho_n = 1 + \frac{t}{12 r_0^2}(n^2-1)$.

$$\text{III} \int_A 2\nu \epsilon_z \epsilon_\theta \, dA = \frac{2\nu}{r_0} \int_{-t/2}^{t/2} (r_0+r)\,dr \int_0^{2\pi} [(1-\tfrac{r}{r_0}) a u_0 + (r_0+r)(u_1+v_1)\, b \cos^2\theta]\, d\theta$$

$$= \frac{2\nu\pi}{r_0} \int_{-t/2}^{t/2} (r_0+r)[2(1+\tfrac{r}{r_0}) a u_0 + (r_0+r)(u_1+v_1) b]\, dr$$

$$= \frac{2\nu\pi r_0 t}{r_0} [2 a u_0 \xi + (u_1+v_1) b \beta$$

$$\text{IV} \int_A -2(1+\nu)\mathcal{E}(\theta)\epsilon_z \, dA = -2(1+\nu) \int_{-t/2}^{t/2} (r_0+r)\,dr \int_0^{2\pi} [a\mathcal{E}_0 + b(r_0+r)\mathcal{E}_1 \cos^2\theta]\, d\theta$$

$$= -2(1+\nu)\pi \int_{-t/2}^{t/2} [2 a \mathcal{E}_0 + b(r_0+r)\mathcal{E}_1](r_0+r)\, dr$$

$$= -2(1+\nu)\pi r_0 t\, [2 a \mathcal{E}_0 + b r_0 \beta \mathcal{E}_1]$$

$$\text{V} \int_A -2(1+\nu)\mathcal{E}(\theta)\epsilon_\theta \, dA = \frac{-2(1+\nu)}{r_0} \int_{-t/2}^{t/2} (r_0+r)\,dr \int_0^{2\pi} \sum_0^\infty \{[1+\tfrac{r}{r_0}(n^2-1)]u_n + n V_n\} \mathcal{E}_n \cos^2(n\theta)\, d\theta$$

$$= -\frac{2(1+\nu)\pi}{r_0} \int_{-t/2}^{t/2} [2(1-\tfrac{r}{r_0}) u_0 \mathcal{E}_0 + \sum_1^\infty \{[1+\tfrac{r}{r_0}(n^2-1)]u_n + n V_n\}\mathcal{E}_n]\, dr$$

$$= \frac{-2(1+\nu)\pi r_0 t}{r_0} [2 \xi u_0 \mathcal{E}_0 + \sum_1^\infty (\rho_n u_n + n V_n)\mathcal{E}_n]$$

$$\text{VI} \int_A 2(1+\nu)\mathcal{E}(\theta)^2 \, dA = 2(1+\nu) \int_{-t/2}^{t/2} (r_0+r)\,dr \int_0^{2\pi} \sum_0^\infty \mathcal{E}_n^2 \cos^2(n\theta)\, d\theta$$

$$= 2(1+\nu)\pi r_0 t\, [2\mathcal{E}_0^2 + \sum_1^\infty \mathcal{E}_n^2]$$

Summing all terms:

$$\frac{2(1-\nu^2)U}{\pi\sigma_0 tE} = \chi = 2a^2 + r_0\delta b^2 + \frac{1}{r_0^2}\left[2\xi u_0^2 + \sum_1^\infty(\xi_n u_n^2 + 2n\rho_n V_n + n^2 V_n^2)\right]$$

$$+\frac{2\nu}{r_0}\left[2au_0\xi + (u_1+v_1)b\beta\right] - 2(1+\nu)\left[2a\varepsilon_0 + b\beta r_0\varepsilon_1\right]$$

$$-\frac{2(1+\nu)}{r_0}\left[2\xi u_0\varepsilon_0 + \sum_1^\infty(\rho_n u_n + nV_n)\varepsilon_n\right]$$

$$+2(1+\nu)\left[2\varepsilon_0^2 + \sum_1^\infty \varepsilon_n^2\right]. \qquad (16)$$

APPENDIX III

Example

The following example will illustrate the method of solution and order of magnitude of results. A 302 stainless steel pipe 10 inches in diameter, with a wall thickness of 0.5 inches is subjected to the temperature distribution shown in Figure 8. Assuming the pipe is free from restraint, what is the stress pattern and how much will the pipe bow?

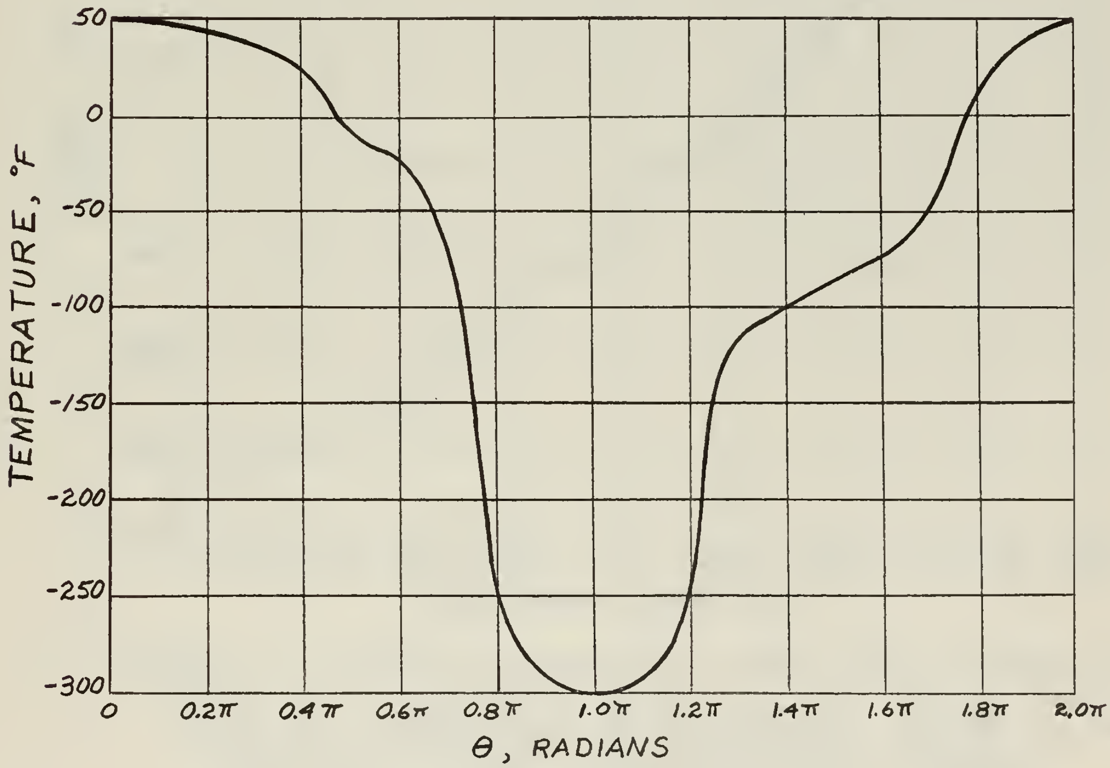

Figure 8. Temperature of Pipe.

The first step is to determine the thermal strain for each temperature. The base temperature in this case is taken as 68°F because the data, see Figure 9, is given in that form. A harmonic analysis is then performed to determine ε_0, ε_1, and $\bar{\varepsilon}_1$. The calculations and results are given in Table II and paragraphs below.

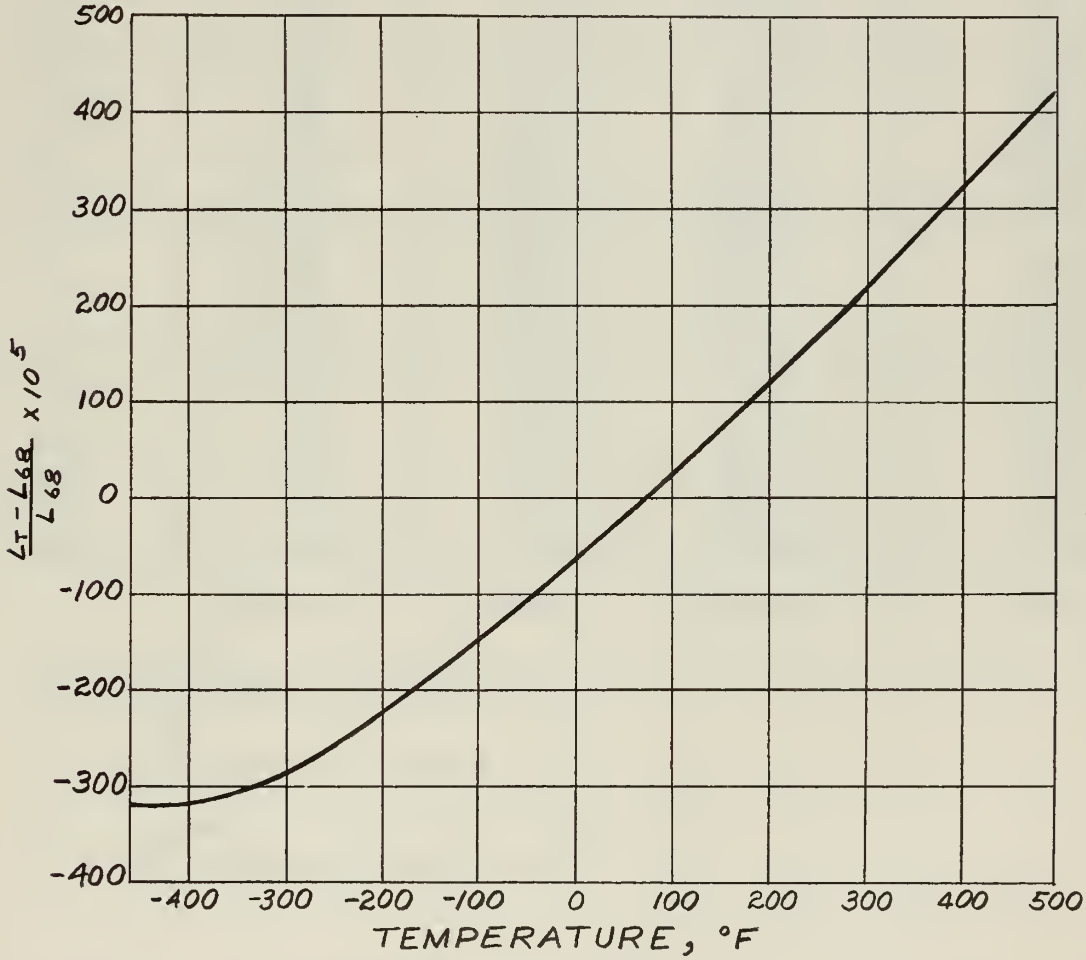

Figure 9. Thermal Expansion of AISI 302 Stainless Steel

From T. F. Durham, R. M. McClintock, and R. P. Reed, "Cryogenic Materials Data Handbook", National Bureau of Standards, PB Report 171809, 1961.

TABLE II

Harmonic Analysis

θ radians	$T(\theta)$ °F	$\mathcal{E}(\theta)$ $\times 10^5$	$\mathcal{E}(\theta)$ cos $\times 10^5$	$\mathcal{E}(\theta)$ sin $\times 10^5$
0.0	50	-20	-20.0	0.0
0.1	48	-21	-20.0	-6.5
0.2	43	-26	-21.1	-15.3
0.3	35	-35	-20.6	-28.3
0.4	23	-45	-14.2	-43.8
0.5	-10	-62	0.0	-62.0
0.6	-20	-80	24.7	-76.2
0.7	-70	-125	73.5	-101.4
0.8	-250	-260	211.0	-153.0
0.9	-295	-285	272.0	-88.2
1.0	-300	-288	288.0	0.0
1.1	-295	-285	272.0	88.2
1.2	-250	-260	211.0	153.0
1.3	-115	-162	95.3	131.2
1.4	-100	-150	46.3	142.5
1.5	-85	-140	0.0	140.0
1.6	-75	-130	-40.2	123.8
1.7	-50	-110	-64.7	89.2
1.8	15	-50	-40.5	29.4
1.9	40	-30	-28.5	9.3
	Total	-2566	1224.0	323.0

$$\mathcal{E}_0 = \frac{1}{20}\sum_0^{1.9\pi} \mathcal{E}(\theta) = -128.3 \times 10^{-5}$$

$$\mathcal{E}_1 = \frac{1}{10}\sum_0^{1.9\pi} \mathcal{E}(\theta)\cos\theta = 122.4 \times 10^{-5}$$

$$\bar{\mathcal{E}}_1 = \frac{1}{10}\sum_0^{1.9\pi} \mathcal{E}(\theta)\sin\theta = 32.3 \times 10^{-5}$$

The bowing of the pipe is

$$K = \frac{\sqrt{\mathcal{E}_1^2 + \bar{\mathcal{E}}_1^2}}{r_0} = \frac{\sqrt{(122.4^{-5})^2 + (32.3^{-5})^2}}{5} = 2.53 \times 10^{-4}/in$$

at $\theta_R = \tan^{-1}\dfrac{\bar{\mathcal{E}}_1}{\mathcal{E}_1} = \tan^{-1}\dfrac{32.3}{122.4} = 14.8°.$

The tangential stress is given by

$$\sigma_\theta = \frac{\nu E r}{(1-\nu^2) r_0} (\varepsilon_1 \cos\theta + \bar{\varepsilon}_1 \sin\theta),$$

therefore, on the mid-thickness curve, r=0, σ_θ =0.

Letting E=30x10^6 psi and ν =0.296,

$$\frac{\nu E}{1-\nu^2} = \frac{0.296(30^{6})}{1-0.0875} = 9.73^{6} \text{ psi}.$$

At the outside diameter r=t/2=0.25 inches.

$$\sigma_\theta = \frac{0.25}{5} 9.73^{6} (122.4 \cos\theta + 32.3 \sin\theta).$$

The results are given in Table III.

The axial stress is given by

$$\sigma_z = -E\{\varepsilon(\theta) - [\varepsilon_0 + (1 + \frac{r}{r_0(1-\nu^2)})(\varepsilon_1 \cos\theta + \bar{\varepsilon}_1 \sin\theta)]\}$$

$$= -30^{6}\{\varepsilon(\theta) - [-128.3^{-5} + (1 + \frac{r}{r_0(1-\nu^2)})(122.4^{-5} \cos\theta + 32.3^{-5} \sin\theta)]\}.$$

On the mid-thickness curve r=0 and $[1 + \frac{r}{r_0(1-\nu^2)}]$ =1.

At the outside diameter r=0.25 and $[1 + \frac{r}{r_0(1-\nu^2)}]$ =1.055.

Axial stress was calculated for each position and the results are given in Table III and also shown in Figure 10.

TABLE III

Tangential and Axial Stress

θ radians	σ_θ, psi at r=0.25	σ_z, psi at r=0	σ_z, psi at r=0.25
0.0	596	4,200	5,700
0.2	575	4,800	8,700
0.4	333	-4,200	-3,000
0.6	-23	-16,500	-16,800
0.8	-390	14,400	13,500
1.0	-596	11,400	9,300
1.2	-575	4,200	2,400
1.4	-333	-13,800	-15,000
1.6	23	2,700	3,000
1.8	390	600	2,700

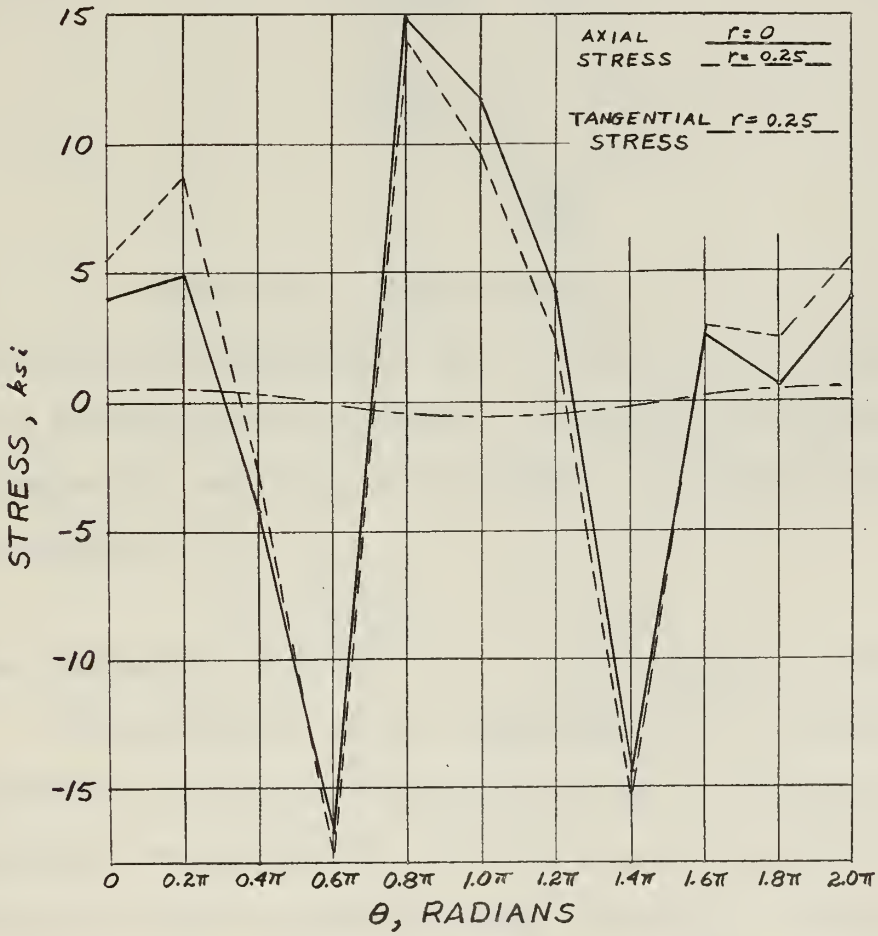

Figure 10. Stress Pattern of Pipe.

The result of the first assumption, that σ_r and $\tau_{r\theta}$ are negligible, is that equilibrium cannot be satisfied. The values of these stresses are now estimated using the conditions of the example. A section is cut out as shown in the free body diagram, Figure 11. These are the positions of maximum and zero tangential stress.

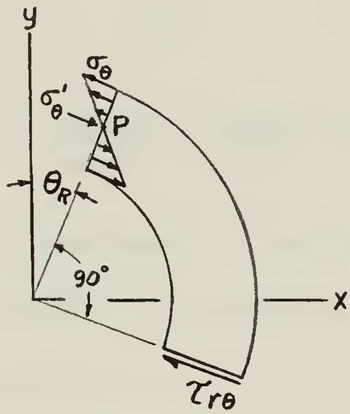

Figure 11. Shear Stress

The stress distribution, σ_θ , causes a net counter-clockwise moment of 25.7 in-lbs. To restore equilibrium, stresses σ_θ' and $\tau_{r\theta}$ are required. Summing moments about the point P,

$$\tau_{r\theta} = \frac{25.7}{0.5(5)} = 10.3 \text{ psi} .$$

The tangential stress, σ_θ' , is also equal to 10.3 psi.

An indication of the magnitude of the radial stress required to maintain equilibrium can be obtained by investigating the condition at $\theta = 195°$, the cold side of the pipe. There the axial stress is about 10,000 psi and will be taken as constant. Figure 12 shows the action of axial and radial stress.

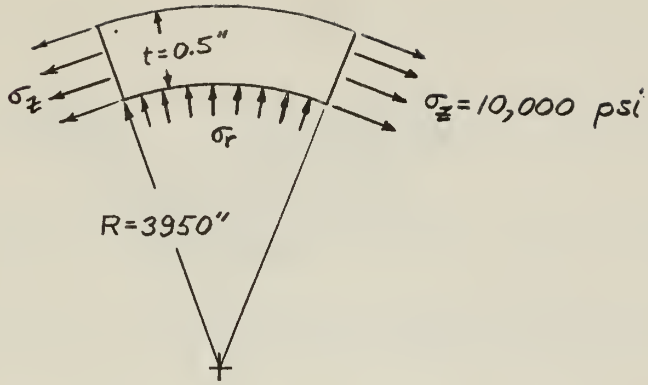

Figure 12. Radial Stress Component Caused by Axial Stress.

The radius of curvature, R, was obtained from the bowing. Summing forces in the radial direction,

$$R\sigma_r \, d\theta - t\sigma_z \, d\theta = 0 \ .$$

Therefore, $\sigma_r = \dfrac{0.5(10,000)}{3950}$ = 1.27 psi, the radial stress caused by axial stress.

The maximum radial stress component resulting from tangential stress can be estimated by considering the stress condition at $\theta = \theta_R$ as shown in Figure 13.

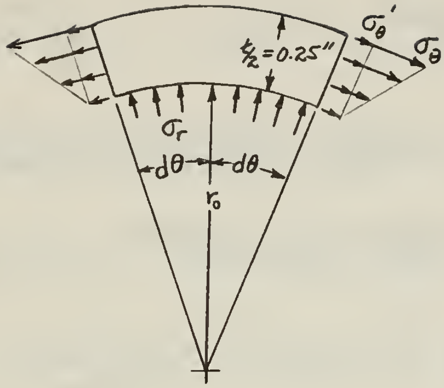

Figure 13. Radial Stress Caused by Tangential Stress.

Taking the sum of the forces in the radial direction we get,

$$2\sigma_r r_o d\theta - 2 \frac{t}{2}(\tfrac{1}{2}\sigma_\theta + \sigma_\theta')d\theta$$

$$\sigma_r = \frac{t}{2r_o}(\tfrac{1}{2}\sigma_\theta + \sigma_\theta')$$

$$\sigma_r = \frac{0.25}{2(5)}\left(\frac{600}{2} + 10\right) = 7.75 \ psi \ .$$

The total radial stress is the sum of the component caused by axial stress and the component caused by tangential stress ($\sigma_r = 1.27 + 7.75 = 9 \ psi$).

It is seen from the above calculations that σ_r and $\tau_{r\theta}$ are small when compared with other stresses, and therefore, neglecting them is realistic.

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