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FUNDAMENTAL RELATIONS IN THE SUPERSONIC FLOW OF A PERFECT GAS AND THE METHOD OF CHARACTERISTICS CALCULATIVE TECHNIQUE

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THE Marquardt CORPORATION
VAN NUYS, CALIFORNIA

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I. SUMMARY

This report represents an attempt to bring together from the fields of thermodynamics, aerodynamics, and mathematics all of the elements of the theory underlying the Method of Characteristics in establishing the fundamental relations in the supersonic flow of a perfect gas. References 1, 2, and 3 have been used extensively in gathering this material.

A brief description of existing characteristics flow programs is also included. More detailed information concerning specific programs is available in the Marquardt Data Processing Department.

II. INTRODUCTION

These programs have been used in the design of high Mach number, internal - external, variable geometry inlets and of high expansion ratio, variable geometry plug nozzles. Test results from both inlets and nozzles have shown satisfactory agreement with predicted theoretical performance.

Extensive use has also been made of these programs in analytical investigations of flow phenomena in supersonic combustion processes.

Presently existing programs will handle isentropic, isenthalpic, irrotational flow for both two-dimensional and axisymmetric designs. Calorific gas imperfections are accounted for to the extent indicated in the body of this report. Flow conditions behind a conical shock can also be computed.

Work is proceeding on other programs which consider the effect of such flow phenomena as chemical reactions in the flow, rotational flow, shock reflections from a boundary, boundary layer, and shock wave-boundary layer interactions.

III. BASIC THERMODYNAMICS

A. General Relations

In order to present the analysis for supersonic flow of a perfect gas, it is first necessary to write down a relation among the state variables P, T, and ρ of perfect gases. Such a relation is called an equation of state.

The equation of state of a perfect gas is

$$P = \rho RT \qquad (1)$$

Where

ρ - Density

P = Pressure

T = Temperature

R = Gas constant

An alternate formulation is

$$PV = RT \tag{2}$$

Where

V = Volume per unit mass

A distinction should be made between thermally perfect and calorifically perfect gas. A thermally perfect gas is one which obeys Equations (1) or (2). A calorifically perfect gas is one whose specific heats are constant. The gas assumed for the existing characteristic programs is thermally perfect and calorifically imperfect. Calorific imperfections are accounted for by providing a curve of the ratio of specific heats (γ) as a function of temperature.

The perfect gas law may be derived from the following mechanical model:

1. The gas is assumed to consist of a set of point masses enclosed in a cubical box.

2. The point masses are in random motion, colliding with each other and the walls of the box in perfectly elastic fashion. From this we can conclude that the change of momentum per unit time at each wall of the box is the same.

3. Define temperature T such that

$$T \simeq \frac{m}{2} V_r^2, \text{ or } \frac{3}{2} kT = \frac{1}{2} mV_r^2,$$

Where

V_r = Root mean square velocity of the point masses

m = Mass of each point mass

k = Boltzmann constant

This establishes a relation between (molecular) kinetic energy and temperature. Real gases can of course not be exactly represented by this model. However, the behavior of real gases can be adequately described by Equations (1) or (2) for most applications. In many problems, the assumption of perfect gas behavior is far less restrictive than certain other assumptions which are made.

B. Equipartition of Energy

In Item 3 above, temperature T is related to the kinetic energy of translation of point masses. If we now consider molecules rather than point masses, we see that there are other modes of motion other than translational, except in the case of inert gases, which contribute to the total energy of the molecule and which can be related to temperature. These modes of motion are rotational motion and vibrational motion.

For example, in the case of a diatomic molecule, the total energy is made up of the following:

1. Translational energy, whereby the center of mass of the molecule moves in the X, Y, and Z directions, (three degrees of freedom)

2. Rotational energy, whereby the molecule rotates about the center of mass, as a rigid rotator, (two degrees of freedom)

3. Vibrational energy, whereby the atoms of the molecule vibrate along the axis, or line connecting the atoms, (one degree of freedom). As a first approximation, we can assume that simple harmonic motion is executed.

In general, the degrees of freedom are allocated as follows:

Translation: 3,

Rotation: 2 for a linear molecule, 3 for a nonlinear molecule, 0 for a monatomic molecule,

Vibration: 3n - 5 for a linear molecule, 3n - 6 for a nonlinear molecule, 0 for a monatomic molecule,

Where

n = Number of atoms in the molecule

The Principle of Equipartition of Energy asserts that with each degree of freedom is associated energy 1/2 KT.

In dealing with the vibrational energy, we assume that the potential energy on the average equals the kinetic energy. This also follows from the assumption of simple harmonic motion.

C. Specific Heat at Constant Volume

From the above, we can deduce an expression for the internal, or molecular energy of a gas. If there are P molecules of gas per unit mass, then

$$E = \frac{[illegible]}{2} = \frac{[illegible]}{2} = E_{trans.} + E_{rot.} + E_{\substack{kinetic \\ vibration}} + E_{\substack{potential \\ vibration}} \qquad (3)$$

Where

E = Internal energy of the gas per unit mass

N = Number of degrees of freedom, plus the vibrational degrees of freedom for potential energy.

(Actually, the internal energy of a real gas also depends very slightly on the volume and temperature.) (We define a calorifically perfect gas as one for which $\gamma = \frac{C_p}{C_v} = \gamma(T)$).

Define $C_v = \frac{E}{T}$ = Specific heat at constant volume for a perfect gas.

Thus, for a diatomic molecule, the number of degrees of freedom is 6, and N is 6 + 1 = 7, so $E = \frac{7}{2} RT$ and $C_v = \frac{7}{2} R$.

D. First Law of Thermodynamics

The first law of thermodynamics states that, for a closed system containing a gas and a quasistatic process,

$$dE = \delta Q_o + \delta W, \tag{4}$$

Where

δQ_o = Quantity of heat exchanged between the system and the surroundings

δW = Amount of work done on the gas by the surroundings

dE = Change in internal energy which results from δQ and δW.

If the internal energy of the gas is a function of the state variables only, dE is a perfect differential, but δW and δQ_o are not necessarily perfect. The symbol δ is used to indicate that these quantities are not functions of only the initial and final states, but also depend upon intermediate conditions.

We now obtain a relation between C_v and C_p, the specific heat at constant pressure. The internal energy of a gas depends upon P, T, and ρ, or P, T, and V. From the equation of state, which we will take to be the perfect gas law, we can express any one of P, T, or V in terms of the other two. Thus E is an explicit function of any two of P, V, or T.

From the first law,

$$\delta W + \delta Q_o = dE = \left(\frac{\partial E}{\partial T}\right)_V dT + \left(\frac{\partial E}{\partial V}\right)_T dV \tag{5}$$

If there is no volume change, then $\delta W = 0$, and $\left(\frac{\partial Q_o}{\partial T}\right)_V = \left(\frac{\partial E}{\partial T}\right)_V$

$$\delta Q_o = dQ_o.$$

Define

$$\left(\frac{\partial Q_o}{\partial T}\right)_V = C_V = \left(\frac{\partial E}{\partial T}\right)_V \tag{6}$$

Rewriting the first law, and using the perfect gas result that $dE = C_V dT$, we obtain

$$C_p = \left(\frac{\partial Q_o}{\partial T}\right)_p = \frac{d(pV)}{dT} + C_v = R + C_v \tag{7}$$

Define

$$\gamma = \frac{C_p}{C_v}$$

E. Enthalpy

The enthalpy, or total heat of a gas is defined to be

$$h = PV + E \tag{8}$$

Again, this is referred to unit mass. From the results for a perfect gas,

$$h = RT + C_v T \tag{9}$$

Here the reference level is taken to be; $h = 0$ when $T = 0$.

F. Entropy

We must now distinguish between reversible and irreversible processes. If when the gas is performing work, or is being worked upon, conditions exist such that the gas is not in equilibrium with the surroundings, the gas will undergo certain net internal accelerations. In order to return to equilibrium, this type of motion must be dissipated as heat in the gas. This amount of wasted heat is added to the term δQ_o in the statement of the first law for reversible processes, to give the new term δQ:

$$\delta Q = \delta Q_o + \delta Q_{diss} \tag{10}$$

Define the entropy (S) by

$$\delta S = \frac{\delta Q}{T}$$

or

$$\delta S \geq \frac{\delta Q_o}{T}$$

Note that δQ_{diss} is not heat which is exchanged between the system and the surroundings, but heat which is generated in the system itself by dissipative processes. In the case of the system doing work adiabatically ($\delta Q_o = 0$), the dissipation due to friction serves to reduce the amount of work capable of being done, and prevents the temperature of the gas from decreasing as much as it would ideally, i.e., if no friction were present.

Entropy is defined in terms of reversible processes. If a system involved in a process undergoes a change from state P_o, T_o, ρ_o to P_f, T_f, ρ_f, the entropy change of the system can be calculated by choosing a series of reversible processes whereby the state variables of the system change from P_o, T_o, ρ_o to P_f, T_f, ρ_f. For the case of a perfect gas, this becomes $\delta S = dS$.

If $\delta Q_o = 0$, i.e., no heat is added from surroundings, the process is adiabatic. If $\delta Q = 0 = \delta Q_o$, the process is reversible and adiabatic, or isentropic.

Again, entropy here is really entropy per unit mass. From the first law

$$\frac{\delta Q}{T} = dS = C_p \frac{dT}{T} - R \frac{dP}{P} \tag{11}$$

This comes about by: $\delta Q + \delta W = dE$, $\delta Q - pdV = C_v dT$, $\delta Q - d(pv) + vdp = C_v dt$, $\delta Q - d(RT) + vdp = C_v dT$, $\delta Q + vdp = C_p dt$.

Note that for dv positive, δW is negative, since δW is work done on the system.

Integrating Equation (11),

$$S = C_p \, LN \, T - R \, LN \, P$$

or

$$S - S_o = C_p \, LN \frac{T}{T_o} - R \, LN \frac{P}{P_o} \qquad (12)$$

This may be rewritten as

$$S = C_p \, LN \frac{T}{P^{R/C_p}}$$

or

$$S = C_p \, LN \frac{T}{P^{\frac{\gamma - 1}{\gamma}}} \qquad (13)$$

For an isentropic process,

$$S = S_o$$

and

$$T \simeq P^{\frac{\gamma - 1}{\gamma}}$$

or

$$P \simeq \rho^{\gamma} \qquad (14)$$

A result which will be used later can be derived from Equation (12). Dividing by C_v,

$$\frac{S - S_o}{C_v} = \gamma \, LN \frac{P \, \rho_o}{\rho \, P_o} - \frac{R}{C_v} \, LN \frac{P}{P_o} = \gamma \, LN \frac{P \, \rho_o}{\rho \, P_o} - (\gamma - 1) \, LN \frac{P}{P_o}$$

and

$$e^{\frac{S-S_o}{C_v}} = \left(\frac{\rho_o}{\rho}\right)^{\gamma} \frac{P}{P_o}$$

or

$$P = \frac{P_o}{\rho_o{}^{\gamma}} \left[e^{\frac{S-S_o}{C_v}} \right] \rho^{\gamma} \tag{15}$$

IV. STEADY STATE EQUATIONS OF MOTION

A. Description of Equations

In describing the equations of motion, we shall refer to two different flow configurations, namely, the two-dimensional case and the axisymmetrical case. In the two-dimensional case, we use the Cartesian coordinates x and y. In the axisymmetrical case, the x-axis is the axis of rotation of the x-y plane, and flow conditions will be the same on any x-y plane which has been rotated at any angle with respect to a stationary line orthogonal to the x-axis.

We shall also use the steady state Eulerian formulation, whereby solid boundaries in the flow are stationary with respect to the observer.

The steady state equation of continuity is, for no sources or sinks,

$$\nabla \cdot (\rho \bar{q}) = 0 \tag{16}$$

Where

ρ = Density

$\bar{q}$ = Velocity vector = (u, v), u = q cos θ, v = q sin θ

Written out, this becomes

$$\rho \left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right] + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + \epsilon \left(\frac{\rho v}{y} \right) = 0 \tag{17}$$

Where

ϵ = 0 for the two-dimensional case

ϵ = 1 for the axisymmetric case

This equation says that mass is preserved at all points in the flow stream.

The Navier-Stokes equation of motion is, in component form, for no dilatation or external forces,

$$\rho \frac{Du}{Dt} = \mu \nabla^2 u - \frac{\partial P}{\partial x} \tag{18}$$

$$\rho \frac{Dv}{Dt} = \mu \nabla^2 v - \frac{\partial P}{\partial y} \tag{19}$$

Where

$$\frac{D}{Dt} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + \frac{\partial}{\partial t}$$

$$u = \frac{dx}{dt}$$

$$v = \frac{dy}{dt}$$

t = Time

μ = Viscosity

The terms $\frac{\partial u}{\partial t}$ and $\frac{\partial v}{\partial t}$ are zero in steady state flow. These equations are the statements of Newton's second law for compressible fluids. In what is to follow, we shall assume $\mu = 0$.

The equation of energy is.

$$\frac{u^2 + v^2}{2} + i = i_o \tag{20}$$

Where

The gas enthalpy per unit mass at any point in the flow stream is

$$i = \frac{\gamma}{\gamma - 1} \frac{P}{\rho} \tag{21}$$

The gas enthalpy for the equilibrium reservoir gas, for which q = (0,0) is

$$i_o = \frac{\gamma}{\gamma - 1} \frac{P_o}{\rho_o} \text{ , or stagnation enthalpy.} \tag{20}$$

If i_o is constant in the entire flow field, the flow is called isenthalpic.

The equation of energy, Equation (20), is valid along the flow streamlines. This equation could also be written as

$$\frac{u^2 + v^2}{2} + C_p T = C_p T_o \tag{20^1}$$

Where $P_o = R \rho_o T_o$.

This equation expresses the interchangeability of random and uniform energy of motion. The term C_pT is related to random motion, and the term $1/2\ (u^2 + v^2)$ is related to uniform motion. If the flow is isenthalpic, the energy equation, Equation (20), is valid at all points in the flow field, and not just along a particular streamline, since i_o is constant.

The temperature T is that which would be measured by an observer moving along the streamline with speed $+\sqrt{u^2 + v^2}$.

The perfect gas law , Equations (1) or (2), also holds for gases with a net velocity. However, in applying it, we must understand that the quantities P, T, and ρ are those which would be measured by an observer moving along with the gas at the same net velocity, i.e., they are static quantities.

Define the quantity a to be

$$a = +\sqrt{\left(\frac{\partial P}{\partial \rho}\right)_S} \tag{22}$$

We will later show that this is the local speed of sound in a gas. For a perfect gas, this becomes

$$a = +\sqrt{\gamma \frac{P}{\rho}} \tag{22^1}$$

since for constant entropy,

$$P = C\rho^{\gamma},\ \frac{dP}{d\rho} = C\,\gamma\,\rho^{\gamma-1} = \gamma \frac{P}{\rho}$$

Another relation which we will need is Crocco's result.

$$\bar{q} \times (\nabla \times \bar{q}) = \mathrm{GRAD}\ i_o - T\ \mathrm{GRAD}\ S \tag{23}$$

B. Special Results

We shall write down some relations which will be needed later. From Equations (22), (23), and (15),

$$\frac{\partial P}{\partial x} = \rho \frac{a^2}{\gamma} \frac{\partial \hat{S}}{\partial x} + a^2 \frac{\partial \rho}{\partial x} \tag{24a}$$

$$\frac{\partial P}{\partial y} = \rho \frac{a^2}{\gamma} \frac{\partial \hat{S}}{\partial y} + a^2 \frac{\partial \rho}{\partial y} \tag{24b}$$

Where

$$\hat{S} = \frac{S}{C_V}$$

C. Results for Isentropic, Isenthalpic Flow

From Equations (22), (22)[1], (20), and (21), and by differentiating Equation (20), and by using Equations (24a) and (24b) with the assumption that S = constant, we obtain

$$u \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial x} + \frac{a^2}{\rho} \frac{\partial \rho}{\partial x} = 0 \tag{25a}$$

$$u \frac{\partial u}{\partial y} + v \frac{\partial v}{\partial y} + \frac{a^2}{\rho} \frac{\partial \rho}{\partial y} = 0 \tag{25b}$$

where

i_o = Constant in the entire flow field.

By substituting Equations (25a) and (25b) into the continuity equation, Equation (17), we obtain

$$\rho \left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right] - \frac{\rho}{a^2} \left[u^2 \frac{\partial u}{\partial x} + uv \frac{\partial u}{\partial y} + uv \frac{\partial v}{\partial x} + v^2 \frac{\partial v}{\partial y}\right] = - \epsilon \frac{\rho v}{y} \tag{26}$$

Equation (26) is the relation which is used in the nozzle flow program which will be described in Section V below.

(Derivation of Equations (24a), (24b))

Define $\hat{S} = S/C_v$.

Then Equation (15) becomes

$$P = \frac{P_o}{\rho_o^{\gamma}} \left[e^{\hat{S}-\hat{S}_o} \right] \rho^{\gamma}$$

$$\frac{\partial P}{\partial x} = \frac{P_o}{\rho_o^{\gamma}} e^{-\hat{S}_o} \left[\rho^{\gamma} e^{\hat{S}} \frac{\partial \hat{S}}{\partial x} + e^{\hat{S}} \gamma \rho^{\gamma - 1} \frac{\partial P}{\partial x} \right]$$

$$= \frac{P_o}{\rho_o^{\gamma}} e^{-\hat{S}_o + \hat{S}} \rho^{\gamma} \frac{\gamma}{\rho} \left[\frac{\rho}{\gamma} \frac{\partial \hat{S}}{\partial x} + \frac{\partial \rho}{\partial x} \right] \qquad (15)^1$$

We also have

$$\frac{\partial P}{\partial x} = \left(\frac{\partial P}{\partial \hat{S}}\right)_{\rho} \frac{\partial \hat{S}}{\partial x} + \left(\frac{\partial P}{\partial \rho}\right)_{\hat{S}} \frac{\partial}{\partial x} = \left(\frac{\partial P}{\partial \hat{S}}\right)_{\rho} \frac{\partial \hat{S}}{\partial x} + a^2 \frac{\partial \rho}{\partial x}$$

Therefore

$$\left[\frac{P_o}{\rho_o^{\gamma}} e^{-\hat{S}_o + \hat{S}} \rho^{\gamma} \frac{\gamma}{\rho} \right] = a^2$$

(Derivation of Equations (25a), (25b))

Start with Equation (20):

$$\frac{u^2 + v^2}{2} + \frac{\gamma}{\gamma - 1} \frac{P}{\rho} = \text{Constant}$$

Differentiating with respect to x,

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} + \frac{\partial}{\partial x}\left[\frac{\gamma}{\gamma-1}\ \frac{P}{\rho}\right] = 0$$

$$\frac{\partial}{\partial x}\left[\frac{\gamma}{\gamma-1}\right]\frac{P}{\rho} = \frac{\gamma}{\gamma-1}\left[\frac{1}{\rho}\frac{\partial P}{\partial x} - \frac{P}{a^2}\ \frac{\partial \rho}{\partial x}\right] = \frac{1}{\rho}\left(\frac{\gamma}{\gamma-1}\right)\left[\frac{\partial P}{\partial x} - \frac{P}{\rho}\ \frac{\partial P}{\partial x}\right]$$

$$= \frac{1}{\rho}\left(\frac{\gamma}{\gamma-1}\right)\left[a^2 - \frac{P}{\rho}\right]\frac{\partial \rho}{\partial x} = \frac{1}{\rho}\left(\frac{\gamma}{\gamma-1}\right)\left[a^2 - \frac{a^2}{\gamma}\right]\frac{\partial \rho}{\partial x}$$

$$= \frac{a^2}{\rho}\left(\frac{\partial \rho}{\partial x}\right)\left[\frac{\gamma}{\gamma-1}\right]\left[1 - \frac{1}{\gamma}\right] = \frac{a^2}{\rho}\ \left(\frac{\partial \rho}{\partial x}\right)$$

We can now write down the final equations for isentropic, isenthalpic compressible fluid flow with unknowns $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, and $\frac{\partial v}{\partial y}$.

From Equation (23),

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \tag{27}$$

Rewriting Equation (26),

$$\rho\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right] - \frac{\rho}{a^2}\left[u^2\frac{\partial u}{\partial x} + uv\frac{\partial u}{\partial y} + uv\frac{\partial v}{\partial x} + v^2\frac{\partial v}{\partial y}\right] = -\epsilon\left(\frac{\rho^v}{y}\right) \tag{26}$$

Also we can write

$$\frac{\partial u}{\partial x}\ dx + \frac{\partial u}{\partial y}\ dy = du \tag{28}$$

$$\frac{\partial v}{\partial x}\ dx + \frac{\partial v}{\partial y}\ dy = dv \tag{29}$$

D. Results for Nonisentropic Flow (Rotational)

Substituting for $\frac{\partial P}{\partial x}$ and $\frac{\partial P}{\partial y}$ from Equations (24a) and (24b) in the Navier-Stokes Equations (18) and (19), we obtain

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \frac{a^2}{\gamma} \frac{\partial \hat{S}}{\partial y} + \frac{a^2}{\rho} \frac{\partial \rho}{\partial x} = 0 \tag{30}$$

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \frac{a^2}{\gamma} \frac{\partial \hat{S}}{\partial y} + \frac{a^2}{\rho} \frac{\partial \rho}{\partial x} = 0 \tag{31}$$

Rewriting the equation of continuity,

$$\rho \frac{\partial u}{\partial x} + \rho \frac{\partial v}{\partial y} + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} = - \epsilon \left(\frac{\rho v}{y}\right) \tag{17}$$

Also,

$$\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = du \tag{32}$$

$$\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy = dv \tag{33}$$

$$\frac{\partial \rho}{\partial x} dx + \frac{\partial \rho}{\partial y} dy = d\rho \tag{34}$$

$$\frac{\partial \hat{S}}{\partial x} dx + \frac{\partial \hat{S}}{\partial y} dy = d\hat{S} \tag{35}$$

The condition that entropy be constant along a streamline except for passage through a strong shock is

$$u \frac{\partial S}{\partial x} + v \frac{\partial S}{\partial y} = 0$$

Note that the equation of energy is not included in this set of equations. The following case of flow with rotation is considered here -- there are variations in the rest enthalpy (i_o) and in the entropy ($\hat{S}$), from streamline to streamline. The entropy is a constant along a particular streamline before and after the shock, while there is a discontinuous increase in entropy on each streamline in crossing the shock front.

E. Weak and Strong Waves

The plug nozzle program (Section V below) carries out the solution of Equations (27), (26), (28), and (29) by a numerical integration using the method of characteristics. Before going into this, however, it is advisable to consider the physical basis for the mathematical development. Indeed, this is really necessary, because certain initial conditions and boundaries must be prescribed before the solution of the equations can be carried out, and these are obtained from physical considerations.

We begin by writing the continuity, Navier-Stokes, and energy equations for one dimension, i.e., the x-direction, in differential form.

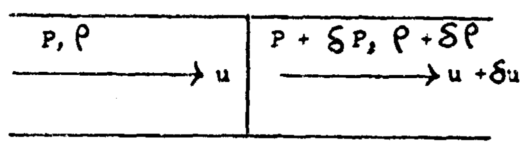
$$\frac{du}{u} + \frac{d\rho}{\rho} = 0 \tag{36}$$

$$udu + \frac{dP}{\rho} = 0 \tag{37}$$

$$udu + C_p dT = 0 \tag{38}$$

F. Speed of a Normal Weak Wave

Assume that we have a flow configuration as shown in the following sketch:



Assume that there exists a disturbance normal to the flow velocity u such that the variables P, ρ, and u undergo small changes δP, $\delta\rho$, and δu, where $\frac{\delta P}{P} << 1$, $\frac{\delta\rho}{\rho} << 1$, $\frac{\delta u}{u} << 1$ and such that the disturbance is stationary.

If these changes are sufficiently small, we may substitute δP for dP, $\delta\rho$ for $d\rho$, and δu for du in Equations (36), (37), and (38) with a small negligible error resulting.

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Combining Equations (37) and (38)

$$\frac{dP}{\rho} = C_p dT = \frac{\gamma}{\gamma - 1} \, d\left(\frac{P}{\rho}\right) \tag{39}$$

so the change in entropy is negligible. Combining Equations (36) and (37)

$$u^2 = \frac{dP}{d\rho} \tag{40}$$

Therefore, for such a disturbance to exist, the flow velocity u must be given by Equation (40).

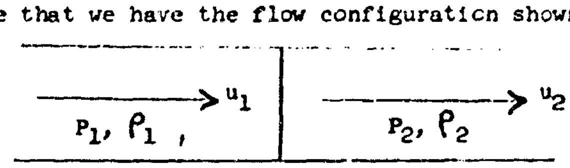
If we impose a velocity u to the left in the above sketch, the discontinuity advances into undisturbed fluid with speed u. The most common example of such a wave is a sound wave, and

$$a = +\sqrt{\left(\frac{\partial P}{\partial \rho}\right)_S}$$

is called the local speed of sound.

G. Rankine-Hugoniot Equations for Normal Strong Waves

Assume that we have the flow configuration shown in the following sketch



Assume that there exists a stationary disturbance such that $\frac{u_2 - u_1}{u_1}$, $\frac{P_2 - P_1}{P_1}$, $\frac{\rho_2 - \rho_1}{\rho_1}$ are not small compared to one.

We refer to this disturbance as a strong wave or shock wave. The Rankine-Hugoniot shock conditions are now expressed as Equations (41), (42), and (43). The mass flow across the wave must be the same as behind the wave, so

$$\rho_1 u_1 = \rho_2 u_2 = m \tag{41}$$

The increase in momentum of the gas per unit time must equal the net force on the gas in the same direction, so

$$- P_2 + P_1 = m(u_2 - u_1) = \rho_2 u_2^2 - \rho_1 u_1^2 \qquad (42)$$

By conservation of total energy (uniform and random), and using the energy equation,

$$\frac{u_1^2}{2} + \frac{\gamma}{\gamma - 1}\frac{P_1}{\rho_1} = \frac{u_2^2}{2} + \frac{\gamma}{\gamma - 1}\frac{P_2}{\rho_2} = \frac{\gamma + 1}{2(\gamma - 1)}(a^*)^2 \qquad (43)$$

where a^* = the speed wherein u and a are equal.

The basic equation for the velocity change across a normal shock wave may be derived as follows. Combine the energy equation, Equation (43), and the momentum equation, Equation (42), to obtain

$$u_1 - u_2 = (u_1 - u_2)\left[\frac{\gamma + 1}{2\gamma}\frac{a^*}{u_1 u_2} + \frac{\gamma - 1}{2\gamma}\right] \qquad (44)$$

The solution to this equation for $u_1 \neq u_2$ is

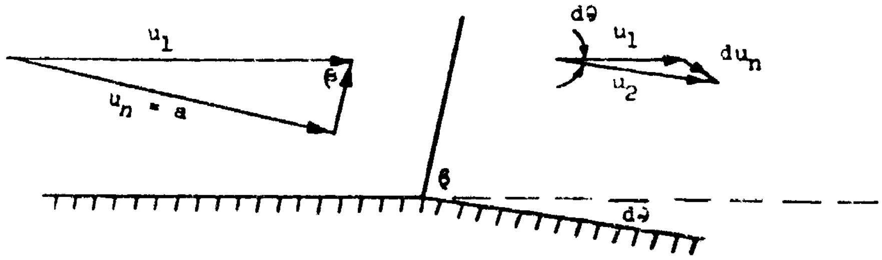
$$u_1 u_2 = a^{*2} \qquad (45)$$

We can conclude from Equation (45) that if u_1 is greater than the speed of sound, then u_2 must be less than the speed of sound.

Define $\frac{u}{a}$ = M = The Mach number for compressible flow.

H. Mach Waves

Assume that we have a weak wave inclined to the direction of flow behind the wave, which is stationary, as shown in the following sketch.



Assume also that du_n is very small compared with u_1. This is defined to be a Mach wave. We know from the study of a weak normal wave that the speed of flow normal to the wave must be a. From the flow diagram,

$$\sin\beta = \frac{a}{u_1} = \frac{1}{M_1} \tag{46}$$

From Equation (46), we may conclude that u_1 must be supersonic, and that there is only one angle β at which the wave may be inclined for initial speed u_1.

With reference to the bent wall boundary, if the flow is required to be parallel to the wall, then the bend creates the disturbance which produces the wave.

The particular wave shown here is an expansion wave. If the bend had been upward, it would have been a compression wave, i.e., the pressure increases. This can be seen from the relations

$$d\theta = \frac{1}{u}\left[du_n \cos\beta\right],$$

and

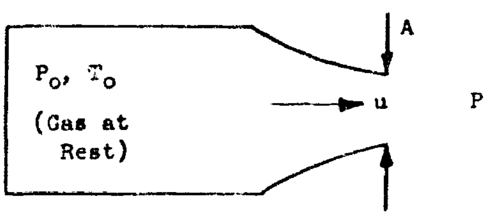
$$d\theta = \sin\beta\sqrt{M^2 - 1}\,\frac{du_n}{u},$$

$$\frac{dP}{P} = -\frac{\gamma u\, du_n \sin\beta}{a^2}.$$

From the first relation, if $d\theta$ is as shown, $du > 0$, from the second relation, $dP < 0$.

I. One-Dimensional Nozzle Flow

We now derive a result for Mach number M where variable cross-sectional area is present.



First we write the continuity equation to take into account the area variation.

$$\frac{d\rho}{\rho} + \frac{du}{u} + \frac{dA}{A} = 0 \qquad (47)$$

Rewriting the Navier-Stokes equation,

$$u\,du + \frac{dP}{\rho} = 0 \qquad (37)$$

Equation (37) may be rewritten as

$$u\,du + a^2 \frac{d\rho}{\rho} = 0 \qquad (37)^1$$

Eliminating $d\rho/\rho$ between Equations $(37)^1$ and (47),

$$\frac{du}{u}(1 - M^2) = -\frac{dA}{A} \qquad (48)$$

If $dA/A = 0$, there are two possibilities: either $M = 1$ or $du = 0$. If, however, $M = 1$ somewhere in the flow, then at that point $dA = 0$. Such a point is called a throat (local minimum in the cross-sectional area A).

We now investigate the mass flow through the configuration shown in the above sketch. Take P to be the external pressure.

$$m = \rho u A = A\sqrt{\frac{2\gamma}{\gamma - 1} P_o \rho_o \left(\frac{P}{P_o}\right)^{\frac{2}{\gamma}} \left[1 - \left(\frac{P}{P_o}\right)^{\frac{\gamma - 1}{\gamma}}\right]} \qquad (49)$$

The mass flow m has a maximum at the point $dm/d\frac{P}{P_o} = 0$, or at

$$\frac{P}{P_o} = \left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma}{\gamma - 1}} \qquad (50)$$

For channel flow, we write down results, assuming entropy constant.

$$\frac{P_o}{P} = (1 + \frac{\gamma - 1}{2} M^2)^{\frac{\gamma}{\gamma - 1}} \qquad (51)$$

$$\frac{T}{T_o} = (1 + \frac{\gamma - 1}{2} M^2)^{-1}$$

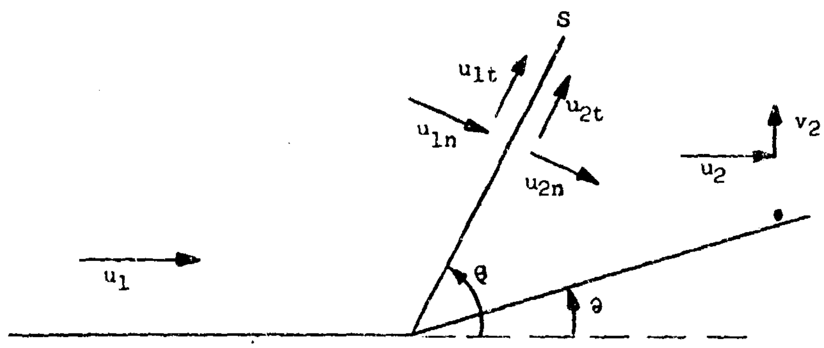
Equation (51) comes directly from the energy equation. If we put M = 1 into Equation (51), we get Equation (50). The channel outlet is the point $dA/A = 0$. Therefore at P corresponding to $dm/d\frac{P}{P_o} = 0$, M = 1 at the outlet.

Now assume that P, the external pressure, is reduced below this value. The mass flow cannot decrease, as Equation (49) would predict, when we lower P, but neither can it increase. We still have M = 1 at the outlet, and P_{outlet} given by Equation (50). Therefore we are led to the conclusion that

$$P \neq P_{outlet} \text{ for } P < P \text{ given by Equation (50)}$$

This means that there is a pressure discontinuity at the outlet (or shock wave) in this case. We may, however, add an expanding section to the nozzle to reach lower pressures beyond the throat. We will then have a convergent-divergent nozzle.

J. Rankine-Hugoniot Equations for Oblique Strong Waves



Continuity:

$$\rho_1 u_1 \sin\beta = \rho_2 (u_2 \sin\beta - v_2 \cos\beta) \tag{52}$$

Conservation of momentum normal to the shock wave:

$$p_1 + \rho_1 u_1^2 \sin^2\beta = p_2 + \rho_2 (u_2 \sin\beta - v_2 \cos\beta)^2 \tag{53}$$

Conservation of momentum parallel to the shock wave:

$$\rho_1 u_1^2 \sin\beta \cos\beta = \rho_2 (u_2 \sin\beta - v_2 \cos\beta)(u_2 \cos\beta + v_2 \sin\beta) \tag{54}$$

Conservation of energy across the wave:

$$\frac{1}{2} u_1^2 + \frac{\gamma}{\gamma-1}\frac{p_1}{\rho_1} = \frac{(\gamma+1)}{2(\gamma-1)} a^{*2} = \frac{1}{2}(u_2^2 + v_2^2) + \frac{\gamma}{\gamma-1}\frac{p_2}{\rho_2} \tag{55}$$

The enthalpy will change when the flow encounters an oblique strong wave.

We just mention the existence of reflected waves, which will occur if a wave is oblique to a straight boundary. The reflected wave keeps the flow parallel to the wall.

In the preceding development, we have classified disturbance waves which can exist in compressible fluids. For our immediate purposes, the class of weak waves is most important. We are now in a position to interpret the results of the numerical integration to follow.

K. The Method of Characteristics for Isentropic, Isenthalpic Flow

By neglecting viscous effects (both in the stream and at the boundaries), small changes in entropy and enthalpy across weak waves, and essentially assuming that C_v and C_p are constant, we arrived at Equations (27), (26), (28), and (29). We further assume that no strong waves will exist in the flow field. If they should occur, their position and shape would have to be determined and the Rankine-Hugoniot equations used. In other words, strong waves must in general be specified as "boundary conditions" in the flow field.

Rewriting Equations (27), (26), (28), and (29),

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0 \tag{27}$$

$$\rho\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right] - \frac{\rho}{a^2}\left[u^2 \frac{\partial u}{\partial x} + uv \frac{\partial u}{\partial y} + uv \frac{\partial v}{\partial x} + v^2 \frac{\partial v}{\partial y}\right] = -\epsilon\left(\frac{\rho v}{y}\right) \tag{26}$$

$$\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = du \tag{28}$$

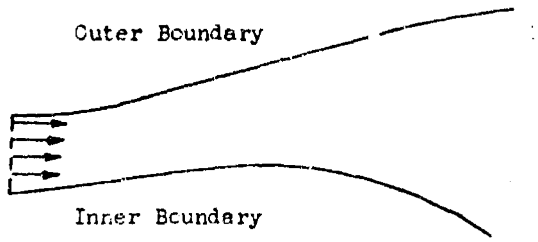
$$\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy = dv \tag{29}$$

We shall refer to these equations shortly.

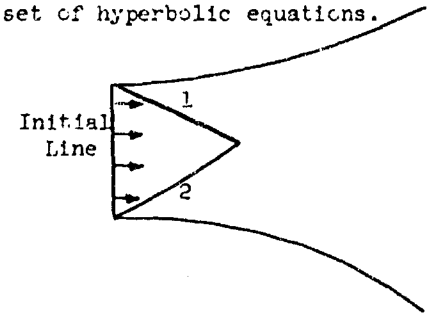
Making the assumptions which we have mentioned, we note that we are dealing with systems of partial differential equations which are first order, that is, the highest order derivatives which appear are first derivatives. (If we included viscous effects, we would have a second order system of equations.) This means that we have to solve an initial value problem, i.e., we specify boundary conditions on only part of the boundary of the flow field. We cannot in general have a closed boundary problem.

In deciding which numerical scheme to use, there exists the possibility of writing difference approximations to our system of equations using a rectangular net. In order for these difference approximations to be valid, we require continuity of higher order derivatives so that the mean value theorem can be applied to the Taylor's series to obtain a remainder term which expresses the size of the truncation error. For accurate solutions, we of course require that the truncation error be at least an order of magnitude smaller than the solutions themselves.

In the case of plug nozzle flow with uniform initial conditions, however, it is evident that if we assume continuity of the first derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ we can obtain only uniform flow throughout the flow stream by the usual processes of numerical integration using rectangular nets.



Consider the case of nozzle flow with uniform initial conditions prescribed along a non-characteristic line in a region of supersonic flow. Equations (26) and (27) are then a set of hyperbolic equations.



By the initial value theorem, the flow in a three-sided region including the initial line will also be uniform (See diagram above). The lines 1 and 2 are characteristic lines. The flow outside of this three-sided region is nonuniform, because of the curvature of the boundary, and the fact that the boundary is a streamline of the flow. Thus the flow changes from uniform to non-uniform across the characteristic lines 1 and 2. Since, as will be indicated, the derivatives at a characteristic are discontinuous, a valid Taylor's series with small remainder cannot be present at the characteristics. A method must be used whereby solutions of the differential system of equations can be patched together along characteristic lines.

Courant points out that the nonlinear equations of hydrodynamics (i.e., Equation (27)) exhibit this behavior of discontinuities being generated in the flow field, even though none are present on the initial line. This cannot happen, however, for linear sets of equations of hyperbolic type; for example, the wave equation

$$a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \frac{\partial^2 u}{\partial t^2}$$

We just mention that one way of eliminating the characteristics is to add a second order viscosity term to the energy and momentum equations.

This second order term has little effect except at places where the derivatives u_x, u_y, v_x, v_y otherwise become discontinuous (at characteristics), then it has the effect of smoothing out the transition lines so that the characteristics vanish.

One method of finding the characteristic lines for Equations (27), (26), (28), and (29) is now illustrated. We can write these equations in the following form:

$$\begin{vmatrix} \rho(1 - \frac{u^2}{a^2}) & - \frac{\rho}{a^2} uv & - \frac{\rho}{a^2} uv & (1 - \frac{v^2}{a^2}) \\ 0 & 1 & -1 & 0 \\ dx & dy & 0 & 0 \\ 0 & 0 & dx & dy \end{vmatrix} \begin{vmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} - \epsilon \frac{\rho v}{y} \\ 0 \\ du \\ dv \end{vmatrix} \tag{56}$$

Solving for $\frac{\partial u}{\partial x}$ by Cramer's rule, we obtain

$$\frac{\partial u}{\partial x} = \frac{(a^2 - v^2)\, dv\, dy - 2\, uv\, dy\, du - (a^2 - v^2)\, dx\, du + \epsilon a^2 \frac{v}{y} dy^2}{a^2 (dx^2 + dy^2) - (udy - vdx)^2} \tag{57}$$

For $\frac{\partial u}{\partial x}$ to be indeterminate, it is necessary and sufficient that both numerator and denominator be zero. This implies that $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ are also indeterminate.

<u>Discussion</u>: Since the denominator is zero, the coefficient matrix is singular, hence of rank less than four. This implies a linear relation among $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, and $\frac{\partial v}{\partial y}$ and hence $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, and $\frac{\partial v}{\partial y}$ are indeterminate.

The condition that the denominator vanish gives the following two characteristic curves, which are obtained directly by solving for dy using the well-known formula for obtaining roots of a quadratic:

$$dy - \lambda^L dx = 0, \qquad \text{(Left characteristic line)} \tag{58}$$

$$dy - \lambda^R dx = 0, \qquad \text{(Right characteristic line)} \tag{59}$$

Where

$$\lambda^L = \frac{uv + a\sqrt{u^2 + v^2 - a^2}}{u^2 - a^2} \tag{60}$$

$$\lambda^R = \frac{uv - a\sqrt{u^2 + v^2 - a^2}}{u^2 - a^2} \tag{61}$$

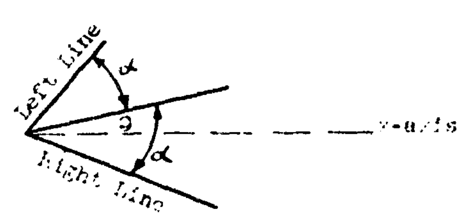
These are the only two characteristic lines which can be found.

From Equations (60) and (61), we see that the characteristics are real if and only if $u^2 + v^2 \geq a^2$, that is, if and only if the flow is supersonic.

We have the following equations, which will be verified:

$$\lambda^L = \tan(\theta + \alpha) \tag{62}$$

$$\lambda^R = \tan(\theta + \alpha) \tag{63}$$



Verification: Assume Equation (62) is true. Then

$$\frac{uv + a\sqrt{q^2 - a^2}}{u^2 - a^2} \overset{?}{=} \tan(\theta + \alpha) = \frac{\tan\theta + \tan\alpha}{1 - \tan\theta\tan\alpha} = \frac{\tan\theta + \frac{a}{\sqrt{q^2 - a^2}}}{1 - \tan\theta\frac{a}{\sqrt{q^2 - a^2}}}$$

$$\frac{q^2 \sin\theta\cos\theta + a^2\sqrt{M^2 - 1}}{q^2\cos\theta^2 - a^2} \overset{?}{=} \frac{\sqrt{M^2 - 1}\sin\theta + \cos\theta}{\sqrt{M^2 - 1}\cos\theta - \sin\theta}$$

$$q^2 \sin\theta\cos^2\theta\sqrt{M^2 - 1} + a^2(M^2 - 1)\cos\theta - q^2\sin^2\theta\cos\theta - a^2\sin\theta\sqrt{M^2 - 1}$$

$$\overset{?}{=} q^2\cos^2\theta\sin\theta\sqrt{M^2 - 1} + q^2\cos^3\theta - a^2\sqrt{M^2 - 1}\sin\theta - a^2\cos\theta$$

$$a^2(M^2 - 1)\cos\theta - q^2\sin^2\theta\cos\theta \overset{?}{=} q^2\cos^3\theta - a^2\cos\theta$$

$$q^2 - q^2\sin^2\theta \overset{?}{=} q^2\cos^2\theta$$

$$q^2 - q^2 = 0$$

Similarly, we could verify Equation (63).

The numerator of the quotient of Equation (57) is set equal to zero. It is

$$(a^2 - v^2)\,dvdy - 2uv\,dydu - (a^2 - v^2)\,dxdu + \epsilon a^2\frac{v}{y}\,dy^2 = 0 \qquad (64)$$

The denominator, set equal to zero, is

$$(a^2 - u^2)\,dy^2 + 2uv\,dxdy + (a^2 - v^2)\,dx^2 = 0 \qquad (65)$$

Multiplying Equation (64) by dx and substituting $(a^2 - v^2)dx^2$ from Equation (65), we obtain

$$(a^2-v^2)\ dvdydx - 2\ uv\ dydudx + (a^2-u^2)\ dy^2du + 2\ uv\ dxdydu + \epsilon a^2 \frac{v}{y}\ dy^2\ dx = 0 \tag{66}$$

or

$$(a^2-v^2)\ dv\ dydx + (a^2-u^2)\ dy^2\ du + \epsilon a^2 \frac{v}{y}\ dy^2\ dx = 0.$$

Ignoring the possible root $dy = 0$, we finally obtain

$$(a^2-v^2)\ dvdx + (a^2-u^2)\ dy\ du + \epsilon a^2 \frac{v}{y}\ dy\ dx = 0 \tag{67}$$

Substituting Equation (58) in Equation (67), and ignoring the possible root $dx = 0$, we obtain

$$(a^2-v^2)\ dv + (a^2-u^2)\lambda^L\ du + \epsilon a^2 \frac{v}{y}\ dy = 0 \tag{68}$$

Substituting Equation (59) in Equation (67), and ignoring the possible root $dx = 0$, we obtain

$$(a^2-v^2)\ dv + (a^2-u^2)\lambda^R\ du + \epsilon a^2 \frac{v}{y}\ dy = 0 \tag{69}$$

Equations (68) and (69), give relations between u, v, and y which must hold along the characteristic lines given by Equations (58) and (59). These are referred to as compatibility equations.

We will now go through a development to put Equation (68) in a more convenient form.

Substituting λ^L from Equation (60) in Equation (68), we obtain

$$- (q^2 \sin\theta \cos\theta + a\sqrt{q^2-a^2})(\cos\theta\, dq - \sin\theta\, q\, d\theta) + (a^2 - q^2 \sin^2\theta)(\sin\theta\, dq + \cos\theta\, q\, d\theta) + \epsilon\, a^2\, q \sin\theta\, \frac{dy}{y} = 0$$

Rearranging terms,

$$\left(\cos\theta + \frac{\sqrt{q^2-a^2}}{a}\sin\theta\right)\left(d\theta - \frac{\sqrt{q^2-a^2}}{a}\,dq\right) + \epsilon \sin\theta\, \frac{dy}{y} = 0 \qquad (70)$$

Assuming that

$$\cos\theta + \frac{\sqrt{q^2-a^2}}{a}\sin\theta \neq 0$$

and multiplying Equation (70) by $\sin\alpha$ we obtain

$$d\theta - \frac{\cot\alpha}{q}\,dq + \frac{\sin\theta\sin\alpha}{\sin(\theta+\alpha)}\,\frac{dy}{y} = 0 \qquad (71)$$

where

$$\tan\alpha = \frac{a}{\sqrt{q^2-a^2}}.$$

By a similar process, Equation (69) becomes

$$d\theta + \frac{\cot\alpha}{q}\,dq - \frac{\sin\theta\sin\alpha}{\sin(\theta-\alpha)}\,\frac{dy}{y} \qquad (72)$$

under the assumption that

$$\cos\theta - \frac{\sqrt{q^2-a^2}}{a}\sin\theta \neq 0$$

We will now go through a development to put Equations (71) and (72) in the form in which it is used in the Plug Nozzle program. Let us first work on the term $\frac{\sin\theta \sin\alpha}{\sin(\theta+\alpha)} \frac{dy}{y}$ of Equation (71).

$$\frac{\sin\theta \sin\alpha}{\sin(\theta+\alpha)} \frac{dy}{y} = \frac{\sin\theta \sin\alpha}{\sin(\theta+\alpha)} \frac{\tan(\theta+\alpha)}{y} dx = \frac{dx}{y}\left[\frac{\sin\theta \sin\alpha}{\cos\theta \cos\alpha - \sin\theta \sin\alpha}\right]$$

$$= \frac{dx}{y}\left[\frac{1}{\cot\theta \cot\alpha - 1}\right] = \frac{dx}{y}\left[\frac{\tan\theta}{\sqrt{M^2-1} - \tan\theta}\right] \tag{73}$$

Now we work on the term $\frac{\sin\theta \sin\alpha}{\sin(\theta-\alpha)} \frac{dy}{y}$ of Equation (72).

$$\frac{\sin\theta \sin\alpha}{\sin(\theta-\alpha)} \frac{dy}{y} = \frac{\sin\theta \sin\alpha}{\sin(\theta-\alpha)} \frac{\tan(\theta-\alpha)}{y} dx = \frac{dx}{y}\left[\frac{\sin\theta \sin\alpha}{\cos\theta \cos\alpha + \sin\theta \sin\alpha}\right]$$

$$= \frac{dx}{y}\left[\frac{1}{\cot\theta \cot\alpha + 1}\right] = \frac{dx}{y}\left[\frac{\tan\theta}{\sqrt{M^2-1} + \tan\theta}\right] \tag{74}$$

Lastly, we derive a new expression for the term $\cot\alpha \frac{dq}{q}$ of Equations (71) and (72). In this work, we shall assume $\gamma = C_p/C_v$ is a constant. From the energy equation

$$q\,dq + C_p\,dT = 0 \tag{75}$$

From the results on isentropic flow, (Equation 51),

$$T = \frac{T_o}{1 + \frac{\gamma - 1}{2} M^2} \tag{76}$$

From Equation (76),

$$dT = - \frac{T_o}{\left(1 + \frac{\gamma - 1}{2} M^2\right)^2} \left[(\gamma - 1) M\right] dM \tag{77}$$

From Equations (77) and (75),

$$\frac{dq}{q} = \frac{T_o C_p}{q^2 \left(1 + \frac{\gamma - 1}{2} M^2\right)^2} (\gamma - 1) M \, d \, M \tag{78}$$

Since $q = Ma$ and $a = \sqrt{\gamma RT}$,

$$\frac{dq}{q} = \frac{\dfrac{T_o C_p}{\left(1 + \frac{\gamma - 1}{2} M^2\right)^2} (\gamma - 1) M \, d \, M}{M^2 \gamma R \dfrac{T_o}{1 + \frac{\gamma - 1}{2} M^2}} = \frac{\dfrac{C_p}{\gamma RM} (\gamma - 1) dM}{\left(1 + \frac{\gamma - 1}{2} M^2\right)}$$

$$= \frac{dM}{M \left(1 + \frac{\gamma - 1}{2} M^2\right)} \tag{79}$$

From these results, Equation (71) assumes the form

$$d\theta - \frac{\sqrt{M^2 - 1} \, d \, M}{M \left(1 + \frac{\gamma - 1}{2} M^2\right)} + \epsilon \left[\frac{\tan \theta}{\sqrt{M^2 - 1} - \tan \theta}\right] \frac{dx}{y} = 0 \tag{71^1}$$

(corresponding to λ^L)

and Equation (72) assumes the form

$$d\theta + \frac{\sqrt{M^2 - 1}\ dM}{M\left(1 + \frac{\gamma - 1}{2} M^2\right)} - \epsilon \frac{\tan \theta}{\sqrt{M^2 - 1} + \tan \theta} \frac{dx}{y} = 0 \qquad (72)^1$$

(corresponding to λ^R)

In deriving Equations $(71)^1$ and $(72)^1$, we have not yet provided for handling the cases

$$dx = 0, \qquad (80)$$

$$dy = 0, \qquad (81)$$

$$\cos \theta + \frac{\sqrt{q^2 - a^2}}{a} \sin \theta = 0, \text{ (Equation } (71)^1\text{)}, \qquad (82)$$

$$\cos \theta - \frac{\sqrt{q^2 - a^2}}{a} \sin \theta = 0, \text{ (Equation } (72)^1\text{)}, \qquad (83)$$

We will now show that Equations (82) and (83) imply that the characteristics lie along the x-axis, or dy = 0. Rewriting Equation (82), and multiplying through by $\sin \alpha$,

$$\cos \theta \sin \alpha + \cos \alpha \sin \theta = 0$$

or

$$\sin (\theta + \alpha) = 0 \qquad (82)^1$$

Doing the same for Equation (83),

$$\cos \theta \sin \alpha - \cos \alpha \sin \theta = 0$$

or

$$\sin (\theta - \alpha) = 0 \qquad (83)^1$$

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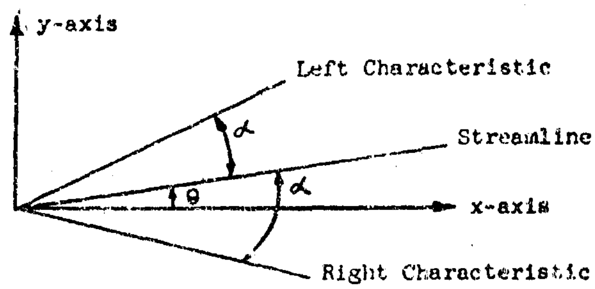
The unique solution to Equation (82)[1] is

$$\theta = -\alpha, \quad \text{(Left line)}, \tag{84}$$

and the unique solution to Equation (83)[1] is

$$\theta = \alpha . \quad \text{(Right line)}. \tag{85}$$

To interpret equations (84) and (85), let us refer to the following diagram:



If $\theta = -\alpha$ for the left line, then obviously the left line lies along the x-axis.

If $\theta = \alpha$ for the right line, then the right line lies along the x-axis.

We might comment here that the notation "left characteristic line", or "left line" denotes the direction that a ball would roll if placed along the line. Likewise, a ball would roll to the right if placed on a "right line".

As a practical matter, we can avoid the condition $dx = 0$ by requiring $M>1$ in the calculations. This point will be clarified later. The condition $dy = 0$ usually does not occur, it if does, there is a special provision which can be carried out in the calculations.

The condition $y = C$ is avoided in the calculations as will be seen later.

V. PLUG NOZZLE PROGRAM

A. General Relations

In the plug nozzle program, we deal with flow which is isentropic and isenthalpic. This implies that in the flow field there are no vorticity, heat transfer, or viscosity effects, and no shocks. The characteristics which apply to this type of flow are expressed by Equations (50) and (65):

$$dy - \tan(\theta + \alpha)\, dx = 0 \qquad \text{(Left line)} \tag{62}$$

$$dy - \tan(\theta - \alpha)\, dx = 0 \qquad \text{(Right line)} \tag{63}$$

The compatibility relations which must hold along the characteristics are

$$d\theta - \frac{\sqrt{M^2 - 1}\; dM}{M\left(1 + \frac{\gamma - 1}{2} M^2\right)} + \epsilon\left[\frac{\tan\theta}{\sqrt{M^2 - 1} - \tan\theta}\right]\frac{dx}{y} = 0 \quad \text{(Along left line)} \quad (71)^1$$

$$d\theta + \frac{\sqrt{M^2 - 1}\; dM}{M\left(1 + \frac{\gamma - 1}{2} M^2\right)} - \epsilon\left[\frac{\tan\theta}{\sqrt{M^2 - 1} + \tan\theta}\right]\frac{dx}{y} = 0 \quad \text{(Along right line)} \quad (72)^1$$

We now make a slight modification to Equations $(71)^1$ and $(72)^1$. In the existing program, we add a small correction factor to the coefficient of dM in Equations $(71)^1$ and $(72)^1$, so that

$$\mp \frac{\sqrt{M^2 - 1}}{M\left(1 + \frac{\gamma - 1}{2} M^2\right)}$$

is replaced by

$$\mp \frac{\sqrt{M^2 - 1}}{M\left(1 + \frac{\gamma - 1}{2} M^2\right)} \left[1 + M\alpha(T, M)\right], \quad \alpha(T, M) = \frac{1}{2\gamma}\, \gamma'(T)\, \frac{\partial T}{\partial M}$$

This comes about by assuming that C_p and C_v are not constant, but $\gamma = \gamma(T)$. We still use the perfect gas law as our equation of state, however.

Derivation of the new expression for the coefficient of dM.

Recall that the dM term was originally $\mp\sqrt{M^2 - 1}\,\frac{dq}{q}$ (Equations (71) and (72)).

By the equation of energy in differential form,

$$C_p dT + q\,dq = 0 \qquad \text{(Along a streamline).} \tag{88}$$

This can be rewritten as

$$\left(\frac{R\,\gamma}{\gamma - 1}\right) \frac{dT}{q^2} = -\frac{dq}{q} \tag*{(88)^1}$$

We have

$$a^2 = \gamma RT$$

$$M^2 = \frac{q^2}{\gamma RT}$$

$$T = \frac{q^2}{\gamma RM^2}$$

$$dT = \frac{2\,q\,dq}{\gamma RM^2} - \frac{2q^2 dM}{\gamma RM^3} - \frac{q^2\,d\gamma}{\gamma^2 RM^2}$$

Equation (88)[1] becomes

$$-\frac{dq}{q} = \frac{1}{q^2}\left(\frac{R\,\gamma}{\gamma - 1}\right)\left\{\frac{2q\,dq}{\gamma RM^2} - \frac{2q^2\,dM}{\gamma\,RM^3} - \frac{q^2\,d\gamma}{\gamma^2\,RM^2}\right\} \tag*{(88)^{11}}$$

From Equation (88)[11], we finally obtain

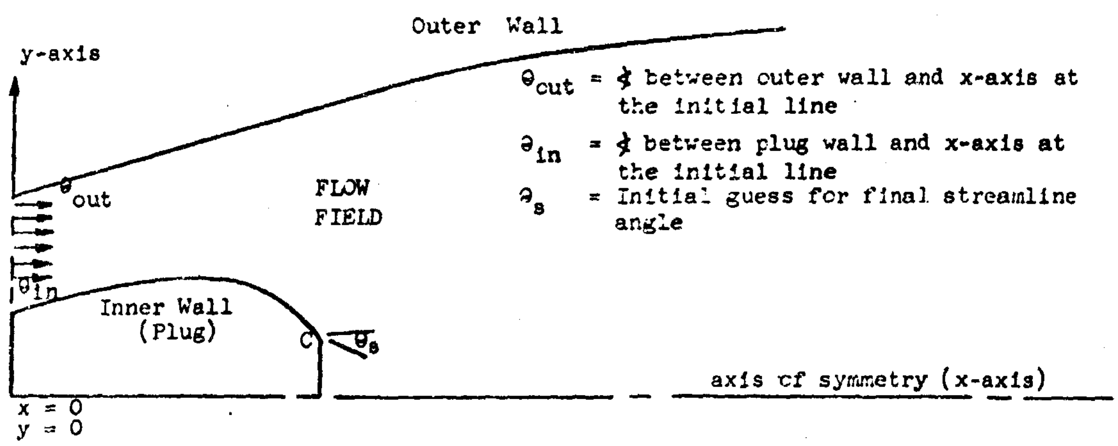
$$\frac{dq}{q} = \frac{dM}{\left(1 + \frac{\gamma - 1}{2} M^2\right)}\left[\frac{1}{M} + \frac{1}{2\gamma}\frac{d\gamma}{dT}\frac{\partial T}{\partial M}\right] \tag{89}$$

In line with this assumption that $C_v = C_v(T)$ and $C_p = C_p(T)$, Equation (51) loses significance, since it is derived assuming C_v and C_p are constant. An expression for $\partial T/\partial M$ which can be derived from the energy equation is

$$\frac{\partial T}{\partial M} = - \frac{TM}{\frac{M^2}{2}\left(1 + \frac{T}{\gamma}\, \gamma'(T)\right) + \frac{1}{\gamma - 1}} \tag{90}$$

However, we still use Equation (51) to obtain initial line temperatures. Also, we neglect certain other effects which should modify our initial assumptions.

The general type of nozzle which will be used is illustrated in the following diagram, which represents a cross section. In the axisymmetrical case, this cross section is any plane which has included the x-axis, or the axis of symmetry. In the two-dimensional case, the cross section is any plane parallel to the x-y plane. The lower half of the nozzle is not shown, since it is a mirror image of the upper half.



The variables x, y, M, and θ are prescribed at discrete points on the initial line as part of the starting, or input data. The initial line, which need not be vertical, is generally assumed to be in the throat, or point of minimum cross-sectional area, and so M along the initial line is usually assigned a constant value just larger than one. A table of (x, y) values describing the outer wall and the plug wall, also the angles θ_{cut} and θ_{in} are also needed.

It is assumed that a continuously differential curve can be passed through the outer and inner wall points, starting at $x = 0$. (From the diagram, it is evident that this is not possible at point c.) We will require also that the streamlines of flow are parallel to the inner and outer wall at points where the derivatives are continuous. At point c, we must handle the situation differently. The methods of procedure for insuring parallel flow at the boundaries and for handling conditions at point c are now described.

B. Physical Considerations at Outer and Inner Wall Boundaries

1. Reflected Mach Waves

We can provide a physical basis for our boundary conditions by showing that the characteristics line segments are also the Mach wave lines. Equation (46) describes the angle of inclination of a Mach wave to the flow direction as a unique function of M.

$$\sin \beta = \frac{1}{M} \tag{46}$$

For the characteristic line, we have

$$\sin \alpha = \frac{a}{q} = \frac{1}{M}, \tag{88}$$

where α = Angle between the characteristic line and the streamline.

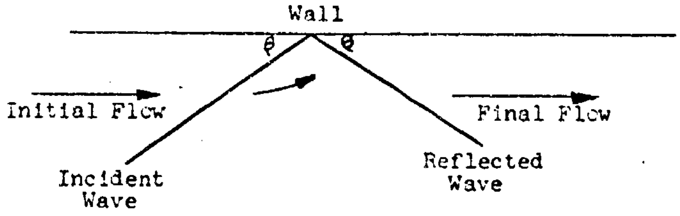
Obviously,

$$\alpha = \beta \tag{89}$$

Note that we are dealing with stationary lines here.

Therefore, the characteristic lines are also Mach lines, i.e., the loci of Mach waves.

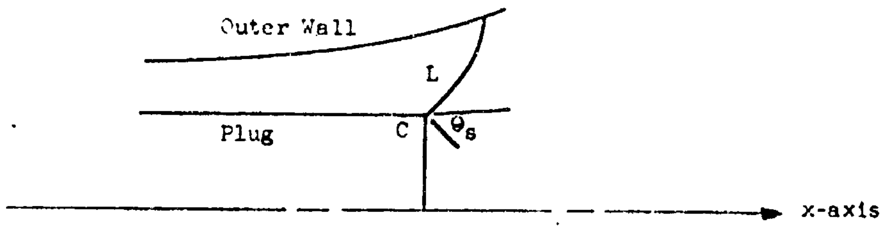
From physical considerations, when a Mach wave is incident on a smooth boundary oblique to the direction of flow, there must be a reflected wave which renders the direction of flow again parallel to the boundary, as shown below.



In the plug nozzle program, the reflected wave will be manifest as a characteristic line originating at the boundary point where the incident characteristic line intersects the boundary. The point of intersection is determined by the calculations.

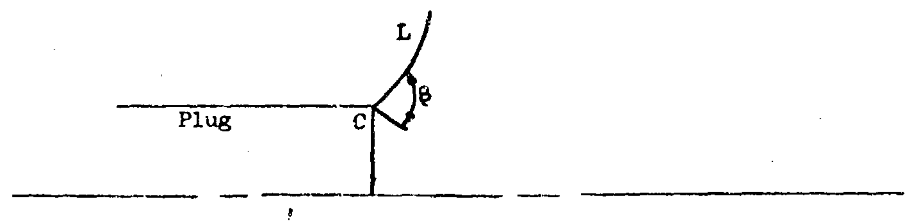
2. Prandtl-Meyer Expansion

In the plug nozzle program, characteristic lines are generated starting at the initial line and moving to the right. Let us consider what happens when we have reached point c on the plug.



Line L is the first left line to reach point c.

By first considering that there is some curvature at point c and then taking the limiting case of a corner at point c, it becomes evident that there must be a fan of characteristics originating at point c and continuing until the final streamline angle θ_s is reached. Pictorially,



The expansion angle is denoted by β.

Now consider what happens at the right characteristics in the fan as we approach point c. Evidently dx approaches zero. Equation (72)[1] then becomes

$$d\theta = \frac{\sqrt{M^2 - 1}\, dM}{M\,(1 + \frac{\gamma - 1}{2} M^2)} \left[1 + M\alpha(T.\ M)\right] \tag{86}$$

and the right characteristics converge to the point c.

Equation (86) gives us a relation between streamline angle and Mach number in the vicinity of point c.

We can obtain a relation between $\theta + \alpha$ and M also.

Since

$$\alpha = \sin^{-1} \frac{1}{M}$$

then

$$d\alpha = - \frac{dM}{M\sqrt{M^2 - 1}}$$

and

$$d(\theta + \alpha) = d\theta + d\alpha = - \left[\frac{\sqrt{M^2 - 1}\, dM}{M\,(1 + \frac{\gamma - 1}{2} M^2)} (1 + M\alpha(T,M) + \frac{dM}{M\sqrt{M^2 - 1}}\right] \tag{87}$$

From Equation (87); we can calculate M for the characteristic left line originating at c at angle $\theta + \alpha$ by numerical integration of Equation (87). In the program, we specify

$$d(\theta + \alpha) = \Lambda(\theta + \alpha) = \Lambda \beta$$

to be a small negative constant angle.

Note that since θ becomes negative, the "ball rolling" definition of left line becomes invalid! Suffice it to say that the left lines which are dealt with here retain the property that the angle α is added to the angle θ to obtain the angle of inclination.

C. General Description of the Calculative Procedure

By referring to the program abstract, we can determine what input data are required for a calculation.

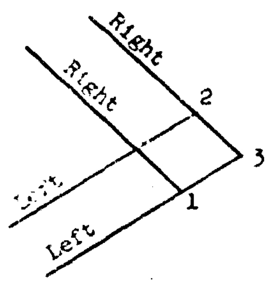
1. Curve Fitting

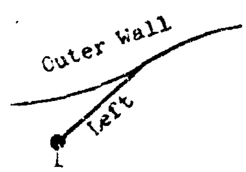
The first procedure is to determine continuously differentiable functions which pass through the specified outer and inner wall points. The slope of, say, the outer wall function at both ends of the nozzle will be that specified by the input data.

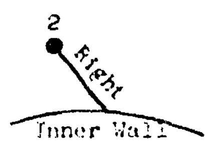
This curve fit is obtained in the following way. We begin at the first outer wall point, where the slope $\theta_{o,1}$ is specified. (The inner wall calculation is carried out in the same way.) Pass a parabola through the first, second, and third specified outer wall points, and determine the slope $\theta_{o,2}$ of this parabola at the second point. Then determine the coefficients of a cubic which passes through the second and first point, and which has slope $\theta_{o,1}$ at the first point and slope $\theta_{o,2}$ at the second point. This cubic is the analytic expression for the outer wall between the first two input points. Obtain a cubic for the second and third points in the way just described, by passing a parabola between the second, third, and fourth points, etc.

2. Characteristics

In generating the characteristic net, we are primarily concerned with finding points of intersection of left lines with right lines, or with points of intersection of left lines with the outer wall, or right lines with the inner wall.







We will now describe in detail the method for calculating T_3, X_3, Y_3, M_3, and θ_3 at point 3 in the diagram above, where we know T_1, X_1, Y_1, θ_1, M_1 and T_2, X_2, Y_2, M_2, θ_2.

Express

$$dx^L = x_3 - x_1, \quad dy^L = y_3 - y_1,$$

$$dx^R = x_3 - x_2, \quad dy^R = y_3 - y_2$$

Rewrite Equations (62), (63), $(71)^1$, and $(72)^1$:

$$dy^L - \lambda^L dx^L = 0, \quad \text{or } y_3 - y_1 - \lambda^L (x_3 - x_1) = 0 \qquad (62)$$

$$dy^R - \lambda^R dx^R = 0, \quad \text{or } y_3 - y_2 - \lambda^R (x_3 - x_2) = 0 \qquad (63)$$

$$\theta_3 - \theta_1 - A^L (M_3 - M_1) + B^L (x_3 - x_1) = 0 \qquad (71)^1$$

$$\theta_3 - \theta_2 + A^R (M_3 - M_2) - B^R (x_3 - x_2) = 0 \qquad (72)^1$$

Where

$$A^L = \frac{\sqrt{M^2 - 1}}{M\left(1 + \frac{\gamma - 1}{2} M^2\right)} \left[1 + M\alpha(T, M)\right]^*$$

$$A^R = \frac{\sqrt{M^2 - 1}}{M\left(1 + \frac{\gamma - 1}{2} M^2\right)} \left[1 + M\alpha(T, M)\right]^*$$

$$B^L = \frac{\epsilon}{y}\left[\frac{\tan\theta}{\sqrt{M^2-1} - \tan\theta}\right]^*$$

$$B^R = \frac{\epsilon}{y}\left[\frac{\tan\theta}{\sqrt{M^2-1} + \tan\theta}\right]^*$$

$$\lambda^L = \tan(\theta + \alpha)^*$$

$$\lambda^R = \tan(\theta - \alpha)^*$$

We also have equation (51) which gives $T = T(M)$.

The star indicates that the starred quantities A^L, A^R, B^L and B^R are to be evaluated at points on the lines connecting X_3 with X_2 and X_1. That is, A^L and B^L are to be evaluated at

$$\theta = \frac{\theta_1 + \theta_3}{2}$$

$$M = \frac{M_1 + M_3}{2}$$

and

$$T = \frac{T_1 + T_3}{2}$$

rather than at, say θ_1, M, and T_1.

This more accurate procedure is called the mean value lattice point method.

Since this implies that Equations (62), (63), $(71)^1$ and $(72)^1$ are nonlinear, we cannot solve directly for X_3 , Y_3 , M_3 , and θ_3, but must use an iterative procedure.

To begin the iteration, evaluate λ^L, A^L, and B^L at M_1, θ_1, and T_1, and λ^R, A^R, and B^R at M_2, θ_2, and T_2. Then solve Equations (62), (63), $(71)^1$ and $(72)^1$ for the initial iterates

$$ {}^i x^3, \quad {}^i y^3, \quad {}^i M^3, \quad \text{and} \quad {}^i \theta^3 \quad \text{where } i = 1. $$

Then evaluate ${}^i T^3$ from Equation (90) using $\Delta M = {}^i M^3 - M_1$ and $\Delta M = {}^i M^3 - M_2$, and expressing

$$ {}^i T^3 = \frac{T_1 + T_2 + {}^i \Delta T^3}{2} $$

where

${}^i \Delta T^3$ = average of the two differences obtained from ${}^i M^3 - M_1$ and ${}^i M^3 - M_2$

Since Equation (51) says

$$ T = \frac{T_T}{\left(1 + \frac{\gamma - 1}{2} M^2\right)} $$

calculating T involves an iteration also.

Now obtain ${}^{i+1} x^3$ and ${}^{i+1} y^3$ from $\quad {}^i \bar{\lambda}^L = 1/2 \left[\tan \left({}^i \theta^3 + {}^i \alpha^3 \right) + \tan (\theta_1 + \alpha_1) \right]$

and

$$ {}^i \bar{\lambda}^R = 1/2 \left[\tan \left({}^i \theta^3 - {}^i \alpha^3 \right) + \tan (\theta_2 - \alpha_2) \right] $$

Obtain

$$ {}^{i+1} M^3, \quad {}^{i+1} \theta^3, \quad {}^{i+1} T^3 $$

from

$$ {}^i \bar{A}^L = 1/2 \left[A^L \left({}^i M^3, {}^i \theta^3, {}^i T^3, {}^i (\gamma' (T)^3 \right) + A^L \left(M_1, \theta_1, T_1, \gamma'(T_1) \right) \right] $$

and so on.

$$B^L = \frac{\epsilon}{y}\left[\frac{\tan\theta}{\sqrt{M^2-1} - \tan\theta}\right]^*$$

$$B^R = \frac{\epsilon}{y}\left[\frac{\tan\theta}{\sqrt{M^2-1} + \tan\theta}\right]^*$$

$$\lambda^L = \tan(\theta + \alpha)^*$$

$$\lambda^R = \tan(\theta - \alpha)^*$$

We also have equation (51) which gives $T = T(M)$.

The star indicates that the starred quantities A^L, A^R, B^L and B^R are to be evaluated at points on the lines connecting X_3 with X_2 and X_1. That is, A^L and B^L are to be evaluated at

$$\theta = \frac{\theta_1 + \theta_3}{2}$$

$$M = \frac{M_1 + M_3}{2}$$

and

$$T = \frac{T_1 + T_3}{2}$$

rather than at, say θ_1, M, and T_1.

This more accurate procedure is called the mean value lattice point method.

Since this implies that Equations (62), (63), $(71)^1$ and $(72)^1$ are nonlinear, we cannot solve directly for X_3, Y_3, M_3, and θ_3, but must use an iterative procedure.

Continue this iteration until

$$\left| {}^{i+1}x_3 - {}^{i}x_3 \right| < \epsilon_x$$

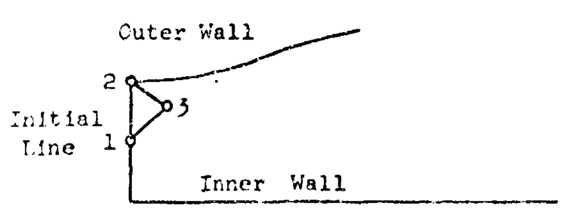
$$\left| {}^{i+1}y_3 - {}^{i}y_3 \right| < \epsilon_y$$

$$\left| {}^{i+1}M_3 - {}^{i}M_3 \right| < \epsilon_M$$

$$\left| {}^{i+1}\theta_3 - {}^{i}\theta_3 \right| < \epsilon_\theta$$

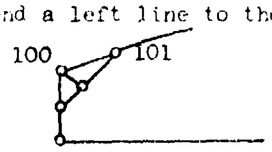
For the case of outer wall or inner wall intersection points, we go through a modification of this type of iteration. However, an outer wall intersection point involves only a left line, i.e., just one set of points T, X, Y, M, θ which are known. An inner wall intersection point involves only a right line. When the iteration for the intersection point (x,y) is complete, tan θ will be dy/dx of the particular cubic that represents the boundary at the point (x, y).

Generally this is the way the characteristic net is begun. Assume that the initial line is neither a left line or a right line. Assume, say, that three points are given as input data to describe the initial line.



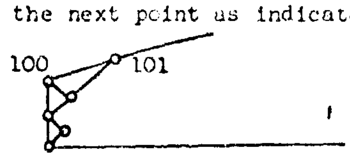
Then we can calculate T_3, X_3, Y_3, M_3, and θ_3 from prescribed conditions at input points 1 and 2. (See above diagram).

We then extend a left line to the outer wall.

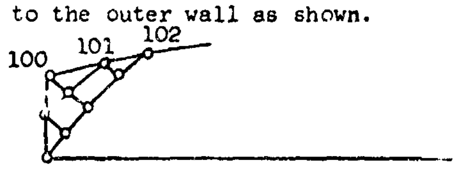


The complete line is labeled the "101" line.

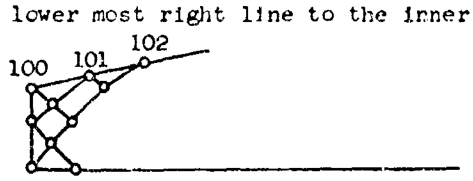
Now calculate the next point as indicated below.



Continue this to the outer wall as shown.

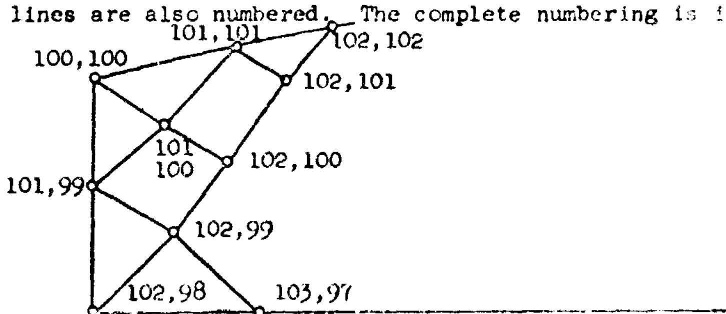


Now extend the lower most right line to the inner wall.



Now start to work line 103 to the outer wall.

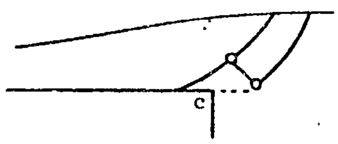
The right lines are also numbered. The complete numbering is indicated here.



The numbering here is of course arbitrary; this was chosen as the system at the time the program was written.

It should be noted that the intersection points 103, 102, or 103, 97 do not as a rule intersect the boundaries at input data points. Also, we cannot trace a streamline by assuming that, say, a streamline can be drawn from point 101, 99 to point 102, 100. We must obtain streamlines by another consideration. All we know in this connection is the streamline angle θ at the two points. Note that the net refinement depends upon the number of specified initial line points.

This process is continued until we have generated a left line past the inner wall cutoff point c. (To obtain this line we must use an imaginary extension of the inner wall boundary past c.)



We then interpolate between left lines n and n+1 to obtain the first special left line which begins at point c. Starting with this line, we develop a fan of left lines about point c, evaluating M on each line using Equation (87).

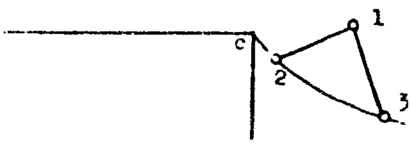
$$d(\theta + \alpha) = -\left[\frac{\sqrt{M^2 - 1}}{M\left(1 + \frac{\gamma - 1}{2} M^2\right)}\left(1 + M\alpha(T, M) + \frac{1}{M\sqrt{M^2 - 1}}\right)\right] dM \qquad (87)$$

where $d(\theta + \alpha) = \Delta(\theta + \alpha)$ is a specified constant.

This expansion is continued until θ corresponds to θ_s, the specified streamline angle at point c. For this correspondence to occur, we must subdivide $\Delta(\theta + \alpha)$ near the angle θ_s.

If the initial guess for θ_s is good, we can trace a streamline starting at point c which is asymptotic to the x-axis.

If it is not good, we can estimate a new θ_s, and try again.



Along this streamline, M_s and T_s are constant, therefore, we need only comput $(X, Y, \theta)_s$ at discrete points, say point 3 above.

In calculating (X, Y, Θ) at point 3, we use the equation for the streamline

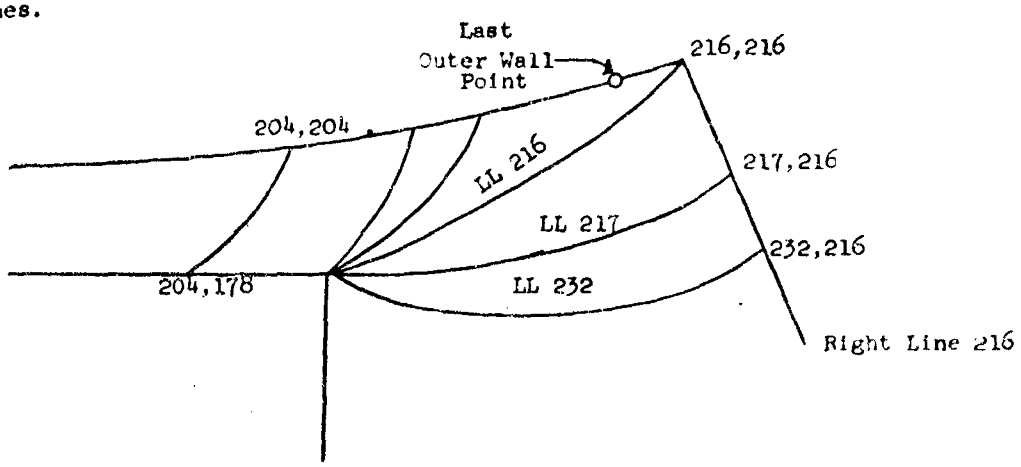
$$dy = \tan \Theta \, dx$$

or

$$(y_3 - y_2) = \tan \left(\frac{\Theta_2 + \Theta_3}{2}\right) (x_3 - x_2)$$

and Equations (63) and (72)[1] for right line < 1, 3 >

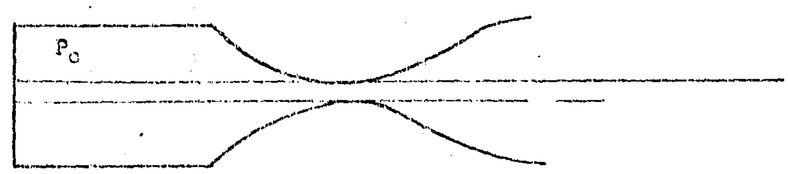
Eventually, we will generate a left line which will extend past the last outer wall data point. When this happens, we extend the outer wall to intersect the line, and then proceed as indicated below, we have omitted the right lines.



3. Nozzle Efficiency

In order to calculate nozzle efficiency, we must first develop a system of equations which describe an "ideal" nozzle.

To do this, we develop briefly a one-dimensional theory which ideally might apply to a nozzle of slowly varying cross-sectional area. The discussion follows Liepmann and Puckett, Aerodynamics of a Compressible Fluid. In this model, the variables M, P, ρ, and T are all constant at a cross section perpendicular to the axis of symmetry, or x-axis. Let us refer back to Equations ([illegible]7), ([illegible]7), (48), (49), and (50), and imagine that we have added an extension to the throat or point where $dA/A = 0$. We have of course no plug or inner boundary to consider here. We now have a convergent-divergent nozzle.



Since the mass flow is the same at all cross sections, we still have

$$A = m \left[\frac{2\gamma}{\gamma - 1} P_o \rho_o \left(\frac{P}{P_o}\right)^{\frac{2}{\gamma}} \left[1 - \left(\frac{P}{P_o}\right)^{\frac{\gamma - 1}{\gamma}} \right] \right] \qquad (49)$$

Let us take $m = \rho^* u^* A^*$ = Mass flow at any cross section perpendicular to the x-axis where the * refers to conditions at the throat.

Equation (49) relates cross-sectional area to pressure and γ.

We have also

$$\frac{\rho_*}{\rho} = \left(\frac{2}{\gamma + 1}\right)^{\frac{1}{\gamma - 1}} \qquad (91)$$

$$a^* = \sqrt{\gamma RT^*} = \sqrt{\gamma RT_o \frac{2}{\gamma + 1}} \qquad (92)$$

$$\left(\frac{A}{A^*}\right)^2 = \frac{\frac{\gamma - 1}{2} \left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma + 1}{\gamma - 1}}}{\left(\frac{P}{P_o}\right)^{\frac{2}{\gamma}} \left[1 - \left(\frac{P}{P_o}\right)^{\frac{\gamma - 1}{\gamma}} \right]} \qquad (93)$$

$$\frac{P}{P_o} = \left(1 + M^2 \frac{\gamma - 1}{2}\right)^{\frac{\gamma}{\gamma - 1}} \qquad (94)$$

From Equations (93) and (94)

$$\left(\frac{A}{A^*}\right)^2 = \frac{1}{M^2}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{\frac{\gamma+1}{\gamma-1}} \tag{95}$$

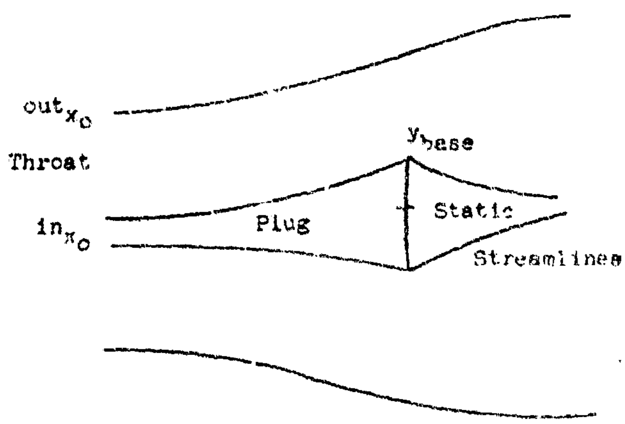
The stream thrust at any point of the nozzle is

$$F = pA + \rho A u^2 = pA\,(M^2\,\gamma + 1)$$

Let us say that we want to evaluate F at a particular area A. If we know A*, we can calculate M from Equation (93), once having determined P from Equation (94). Then we have

$$\frac{F}{p_o} = \frac{M^2\,\gamma + 1}{\left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{\gamma}{\gamma-1}}}\,\pi y_e^2$$

where y_e = the value y, the perpendicular distance from the exit x_e to the wall.



Now we develop the thrust developed by the plug nozzle. This will be done at discrete points x_e corresponding to the intersections of left lines with the outer wall. The thrust will be expressed as the sum of four terms

$$\frac{1}{P_o} = \left[F_1 + F_2 + F_{3e} + F_4 \right] = \frac{1}{P_o} F_{PNe}$$

We assume that the flow is isentropic, so the stagnation pressure P_o is the same throughout the field of flow. Note that the equations apply only to the axisymmetric case.

The Expression for F_1:

This term is the mass flow component parallel to the x-axis. It is

$$\frac{F_1}{P_o} = \frac{(M^2 \gamma + 1)}{\left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{\gamma}{\gamma - 1}}} \pi \left({}^{out}y_o^{\,2} - {}^{in}y_o^{\,2} \right)$$

for flow parallel to the x-axis.

Where M is the Mach number at the initial point X_o (M is assumed to be constant on this cross section).

The Expression for F_2:

This term is the force on the base of the plug. We can calculate P in the static region from Equation (94), since M is constant on the final streamline. We obtain

$$\frac{F_2}{P_o} = \frac{\pi y^2_{base}}{\left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{\gamma}{\gamma - 1}}}$$

The Expression for F_{3e}:

This term expresses the force on the outer wall which adds to or subtracts from the thrust.

$$\frac{F_{3e}}{P_o} = \pi \sum_{y=0}^{y=y_e} (y_{j+1}^2 - y_j^2) \frac{P_j + P_{j+1}}{2 P_o} \simeq \frac{1}{P_o} \int_{y=0}^{y=y_e} p \, d A$$

where y_j are boundary points (outer wall) and P_j are pressures at intersections of left lines with the outer wall. P_j can be computed from M_j by Equation (94).

The Expression for F_4:

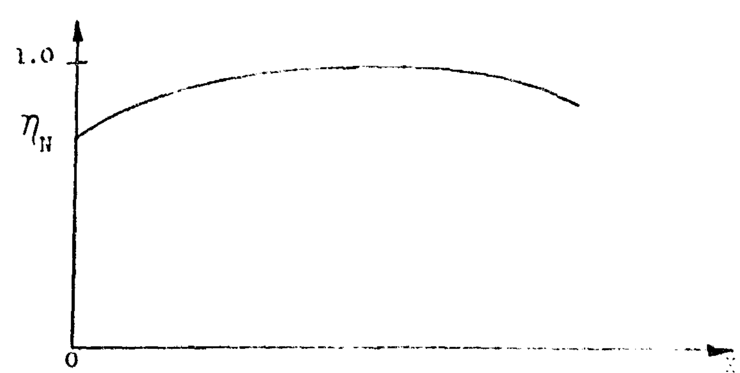
This term is similar to F_3 except it expresses the force component on the inner wall.

$$\frac{F_4}{P_o} = \pi \sum_{y=0}^{y=y_{base}} (y_i^2 - y_{i+1}^2) \frac{P_i + P_{i+1}}{2 P_o}$$

Here P_i are pressures at intersections of right lines with the inner wall. Define the efficiency to be

$$\eta_N = \frac{F_{PNe}}{P_o} \frac{P_o}{F}$$

Since we have η_N at discrete points, we could "break off" the outer wall at one of these discrete points, and have the efficiency for the chopped-off nozzle. However, from physical considerations, we must not chop off the outer wall to the left of the right line which hits the inner wall boundary.



VI. REFERENCES

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APPENDIX A

SUMMARY OF NOMENCLATURE

Symbol	Description
α	Mach angle
a	Speed of sound
a*	Critical speed
C_v	Specific heat at constant volume
C_p	Specific heat at constant pressure
E	Internal energy per unit mass
ϵ	= 1 if axisymmetrical geometry, = 0 if two-dimensional
γ	Ratio of C_p to C_v
h or i	Enthalpy per unit mass
k	Boltzmann constant
m	Mass in Section III-A, mass flow otherwise
M	Mach number
P	Pressure
ρ	Density, mass per unit volume
Q	Heat energy per unit volume
$\bar{q}$	Flow velocity = (u, v)
q	Flow speed = $\sqrt{u^2 + v^2}$
R	Gas constant
S	Entropy per unit mass
T	Temperature
t	Time
θ	Flow angle
μ	Viscosity
u	Component of velocity in the x-direction
v	Component of velocity in the y-direction
V	Volume per unit mass
W	Work
x, y	Coordinates of a point in the Euclidean plane