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ANALYTIC SOLUTIONS OF THE  
ARROW, HARRIS, AND MARSCHAK  
DYNAMIC INVENTORY MODEL

WALTER S. HAMILTON

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ANALYTIC SOLUTIONS OF THE ARROW, HARRIS,  
AND MARSCHAK DYNAMIC INVENTORY MODEL

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Walter S. Hamilton

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Walter S. Hamilton

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Lieutenant Commander, United States Navy

Submitted in partial fulfillment  
of the requirements for the

OPERATIONS ANALYSIS CURRICULUM

United States Naval Postgraduate School  
Monterey, California

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## ABSTRACT

A general survey is undertaken of the techniques used in determining loss functions to be minimized for the purpose of finding optimal  $(s,S)$  values in the Arrow, Harris, and Marschak dynamic inventory model. A brief discussion of the cost functions and the aspects of demand, backlogging, lag time, and discount rate are presented in the interest of better understanding of their use in the real world. Analytic solutions to the model have been derived by two different, but related, methods involving stationary costs. The first method involves direct use of the cost functions in a Markov process to arrive at an integral equation of renewal which is developed into a stationary loss function by an Abelian limit theorem. The second method involves the determination of the stationary distribution of stock level through renewal theory ; the loss function for a representative period is brought into the computations independently in determining the expected loss with respect to the stationary distribution of stock level. The two methods produce the same result.

A digital computer simulation of the Arrow, Harris, and Marschak dynamic model has been outlined by M. Geisler [2]. Some highlights of Geisler's work have been reviewed here.

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## PREFACE

Inventory control organizations of the Navy have contracted for numerous studies into the decision problems involved in dynamic inventory for the multi-echelon systems as found in the Navy. Also, a contract for the construction of a computerized dynamic model was first let in early 1955 to the Stanford Research Institute. This work continues today with improvements being added as break-throughs develop in the study of the nature of complex Navy inventory operations. Industry has no parallel in complexity and magnitude. Models in use prior to the early part of 1955 were more applicable to reasonable-size firms. Consequently, the boundaries of development of inventory theory are now being pushed by the military organization in keeping with the need for greater efficiency and economy. The magnitude of the difficulty in finding models that closely approximate the military inventory management problem are appreciated by relatively few. The optimization of buying, allocating, and redistributing based on complex statistical fluctuations in demand, order arrival, and military loss due to shortage in a multi-echelon structure of centers and depots is an almost insurmountable obstacle to solution for a large part of the total carried inventory.

The exposition presented in this paper has been kept relatively simple. Its purpose is to illuminate the basic

concepts of the most simple dynamic inventory problem as first presented by Arrow, Harris, and Marschak and, thereby, gain sufficient insight for proceeding into more complex dynamic models.

Source material for this paper has consisted primarily of references [1],[2],[5], and notes taken during a course of instruction in logistics given by Professor Thomas E. Oberbeck at the United States Naval Postgraduate School. I wish to express my gratitude to Professor Oberbeck for his support and encouragement and to my wife, Helen, for her loyal devotion to the typing and clerical tasks.

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## TABLE OF SYMBOLS

(Listed in the order of their use in the text)

|                |  |
|----------------|--|
| $s$            | reorder point of the stock level   |
| $S$            | upper limit of the stock level   |
| $y_t$          | stock level at the beginning of period $t$   |
| $c$            | ordering cost per unit of item ordered   |
| $z$            | reorder quantity in general  |
| $l(y_n)$       | loss or cost function for the $n^{\text{th}}$ period   |
| $l_\infty$     | stationary or equilibrium cost function  |
| $l(y)$         | same as $l(y_n)$ without regard to time  |
| $g(y)$         | stationary density function of stock level   |
| $f(\xi)$       | density function of demand   |
| $x_t$          | stock level at beginning of period $t$ after reordering  |
| $z_t$          | reorder quantity at beginning of period $t$  |
| $p$            | penalty cost per unit of item not in stock   |
| $h$            | holding cost per unit of item in storage   |
| $F(\xi)$       | cumulative distribution function of demand   |
| $L(y_0)$       | minimum discounted expected loss over an infinite time period with initial stock level, $y_0$                                  |
| $\alpha$       | discount rate which discounts future losses to present value   |
| $F_n(\xi)$     | $n$ -fold convolution for distribution of demand   |
| $v(y)$         | a random variable having non-negative integer values which specify the number of periods less one in a reorder cycle.          |
| $\gamma_{s-s}$ | random variable of the excess distribution or the amount by which stock level goes below reorder point $s$ prior to reordering |

$\phi_{v(y)}$  a random variable representing the sums of demand just prior to  $\phi_{v(y)+1}$  which causes a reorder.

$\Pr[\gamma_{s-s} > u]$  probability that excess of quantity of stock below the reorder point  $s$  is greater than some value  $u$ .

## 1. Introduction

The history and general characteristics of the inventory problem under various operating conditions have been adequately discussed in [1]. The  $(s,S)$  policy<sup>1</sup> is widely used in practice and has been subject to extensive analytical study. This ordering rule is simple and has the following advantages: (i) considerable intuitive appeal, (ii) large amounts of data available, (iii) a certain ease of probabilistic computation and analysis and (iv) existence of analytic proofs as to its optimality under a wide range of conditions. Note that there are two approaches to the inventory problem. One is to choose a simple policy (e.g.  $(s,S)$  type) and proceed with the analysis of specific  $(s,S)$  policies, and the other approach is solve the decision problem through determination of the characteristic of the optimal ordering policy which minimizes or maximizes an appropriate objective function. Finding the characteristic of the optimal policy is very difficult. Thus the usual approach is to choose the practical and sound  $(s,S)$  policy and then proceed to find within this policy class a specific policy which minimizes the expected long run costs (i.e., specific values of  $s$  and  $S$ ). Then the study of the inventory model becomes a study of the associated stochastic process which is generally a Markov process where the states of the system (current stock level) depend on the past only through the present.

<sup>1</sup>Often called the two-bin policy.

The Arrow, Harris, and Marshak dynamic inventory model (hereafter called the AHM Model) is the generic name often given in the literature to identify the  $(s,S)$  policy model where ordering takes place at the beginning of the period when the stock level is less than the level  $s$  (i.e.  $y_t < s$ ) and demand is identically and independently distributed in each time period. There may or may not be a specified delivery lag. In this thesis, deliveries will be assumed to arrive immediately after being ordered. It is assumed that the reader is familiar with the AHM dynamic model of reference [5]. Some simple restrictions on the cost functions (having to do with the convexity of the expected values of the cost functions) according to Scarf [10] are the only restrictions required in order that the optimal policy for the AHM dynamic model be of the  $(s,S)$  type. Thus over a wide range of conditions (including any demand distribution functions), there can be confidence in using  $(s,S)$  policies. Sections 2 and 3 are devoted to a resumé of the cost functions and other input factors in order to develop a better understanding of their meaning in analysis and their relationship to the real world.



## 2. Cost Functions.

The inventory problem has three cost functions which can have four forms (linear, convex, concave, or mixed).

### a. Ordering Cost.

Suppose  $z$  represents the amount of stock to be ordered and  $c$  the cost per unit ordered. A linear ordering cost is directly proportional to the amount ordered, i.e.  $c \cdot z$ . The case where  $c(z)$  is concave is a common situation in buying since this means that each additional item costs less. A special case of concavity exists when  $c(z)$  is composed of  $c \cdot z + K$  where  $K$  is the administrative costs of processing an order. The convex case is unlikely to occur since it means that each additional item costs more. The mixed case where pieces are made up of linear and concave portions is practical.

### b. Storage Cost.

Usually storage costs are proportional to the size of stock on hand, but the cost may increase more rapidly if the storehouse capacity is exceeded and additional space must be rented. Storage costs may include such things as stock maintenance, storage rental, stock obsolescence or spoilage, stock repair, etc.

### c. Penalty Cost.

These costs arise when demand exceeds supply

as opposed to storage costs which arise due to supply exceeding demand. The form of the penalty cost function varies widely since so many conditions exist. The simplest is the linear case. A realistic case for industry is the convex function since a small shortage has small consequences but a large shortage produces increasingly greater difficulties for the customers. In some military applications certain items directly affect the mission and a shortage of these items will cause failure of the mission; consequently the shortage function could be considered a positive constant when a shortage develops and zero otherwise. In reality, the military supply items vary in importance and end-use. This is why a military supply system is hardput to define the military essentiality of all items. So far, the military essentiality designation has been applied to the allowance lists of designated submarines; however there is a move afoot to expand this program to include all naval vessels. The obvious difficulty in this project lies in reviewing and updating the allowance item essentiality designation.

### 3. Other Factors.

Inventories are held for the ultimate purpose of satisfying demand, and money laid out for orders and items of inventory will not be returned for awhile. The alternative to having money tied up in inventory is to have it invested in securities, bonds, or similar instruments. Long-term government bonds are a good standard for comparison since they are both secure and yield a reasonable return of 4%. Lag time is the time between ordering and receiving, and it directly affects the amount of stock to be carried. Backlogging is the procedure whereby demands in excess of stock (unfilled orders) are kept on the books until they can be satisfied thereby giving meaning to negative inventory values.

#### a. Discount Rate.

A return of \$1.04 at the end of a year is equivalent to \$1.00 invested today in inventory. The discount rate is  $\alpha = 1/1.04$  which is an equivalent way of considering the comparative return. Thus a \$1.00 return in a year is equivalent to  $\alpha$  dollars invested in inventory today. Considering that the \$1.00 return on investment is re-invested each year or remains invested at 4%, then  $\alpha$ ,  $\alpha^2$ ,  $\alpha^3$ ,  $\dots$  represents the discount rate for \$1.00 after one year, two years, three years, etc. (i.e.  $\alpha^n$  is the present value of \$1.00 which has been tied-up in inventory during the  $n^{\text{th}}$  period).

For all practical purposes  $\alpha = 1$  when the time horizon is short.

b. Demand.

Demand is usually assumed to be independent of the decision makers control; however, this is not always the case in industry (e.g. demand is directly related to advertising). In military inventory it seems safe to assume that demand is independent of the decision maker. Demand may be regarded as deterministic in future time periods. However, it is more realistic to regard demand as probabilistic. Although the demand probability distribution may be known to change such as in seasonal fluctuations or long-term trends, the state of methodology in stochastic processes for inventory control requires the assumption that demand is identically distributed in the time periods. Because of this restriction, the deterministic case is sometimes a better approximation for demand when it is rapidly changing over time (i.e. the deterministic case fixes demand in the various time periods according to some previous knowledge of its behavior).

Another way of looking at the aforementioned probabilistic case is to allow the size of each demand to be a random variable. Consequently, this is a continuous time stochastic process which is more appropriate when orders can be placed at any moment of time. For

application of this concept, the assumption is made that demands are independent and identically distributed random variables.

When nothing is known of demand one may apply decision-making under uncertainty as developed by Wald [3] which involves both estimation and decision-making as successive observations of demand are taken, a complicated sequential decision problem. Y. Fukuda provides a qualitative approach to this solution [4].

c. Lag Time.

There are three cases of lag time to consider. In this thesis the time between order and delivery is assumed to be zero in order not to over-complicate the computational techniques to be displayed; however the extension of this model to the case of fixed lag time between order and delivery is presented by Karlin and Scarf [1] and has the same characteristics as the dynamic inventory model with no lag time when both models assume backlogging (see subsection 3d). The case where lag time is a random variable has been treated by Scarf [1] (see chapter 16).

It should be noted that when demand is deterministic, the consideration of fixed lag time is inconsequential since we could order sufficiently in advance to cover the known demands. In all cases the



lag time directly affects the amount of stock held (i.e. an increase in inventory to protect against shortages until deliveries arrive). There is one practical consideration in the dynamic model where lag time is a random variable for which no solution has been found through the dynamic programming formulation at the present time and that is the condition where orders arrive out of sequence. This is referred to as cross over. Remember that order quantity is a random variable.

#### d. Backlogging.

When demand for an item exceeds supply, then the policymaker may do one of several things: (i) make a premium order thereby taking a penalty in the amount of the premium costs, (ii) do nothing and allow the customer to go elsewhere for satisfaction thereby taking a penalty in customer goodwill, (iii) backlog the order awaiting arrival of orders to satisfy the unfilled demand thereby taking a penalty that may relate to both customer goodwill and/or increased ordering costs. In this case a negative inventory becomes meaningful.



#### 4. Methods of Solution Available.

At this juncture there exists a stochastic process based on the  $(s,S)$  type policy which may now be treated in one of two ways.

##### a. First Way.

The first way consists of viewing the transient behavior of the stochastic process through  $n$ -periods or an infinite number of periods. This essentially is the original approach used by Arrow, Harris, and Marschak [5], [9] in setting up the loss function for a general period  $t$ , then summing the expected losses over a given time horizon with respect to the random variable of stock level. The distribution of stock level is deduced through a functional relation with the random variable of demand. This method will be outlined in section 5 where it will be pushed to an approximate solution through use of an integral equation of renewal theory [1], [5].

##### b. Second Way.

The second method of treatment deals with the stationary phenomena of the process. If  $y_n$  represents the stock level at time  $n$ , then subject to mild stability requirements it can be shown [1] (refer to pages 234-237 and pages 292-297) that  $y_n$  possesses the so-called ergodic property (i.e. the distribution function for  $y_n$  converges to a limiting distribution function) when the optimal policy is of the  $(s,S)$  type. Thus the limiting or stationary distribution is a function of

the policy and the demand distribution, and it is independent of the initial stock level. The beauty of this method is that the cost functions have not entered into the problem up to this point. Using the loss function representation for a general period  $n$ , denoted  $\ell(y_n)$ , and dropping the dependence on  $n$ , the expected value of  $\ell(y)$  with respect to the stationary density yields the equilibrium average loss, denoted as follows:

$$\ell_{\infty} = \int \ell(y) g(y) dy$$

where  $g(y)$  denotes the stationary distribution for the stock level.

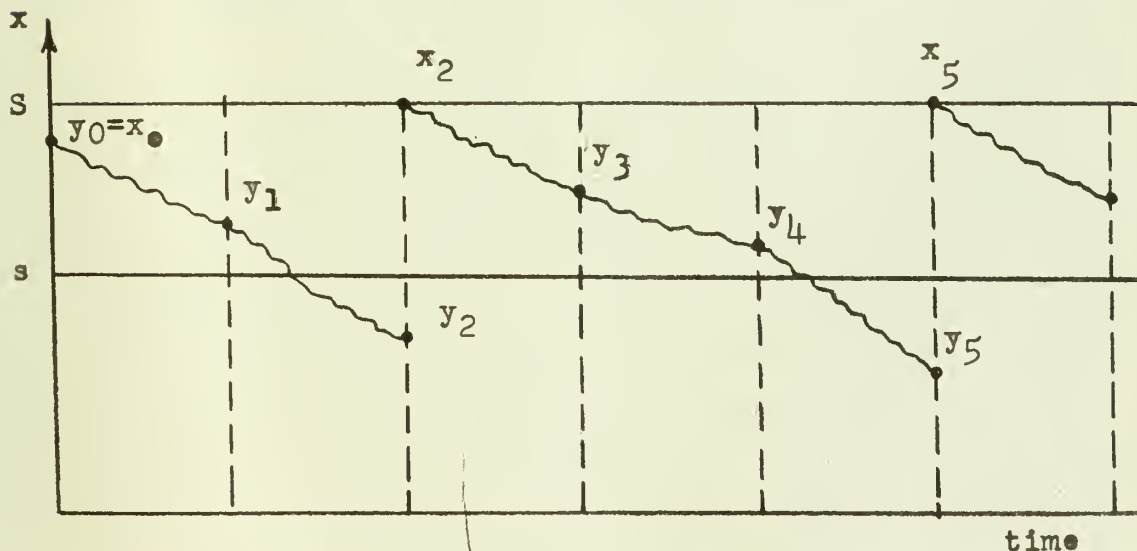
The stationary distributions discussed in subsection 2b can be found through the application of renewal theory [1] (refer to, chapter 15) or through the application of differential equations [1] (refer to, chapter 14) when the demand distribution function is a member of the Gamma family of distributions.

## 5. Original Method of Solution.

It is proposed that the original Arrow, Harris, and Marschak dynamic model be solved for the optimal values of  $(s, S)$  where the demand density function is specified as  $f(\xi) = e^{-\xi}$ . The detailed computations are straightforward and can be found in Appendix I. The rule of action for time period  $t$  with  $y_t$  as the stock level,  $x_t$  as the stock level including replenishment, and  $\xi_t$  as the demand are represented both analytically and graphically as follows:

If  $y_t > s$ , then  $z_t = 0$  (and  $x_t = y_t$ )

If  $y_t \leq s$ , then  $z_t = S - y_t$  (and  $x_t = S$ )



Stock Level as a Function of Time

Figure 1

Let  $c(\cdot)$ ,  $p(\cdot)$ , and  $h(\cdot)$  be the ordering, penalty, and holding cost functions respectively. This model has cost

functions defined as follows:

$$c(\circ) = K$$

$$p(\circ) = p$$

$$h(\circ) = h \times y_t$$

Under an  $(s, S)$  policy with the initial stock position,  $y_0$ , specified, there is a Markov process on the sequence of values for the stock level. The loss,  $\ell(y_t)$ , in any period  $(t, t+1)$  depends on the stock level,  $y_t$ , at the beginning of the period, the cost functions, and the demand distribution function,  $F(\xi)$ :

$$(5.1) \quad \ell(y_t) = \begin{cases} h \times y_t + p[1 - F(y_t)] & \text{for } s < y_t \leq S \\ h \times S + p[1 - F(S)] & \text{for } y_t \leq s \end{cases}$$

An important feature of the function  $\ell(y_t)$ , which will be used later in finding the optimal  $(s, S)$  values, is that the function involves  $s$  and  $S$  as parameters and is constant with respect to  $y_t \leq s$ ; thus  $\ell(0) - \ell(S) = K$ . Consider  $L(y_0)$  as the minimum discounted expected loss which will be incurred during an infinite time period if  $y_0$  is the initial stock level and an optimal ordering rule is used throughout this infinite time period. If an order at the first stage brings the stock level up to an amount  $y$  and an optimal ordering policy is followed in the second stage onward, then the expected loss from the second stage onward discounted to the present is:

$$\alpha \left\{ \int_{0-}^S L(S-\xi) dF(\xi) + L(0)[1-F(S)] \right. \quad \text{if } y \leq s$$

(5.2)

$$\alpha \left\{ \int_{0-}^y L(y-\xi) dF(\xi) + L(0)[1-F(y)] \right. \quad \text{if } y > s$$

Consequently, the minimum expected loss is achieved by the policy that minimizes the sum of the expected loss for the first period and the discounted minimum expected losses in the future:

$$(5.3) \quad L(y) = l(y) + \alpha \int_{0-}^S L(S-\xi) dF(\xi) + \alpha L(0)[1-F(S)] \quad \text{if } y \leq s$$

$$(5.4) \quad L(y) = l(y) + \alpha \int_{0-}^y L(y-\xi) dF(\xi) + \alpha L(0)[1-F(y)] \quad \text{if } y > s$$

These are the same functional equations (4.11) and (4.12) of [5] showing the recursive properties of the dynamic model. The lower limit of integration is set at 0- in order to insure that the random variable for demand,  $\xi$ , does not have a positive probability at  $\xi=0$ , and, thereby, avoid a discontinuity in the Stieltjes integral

The exposition in this section could be shown for the case in which the demand distribution is discrete and all functions are defined on integer values; the integrals would be replaced by appropriate summations.



It was noted earlier in equation (5.1) that  $\ell(y)$  is independent of  $y$  for  $0 \leq y \leq s$ . Note in equation (5.3) that  $L(y)$  is independent of  $y$  for  $0 \leq y \leq s$ . Putting  $y=0$  in (5.3) and  $y=s$  in (5.4), then subtracting the latter from the former produces:

$$(5.5) \quad L(0) - L(s) = \ell(0) - \ell(s) = K$$

Equation (5.4) can be solved for the function  $L(y)$  by considering  $L(0)$  as an unknown parameter, then (5.5) can be used to find  $L(0)$ . But before this solving takes place, (5.4) is changed into the usual form of the integral equation of renewal theory whose solution is found in terms of  $L$  and  $\ell$ . It is this solution that is used to find  $L(0)$ . Substitutions, change of variable, and renewal summation of convolutions for demand provide the means for finding a solution [5].

The steps begin by breaking up the integral on the right side of (5.4):

$$\begin{aligned} \int_0^y L(y-\xi) dF(\xi) &= \int_0^{y-s} L(y-\xi) dF(\xi) + \int_{y-s}^y L(y-\xi) dF(\xi) \\ &= \int_0^{y-s} L(y-\xi) dF(\xi) + L(0) \int_{y-s}^y dF(\xi) \end{aligned}$$

In the last term  $L(0) = L(y-s)$ . Substitute this



into (5.4), then in the next step substitute  $\eta = y-s$  and

$$L(y) = L(\eta+s) = \lambda(\eta):$$

$$(5.4)' \quad L(y) = \ell(y) + \alpha \int_0^{y-s} L(y-\xi) dF(\xi) + \alpha L(0) \int_{y-s}^y dF(\xi) + \alpha L(0) [1-F(y)]$$

$$\lambda(\eta) = \ell(\eta+s) + \alpha \int_0^{\eta} \lambda(\eta-\xi) dF(\xi) + \alpha L(0) \{F(y) - F(\eta) + [1-F(y)]\}$$

$$\lambda(\eta) = \ell(\eta+s) + \alpha \int_0^{\eta} \lambda(\eta-\xi) dF(\xi) + \alpha L(0) [1-F(\eta)] \quad \text{for } \eta > 0$$

This is the integral equation of renewal theory. The  $n$ -fold convolution of the distribution of demand is  $F_n(\xi)$  which is used in the solution of (5.4) as follows:

$$(5.6) \quad H_{\alpha}(\xi) = \sum_{n=1}^{\infty} \alpha^n F_n(\xi)$$

where the convolutions are defined as follows:

$$F_1(\xi) = F(\xi)$$

$$F_n(\xi) = \int_0^{\xi} F_{n-1}(\xi-u) dF(u) \quad \text{for } n=2, 3, \dots$$

The solution to (5.4) in terms of  $L$  and  $\ell$  is shown on page 33 of [5]:

$$(5.8) \quad L(0) = \frac{K + l(s) + \int_0^{S-s} l(s-\xi) dH_{\alpha}(\xi)}{(1-\alpha)[1+H_{\alpha}(S-s)]}$$

As pointed-out on page 34 of [5], the determination of the optimal policy which minimizes  $L(0)$  is tantamount to finding the optimal policy that minimizes  $L(y_0)$  provided that the optimal value of  $s$  is greater than zero. The finding of the optimal  $(s, S)$  policy required that the system of equations  $\frac{\partial}{\partial s} L(0) = 0$  and  $\frac{\partial}{\partial \Delta} L(0) = 0$  be solved in terms of  $S$  and  $\Delta = S-s$ . Also  $\frac{\partial^2}{\partial s^2} L(0)$  must be positive for the minimum to exist. The appropriate equations are:

$$(5.9) \quad \frac{\partial}{\partial s} L(0) = h - p \times f(s) + \int_0^{S-s} [h - p \times f(s-\xi)] dH_{\alpha}(\xi) = 0$$

$$\frac{\partial}{\partial \Delta} L(0) = h(S-s) - p[f(S) - F(s)] + K + \int_0^{S-s} [h - p \times f(s-\xi)] H_{\alpha}(\xi) d\xi$$

It is assumed that  $H_{\alpha}(\xi) = \int_0^{\xi} h_{\alpha}(t) dt$ .

One can find  $h_{\alpha}(\xi)$  for many demand density functions which have practical application to actual demand behavior. However, solving (5.9) for the optimal  $(s, S)$  values may require numerical methods.

The other alternative to this method is a simulation

of (5.8) using predetermined functions of  $dH_\alpha(\xi) = \sum_{n=1}^{\infty} \alpha^n F_n(\xi)$ . For some guidelines on the number of runs necessary for a 95% confidence level, see Geisler [2]. An attempt was made to set up this simulation on the CDC 1604 computer using FORTRAN machine language; however, unfamiliarity with FORTRAN caused the effort to be time consuming both in preparation of the program and in debugging. It is suggested that the next undertaking of this problem begin very early in the learning phase of programming and that another machine language, such as SCRAP, be investigated for better adaptability of this problem.

As foretold in section 1 the equation is now transformed to a stationary cost equation which implies that the loss in period  $t$ ,  $l_t$ , approaches a limiting value  $l_\infty$  as  $t \rightarrow \infty$  and is independent of  $y_0$ , the initial stock level. By use of a standard Abelian theorem [10] and as heuristically outlined on pages 37-38 of [5], the right side of equation (5.8) may be multiplied by  $(1-\alpha)$ , and letting  $\alpha \rightarrow 1$  as suggested by  $\lim_{\alpha \rightarrow 1} (1-\alpha)L(0) = l_\infty$ , the stationary loss equation is obtained:

$$l_\infty = \frac{K + l(s) + \int_0^{s-s} l(s-\xi) dH(\xi)}{1 + H(s-s)}$$

A solution using demand density function  $f(\xi) = e^{-\xi}$  will

be found so that a comparison can be made with the solution in section 6. On page 39 of [5], demand densities of the type

$$f(\xi) = \frac{\beta^k}{k} \int_0^\xi u^{k-1} e^{-\beta u} du, \quad k > 0 \quad \text{and} \quad \beta > 0$$

(where  $k$  is an integer) are shown to give the following results:

$$H_\alpha(\xi) = \frac{\beta \alpha^{1/k}}{k} \int_0^\xi e^{-\beta u} \left[ \sum \omega_j e^{\omega_j \alpha^{1/k} \beta u} \right] du$$

The gamma type density function is chosen since the product of the characteristic function  $n$ -times yields another gamma density with parameters  $nk$  and  $\beta$  (the sum or addition of  $n$  independent identical random variables is the product of the characteristic function  $n$ -times). Here  $\omega_1, \omega_2, \dots, \omega_k$  are the  $k^{\text{th}}$  roots of unity. For  $f(\xi) = e^{-\xi}$ , the parameters are  $k=1$  and  $\beta=1$  (thus  $\omega_1 = 1$ ). Since  $\alpha=1$ , then

$$H(\xi) = \int_0^\xi e^{-u} (e^u) du = \xi$$

and thus,

$$l_\infty = \frac{K + hS + p e^{-s} + \frac{1}{2}h(S^2 - s^2)}{1 + S - s}$$

As shown in Appendix I for fixed  $\Delta = S-s$ , the minimum occurs at

$$S = \ln \frac{p}{h} - \ln(1+\Delta) + \Delta$$

$$s = \ln \frac{p}{h} - \ln(1+\Delta)$$

However, it is useful for comparison with other methods of solution to find the optimal value of  $\Delta$ . From Appendix I,

$$\Delta = \sqrt{\frac{2K}{h}}$$

$$e^{-s} = \frac{h + \sqrt{2Kh}}{p}$$

which will turn out to be the same solution in section 6 wherein the stationary distribution for stock level was used.

## 6. Solution Using Stationary Distributions of Stock Level.

As noted in chapter 15 of [1],  $l_{\infty} = \lim_{\alpha \rightarrow 1} (1-\alpha)L(y)$  may be also determined by finding the stationary distribution of the stock level and afterwards calculating the costs for a single stage problem where the stock level is a random variable having the stationary distribution. The stationary distribution may be obtained for a general demand density function  $f(\xi)$  and is composed of two sections. For stock level,  $y$ , in the range  $[s, S]$ ,

$$(6.1) \quad g(y) = C \sum_{n=1}^{\infty} F_n(S-y) \quad \text{for } s \leq y \leq S$$

The constant  $C$  is determined by the appropriate boundary conditions and  $F_n$  is the  $n$ -fold convolution of the demand distribution function (cf. equation 5.6). The stationary density function for  $y < s$  is a multiple of the so-called excess distribution for the renewal process of independent random variables  $\xi_1$ . Although (6.1) may be difficult to find, the distribution for the excess variable is more readily obtained by use of Laplace transforms. The excess variable can be characterized as

$$\xi_1 + \xi_2 + \dots + \xi_{v(y)+1} - (S-s) = \gamma_{S-s}$$

where  $v(y)$  is a random variable whose value may be an integer from the set  $(0, 1, 2, 3, \dots)$  satisfying the



following inequalities:

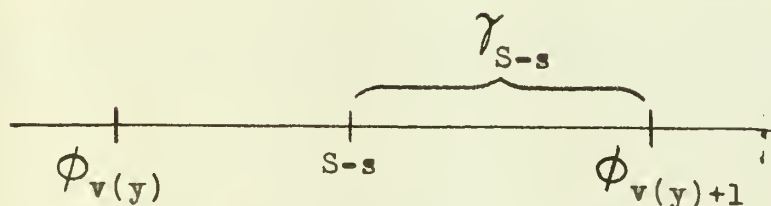
$$\begin{aligned}\xi_1 + \xi_2 + \dots + \xi_{v(y)} &< S-s \\ \xi_1 + \xi_2 + \dots + \xi_{v(y)+1} &\geq S-s\end{aligned}$$

Letting

$$\phi_{v(y)} = \xi_1 + \xi_2 + \dots + \xi_{v(y)} \quad \text{and}$$

$$\phi_{v(y)+1} = \xi_1 + \xi_2 + \dots + \xi_{v(y)+1}$$

the excess variable of  $y$  may be pictured as follows:



Excess Variable of  $S-s$

Figure 2

The renewal type equation follows from this definition of

$$\gamma_{S-s} [1],$$

$$\Pr [\gamma_{S-s} > u] = \int_0^{S-s} \Pr [\gamma_{S-s-y} > u] dF(y) + [1-F(S-s-u)].$$

A renewal process is defined as a stochastic process composed of sums of positive identically distributed and independent random variables. This much of the theory

is presented in order to provide some intuitive appeal. Those interested in the detailed study of renewal processes are referred to chapter 15 of [1].

When the demand density functions are members of the gamma family, the stationary density function of stock level may be derived alternately through the use of differential equations as in chapter 14 of [1]. The stationary distribution,  $g(y)$ , for the case where  $f(\xi) = e^{-\xi}$  is taken from [1] and is found to be

$$g(y) = \begin{cases} \frac{1}{1 + S - s} & \text{for } s < y \leq S \\ \frac{1}{1 + S - s} e^{-(s-y)} & \text{for } y \leq s \end{cases}$$

The stationary loss is a straight forward calculation

$$l_{\infty} = \int l(y) g(y) dy$$

where the one period loss in our case is set-down in (5.1) as

$$l(y) = \begin{cases} hy + p[1-F(y)] & \text{for } s < y \leq S \\ hS + p[1-F(S)] & \text{for } y \leq s \end{cases}$$

Since  $F(\xi) = 1 - e^{-\xi}$ , then

$$l(y) = \begin{cases} hy + p[e^{-y}] & \text{for } s < y \leq S \\ hS + p[e^{-S}] + K & \text{for } y \leq s \end{cases}$$

Since our model allows backlogging, negative stock levels

may exist, and then

$$\begin{aligned}
 l_{\infty} &= \int_{-\infty}^s (hS + p e^{-S} + K) \frac{1}{1+S-s} e^{-(s-y)} dy + \int_s^S (hy + p e^{-y}) \frac{1}{1+S-s} dy \\
 &= \frac{K + hS + p e^{-s} + \frac{1}{2}h(S^2 - s^2)}{1 + S - s}
 \end{aligned}$$

This is the same stationary cost function as (5.10), and the solutions are the same.

## 7. The Simulation Problem.

The great advantage in the use of renewal theory is that the cost functions enter separately into the solution and in addition they enter into a simple expected value form with respect to the stationary density of stock level. Thus the cost functions may be more complicated without unduly burdening the solution. This approach also provides a more flexible method for simulation wherein the program will not be tied down to a specific set of cost functions as would be the case in a simulation of the equation for  $L(0)$ , pointed out earlier in section 5.

Geisler's report [2] provides a statistical evaluation of the sample size required for a desired level of precision for estimating shortages and overages (i.e. stock level above zero) in the AHM dynamic model for both lag and no lag in delivery with linear penalty and holding cost and ordering cost,  $K + c \cdot z$ . The precision represents a way of finding sample size when the mean number of shortages or overages is within  $K\%$  of the true value with 95% confidence. Part D of Geisler's report has sections III and IV which analyze the given model by inventory cycles and by time periods, respectively. The time period model is what has been discussed in this thesis. The inventory cycle is the number of periods between reorder where the number of periods is a random variable. The concepts in renewal theory

are covered in chapter 15 of [1], sections III and IV.

Sections III and IV outline the analytic solutions. Subsection III-3.b shows the specific expected costs with respect to the stationary density of stock level when  $f(\xi) = \lambda e^{-\lambda \xi}$ . In subsection IV-2, the expected values of shortage level, stock level (overage), and reorder quantity per period are given on pages 79, 85, and 89 respectively. To get the stationary loss (cost) multiply, respectively by  $p$ ,  $h$ , and  $c$ . To the reorder cost  $\frac{1}{\lambda} \cdot c$  must be added  $\frac{K}{\lambda(1+\lambda\Delta)}$ . Consequently Geisler's report has provided an exercise reference in the study of renewal theory and its utilization in finding an analytic solution to the AHM dynamic model with less restrictive cost functions. When the stationary costs have been derived, either directly or by use of numerical methods, a simulation can be undertaken to find the optimal  $(s, S)$  policy which minimizes the stationary loss function. Section V of part D can be used in the simulation to determine sample sizes.

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# APPENDIX I

## 9. Computations for the Original AHM Solution of Section 5.

Starting with the case where  $f(\xi) = e^{-\xi}$ , the values of  $(s, S)$  in functional form will be found that minimize

$$l_{\infty} = \frac{K + hS + p e^{-s} + \frac{1}{2}h(S^2 - s^2)}{1 + S - s}$$

Let  $\Delta = S - s$ .

$$\textcircled{1} \quad \frac{\partial}{\partial S} l_{\infty} = \frac{(1 + S - s)(h + hS) - [K + hS + p e^{-s} + \frac{1}{2}h(S^2 - s^2)]}{(1 + S - s)^2}$$

$$\textcircled{2} \quad \frac{\partial}{\partial s} l_{\infty} = \frac{(1 + S - s)(-p e^{-s} - hs) - [K + hS + p e^{-s} + \frac{1}{2}h(S^2 - s^2)](-1)}{(1 + S - s)^2}$$

$$\textcircled{1} \quad h + h\Delta + hS + hS\Delta - K - hS - p e^{-s} - \frac{1}{2}\Delta h(S + s) = 0$$

$$h + h\Delta + \frac{1}{2}\Delta h(S - s) - K - p e^{-s} = 0$$

$$-K - p e^{-s} + h + \Delta h + \frac{1}{2}\Delta^2 h = 0$$

$$\textcircled{2} \quad -p e^{-s} - \Delta p e^{-s} - hs - \Delta hs + K + hS + p e^{-s} + \frac{1}{2}\Delta hS + \frac{1}{2}\Delta hs = 0$$

$$-\Delta p e^{-s} + h(S - s) + \frac{1}{2}\Delta hS - \frac{1}{2}\Delta hs + K = 0$$

$$-K = -\Delta p e^{-s} + h\Delta + \frac{1}{2}\Delta h(S - s)$$

$$-K = -\Delta p e^{-s} + h\Delta + \frac{1}{2}h\Delta^2$$

Substituting  $\textcircled{2}$  into  $\textcircled{1}$  for  $-K$ :

$$(-\Delta p e^{-s} + h\Delta + \frac{1}{2}h\Delta^2) - p e^{-s} + h + h\Delta + \frac{1}{2}h\Delta^2 = 0$$

$$-p e^{-s}(1 + \Delta) + h(\Delta^2 + 2\Delta + 1) = 0$$

$$p e^{-s}(1 + \Delta) = h(1 + \Delta)^2$$

$$\frac{p}{h} e^{-s} = 1 + \Delta$$

$$e^s = \frac{p}{h(1+\Delta)}$$

$$s^* = \ln \frac{p}{h} - \ln(1+\Delta)$$

$$s^* = \ln \frac{p}{h} - \ln(1+\Delta) + \Delta$$

$s^*$  and  $S^*$  denote  
optimal values

These results agree with page 40 of [5] where  $\Delta$  has a fixed value. In order to solve explicitly for  $\Delta$ , a mathematical device must be used. This device consists of removing the variable  $h$  by taking the loss function with respect to  $h$  after certain substitutions into  $l_\infty$ . Substitute the following into  $l_\infty$ :

$$p e^{-s} = h(1+\Delta^*)$$

$$s^* = \ln \frac{p}{h(1+\Delta^*)}$$

$$s^* = \ln \frac{p}{h(1+\Delta^*)} + \Delta^*$$

$$l_\infty = \frac{K+h \left[ \ln \frac{p}{h(1+\Delta^*)} + \Delta^* \right] + h(1+\Delta^*) + \frac{1}{2}h \Delta^* \left[ 2 \ln \frac{p}{h(1+\Delta^*)} + \Delta^* \right]}{1+\Delta^*}$$

$$l_\infty = \frac{K+h \left[ \frac{1}{2} \Delta^* + 2 \Delta^* + 1 \right] + h(1+\Delta^*) \ln \frac{p}{1+\Delta^*}}{1+\Delta^*}$$

$$\frac{l_{\infty}}{h} = l'' = \frac{K''}{1+\Delta^*} + l' \times \frac{\frac{1}{2} \Delta^{*2} + 2 \Delta^* + 1}{1+\Delta^*} + \ln p'' - \ln(1+\Delta^*)$$

where  $K'' = \frac{K}{h}$  and  $p'' = \frac{p}{h}$ . A check of the consistency in the

units of the equation shows that  $l' = (\text{one charge-unit/period})$   
 $= (\text{storage cost/unit-period})(\text{no. of units stored, averaged})$ .

In the first term the order handling charge is divided equally among the number of units in the dynamic range of  $+1$ . The third term is the penalty premium expressed in  $(l' \text{ units} \times 1 \text{ Unit})$  and averaged. This analysis of consistency helps verify the propriety of the substitutions. Now minimize with respect to  $\Delta^*$ :

$$\begin{aligned} \frac{\partial}{\partial \Delta^*} l'' &= \frac{-K''}{(1+\Delta^*)^2} + \frac{(1+\Delta^*)(2+\Delta^*) - (\frac{1}{2} \Delta^{*2} + 2 \Delta^* + 1)}{(1+\Delta^*)^2} - \frac{1+\Delta^*}{(1+\Delta^*)^2} \\ -K'' + [\Delta^{*2} + 3 \Delta^* + 2 - \frac{1}{2} \Delta^{*2} - 2 \Delta^* - 1] - 1 + \Delta^* &= 0 \\ -K'' + \frac{1}{2} \Delta^{*2} &= 0 \\ \Delta^* &= \sqrt{2K''} = \sqrt{\frac{2K}{h}} \end{aligned}$$

To set the  $(s^*, S^*)$  values after  $\Delta^*$  is determined, an equation for  $s^*$  is used:

$$\begin{aligned} p e^{-s} &= h(1+\Delta^*) \\ e^{-s} &= \frac{h}{p} \left( 1 + \sqrt{\frac{2K}{h}} \right) = \frac{h + \sqrt{2Kh}}{p} \end{aligned}$$

This completes the analysis.





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Analytic solutions of the Arrow, Harris,



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