[redacted]

BRL

REPORT NO. 719
MAY 1950

SUPERSONIC FLOW OVER BODIES OF REVOLUTION
(WITH SPECIAL REFERENCE TO HIGH SPEED COMPUTING)

R. F. Clippinger
N. Gerber

BALLISTIC RESEARCH LABORATORIES

ABERDEEN PROVING GROUND, MARYLAND

B A L L I S T I C   R E S E A R C H   L A B O R A T O R I E S

REPORT NO. 719

11 May 1950, 12 67p.

14 BRL-719

6

SUPERSONIC FLOW OVER BODIES OF REVOLUTION
(WITH SPECIAL REFERENCE TO HIGH SPEED COMPUTING),

10 R. F. Clippinger

N. Gerber ,

16

[redacted] TB3-0108H [redacted]

A B E R D E E N   P R O V I N G   G R O U N D,   M A R Y L A N D

B

CONTENTS

B A L L I S T I C   R E S E A R C H   L A B O R A T O R I E S

REPORT NO. 719

RFClippinger/NGerber/atg
Aberdeen Proving Ground, Md.
May 1950

SUPERSONIC FLOW OVER BODIES OF REVOLUTION
(WITH SPECIAL REFERENCE TO HIGH SPEED COMPUTING)

ABSTRACT

With the advent of large-scale high-speed computing machines, it has become feasible to solve certain supersonic flow problems by numerical methods using the exact hydrodynamical equations instead of resorting to linearization or graphical methods. This report describes in detail one such numerical method; namely, an efficient form of the method of characteristics.

Characteristic equations are derived for supersonic, steady, inviscid, isoenergetic flows in terms of a variety of dependent variables. The computation described is applicable to non-yawed bodies of revolution having pointed noses and fairly arbitrary contours, which lie in a uniform stream moving fast enough to produce a shock-wave at the nose and maintain supersonic flow everywhere. The computational procedure is divided into several parts: Taylor-Maccoll, corner, contour, general, and shock processes. Equations and boundary conditions are given for each of these procedures.

A discussion is given of several methods of numerically solving systems of 1st order ordinary differential equations, such as are encountered in the Taylor-Maccoll and corner processes. The other computations involve approximating partial derivatives by difference quotients and solving on a finite grid of points. Solutions are derived for the cases in which the derivatives are approximated to 1st, 2nd, and 3rd orders.

An empirical study is made of the error due to the introduction of finite differences. This is based on the results of a particular calculation performed on the ENIAC. It is shown that a knowledge of the nature of the errors leads to a procedure for extrapolation to zero grid size, which reduces by a factor of ten the total labor required to obtain a solution correct to about four significant figures.

SUPERSONIC FLOW OVER BODIES OF REVOLUTION
(WITH SPECIAL REFERENCE TO HIGH SPEED COMPUTING)

Section 1. Introduction

The mathematical basis for computing the velocity, density, and pressure distribution of air flowing faster than sound over plane bodies and bodies of revolution has been laid by Riemann[1.1], Picard[1.2], Hadamard[1.3], Goursat[1.4], Lewy[1.5], Friedrichs and Lewy[1.6], Frankl and Aleksieva[1.7], Courant and Lax[1.8], and others. The problem is that of solving a non-linear system of hyperbolic partial differential equations with boundary conditions given on a known curve (the body contour) and on a curve not known in advance (the shock wave).

Methods have been known for fifteen years for solving the exact equations (without friction, with rotation) for supersonic flow about plane and axial bodies. Heretofore only slight use has been made of them, however, because of the extreme tediousness of the numerical computations. Instead, the solution of supersonic flow problems has proceeded along two main lines: (1) graphical and semi-graphical procedures, developed especially by Prandtl, Busemann, Sauer, Tollmien, Guderley, and others of the German school; and (2) linearization of the hydrodynamical equations. Linear problems are easier to solve; whole families of solutions may often be obtained exhibiting the variation of the solutions with important parameters. Indeed, even if the linear problem has been obtained by neglecting some moderately large terms, the solution is often very valuable qualitatively in guiding the intuition.

1.1 R. Courant and D. Hilbert, Methoden der Mathematischen Physik, II; p. 311. Julius Springer, Berlin, 1937.
1.2 E. Picard, Traite d'analyse, II. Paris (3rd ed. 1926)
1.3 J. Hadamard, Lecons sur le Problem de Cauchy; p. 487. Paris, 1932
1.4 E. Goursat, Cours d'Analyse, II; p. 360. Paris (4th ed. 1924)
1.5 H. Lewy, "Ueber das Anfangswertproblem bei einer hyperbolischen nichtlinearen partiellen Differentialgleichung zweiter Ordnung mit zwei unabhängigen Veränderlichen," Mathematische Annalen, vol. 98 (1927), pp. 179-191.
1.6 K. Friedrichs and H. Lewy, "Das Anfangswertproblem einer beliebigen nichtlinearen hyperbolischen Differentialgleichung beliebiger Ordnung in zwei Variabeln. Existenz, Eindeutigkeit und Abhängigkeitsbereich der Lösung," Mathematische Annalen, vol. 99 (1928), pp. 220-221.
1.7 F. Frankl and P. Aleksieva, "Two Boundary Value Problems from the Theory of Hyperbolic Partial Differential Equations with Applications to Supersonic Gas Flow", Rec. Math. Mosc., T. 41:3 (1934). (Also BRL Report X-123; Aberdeen Proving Ground, Maryland.)
1.8 R. Courant and P. Lax, "On Nonlinear Partial Differential Equations with Two Independent Variables", Communications on Pure and Applied Mathematics, Vol. II, nos. 2-3 (1949); p. 255.

PRECEDING PAGE BLANK

With the advent of high speed computing devices such as the ENIAC now operating at the BRL or the EDVAC being installed at the BRL, a shift of emphasis will take place. A greater effort will be devoted to the solution of the exact equations. It will be possible to solve these equations so rapidly that parameters may again be introduced. Since the machines are even able to think in an elementary way, they can be made to solve in a numerical manner such problems in the Calculus of Variations as determining the head shape of given diameter and head length which will lead to minimal head drag.

This report has been written in an effort to accelerate the change in emphasis. It includes some results obtained using the ENIAC. It is expected that more ENIAC and EDVAC results will appear in later reports.

Section 2. Fundamental Equations

Introduction

The problems considered in this report are all of the following type: air flows steadily and supersonically, from a region of uniformity, past a body which may be plane or have symmetry of revolution. If there is a shock wave, the Mach number is assumed large enough and the initial flow deviation small enough so that the shock front is attached to the body at a known point, and the velocity is everywhere supersonic. Air is considered a perfect gas, body forces and friction (therefore heat conduction) are neglected, but rotation of the flow caused by a curved shock wave is allowed.

With these restrictions the continuity, energy, and Euler equations are [2.1]

$$(2.1) \qquad \nabla \cdot (\rho \bar{Q}) = 0,$$

$$(2.2) \qquad Q^2 + \frac{2}{(\gamma - 1)} A^2 = C^2,$$

$$(2.3) \qquad (\bar{Q} \cdot \nabla)\, \bar{Q} = -\frac{1}{\rho} \nabla p,$$

where $\bar{Q}$, ρ, p, γ, and A are the velocity vector, density of the air, pressure of the air, ratio of specific heats, and velocity of sound, respectively.

Equation (2.2) shows that as the velocity of sound approaches zero the velocity approaches a limit C. Equation (2.2) holds across a shock wave[2.2] also, and therefore C is the same for all parts of the fluid.

2.1 R. Courant and K. O. Friedrichs, Supersonic Flow and Shock Waves; pp. 14, 22, 15. Interscience Publishers Inc., New York, 1948
2.2 W. F. Durand, Aerodynamic Theory; Vol. III, p. 217.

Accordingly, we shall take it as the unit velocity, setting

$$(2.4) \qquad q \equiv Q/C, \quad a \equiv A/C.$$

With this notation equations (2.1), (2.2) and (2.3) become

$$(2.5) \qquad \nabla \cdot (\rho \bar{q}) = 0,$$

$$(2.6) \qquad q^2 + \frac{2}{\gamma - 1} a^2 = 1,$$

and

$$(2.7) \qquad (\bar{q} \cdot \nabla) \bar{q} = - \frac{a^2}{\gamma p} \nabla p .$$

For a perfect steady gas without friction and body forces, it can be shown that in those regions where there is no shock-wave the constancy of the entropy on streamlines follows from equations (2.1), (2.2) and (2.3). Hence

$$(2.8) \qquad \bar{q} \cdot \nabla S = 0 \quad \text{where } p/\rho^{\gamma} = e^{S/c_v}$$

or

$$(2.9) \qquad \rho (\bar{q} \cdot \nabla p) = \gamma p (\bar{q} \cdot \nabla \rho)$$

Special Coordinate Systems

In the case of flow past or through a body of revolution we shall introduce the axis of symmetry as the x-axis with the orientation of the free stream velocity vector $\bar{q}_1$. We shall let the y-axis be a line through the leading point A, perpendicular to the axis. We assume all velocity vectors lie in planes through the x-axis and have components u and v parallel to the x and y axes which are independent of the angle about the x-axis.

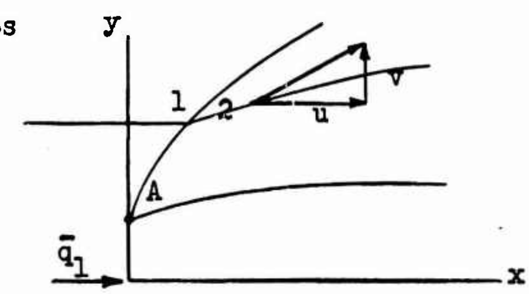


Figure 2.1

Except for an arbitrary translation along the y axis the disposition of axes is the same for plane flow.

With these definitions equations (2.7) and (2.9) become, both for plane and axial flow,

$$(2.10)\begin{cases} u\,u_x + v\,u_y = -(a^2\,p_x)/\gamma p, \\ u\,v_x + v\,v_y = -(a^2\,p_y)/\gamma p, \end{cases}$$

and

$$(2.11)\quad \rho(u\,p_x + v\,p_y) = \gamma p(u\,\rho_x + v\,\rho_y);$$

where

$$u_x \equiv \frac{\partial u}{\partial x}, \quad p_y \equiv \frac{\partial p}{\partial y}, \text{ etc.}$$

The equation of continuity is

$$(2.12)\quad u\,\rho_x + v\,\rho_y + \rho(u_x + v_y + \epsilon\, v/y) = 0,$$

where $\epsilon =$ 0 or 1 for plane or axial flow.

Substituting for p_x, p_y and $u\,\rho_x + v\,\rho_y$ from equation (2.10) and (2.12) into equation (2.11) we obtain an equation

$$(2.13)\quad H\,u_x + K(u_y + v_x) + L\,v_y + \epsilon\,\frac{a^2 v}{y} = 0,$$

where

$$H = a^2 - u^2,\ K = -\,u\,v,\ L = a^2 - v^2,$$

independent of p and ρ. This equation holds whether the flow is rotational or not. In addition, if the flow is irrotational

$$(2.14)\quad \nabla \times \bar{q} = 0,$$

or

$$(2.15)\quad v_x - u_y = 0.$$

Shock Wave

If there is a shock-wave somewhere in the flow, the following equations arising from the conservation of mass, energy, and momentum hold across it[2.2]:

$$(2.16)\quad \frac{p_2}{p_1} = \frac{2\,\gamma\,M_1^2 \sin^2\theta_w - (\gamma - 1)}{\gamma + 1}$$

$$(2.17)\quad \frac{\rho_2}{\rho_1} = \frac{q_1 \sin\theta_w}{q_2 \sin(\theta_w - \theta)} = \frac{(\gamma + 1)\,M_1^2 \sin^2\theta_w}{(\gamma - 1)\,M_1^2 \sin^2\theta_w + 2}$$

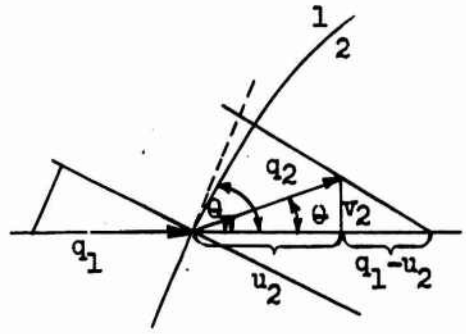


Figure 2.2

In addition the component of velocity parallel to the shock wave is unchanged.

$$(2.18) \qquad q_1 \cos \theta_w = q_2 \cos (\theta_w - \theta).$$

In these equations θ_w is the angle between the velocity vector on the side denoted by subscript one in front of the shock-wave, and the shock-wave, M_1 is q_1/a, and θ is the angle between $\bar{q}_1$ and $\bar{q}_2$. From these equations it is possible to eliminate p_2/p_1 and ρ_2/ρ_1 obtaining the relation

$$(2.19) \quad v_2^2\left[\frac{\gamma-1}{\gamma+1}\,\frac{1}{q_1}+\frac{2q_1}{\gamma+1}-u_2\right] = (q_1-u_2)^2\left[u_2-\frac{\gamma-1}{(\gamma+1)q_1}\right].$$

An additional relation may be read from Figure 2.2

$$(2.20) \qquad \frac{dy}{dx} = \frac{q_1 - u_2}{v_2},$$ where $\frac{dy}{dx}$ is the slope of the shock-wave.

These two equations are the boundary conditions which must be satisfied at the shock-wave. We shall call them the shock conditions.

Rotation Introduced by the Shock Wave

From equations (2.16) and (2.17) it is apparent that the entropy jump across the shock,

$$(2.21) \qquad \Delta s \equiv s_2 - s_1 = c_v \log \left[\frac{p_2}{p_1}\Big/\left(\frac{\rho_2}{\rho_1}\right)^{\gamma}\right],$$

is a function of θ_w. Therefore the entropy is constant behind the shock-wave only if the shock is straight in any x-y plane.

Let the subscripts 0 and 3 refer to stagnation before and behind the shock-wave if there is one. Then from equations (2.8) and (2.6)

$$p/\rho^{\gamma} = p_3/\rho_3^{\gamma},$$

and $\dfrac{\gamma p/\rho}{\gamma p_3/\rho_3} = 1 - q^2$;

or

$$(2.22) \qquad p = p_3(1 - q^2)^{\frac{\gamma}{\gamma-1}} = p_3\, B(q),$$

and

$$(2.23) \qquad \rho = \rho_3(1 - q^2)^{\frac{1}{\gamma-1}} = \rho_3\, A(q),$$

where p_3 and ρ_3 vary from streamline to streamline.
Let us attempt to define a stream function $z(x,y)$ by the equations

$$(2.24) \qquad z_x = - A\, y^{\epsilon}\, v$$

and

$$(2.25) \qquad z_y = A\, y^{\epsilon} u,$$

where ϵ is as defined before.
Clearly from this definition ∇z is perpendicular to $\bar{q}$. The definition is only justified however if

$$(2.26) \qquad (z_x)_y = (z_y)_x .$$

To prove this, consider the continuity equation

$$\nabla \cdot (\rho \bar{q}) = \nabla \cdot \left[\rho_3 (z)\, A\, \bar{q}\right] = 0.$$

Differentiating:

$$\frac{A d\rho_3}{dz} (\nabla z \cdot \bar{q}) + \rho_3 \nabla \cdot (A\, \bar{q}) = 0.$$

But $\nabla z \cdot \bar{q} = 0$ and therefore

$$\nabla \cdot (A\, \bar{q}) = 0,$$

or

$$\frac{\partial}{\partial x} (A\, y^{\epsilon} u) + \frac{\partial}{\partial y} (A\, y^{\epsilon} v) = 0.$$

Referring to equations (2.24) and (2.25) we see that (2.26) is satisfied.
From the equation (2.2) we see that

$$(2.27) \qquad \frac{p_3}{p_0} = \frac{\rho_3}{\rho_0} = h(z).$$

Thus we have, substituting in (2.22) and (2.23)

$$(2.28) \qquad p = p_o\, h(z)\, B(q),$$

and

$$(2.29) \qquad \rho = \rho_o\, h(z)\, A(q).$$

These equations hold everywhere if we let $h(z) = 1$ before the shock.

We may now obtain a second equation to replace equation (2.15) when the flow contains a shock-wave. To do this we differentiate equation (2.28) logarithmically:

$$\frac{p_x}{p} = \frac{h'}{h} z_x - \frac{\gamma(u\,u_x + v\,v_x)}{\frac{\gamma-1}{2}(1-q^2)} ,$$

or

$$\frac{p_x}{p} = - y^{\epsilon} A\, v \frac{h'}{h} - \frac{\gamma}{a^2}(u\,u_x + v\,v_x);$$

but from equation (2.10)

$$\frac{p_x}{p} = - \frac{\gamma}{a^2}(u\,u_x + v\,u_y),$$

and, therefore,

$$\frac{a^2}{\gamma}\frac{h'}{h} A\, y^{\epsilon} v + u\,u_x + v\,v_x = u\,u_x + v u_y ,$$

or finally

$$u_y - v_x = y^{\epsilon} \frac{a^2 A}{\gamma}\frac{h'}{h} . \tag{2.30}$$

This equation reduces to (2.15) if there is no curved shock-wave since $h'(z)$ is then zero.

In order to use equation (2.30) it is desirable to express $-(\gamma-1)h'/(\gamma h)$ in terms of velocity components. This may be done using equations (2.16), (2.17), (2.18), (2.19), and (2.20). Let us write

$$\rho_3 = (\rho_2/\rho_1)\left[(\rho_1/\rho_0)/(\rho_2/\rho_3)\right]\rho_0 .$$

From equation (2.29)

$$\frac{\rho_1}{\rho_0} = (1 - q_1{}^2)^{\frac{1}{\gamma-1}} ,$$

and

$$\frac{\rho_2}{\rho_3} = (1 - q_2^2)^{\frac{1}{\gamma-1}} ;$$

similarly, using equations (2.17), (2.6), (2.19), and (2.20)

$$\frac{\rho_2}{\rho_1} = \frac{\gamma+1}{\gamma-1}\,\frac{(u_2 - b)}{(u_2 - \frac{1}{q_1})} ,$$

where

$$b \equiv \frac{\gamma-1}{\gamma+1}\frac{1}{q_1} + \frac{2}{\gamma+1} q_1 .$$

Furthermore, using equation (2.19)

$$1 - q_2^2 = 1 - u_2^2 - v_2^2$$

$$= \frac{2q_1}{\gamma+1}\,\frac{(u_2 - \frac{1}{q_1})(\gamma u_2 - e)}{(u_2 - b)} ,$$

where

$$e \equiv \frac{\gamma+1}{2} q_1 + \frac{\gamma-1}{2} \frac{1}{q_1} ,$$

Therefore

$$h \equiv \frac{\rho_3}{\rho_0} = \frac{A(q_1)(\gamma+1)^{\gamma/(\gamma-1)}}{(\gamma-1)(2q_1)^{1/(\gamma-1)}} \left[\frac{(u_2 - b)}{(u_2 - \frac{1}{q_1})(\gamma u_2 - e)^{1/\gamma}} \right]^{\gamma/(\gamma-1)} ,$$

so that

$$(2.31) \qquad -\frac{(\gamma-1)}{\gamma} \frac{h'}{h} = \frac{(u_2-q_1)^2}{(u_2-b)(u_2-\frac{1}{q_1})(\gamma u_2-e)} \frac{du_2}{dz} .$$

We shall call this quantity $g(z)$ and rewrite equation (2.30)

$$(2.32) \qquad v_x - u_y = \frac{y^{\epsilon} B g}{2}$$

We summarize these results in equations

2.I

a) $H u_x + K(u_y + v_x) + L v_y + \epsilon \frac{a^2 v}{y} = 0$

$H \equiv a^2 - u^2, \; K \equiv -uv, \; L \equiv a^2 - v^2, \quad \epsilon = \begin{cases} 0 & \text{plane flow} \\ 1 & \text{axial flow} \end{cases}$

b) $v_x - u_y \equiv \frac{y^{\epsilon} B g}{2}$

$B \equiv (1 - q^2)^{\gamma/(\gamma-1)}$

b') $g \equiv \dfrac{(u_2 - q_1)^2}{(u_2-b)(u_2-\frac{1}{q_1})(\gamma u_2-e)} \dfrac{du_2}{dz}$

$b \equiv \left(\frac{\gamma-1}{\gamma+1}\right) \frac{1}{q_1} + \left(\frac{2}{\gamma+1}\right) q_1, \qquad e \equiv \left(\frac{\gamma+1}{2}\right) q_1 + \left(\frac{\gamma-1}{2}\right) \frac{1}{q_1}$

c) $dz = y^{\epsilon} A(-v\,dx + u\,dy)$

d) $p = p_0 h B$

e) $\rho = \rho_0 h A$

$A \equiv (1 - q^2)^{1/(\gamma-1)}$

$h = \dfrac{A(q_1) \; (\gamma+1)^{\gamma/(\gamma-1)}}{(\gamma-1)(2q_1)^{1/(\gamma-1)}} \left[\dfrac{u_2 - b}{(u_2-\frac{1}{q_1})(\gamma u_2-e)^{1/\gamma}} \right]^{\gamma/(\gamma-1)}$

f) $a^2 = \frac{\gamma-1}{2}(1 - q^2)$

g) $v_2^2(b - u_2) = (q_1 - u_2)^2(u_2 - d)$

$d \equiv \left[(\gamma-1)/(\gamma+1)\right] \frac{1}{q_1}$

h) $\frac{dy}{dx} = (q_1 - u_2)/v_2$

i) $y = F(x)$ equation of contour of given body

j) $v = u\,F'(x)$

k) $z = 0$

Characteristic Equations

Although it would be possible to solve directly by numerical methods equations 2.Ia,b,c subject to the boundary conditions 2.Ig,h,i,j, and k (being careful to satisfy the cause and effect condition of Courant and Lewy) we have preferred to work instead with characteristic equations to be derived below.

Hyperbolic partial differential equations 2.4 differ from elliptic equations in that the solutions may have derivatives of certain orders which are discontinuous across certain curves (if there are two independent variables) called characteristics. In the case of supersonic flow these curves are called Mach lines in honor of the physicist who discovered them with shadowgraphs. It is shown[2.3] that if the system is of second order there are two characteristics through each point of any region where the differential equations are hyperbolic; that is, the flow supersonic. Following the procedure of Frankl and Aleksieva[1.7] we shall introduce these characteristics as a basis for a curvilinear coordinate system.

Let $\alpha(x,y)$ = constant be the equation of a family of curves and $\beta(x,y)$ = constant be the equation of another. Let α and β have continuous derivatives with respect to x and y. If we introduce α and β as independent variables[2.4], assuming the Jacobian α and β with respect to x and y is zero or infinite only at isolated points, equations 2.Ia and b become:

2.3 R. Courant and D. Hilbert, Methoden der Mathematischen Physik, v. II.
2.4 The method of characteristics as used by Sauer and Tollmien does not use coordinates constant as characteristics but uses instead an infinity of affine coordinate systems based on straight lines parallel to tangents to two characteristics.

$$(2.33)\quad (H\alpha_x+K\alpha_y)u_\alpha+(K\alpha_x+L\alpha_y)v_\alpha=-(H\beta_x+K\beta_y)u_\beta-(K\beta_x+L\beta_y)v_\beta-\frac{\epsilon a^2 v}{y}$$

$$(2.34)\quad -\alpha_y u_\alpha+\alpha_x v_\alpha=\beta_y u_\beta-\beta_x v_\beta+\frac{y\epsilon Bg}{2}\ .$$

If u and v were given continuously differentiable functions of β on a surface α = constant, equations (2.33) and (2.34) could be solved for u_α and v_α if and only if

$$(2.35)\quad \begin{vmatrix} H\alpha_x+K\alpha_y & K\alpha_x+L\alpha_y \\ -\alpha_y & \alpha_x \end{vmatrix} \neq 0.$$

In this case u_α and v_α would also be continuous. If $u(\beta)$, $v(\beta)$, $\alpha(x,y)$, $\beta(x,y)$ possess higher order continuous derivatives, it will be seen by differentiation that the higher derivatives of u and v, and u_β, v_β, etc. with respect to α are determined by equations (2.33), (2.34), and their derivatives. Therefore, if α = constant is a characteristic, equation (2.35) must fail:

$$(2.36)\quad H\alpha_x^2+2K\alpha_x\alpha_y+L\alpha_y^2=0.$$

Similarly it will be found that $\beta(x,y)$ must also satisfy equation (2.36) in order to be a characteristic.

To be precise we may define $\alpha(x,y)$ and $\beta(x,y)$ by the conditions

$$(2.37)\quad H\alpha_x+(K-R)\alpha_y=0,$$

and

$$(2.38)\quad H\beta_x+(K+R)\beta_y=0,$$

$$(2.39)\quad R=a\sqrt{q^2-a^2}$$

$$HL=K^2-R^2$$

with some boundary conditions to be stated later.

From the equations

$$\begin{cases} x_\alpha\ \alpha_x+x_\beta\ \beta_x=1\ , \\ y_\alpha\ \alpha_x+y_\beta\ \beta_x=0\ , \\ x_\alpha\ \alpha_y+x_\beta\ \beta_y=0\ , \\ y_\alpha\ \alpha_y+y_\beta\ \beta_y=1\ ; \end{cases}$$

we find that

$$
\begin{aligned}
x_\alpha &= \beta_y / \Delta \quad , \\
x_\beta &= -\alpha_y / \Delta \quad , \\
(2.39) \qquad y_\alpha &= -\beta_x / \Delta \quad , \\
y_\beta &= \alpha_x / \Delta \quad , \\
\Delta &= \alpha_x \beta_y - \alpha_y \beta_x .
\end{aligned}
$$

and, substituting into (2.37) and (2.38) we get

$$(2.40) \qquad H\, y_\alpha - (K + R)\, x_\alpha = 0, \text{ or } (K - R)\, y_\alpha - L\, x_\alpha = 0$$

and

$$(2.41) \qquad H\, y_\beta - (K - R)\, x_\beta = 0,$$

$$(2.42) \qquad H\, u_\alpha + (K - R)\, v_\alpha + x_\alpha\Big(\epsilon \frac{a^2 v}{y} + \frac{BRy^\epsilon g}{2}\Big) = 0,$$

and

$$(2.43) \qquad (K - R) u_\beta + L\, v_\beta + y_\beta \Big(\epsilon \frac{a^2 v}{y} - \frac{BRy^\epsilon g}{2}\Big) = 0.$$

Together with equations (2.24) and (2.25) and the boundary conditions 2.Ig, h,i,j, equations (2.40)-(2.43) may be used as a basis for computing numerically plane or axial flows. If the flow is plane and irrotational, then it is preferable to introduce the velocity components as independent variables because the equations then become linear.

Other variables which are better adapted to computation of certain flows are q, the magnitude of the velocity, and Θ, the angle between the x-axis and velocity vector, or $\sqrt{(q^2 - a^2)/a^2} = p$ and $\tan \Theta = t$; it may sometimes be advantageous to couple to z, a function ϕ defined by the equations

$$(2.44) \qquad \phi_x = G\, u$$

$$(2.45) \qquad \phi_y = G\, v$$

and G must be chosen so that $(\phi_x)_y = (\phi_y)_x$. ϕ reduces to the velocity potential for irrotational flow. For future reference, we include characteristic equations in these variables in our summary.

$$
\text{a)} \quad
\begin{aligned}
& H \alpha_x + (K-R)\, \alpha_y = 0 \\
& H \equiv a^2 - u^2 \quad , \\
& K \equiv -uv \; , \\
& R \equiv a \sqrt{q^2 - a^2}.
\end{aligned}
$$

$$
2.\mathrm{II}\begin{cases}
\text{b)} & H\beta_x + (K+R)\beta_y = 0 \\
\text{c)} & (K-R)\, y_\alpha - L\, x_\alpha = 0 \text{ or } y_\alpha = \lambda x_\alpha \\
 & \quad L \equiv a^2 - v^2 \\
 & \quad \lambda \equiv L/(K-R) \\
\text{d)} & H\, y_\beta - (K-R)x_\beta = 0 \quad \text{or } \omega y_\beta = x_\beta \\
 & \quad \omega \equiv H/(K-R) \\
\text{e)} & H\, u_\alpha + (K-R)\, v_\alpha + x_\alpha\left(\epsilon \dfrac{a^2 v}{y} + \dfrac{BR\, y^\epsilon g}{2}\right) = 0 \\
 & \omega u_\alpha + v_\alpha + x_\alpha \left(\epsilon \dfrac{P}{K-R} + \dfrac{Q}{K-R}\right) = 0 \\
 & \quad P \equiv \dfrac{a^2 v}{y}, \quad Q \equiv \dfrac{BR\, y^\epsilon g}{2} \\
\text{f)} & (K-R)\, u_\beta + L\, v_\beta + y_\beta \left(\epsilon \dfrac{a^2 v}{y} - \dfrac{BR\, y^\epsilon g}{2}\right) = 0 \\
 & u_\beta + \lambda v_\beta + y_\beta \left(\epsilon \dfrac{P}{K-R} - \dfrac{Q}{K-R}\right) = 0 \\
\text{g)} & a\, y^\epsilon A \phi_\beta - G\, z_\beta \sqrt{q^2 - a^2} = 0 \\
\text{g')} & \phi_x = Gu, \quad \phi_y = Gv \\
\text{h)} & a\, y^\epsilon A \phi_\alpha + G\, z_\alpha \sqrt{q^2 - a^2} = 0 \\
\text{h')} & z_x = -y^\epsilon A v, \quad z_y = y^\epsilon A u \\
\text{i)} & q_\alpha \dfrac{\sqrt{q^2 - a^2}}{aq} + \theta_\alpha + z_\alpha \left[\epsilon \dfrac{\sin\theta}{qAy^2} + \dfrac{1-q^2}{q^2}\sqrt{\dfrac{q^2-a^2}{a^2}}\left(\dfrac{g}{2}\right)\right] = 0 \\
\text{j)} & -q_\beta \dfrac{1}{q}\sqrt{\dfrac{q^2-a^2}{a^2}} + \theta_\beta + z_\beta \left[\epsilon \dfrac{\sin\theta}{qAy^2} - \dfrac{1-q^2}{q^2}\sqrt{\dfrac{q^2-a^2}{a^2}}\left(\dfrac{g}{2}\right)\right] = 0 \\
\text{k)} & \phi_\alpha + \dfrac{q\sqrt{q^2-a^2}\; y_\alpha}{a\cos\theta - \sqrt{q^2-a^2}\sin\theta} = 0, \quad z_\alpha = \dfrac{yAq}{\sqrt{1+t^2}}(y_\alpha - t x_\alpha) \\
\text{l)} & \phi_\beta - \dfrac{q\sqrt{q^2-a^2}\; y_\beta}{a\cos\theta + \sqrt{q^2-a^2}\sin\theta} = 0, \quad z_\beta = \dfrac{yAq}{\sqrt{1+t^2}}(y_\beta - t x_\beta) \\
\text{m)} & 2q^2 G_z = G\, g(1-q^2)
\end{cases}
$$

$$\begin{cases} \text{n)} \quad y_\alpha (t + p) = x_\alpha (tp - 1), \quad t = \tan\theta, \quad p = \sqrt{\dfrac{q^2 - a^2}{a^2}} \\ \text{o)} \quad y_\beta (p - t) = x_\beta (tp + 1) \\ \text{p)} \quad dz = \dfrac{yAq}{\sqrt{1+t^2}} (dy - tdx), \quad q^2 = \dfrac{\overset{*}{q}{}^2 (1 + p^2)}{1 + \overset{*}{q}{}^2 p^2} \\ \qquad \overset{*}{q}{}^2 = \dfrac{\gamma - 1}{\gamma + 1} \\ \text{q)} \quad (1 + t^2) f(p) p_\alpha + t_\alpha + \dfrac{t}{y}(y_\alpha - tx_\alpha) = 0, \quad p^2 = \dfrac{q^2 - \overset{*}{q}{}^2}{\overset{*}{q}{}^2 (1 - q^2)} \\ \text{r)} \quad -(1 + t^2) f(p) p_\beta + t_\beta + \dfrac{t}{y}(y_\beta - tx_\beta) = 0 \\ \qquad f(p) = \dfrac{(1 - \overset{*}{q}{}^2) p^2}{(1+p^2)(1+\overset{*}{q}{}^2 p^2)} \\ \qquad S \equiv \dfrac{\epsilon P}{K-R} = \dfrac{\epsilon a^2 v}{y(K-R)}, \quad T \equiv \dfrac{Q}{K-R} = \dfrac{BR\, y^{\epsilon} g}{2(K-R)} \end{cases}$$

Section 3. Numerical Solution of Boundary Value Problem

Introduction

Typical of the characteristic equations which may be used to compute the supersonic steady frictionless flow past a given plane or axisymmetric body are equations 2.II c,d,e,f, and 2.Ic with boundary conditions 2.I g, h,i,j. Accordingly we shall describe the procedure we use in terms of these equations. This is no restriction, of course, and any of the other sets of equations may be used, e.g., 2.IIn, o, p, q, r with boundary condition deduced from 2.Ig,h,i,j.

We shall consider the case of axisymmetric rotational flow; the cases of axisymmetric potential flow, plane rotational or plane potential flow may be treated the same way with several simplifications.

Consider then a supersonic flow, uniform at infinity, past a body of revolution ABDEF (Figure 3.1). The fact that we have assumed the flow supersonic implies a restriction relating the free stream Mach number and nose angle of the body. Indeed if the nose is blunt, or if it is pointed but the free stream Mach number is less than some number greater than one, it is known that the shock wave is detached from the body and crosses the axis normal to it. From equation (2.17) it follows that q_2 would be less than $[(\gamma-1)/(\gamma+1)]^{\frac{1}{2}}$, i.e. subsonic. For cones there is a half-vertex angle (about 52° 34' in air) above which the shock-wave is detached at all Mach numbers. For each smaller cone-angle there is a Mach number above which the shock is attached to the vertex and there are two conical shock-waves, each of which corresponds to a mathematical solution of the flow problem.[3.1]

Intuitively it is clear that for other bodies of revolution than cones there exist one or more solutions (probably two) with attached shock if the nose angle is small enough, the curvature negative or zero, and the Mach number large enough. As far as we know, this has not been demonstrated mathematically, although the paper by Frankl and Aleksieva[1.7] contains a theorem which the authors believe could be extended to do it. We shall assume that it is true and that we have this case before us. We shall assume furthermore that the solution is a continuous function of the boundary $y = F(x)$ in the sense that if we replace a small section of the nose by a straight line A'B tangent to it at the point B of juncture and

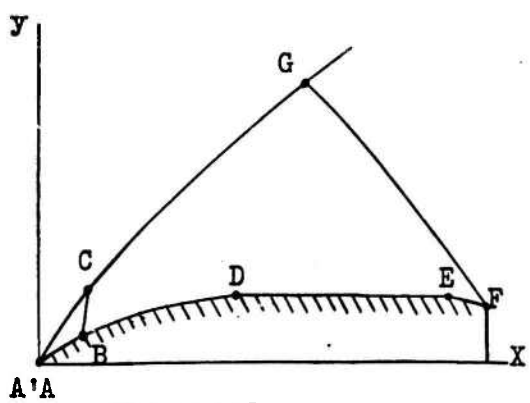


Figure 3.1

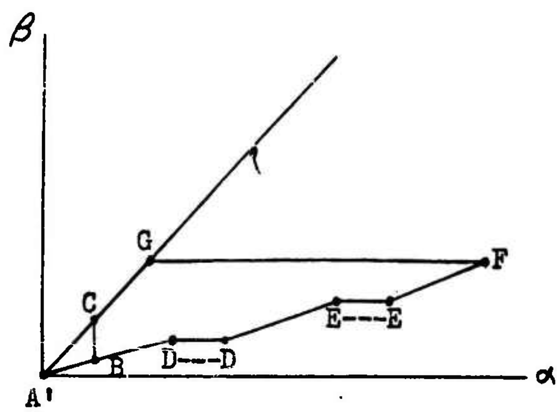


Figure 3.2

let B approach A, the flow about A'BDEF will approach the flow about ABDEF i.e.

$$\lim_{B \to A} x'(\alpha, \beta) = x(\alpha, \beta),$$

$$\lim_{B \to A} u'(\alpha, \beta) = u(\alpha, \beta)$$

etc.

In summary we assume that the given body of revolution has a contour characterized by the equation $y = F(x)$; that F'' exists except at isolated points, and is less than or equal to zero everywhere between $x = 0$ and $x = x_F$, except at a finite number of points such as D and E where $F'(x)$ may be discontinuous; that $F'(0) \ll \tan 52^0$; and that M_1 is large enough so that the shock-wave is attached.

3.1 G. I. Taylor and J. W. Maccoll, Proc. Roy. Soc. of London, Series A, vol 139, 1933; p. 278-299.

Approximations at the Nose

At the nose, in accordance with an assumption stated above we replace the contour to the left of B by a straight line tangent to the given contour at B.

Since the flow at a point P is independent[3.2] of changes made in the region bounded by the two half characteristics farthest from the velocity vector at P, the flow in the region A'BC, bounded by A'B, a characteristic BC, and the shock A'C, is precisely Taylor-Maccoll[3.1] flow over a cone. That is, u and v are constant on lines

$$(3.1) \quad \frac{y}{x - x_{A'}} = t$$

through the nose A'. We may therefore seek immediately the values of x,t,u,v,z for equally spaced values of some variable such as y along the characteristic BC. The differential equations for u,v,and t may be found more readily from equation 2.Ia,b, 2.IId than from Taylor and Maccoll's equations. Since u and v depend only on t,

$$u_x = \frac{du}{dt} t_x = - \frac{du/dy}{dt/dy} \frac{t^2}{y} = - \frac{u'}{t'} \frac{t^2}{y}$$

where we denote by a prime, d/dy. Similarly

$$u_y = \frac{u'}{t'} \frac{t}{y} ,$$

$$v_x = - \frac{v'}{t'} \frac{t^2}{y} ,$$

and

$$v_y = \frac{v'}{t'} \frac{t}{y} .$$

Therefore from 2.Ia,b (g is zero in A'BC since the shock is straight)

$$u'(K - tH) + v'(L - tK) + \frac{a^2 vt'}{t} = 0,$$

$$u' + v't = 0.$$

In addition it follows from equations (3.1) and 2.IId that

$$(3.2) \qquad t' = \frac{t(K - R - tH)}{y(K - R)} .$$

Using equation (3.2), the equations above (3.2) may be solved for u' and v':

$$(3.3) \qquad u' = \frac{-a^2 vt}{y[t(K - R) - L]} ,$$

3.2 R. Courant and D. Hilbert, Methoden der Mathematischen Physik, Vol. II, p. 307

$$(3.4) \qquad v' = \frac{a^2 v}{y\,[t(K - R) - L]} \,.$$

To these equations we may add three initial conditions

$$(3.5) \qquad t_o = \frac{y_B}{x_B - x_{A'}} \,,$$

$$(3.6) \qquad u_o = q_o / \sqrt{1 + t_o^2} \,,$$

and

$$(3.7) \qquad v_o = q_o t_o / \sqrt{1 + t_o^2} \,.$$

Integration is to stop when u, v and t satisfy equation

$$(3.8) \qquad (\gamma+1)t\,(u+vt)(tu-v) = (\gamma-1)\left[1 + t^2 - (u + vt)^2\right]$$

derived from shock conditions 2.I g and h by elimination of q_1. q_1 is then determined from equation

$$(3.9) \qquad q_1 = u + vt.$$

The value of q_1 so obtained will vary with q_o. Therefore in order to obtain the flow at a prespecified Mach number or value of q_1 we shall have to modify q_o in a way governed by the variation of q_1.

Distributing the data at equal intervals of y yields a poor distribution in the hodograph plane. It is better to use something dependent on the velocity as independent variable, e.g. $\frac{v}{u} = T$. If this be done, it may be shown as above that the differential equations are

$$(3.2') \quad \begin{cases} u' = \dfrac{du}{dT} = \dfrac{-yu}{x+yT} \\ z' = -\,y\,a\,A\,f(va + u\sqrt{q^2 - a^2}) \\ x' = (a^2 - u^2)f \\ y' = -\,(uv + a\sqrt{q^2 - a^2})\,f \\ v = uT, \end{cases} \qquad f = \frac{-yu\left(\frac{L}{K-R}\,x - y\right)}{a^2 v(x+yT)}$$

with initial conditions

$$(3.5') \qquad \begin{aligned} T_o &= y_B/(X_B - X_{A'}) \,, \\ u_o &= q_o / \sqrt{1 + T_o^2} \,, \\ z_o &= 0 \,, \\ x_o &= x_B \,, \end{aligned}$$

and

$$y_o = y_B \,;$$

and terminal conditions to determine q_0 and q_1 :

$$(3.6') \qquad (ux+vy)\left[(y^2+x^2)u - x(ux+vy)\right] = \left(\frac{\gamma-1}{\gamma+1}\right)x\left[x^2+y^2-(ux+vy)^2\right]$$

$$q_1 = \frac{ux + vy}{x} \quad .$$

If the problem is to be done by hand, equations (3.2), (3.3), and (3.4) may be solved first; α and β found later. With the ENIAC, however, this would waste time since more cards would have to be read. From equation 2.Ic we find that

$$(3.10) \qquad z' + \frac{yA}{K-R}\left[u(K-R) - vH\right] = \frac{yAa}{L}\left[a u - v\sqrt{q^2-a^2}\right].$$

Freedom in Choice of α and β .

As for β , let us consider more generally the determination of α and β throughout the whole region in which we shall seek the values of x, y, u, v, z. Suppose that we assign β arbitrarily on BC increasing from B to C (Figure 3.3). Assuming, as we have, that the characteristics have no envelopes and that therefore there is one characteristic of each family through each point, the values of β in BC determine the values of β through the region BCD but have no effect on the values of α in that region. Accordingly we are free to make α an arbitrary increasing function on BD. This will determine α in the region BDEC but will not affect the values of β in CDE. (E may be at infinity since DE may not intersect the shock wave.) Step by step it is seen that β may be assigned arbitrarily (we shall make it increase from B to H) along BCEH and α may be assigned arbitrarily along BDFI (we shall make it increase from D to I). Indeed we may make $\alpha = \bar{\alpha}(\beta)$, $\beta = \bar{\alpha}^{-1}(\alpha)$ along BDFI and $\beta = \bar{\beta}(\alpha)$, $\alpha = \bar{\beta}^{-1}(\beta)$ along CEH where $\bar{\alpha}(\beta)$ and $\bar{\beta}(\alpha)$ are non-decreasing functions. In order to make the map of the region BDFIHECB on to a portion of the $\alpha\beta$ plane a one to one map, it is necessary to forbid the maps of BDFI and CEH to have any points in common. Because there are two arbitrary functions at one's disposal in the assignment of parameters there are many choices available. One could let $\beta = y$ on BCEH and $\alpha = x$ on BDFI; one

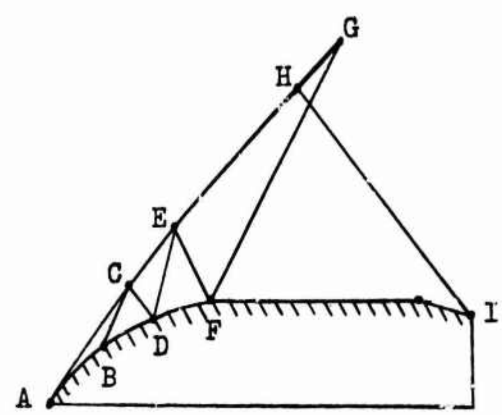


Figure 3.3

could let $\alpha = \beta = x$ on BDFI or on CEHG (in the latter case, this leaves α undefined downstream of the characteristic $\alpha = \infty$ which may intersect the body somewhere between D and I); one could let $\beta = y$ on CEHG, or if s is arc length along BDFI, s' is arc length along CEHG and σ is curvature along either, one could let

$$\frac{d\alpha}{ds} = \alpha_0 + \alpha_1 \, \sigma(s) \qquad \text{on BDFI} ,$$

$$\frac{d\beta}{ds'} = \beta_0 + \beta_1 \, \sigma(s') \qquad \text{on CEHG}$$

where $\alpha_0, \alpha_1, \beta_0, \beta_1$ are positive constants, so computations carried out on a square grid in the α , β plane would correspond in the physical plane to a fine grid where the velocity is changing rapidly and a large grid where it is changing slowly. Other choices will suggest themselves to the reader. To facilitate the simultaneous programming of whole families of flows for ENIAC and other machines we shall in this report usually map the shock-wave onto a line of slope one and the given contour onto lines of slope $\frac{1}{2}$ or 1.

Corners

From a point on the given contour such as D, F or I on Figure 3.4 where the slope is decreasing but discontinuous, it is known that a family of characteristics α = constant emanates, but only one characteristic β = constant. Therefore points D, F and I must map onto horizontal line segments D'...D', F'....F' and I'...I'. Once again the assignment of α is arbitrary.

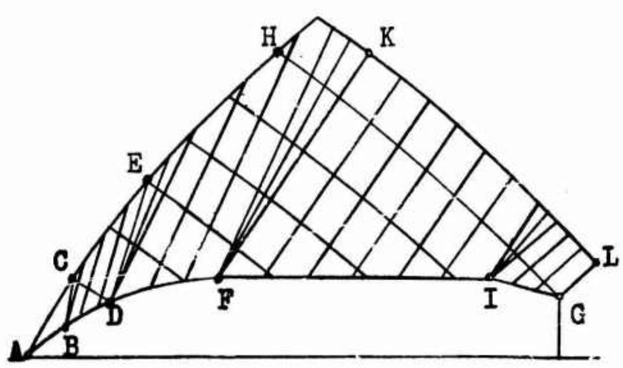


Figure 3.4

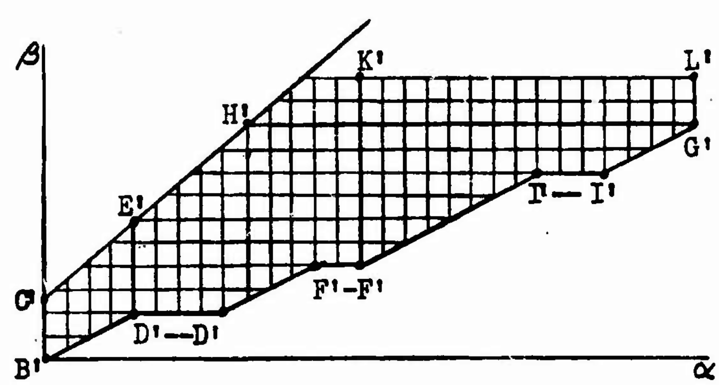


Figure 3.5

Terminal Boundary to Flow Computations

Since the velocity at G is independent of the velocity at points to the right of characteristic HG, there is no reason to compute the flow beyond this curve if, for example, the pressure distribution along A'BDFIG is desired. The flow can be computed in the region HGF bounded by the characteristic HG. It will have to be computed also in some region HGLK in order to determine the flow in the base region. At the present, there is no known satisfactory way of doing this because the non-viscous steady flow model is not applicable to the wake, as a glance at a shadowgraph of a wake will suggest. Thus the object of this report will be to determine the velocity at a net-work of points in the region R or R' shown in Figures 3.4 and 3.5.

Returning to the determination of the flow along the characteristic BC, we shall simply set $\alpha = 0$ and $\beta = \beta_o(y - y_B)$. Summarizing: along BC

a) $\dfrac{dt}{dy} = \dfrac{t(K-R-tH)}{y\,(K - R)}$

b) $t = \dfrac{y}{x-x_{A'}}$

c) $\dfrac{du}{dy} = \dfrac{-a^2 v t}{y\,[t(K-R)-L]}$

d) $\dfrac{dv}{dy} = \dfrac{a^2 v}{y\,[t(K-R)-L]}$

e) $\dfrac{dz}{dy} = \dfrac{yA}{K-R}\,[u(K-R)-vH]$

f) $\beta = \beta_o(y - y_B)$

g) $\alpha = 0$

h) $t_o = \dfrac{y_B}{X_B-X_{A'}}$, $\quad u_o = \dfrac{q_o}{\sqrt{1 + t_o^{\,2}}}$, $\quad v_o = \dfrac{q_o t_o}{\sqrt{1+t_o^{\,2}}}$

i) $(\gamma+1)t_w(u_w+v_w t_w)(t_w u_w-v_w) = (\gamma-1)\left[(1+t_w^2)-(u_w+v_w t_w)^2\right]$

j) $q_1 = u_w + v_w t_w$

k) $u' = \dfrac{du}{dT} = -\,yu/(x + yT)$

l) $z' = -ya\,Af(v\,a + u\sqrt{q^2 - a^2})$

m) $x' = (a^2 - u^2)f$

n) $y' = -\,(uv + a\sqrt{q^2 - a^2}\,)\,f$

$$
3.\mathrm{I}\begin{cases}
\text{o)} & v = u\,T \\
\text{p)} & f = -\,y\,u\,\left[\left\{L/(K-R)\right\}\,x - y\right]\Big/\left[a^2 v(x + yT)\right] \\
\text{q)} & q^2 = u^2 + v^2, \quad a^2 = \dfrac{\gamma - 1}{2}\,(1 - q^2) \\
\text{r)} & H = a^2 - u^2, \quad K = -uv, \quad L = a^2 - v^2, \quad R = a\sqrt{q^2 - a^2} \\
\text{s)} & A = (1 - q^2)^{\frac{1}{\gamma - 1}} \\
\text{t)} & \begin{cases} T_o = y_B/(x_B - x_{A'}) \\ u_o = q_o/\sqrt{1 + T_o^{\,2}} \\ z_o = 0 \\ x_o = x_B \\ y_o = y_B \end{cases} \\
\text{u)} & \beta = (T - T_o)/(T_1 - T_o) \\
\text{v)} & (ux + vy)\left[(y^2 + x^2)u - x(ux + vy)\right] = \left(\dfrac{\gamma - 1}{\gamma + 1}\right) x \left[x^2 + y^2 - (ux + vy)^2\right] \\
\text{w)} & q_1 = (ux + vy)/x
\end{cases}
$$

t, u, v, z, x, β are determined at equally spaced intervals of y by solving equations 3.I a,b,c,d,e,f with initial conditions 3.Ih and terminal conditions 3.I i,j; or u,z,x,y,v, β are determined at equally spaced intervals of T by solving equations 3.I k--n, u, with initial conditions 3.It and terminal conditions, 3.I v,w.

Contour Process

Once the initial data has been determined, flow variables may be found at the intersection D_1 of the given contour and a characteristic β = constant through a point B_1 on BC. The process involved in doing this we

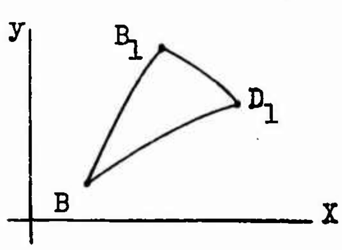


Figure 3.6

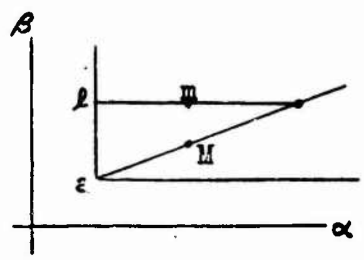


Figure 3.7

shall call the contour process. It will be used every time the flow variables are to be found at a new point on the contour.

Since only α varies in passing from B_1 to D_1 two of the equations to be used are 2.IIc and 2.IIe

$$(3.11) \qquad y_\alpha = \lambda x_\alpha ,$$

and

$$(3.12) \qquad \omega u_\alpha + v_\alpha + \frac{x_\alpha}{K-R}(\epsilon P + Q) = 0.$$

Actually Q will be zero since the shock is straight near $z = 0$. The other three equations are the boundary conditions 2.I i,j

$$(3.13) \qquad y = F(x),$$

$$(3.14) \qquad v = uF'(x) = uG(x)$$

and the condition

$$(3.15) \qquad z = 0.$$

A simple procedure which can be used to find the flow variables at D_1 is simply to replace $\frac{\partial x}{\partial \alpha}$ by $\frac{x_{D_1} - x_{B_1}}{\Delta \alpha}$ etc., obtaining linear equations for x, y, u, v. The justification for this procedure is given in Frankl and Aleksieva's paper which applies to our problem once we assume that the flow on BC is correctly determined. Instead of using equation (3.13) as it is we prefer to differentiate it in order to obtain a linear equation in x_{D_1} and y_{D_1}

$$(3.13') \qquad \frac{dy}{dx} = G(x).$$

Let us denote the map of B_1, B and D_1 in the $\alpha\,\beta$ plane by ℓ , a, and no name, (Figure 3.7) and denote the corresponding x,y,u,v,z by x_ℓ, y_ℓ, u_ℓ, v_ℓ, z_ℓ, x_a, y_a, u_a, v_a, z_a, x, y, u, v, z. Then we get, using the suggested procedure, the following linear equations

$$(3.16) \qquad y - y_a = F'(x_a)(x - x_a) ,$$

$$(3.17) \qquad y - y_\ell = \lambda_\ell (x - x_\ell)' ,$$

$$(3.18) \qquad v = u\,G ,$$

$$(3.19) \qquad \omega_\ell (u - u_\ell) + v - v_\ell + \epsilon \frac{P_\ell}{K_\ell - R_\ell}(x - x_\ell) = 0,$$

and

$$(3.20) \qquad z = 0 ;$$

which, solved for x, y, u, v, z give

$$(3.21) \qquad x = \left[y_\ell - y_a + F'_a x_a - \lambda_\ell x_\ell\right] / (F'_a - \lambda_\ell),$$

$$(3.22) \qquad y = \left[y_\ell F'_a - \lambda_\ell y_a + \lambda_\ell F'_a(x_a - x_\ell)\right] / (F'_a - \lambda_\ell),$$

$$(3.23) \qquad z = 0 ,$$

$$(3.24) \qquad u = \left[\omega_\ell u_\ell + v_\ell - \epsilon \frac{P_\ell}{K_\ell - R_\ell} (x - x_\ell) \right] / (\omega_\ell + G),$$

and

$$(3.25) \qquad v = G \left[\omega_\ell u_\ell + v_\ell - \epsilon \frac{P_\ell}{K_\ell - R_\ell} (x - x_\ell) \right] / (\omega_\ell + G).$$

This represents the simplest possible contour process. Clearly many refinements may be made. For example, it would be preferable to evaluate λ, ω and P at m, the midpoint of ℓ and the desired point, and G at M, the midpoint of a and the desired point. This may be done by extrapolation or by integration. We shall reserve discussion of such refinements for the next section.

General Process

Once the flow variables have been found at D_1 or any other point on the contour, the next step is to find them at a point P at the intersection of a characteristic β = constant through B_2 on the initial line and a characteristic α = constant through D_1. More generally, given the flow variables at any two points ℓ and u not on the same characteristic we may find them at the intersection P of characteristics β = constant and α = constant through ℓ and u respectively. We shall call the process for

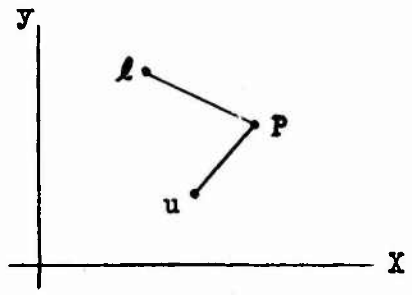


Figure 3.8

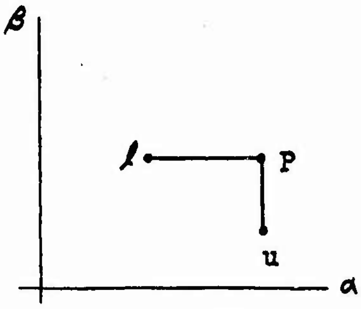


Figure 3.9

doing this the general process.

The equations to be used to this end are equations 2.II cdefh'

$$(3.26) \qquad y_\alpha = \lambda x_\alpha ,$$

$$(3.27) \qquad \omega y_\beta = x_\beta ,$$

$$(3.28) \qquad \omega u_\alpha + v_\alpha + x_\alpha \left(\epsilon \frac{P}{K-R} + \frac{Q}{K-R} \right) = 0 ,$$

$$(3.29) \qquad u_\beta + \lambda v_\beta + y_\beta \left(\epsilon \frac{P}{K-R} - \frac{Q}{K-R} \right) = 0,$$

and

$$(3.30) \qquad dz = yA\,(-v\,dx + u\,dy).$$

The method again is to replace partial derivatives by difference quotients. There is no need for the grid sizes in α and β to be equal since the difference equations do not contain $\Delta\alpha$ or $\Delta\beta$:

$$(3.31)\quad y - \bar{\lambda} x = y_{\ell} - \bar{\lambda} x_{\ell} ,$$
$$(3.32)\quad \bar{\omega} y - x = \bar{\omega} y_u - x_u ,$$
$$(3.33)\quad \bar{\omega} u + v = \bar{\omega} u_{\ell} + v_{\ell} - (x - x_{\ell})(\bar{S} + \bar{T}) ,$$
$$(3.34)\quad u + \bar{\lambda} v = u_u + \bar{\lambda} v_u - (y - y_u)(\bar{S} - \bar{T}) ,$$

and

$$(3.35)\quad 2z = z_{\ell} + z_u + \bar{y}\bar{A}\left[-(2x - x_{\ell} - x_u)\bar{v} + (2y - y_{\ell} - y_u)\bar{u}\right] ,$$

where $S = \frac{\epsilon P}{K-R}$ and $T = \frac{Q}{K-R}$.

Instead of evaluating the quantitites $\bar{\lambda}$, $\bar{y}$, $\bar{A}$, etc. at the midpoints of ℓ and P, and u and P we suggest in this simple general process evaluating them at the midpoint of ℓ and u; i.e.,

$$\bar{y} = \frac{y_{\ell} + y_u}{2}$$

$$\bar{u} = \frac{u_{\ell} + u_u}{2}$$

$$\bar{q}^2 = \bar{u}^2 + \bar{v}^2$$

$$\bar{A} = A(\bar{q}^2),$$

etc. This simplifies the computation and is all that is justified until a more careful procedure is described in the next section. Solution of equations 3.31-3.35 yields x, y, u, v, z at P.

$$(3.36)\quad x = \left[\bar{\omega}(y_{\ell} - \bar{\lambda} x_{\ell}) - (\bar{\omega} y_u - x_u)\right] /(1 - \bar{\lambda}\bar{\omega}),$$
$$(3.37)\quad y = \left[(y_{\ell} - \bar{\lambda} x_{\ell}) - \bar{\lambda}(\bar{\omega} y_u - x_u)\right] /(1 - \bar{\lambda}\bar{\omega}),$$
$$(3.38)\quad u = \left[\left\{u_u + \bar{\lambda} v_u - (y-y_u)(\bar{S}-\bar{T})\right\} - \bar{\lambda}\left\{\bar{\omega} u_{\ell} + v_{\ell} - (x-x_{\ell})(\bar{S}+\bar{T})\right\}\right] / (1- \bar{\lambda}\bar{\omega}),$$
$$(3.39)\quad v = \left[\left\{\bar{\omega} u_{\ell} + v_{\ell} - (x-x_{\ell})(\bar{S}+\bar{T})\right\} - \bar{\omega}\left\{u_u + \bar{\lambda} v_u - (y-y_u)(\bar{S}-\bar{T})\right\}\right] / (1- \bar{\lambda}\bar{\omega}),$$
$$(3.40)\quad z = \tfrac{1}{2}\left[z_{\ell} + z_u + \bar{y}\bar{A}\left\{-(2x-x_{\ell}-x_u)\bar{v} + (2y - y_{\ell} - y_u)\bar{u}\right\}\right] .$$

Any hand computation must be accompanied by checks. The values of H, K, L and R may be checked by the identity

$$HL \equiv K^2 - R^2 ;$$

the solutions (3.36) - (3.40) by substitution into some of equations (3.31)-(3.34) the value of z by the formula

$$z = z_{\ell} + \bar{y}\bar{A}\left[-(x - x_{\ell})\bar{v} + (y - y_{\ell})\bar{u}\right]$$

etc.

Shock Process

Similar to the contour process though more complicated is the other boundary process which gives the flow variables at a new point on the shock wave. In this case the flow variables are known at a point a on the shock wave and another point u lying on a characteristic β = constant through a. The point P lies on the shock wave and a characteristic α = constant through u.

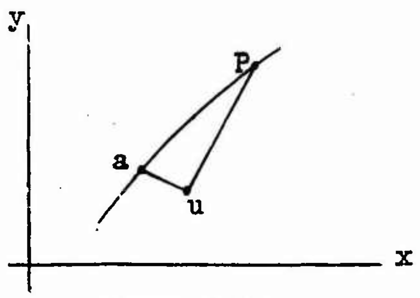


Figure 3.10

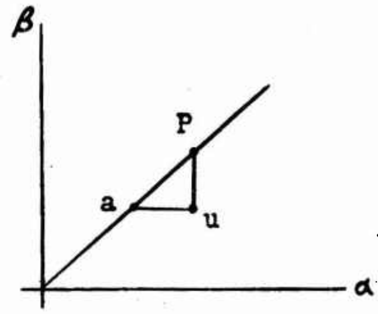


Figure 3.11

The equations we shall use are 2.II d,f, h' and 2.I g,h,b'

$$(3.41) \qquad \omega y_\beta = x_\beta \, ,$$

$$(3.42) \qquad \frac{dy}{dx} = \frac{q_1 - u}{v} \, ,$$

$$(3.43) \qquad u_\beta + \lambda v_\beta + y_\beta \left(\frac{\epsilon P - Q}{K-R} \right) = 0,$$

$$(3.44) \qquad v^2(b - u) = (q_1 - u)^2(u - d), \text{ or } \frac{dv}{du} = v'(u,v),$$

$$(3.45) \qquad dz = yA\,(- v\,dx + u\,dy),$$

and

$$(3.46) \qquad g \equiv \frac{(u - q_1)^2}{(u-b)(u-1/q_1)(\gamma u-e)} \quad \frac{du}{dz} \, .$$

The procedure is as before; the coefficients in equations (3.41) and (3.43) may be evaluated at u, the other at a. The results for x, y, u, v, z are

$$(3.47) \qquad x = \left[\omega_u h_a x_a - x_u + \omega_u (y_u - y_a)\right] / (\omega_u h_a - 1),$$

$$(3.48) \qquad y = \left[h_a \omega_u y_u - y_a - h_a (x_u - x_a)\right] / (\omega_u h_a - 1),$$

$$(3.49) \qquad u = \left[\lambda_u f_a u_a + u_u + \lambda_u (v_u - v_a) - (y - y_u)(S_u - T_u)\right] / (\lambda_u f_a + 1),$$

$$(3.50) \qquad v = \left[f_a \lambda_u v_u + v_a + f_a (u_u - u_a) - f_a (y - y_u)(S_u - T_u)\right] / (\lambda_u f_a + 1),$$

$$(3.51) \qquad z = z_a + y_a A_a \left[-v_a (x - x_a) + u_a (y - y_a)\right] ,$$

where

$$h_a = (q_1 - u_a)/v_a$$

$$f_a = \left[(q_1 - u_a) \left\{ (q_1 - u_a) + 2(d - u_a)\right\} + v_a^2\right] / \left[2v_a (1 + \lambda_u f_a)\right] \, .$$

When the flow variables are determined at P, g may be computed at the midpoint of aP by equation (3.46) using

$$\frac{du}{dz} = \frac{u - u_a}{z - z_a} \quad .$$

Corner Process

We have seen that a corner such as D (Figure 3.2) maps into a line-segment D---D in the $\alpha\,\beta$ plane. Along DD u, v, and α vary although x, y,z are fixed. α may be related to u and v in any practical way such as

$$(3.52) \quad \alpha = \frac{(\alpha_1 - \alpha_2)u_1\,u_2}{v_1\,u_2 - v_2\,u_1}\;\frac{v}{u} + \frac{\alpha_2 v_1 u_2 - \alpha_1 v_2 u_1}{v_1 u_2 - u_1 v_2} \quad ,$$

a linear function of the tangent of the velocity inclination.

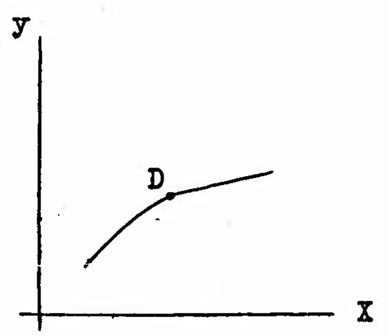


Figure 3.12

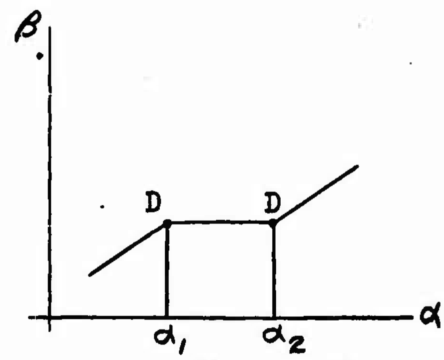


Figure 3.13

Since x is constant at D,

$$x_\alpha = 0 \;,$$

and thus from 2.IIe

$$(3.53) \quad \omega u_\alpha + v_\alpha = 0 \;,$$

or

$$(3.54) \quad \omega du + dv = 0 \quad .$$

This is an ordinary differential equation in u and v whose solution is an epicycloid in the u,v plane. The solution has been tabulated in various places[3.3]. Thus it is simple to obtain a set of points along D---D. However, if more accuracy is required than three significant figures, and if it is desirable to space the data at equal increments in v/u, then it is simpler to modify equation (3.53) introducing v/u = t as independent variable

3.3 E.g., N.A.C.A. Technical Note 1428, Dec. 1947, Notes and Tables for Use in the Analysis of Supersonic Flow.

$$(3.55) \qquad \frac{du}{dt} = \frac{-u}{\omega + t} ,$$

and solve directly.

Section 4. Refined Numerical Methods

Introduction

In section 3 the problem of determining numerically the supersonic steady frictionless flow past a body of revolution was solved in a simple way. Having assumed that the flow near the nose could be reasonably well approximated by the Taylor-Maccoll flow with the smaller shock wave angle, the paper of Frankl and Aleksieva[1.7] proves that there exists a unique solution and that the approximation outlined in the last section converges to this solution as the mesh size approaches zero (if we ignore round-off errors).

However, in the last section, little attention was paid to the problem of getting the best approximation for a given amount of labor by hand or by machine. This is a problem which can never be completely solved. Nevertheless, in this section we shall examine some aspects of the problem and show how some of the computations previously described can be done to a given accuracy more easily. We shall indicate in several places how a problem should be treated depending on whether it is done by hand, or by a fast machine of small memory, or by a fast machine of large memory.

Systems of Ordinary Differential Equations with Initial Conditions

Given a system

$$(4.1) \qquad y_i' = f_i(y_1, \ldots, y_m, x), \qquad i = 1, 2, \ldots, m, \qquad y_i = y_i(x),$$

of ordinary differential equations with initial conditions

$$(4.2) \qquad y_i(x_o) = y_{i_o} .$$

The numerical method of solution most commonly used for hand computation at the BRL is due to Moulton[4.1], although it differs only a little from a method used earlier by Adams. Like most numerical methods it assumes that the solutions may be closely approximated on short intervals by polynomials of suitable degree. The polynomial of degree n passing through n+1 points has been found in various forms by Gregory (1670), Newton (1687), Waring (1779), Lagrange (1795). If Gregory's formula for

$$(4.3) \qquad f_i(y_1, \ldots, y_m, x) = F_i(x)$$

4.1 Bennett, Milne, and Bateman, "Numerical Integration of Differential Equations", Bulletin of the National Research Council No. 92, Nov. 1933, pp. 74, 75, 80.

be integrated from x_n to x_{n+1} the following formulae result:

$$(4.4) \quad \int_{x_n}^{x_{n+1}} F_i(x)dx = h\left[F_{i,n+1} - \tfrac{1}{2}\nabla F_{i,n+1} - \tfrac{1}{12}\nabla^2 F_{i,n+1}\right] +$$

$$+ h\left[-\tfrac{1}{24}\nabla^3 F_{i,n+1} - \tfrac{19}{720}\nabla^4 F_{i,n+1} - \tfrac{3}{160}\nabla^5 F_{i,n+1}\right] +$$

$$+ h\left[-\tfrac{863}{60480}\nabla^6 F_{i,n+1} - \cdots\right] \quad ,$$

where

$$(4.5) \quad \begin{cases} h = x_{n+1} - x_n \\ F_{i,n+1} = f_i(y_{1,n+1}, y_{2,n+1}, \cdots y_{m,n+1}, x_{n+1}) \\ \nabla F_{i,n+1} = F_{i,n+1} - F_{i,n} \\ \nabla^2 F_{i,n+1} = \nabla\left[\nabla F_{i,n+1} - \nabla F_{i,n}\right] = F_{i,n+1} - 2F_{i,n} + F_{i,n-1} \\ \quad \vdots \qquad \vdots \qquad \vdots \\ \nabla^j F_{i,n+1} = \nabla\left[\nabla^{j-1} F_{i,n+1} - \nabla^{j-1} F_{i,n}\right] . \end{cases}$$

Moulton's method is the following: values of $y_{i,1}$, $F_{i,1}$, $y_{i,2}$, $F_{i,2}, \ldots, y_{i,j}$, $F_{i,j}$ are obtained corresponding to $x_1, x_2, \ldots, x_j$ by special means such as Taylor's series. These are arranged on a computing form as follows:

(4.6)

x_0	$y_{1,0}$	$F_{1,0}$					$y_{2,0}$	$F_{2,0}$					$\cdots$	$y_{m,0}$	$F_{m,0}$				
x_1	$y_{1,1}$	$F_{1,1}$	$\nabla F_{1,1}$				$y_{2,1}$	$F_{2,1}$	$\nabla F_{2,1}$				$\cdots$	$y_{m,1}$	$F_{m,1}$	$\nabla F_{m,1}$			
x_2	$y_{1,2}$	$F_{1,2}$	$\nabla F_{1,2}$	$\nabla^2 F_{1,2}$			$y_{2,2}$	$F_{2,2}$	$\nabla F_{2,2}$	$\nabla^2 F_{2,2}$			$\cdots$	$y_{m,2}$	$F_{m,2}$	$\nabla F_{m,2}$	$\nabla^2 F_{m,2}$		
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$		$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$		$\cdot$		$\cdot$	$\cdot$	$\cdot$	$\cdot$		
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$		$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$		$\cdot$		$\cdot$	$\cdot$	$\cdot$	$\cdot$		
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$		$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$		$\cdot$		$\cdot$	$\cdot$	$\cdot$	$\cdot$		
x_j	$y_{1,j}$	$F_{1,j}$	$\nabla F_{1,j}$	$\nabla^2 F_{1,j}$	$\cdots$	$\nabla^{j-1} F_{1,j}$	$y_{2,j}$	$F_{2,j}$	$\nabla F_{2,j}$	$\nabla^2 F_{2,j}$	$\cdots$	$\nabla^{j-1} F_{2,j}$	$\cdots$	$y_{m,j}$	$F_{m,j}$	$\nabla F_{m,j}$	$\nabla^2 F_{m,j}$	$\cdots$	$\nabla^{j-1} F_{m,j}$

A trial value for $F_{i,j+1}$ is then secured by extrapolation or otherwise, and $y_{i,j+1}$ is computed from the formula

$$(4.7) \qquad y_{i,j+1} = y_{i,j} + \int_{x_j}^{x_{j+1}} F_i dx,$$

using equation (4.4).

A corrected value of $F_{i,j+1}$ is then found from equation (4.1) and the process repeated until no change occurs. The values of x_{j+1}, $y_{1,j+1}$, $F_{1,j+1}$, ..., $\nabla^{j-1} F_{1,j+1}, \ldots, \nabla^{j-1} F_{m,j+1}$ are entered in the table. Values of $y_{i,j+2}$, $y_{i,j+3}$, etc., are computed in the same way until the end of the interval is reached. Practically, differences beyond the third are not often used so that the solution is approximated by a fourth degree polynomial. We shall refer to it then as a fourth order method. More generally we shall say that if j-1 differences are used the method is a jth order method, because if the functions y_i were analytic we should be making truncation errors of the form $\left[d^{j+1}y/dx^{j+1}\right] \left[\delta^{j+1}/(j+1)!\right]$ in each interval. It is customary to choose j conveniently and then to adjust the size of the interval so that $\nabla^j F_{i,k}$ are negligible. This may require several changes of interval size during the computation. It will be noted that if it be necessary to halve the interval, then auxiliary points must be inserted using interpolation formulae. As a guard against errors, hand computations are frequently checked using equation (4.7) over larger intervals and, for example, Simpson's Rule and/or Weddle's Formula.

Moulton's method, though convenient for hand computation, has three defects for machine computation which are avoided by other methods. These are: (a) the number of quantities which must be remembered in going from x_k to x_{k+1}, namely, $[(j+1)m+1]$; (b) the fact that early steps are different from later steps; and (c) the necessity of using interpolation formulae in reducing the interval size. In setting up a problem for a machine, if there are N registers available to remember numbers, then $1+(j+1)m$ must be less than or equal to N. Thus if a fourth order approximation is to be used, then m must be less than or equal to $(N-1)/5$. For example, N will be about twelve to fourteen for the ENIAC if ten figures are to be used. Let us in future discussions let $N = 13$; thus only a second order system could be handled by Moulton's method on the ENIAC.

Most of the other common methods such as those of Adams, Steffensen and Milne are subject to the same objections for machine work and accordingly will not often be used for high-speed machines.

The best known methods which are less open to objections (a), (b), and (c) are those of Runge and Kutta[4.3]. The step from x_k to x_{k+1} is the same for all k with these methods. For example, their fourth order method is described almost completely by the formulae

$$(4.8)\begin{cases} y_{i,n+1} = y_{i,n} + (k_{i,1} + 2k_{i,2} + 2k_{i,3} + k_{i,4})/6, \\ \text{where} \\ k_{i,1} = h\, f_i\,(x_n,\, y_{1,n},\, \cdots\, y_{m,n}) \\ k_{i,2} = h\, f_i\,(x_n + \tfrac{1}{2}h,\, y_{1,n} + \tfrac{1}{2}k_{1,1},\, \cdots,\, y_{m,n} + \tfrac{1}{2}k_{m,1}) \\ k_{i,3} = h\, f_i\,(x_n + \tfrac{1}{2}h,\, y_{1,n} + \tfrac{1}{2}k_{1,2},\, \cdots,\, y_{m,n} + \tfrac{1}{2}k_{m,2}) \\ k_{i,4} = h\, f_i\,(x_n + h,\, y_{1,n} + k_{1,3},\, \cdots,\, y_{m,n} + k_{m,3}) \end{cases}$$

$$(i = 1, 2, \ldots, m).$$

This method is free from defects (b) and (c). As for (a) there is one stage of the computation when x_n, $y_{1,n}$, ..., $y_{m,n}$, x_n, $y_{1,n} + (k_{1,1}+2k_{1,2})/6$, $y_{2,n} + (k_{2,1} + 2k_{2,2})/6$, ..., $y_{m,n} + (k_{m,1} + 2k_{m,2})/6$, $k_{1,2}$, ..., $k_{m,2}$, $k_{1,3}$, $k_{2,3}$, ..., $k_{m-1,3}$ must be remembered, i.e., 1 + 4m quantities. In this case, therefore, if N = 12, m = 3; so a system of three equations could be handled by the ENIAC if the Runge-Kutta fourth order method is used and 13 registers are available. This method is therefore superior to the previously described methods for machines of limited storage space. It will be observed that the functions f_i are used four times in going from x_n to x_{n+1} as compared to once by the Moulton and similar methods. Usually the formation of these functions is the most difficult part of the problem and thus the Runge-Kutta method is in most cases more tedious for hand computation. However, for machines such as the ENIAC where computing time may be a small fraction of total time spent on a problem, and where it is nearly as easy to instruct it to form f_i four times as one, this is no drawback.

Another method which is somewhat similar to the Runge-Kutta method in adaptability to high speed machines but superior in regard to space, is described below. We have not found it in the literature, (although it may have been known to Gregory, Euler, Newton, or Lagrange). Therefore we shall derive it in detail. In this development we shall always assume that i runs from 1 to m. Let us assume that the functions f_i are analytic; then y_i are also analytic. Let us denote $y_i(x_n + \frac{h}{2})$ by $\bar{y}_i$, $\frac{dy_i}{dx}(x_n + \frac{h}{2})$ by $\bar{y}_i'$, etc. Expanding $y_{i,n}$, $y_{i,n}'$, $y_{i,n+1}$, $y_{i,n+1}'$ about $x_n + \frac{h}{2}$ we have:

4.3 Bennett, Milne, and Bateman, Op. cit., pp. 77-80

$$
(4.9)\begin{cases}
y_{i,n+1} = \bar{y}_i + \bar{y}_i'\dfrac{h}{2} + \bar{y}_i''\dfrac{h^2}{8} + \bar{y}_i'''\dfrac{h^3}{48} + \bar{y}_i''''\dfrac{h^4}{384} + 0(h^5) \\
y_{i,n} = \bar{y}_i - \bar{y}_i'\dfrac{h}{2} + \bar{y}_i''\dfrac{h^2}{8} - \bar{y}_i'''\dfrac{h^3}{48} + \bar{y}_i''''\dfrac{h^4}{384} + 0(h^5) \\
y_{i,n+1}' = \bar{y}_i' + \bar{y}_i''\dfrac{h}{2} + \bar{y}_i'''\dfrac{h^2}{8} + \bar{y}_i''''\dfrac{h^3}{48} + 0(h^4) \\
y_{i,n}' = \bar{y}_i' - \bar{y}_i''\dfrac{h}{2} + \bar{y}_i'''\dfrac{h^2}{8} - \bar{y}_i'''' + \dfrac{h^3}{48} + 0(h^4),
\end{cases}
$$

where $0(h^n)/h^n$ is a quantity which approaches a finite limit as h approaches 0.

Adding and subtracting the first and second pairs of equations:

$$
(4.10)\begin{cases}
\text{a)} \quad y_{i,n+1} - y_{i,n} = \bar{y}_i'h + \bar{y}_i'''\dfrac{h^3}{24} + 0(h^5) \\
\text{b)} \quad y_{i,n+1} + y_{i,n} = 2\bar{y}_i + \bar{y}_i''\dfrac{h^2}{4} + 0(h^4) \\
\text{c)} \quad y_{i,n+1}' - y_{i,n}' = \bar{y}_i'' h + 0(h^3) \\
\text{d)} \quad y_{i,n+1}' + y_{i,n}' = 2\bar{y}_i + \bar{y}_i'''\dfrac{h^2}{4} + 0(h^4) .
\end{cases}
$$

Assume now that $y_{i,n}$ and x_n are known and that $y_{i,n+1}$ is known to order j; i.e.,

$$y_{i,n+1} = y_{i,n+1,j} + 0(h^{j+1})$$

where j = 0, 1, 2, or 3. Then the following sequence of computations will yield $y_{i,n+1,j+1}$

$$
(4.11)\begin{cases}
\text{a)} \ y_{i,n}' = f_i(y_{1,n}, y_{2,n}, \dots, y_{m,n}\ x_n) & \text{equa. (4.1)} \\
\text{b)} \ y_{i,n+1,j}' = f_i(y_{1,n+1,j}, \dots, y_{m,n+1,j}, x_n+h) + 0(h^{j+1}) & \text{equa. (4.1)} \\
\text{c)} \ \bar{y}''_{i,j}h = y_{i,n+1,j}' - y_{i,n}' + 0(h^j) & \text{equa. (4.10)c} \\
\text{d)} \ 2\bar{y}_{i,j} = y_{i,n+1,j} + y_{i,n} - \bar{y}''_{i,j-2}\, h^2/4 + 0(h^{j+1}) & \text{equa. (4.10)b} \\
\text{e)} \ \bar{y}_{i,j}' = f_i(\bar{y}_{1,j}, \bar{y}_{2,j}, \dots, \bar{y}_{m,j}, x_n + h) + 0(h^{j+1}) & \text{equa. (4.1)} \\
\text{f)} \ \bar{y}'''_{i,j-2}\, h^2/4 = y_{i,n+1,j}' + y_{i,n}' - 2\bar{y}_{i,j}' + 0(h^{j+1}) & \text{equa. (4.10)d} \\
\text{g)} \ y_{i,n+1,j+1} = y_{i,n} + \bar{y}_{i,j}'h + \bar{y}'''_{i,j-2}\, h^3/24 + 0(h^{j+2}) & \text{equa. (4.11)a}
\end{cases}
$$

By repeating this sequence of operation 4 times $y_{i,n+1}$ will be obtained to fourth order. If a higher order approximation is desired, it is necessary to adjoin more points, or differentiate the differential equations to have more equations. Since fourth order is a convenient approximation, we shall not discuss the other possibilities.

Let us compare this method now with the previously discussed ones. Here, as in the Runge-Kutta method, there is no difference between the step from x_0 to x_1 and the step from x_n to x_{n+1}; nor is there any difficulty in changing the interval. As for the tax on memory, the moment of greatest storage requirement occurs when x_n, $y_{i,n}$, $y_{i,n+1,j}$ and $\bar{y}_{i,j}$ must all be held in order to form $\bar{y}'''_{i,j-2}\, h^2/4$ ($y_{i,n+1,j}$ may be substituted for $y_{i,n+1,j-1}$ as soon as formed). Thus the inequality $1 + 3m \leqq M$ must be satisfied. For the ENIAC, therefore, $m \leqq 4$; i.e., it is possible to approximate to fourth order a system of four equations.

The price for being able to solve 4 instead of 3 equations is the formation of f_i 20 times instead of 4 as in the Runge-Kutta method; just as the price for being able to solve 3 instead of 2 equations is the formation of f_i 4 times instead of one.

If the process described above is stopped at the second iteration, it amounts to the Heun method:

$$(4.12)\quad y_{i,n+1}=y_{i,n}+\frac{h}{2}\left[f_i(y_{1,n},\dots,y_{m,n},x_n)+f_i(y_{1,n}+hf_1\{y_{1,n},\dots,y_{m,n},x_n\},\dots,x_{n+1})\right].$$

This method yields only a 2nd order approximation to the solution but has the advantage that only 1 + 2m registers are required for dead storage, so that a system of 6 equations may be handled on the ENIAC. This is actually the method which has often been used in the past for the ENIAC. It is clear, however, that if four or less equations are involved, or if eight or less are involved and only five significant figures are to be carried through the computation, then it is highly desirable to use the fourth order approximation described above; for larger steps may be used for a given truncation error, and there will be correspondingly less round-off error (only the round-off error of the last iteration counts). The following table summarizes approximate estimates for various methods of solving ordinary differential equations with the emphasis on computing machines of small memory capacity.

Method	Moulton Adams-Bashforth Stephens Milne	Runge-Kutta	Clippinger	Heun	Analytic Continuation	Successive approximation
Hand computation	Excellent	Good	Poor	Fair	Not usually practical	Occasionally good
Machine computation	Poor	Good	Good	Good	Not usually applicable	Not adapted for machines of small memory
Different procedure at start	Yes	No	No	No	No	
Interpolates to reduce interval	Yes	No	No	No	No	
Number of registers in internal memory required for storage for system of m equations and 4th order approximation	1 + 5m	1 + 4m	1+3m	(1+2m, but gives 2nd order approximation, not 4th)		
Number of equations that can be solved to 4th order with ENIAC	2	3	4	(6, but 2nd order)		

<u>Higher Order Approximations to the Solution of Systems of Hyperbolic Partial Differential Equations</u>

Just as it was possible to approximate the solution of a system of ordinary differential equations by a polynomial of arbitrary order over a given interval, so it is possible to do the same for a system of partial differential equations. We shall not attempt to give any general theory; however, certain general observations may be made. Suppose we have a system of m-2 quasi-linear hyperbolic first order partial differential equations in two independent variables u_1 and u_2, and m-2 dependent variables $u_3, \dots, u_m$. Suppose there are m-2 families of real characteristic curves, only two families being distinct; then characteristic variables α and β may be introduced so that the system takes the canonical form:

$$(4.13)\quad \begin{cases} \displaystyle\sum_{i=1}^{m} a_{ij} \frac{\partial u_i}{\partial \alpha} = b_j & j = 1, 2, \ldots, \ell \\ \displaystyle\sum_{i=1}^{m} a_{ij} \frac{\partial u_i}{\partial \beta} = b_j & j = \ell + 1, \ldots, m \ . \end{cases}$$

If now the a_{ij} and b_j are analytic functions of $u_1, \ldots, u_m$ and the u's are all known at a set of n points P_1, P_2, etc., in the α, β plane, the problem of finding the u's at a point P nearby may be attacked as follows: Assuming that the u's are analytic, we may expand them at $P_1, P_2, \ldots, P_n$, etc., about some convenient point $\bar{P}$. These mn equations plus the m differential equations at $\bar{P}$ plus the $(2^{k+1} - 2)m$ differential equations at $\bar{P}$ obtained by k-fold differentiation yield $(2^{k+1} + n-1)m$ algebraic or transcendental equations for $\bar{u}$, $\partial \bar{u}_i/\partial\alpha$, $\partial \bar{u}_i/\partial\beta$, $\partial^2 \bar{u}_i/\partial\alpha\partial\alpha$, $\partial^2 \bar{u}_i/\partial\alpha\partial\beta$, etc. They may in general be solved

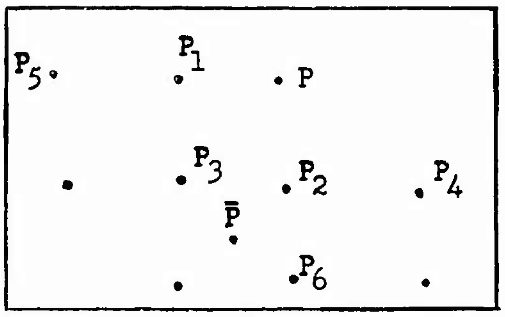


Figure 4.1

for $(2^{k+1}+n-1)m$ of these derivatives in terms of the known u's at P_1, $P_2, \ldots P_n$. If these then be substituted in the Taylor expansions for the u_i at P about $\bar{P}$, a polynomial approximation of some order j is obtained. Since there are

$$m(1+2+\ldots+j) = mj(j+1)/2$$

derivatives of u's at $\bar{P}$ of order j or less, j would in general be the largest interger less than or equal to

$$J = -\frac{3}{2} + \frac{\sqrt{9+8(2^{k+1}+n-2)}}{2} \ .$$

For example if the differential equations are not differentiated and two points are used, J = 1 and we are assured of being able to obtain a first order approximation to the u's at P.

However, it may well be that by choosing the points P_1, P_2, etc., properly the equations obtained in $\bar{u}_i$, $\partial \bar{u}_i/\partial\alpha$, etc., contain fewer independent quantities to be eliminated than $mj(j+1)/2$. For example, using four points P_1, P_2, P_3, P arranged at the corners of a rectangle with sides parallel to the α, β axes and with $\bar{P}$ chosen at the center, the expressions

$$\bar{u}_i + (\Delta\alpha)^2 \frac{\partial^2 \bar{u}_i}{\partial\alpha^2} + (\Delta\beta)^2 \frac{\partial^2 \bar{u}_i}{\partial\beta^2}$$

occur in the expansion of u_i at each of the four points; thus the number of quantities of order two or less at $\bar{P}$ to be eliminated is reduced from 6 to 4 and so, using only the original differential equations a second order approximation to the u's at P may be found. Until the reduction in number of independent quantities was noted, one would have predicted that there were only enough equations to provide a first order approximation to the u's at P.

In the process of extending the functions u_i from points P_1, ..., P_n to point P we have used (with reservations) expansions of u_i at P_ℓ about $\bar{P}$ and the differential equations and their $2^{k+1}-2$ sets of derived equations at $\bar{P}$. $(2^{k+1}-1)mn$ additional equations in the derivatives of u_i at $\bar{P}$ may be obtained by expanding the derivatives of u_i up to order k about $\bar{P}$ and substituting in the $(2^{k+1}-1)m$ differential equations at each point P_1, P_2, ..., P_n. Except for checking, these equations, however, are usually of academic interest only, since they have essentially been used already in the earlier determination of the u's at P_1, P_2, etc.

The assumption that the u's are analytic must be considered for each problem. It is well known that the solution of a system of hyperbolic differential equations with analytic coefficients need not be analytic. (E.g., the two-dimensional wave equation $\partial^2 u/\partial x^2 = \partial^2 u/\partial y^2$ is equivalent to the system $p_x = q_y$, $p_y = q_x$, where $p = u_x$, $q = u_y$. The general solution is $p = f'(x+y) + g'(x-y)$, $q = f'(x+y)-g'(x-y)$, where $f(\alpha)$ and $g(\beta)$ are any functions with continuous second derivatives, and $f'(\alpha) = df/d\alpha$, $g'(\beta) = dg/d\beta$.) In fact, the characteristics may be defined as curves along which discontinuities of derivatives of some order may occur even though the solutions y_i are themselves continuous. As an example for the aerodynamicist, consider the flow over a body of revolution with a contour having discontinuous slopes as in

Figure 4.2. Consider any curve ABC intersecting at a non-zero angle the characteristics which bound the expansion regions. The velocity components and the pressure and density of the air all have discontinuous derivatives at P_1, P_2, P_3, P_4 with respect to arc length along this curve.

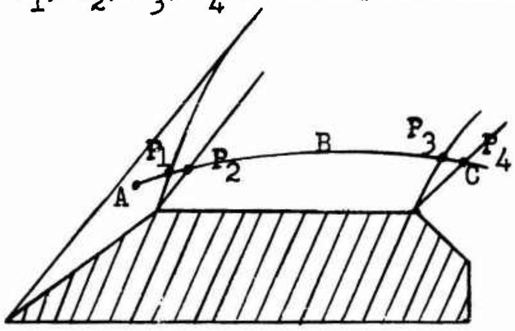


Figure 4.2

On the other hand, if functions u_i have analytic initial values given along an analytic curve which is not a characteristic, then the u's will be analytic in the region of determinacy. More general theorems of this nature are available.[4.4]

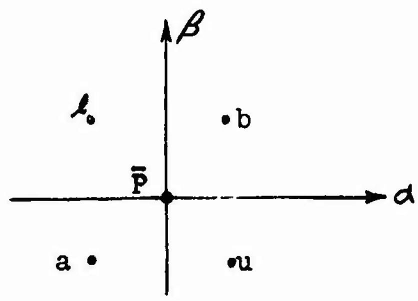


Figure 4.3

Let us return now to the general process described in section 3, and rediscuss it in the spirit of the above general remarks. As in section 3, we assume that we are to determine a solution of equations

$$
(4.14)\begin{cases} \text{a)} \quad (K-R)y_\alpha - Lx_\alpha = 0 & \text{or } y_\alpha = \lambda x_\alpha \\ \text{b)} \quad H y_\beta - (K-R)x_\beta = 0 & \text{or } \omega y_\beta = x_\beta \\ \text{c)} \quad Hu_\alpha + (K-R)v_\alpha + x_\alpha\left[\epsilon \dfrac{a^2 v}{y} + (BRy^\epsilon g)/2\right] = 0 & \end{cases}
$$

4.4 R. Courant and D. Hilbert, Methoden der Mathematischen Physik, vol. II, Chapter 5.
F. Frankl and P. Aleksieva, Op. cit., p. 793.
H. Lewy, Op. cit., p. 179.

$$
(4.14)\ \text{cont'd}\quad
\begin{cases}
\text{d)}\quad (K-R)u_\beta + Lv_\beta + y_\beta\left[\epsilon\dfrac{a^2 v}{y} - (BRy^\epsilon g)/2\right] = 0 \\
\text{e)}\quad z_x = -y^\epsilon Av \qquad z_y = y^\epsilon Au \\
\text{with boundary conditions} \\
g = \dfrac{(u_2 - q_1)^2}{(u_2-b)(u_2-\frac{1}{q_1})(\gamma u_2 - e)} \qquad \dfrac{du_2}{dz} \\
v_2^2(b-u_2) = (q_1 - u_2)^2(u_2 - d) \\
\dfrac{dy}{dx} = (q_1 - u_2)/v_2 \\
y = F(x) \\
v = uF'(x) \\
z = 0
\end{cases}
$$

We assume that x, y, u, v, z are known at ℓ and u in the α, β plane and are to be found at b, the intersection of characteristics through ℓ and u. Let us translate the α, β axes to $\bar{P}$, the midpoint of ℓ and u. Let us expand x at ℓ, b, and u about $\bar{P}$, indicating quantities at $\bar{P}$ by bars and quantities at b by no subscript, $\Delta\alpha$ and $\Delta\beta$ by α and β respectively, derivatives with respect to α by subscript α, etc.:

$$
(4.15)\quad
\begin{cases}
x = \bar{x} + \alpha\bar{x}_\alpha + \beta\bar{x}_\beta + \frac{1}{2}(\alpha^2\bar{x}_{\alpha\alpha} + \beta^2\bar{x}_{\beta\beta}) + \alpha\beta\bar{x}_{\alpha\beta} + O(h^3) \\
x_\ell = \bar{x} - \alpha\bar{x}_\alpha + \beta\bar{x}_\beta + \frac{1}{2}(\alpha^2\bar{x}_{\alpha\alpha} + \beta^2\bar{x}_{\beta\beta}) - \alpha\beta\bar{x}_{\alpha\beta} + O(h^3) \\
x_u = \bar{x} + \alpha\bar{x}_\alpha - \beta\bar{x}_\beta + \frac{1}{2}(\alpha^2\bar{x}_{\alpha\alpha} + \beta^2\bar{x}_{\beta\beta}) - \alpha\beta\bar{x}_{\alpha\beta} + O(h^3)
\end{cases}
$$

where h is the larger of α and β. Adding and subtracting the third equation to and from the second, we obtain

$$
(4.16)\quad
\begin{cases}
x_\ell + x_u = 2\bar{x} + \alpha^2\bar{x}_{\alpha\alpha} + \beta^2\bar{x}_{\beta\beta} - 2\alpha\beta\bar{x}_{\alpha\beta} + O(h^3) \\
y_\ell + y_u = 2\bar{y} + \alpha^2\bar{y}_{\alpha\alpha} + \beta^2\bar{y}_{\beta\beta} - 2\alpha\beta\bar{y}_{\alpha\beta} + O(h^3) \\
u_\ell + u_u = 2\bar{u} + \alpha^2\bar{u}_{\alpha\alpha} + \beta^2\bar{u}_{\beta\beta} - 2\alpha\beta\bar{u}_{\alpha\beta} + O(h^3) \\
v_\ell + v_u = 2\bar{v} + \alpha^2\bar{v}_{\alpha\alpha} + \beta^2\bar{v}_{\beta\beta} - 2\alpha\beta\bar{v}_{\alpha\beta} + O(h^3) \\
z_\ell + z_u = 2\bar{z} + \alpha^2\bar{z}_{\alpha\alpha} + \beta^2\bar{z}_{\beta\beta} - 2\alpha\beta\bar{z}_{\alpha\beta} + O(h^3)
\end{cases}
$$

$$
(4.17)\begin{cases}
x_d = x_\ell - x_u = -2\alpha\bar{x}_\alpha + 2\beta\bar{x}_\beta + O(h^3) \\
y_d = y_\ell - y_u = -2\alpha\bar{y}_\alpha + 2\beta\bar{y}_\beta + O(h^3) \\
u_d = u_\ell - u_u = -2\alpha\bar{u}_\alpha + 2\beta\bar{u}_\beta + O(h^3) \\
v_d = v_\ell - v_u = -2\alpha\bar{v}_\alpha + 2\beta\bar{v}_\beta + O(h^3) \\
z_d = z_\ell - z_u = -2\alpha\bar{z}_\alpha + 2\beta\bar{z}_\beta + O(h^3)
\end{cases}
$$

Substituting from equations (4.14) into equations (4.17), we obtain the two pairs of equations

$$
(4.18)\begin{cases}
x_d = -2\alpha\bar{x}_\alpha + \bar{\omega}\, 2\beta\bar{y}_\beta + O(h^3) \\
y_d = -\bar{\lambda} 2\alpha\bar{x}_\alpha + 2\beta\bar{y}_\beta + O(h^3) \\
u_d = -\left[(\epsilon\bar{P}-\bar{Q})/(\bar{K}-\bar{R})\right] 2\beta\bar{y}_\beta - 2\alpha\bar{u}_\alpha - \bar{\lambda} 2\beta\bar{v}_\beta + O(h^3) \\
v_d = \left[(\epsilon\bar{P}+\bar{Q})/(\bar{K}-\bar{R})\right] 2\alpha\bar{x}_\alpha + \bar{\omega} 2\alpha\bar{u}_\alpha + 2\beta\bar{v}_\beta + O(h^3)
\end{cases}
$$

for the pairs of unknowns $2\alpha\bar{x}_\alpha$, $2\beta\bar{y}_\beta$ and $2\alpha\bar{u}_\alpha$, $2\beta\bar{v}_\beta$. As for the coefficients, we observe that from equations (4.16)

$$
(4.19)\quad f(\bar{x},\bar{y},\bar{u},\bar{v},\bar{z}) = f\left[\frac{x_\ell + x_u}{2} + O(h^2), \frac{y_\ell + y_u}{2} + O(h^2), \ldots, \frac{z_\ell + z_u}{2} + O(h^2)\right]
$$

$$
= f\left(\frac{x_\ell + x_u}{2}, \frac{y_\ell + y_u}{2}, \ldots, \frac{z_\ell + z_u}{2}\right) + O(h^2)
$$

if f is analytic (class C_1). Therefore to the order indicated the coefficients may be replaced by functions of the means of x, y, u, v, z at ℓ and u. The determinant of the coefficients in each pair of equations is

$$
1 - \bar{\lambda}\,\bar{\omega} + \frac{2a\sqrt{q^2 - a^2}}{uv + a\sqrt{q^2 - a^2}} ;
$$

it would only be zero if the Mach number were one or infinity, and infinite if $uv + a\sqrt{q^2 - a^2} = 0$. We rule out these three cases. With this restriction, equations (4.18) may be solved with third order errors:

$$
(4.20)\begin{cases}
2\alpha\bar{x}_\alpha = \left[\bar{\omega} y_d - x_d\right] / \left[1 - \bar{\lambda}\,\bar{\omega}\right] + O(h^3) \\
2\beta\bar{y}_\beta = \left[y_d - \bar{\lambda} x_d\right] / \left[1 - \bar{\lambda}\,\bar{\omega}\right] + O(h^3)
\end{cases}
$$

$$
(4.20)\ \text{(cont'd)}\begin{cases} 2\alpha\bar{u}_\alpha = -\left[(1-\bar{\lambda}\bar{\omega})(\bar{\lambda}v_d + u_d) + \{\bar{S}-\bar{T}-\bar{\lambda}\,\bar{\omega}\,(\bar{S}+\bar{T})\}y_d + 2T\bar{\lambda}x_d\right]/\left[1-\bar{\lambda}\bar{\omega}\right]^2 + O(h^3) \\ 2\beta\bar{v}_\beta = \left[(1-\bar{\lambda}\,\bar{\omega})(v_d+\bar{\omega}u_d) - 2T\bar{\omega}y_d + \{\bar{S}+\bar{T}-\bar{\lambda}\,\bar{\omega}(\bar{S}-\bar{T})\}\,x_d\right]/\left[1-\bar{\lambda}\,\bar{\omega}\right]^2 + O(h^3) \end{cases}
$$

where $S = \dfrac{\epsilon a^2 v}{y(K-R)}$ and $T = \dfrac{By^{\epsilon}Rg}{2(K-R)}$.

These quantities may now be substituted into the expansions for x, y, u, v, z; but the results

$$
x = \bar{x} + \tfrac{1}{2}\left[2\,\bar{\omega}y_d - (1+\bar{\lambda}\,\bar{\omega})x_d\right]/\left[1-\bar{\lambda}\bar{\omega}\right]
$$

$$
y = \bar{y} + \tfrac{1}{2}\left[(1+\bar{\lambda}\,\bar{\omega})y_d - 2\bar{\lambda}x_d\right]/\left[1-\bar{\lambda}\,\bar{\omega}\right]
$$

$$
(4.21)\quad u = \bar{u} + \tfrac{1}{2}\left[\{(1+\bar{\lambda}\bar{\omega})u_d + 2\,\bar{\lambda}\,v_d\} - 2\,\{(y-y_u)(\bar{S}-\bar{T}) - \bar{\lambda}\,(x-x_\ell)(\bar{S}+\bar{T})\}\right]/\left[1-\bar{\lambda}\bar{\omega}\right]
$$

$$
v = \bar{v} + \tfrac{1}{2}\left[\{2\bar{\omega}u_d + (1+\bar{\lambda}\bar{\omega})v_d\} - 2\,\{(x-x_\ell)(\bar{S}+\bar{T}) - \bar{\omega}(y-y_u)\cdot(\bar{S}-\bar{T})\}\right]/\left[1-\bar{\lambda}\bar{\omega}\right]
$$

$$
z = \bar{z} + \bar{y}\bar{A}\ \left[-(x-\bar{x})\bar{v} + (y-\bar{y})\bar{u}\right],
$$

where $\bar{x} = \frac{1}{2}(x_\ell + x_u)$, $\bar{y} = \frac{1}{2}(y_\ell + y_u)$, $\bar{u} = \frac{1}{2}(u_\ell + u_u)$, $\bar{v}=\frac{1}{2}(v_\ell + v_u)$,
are correct only to first order since $\frac{1}{2}\left[\alpha^2 x_{\alpha\alpha} + \beta^2 x_{\beta\beta} + 2\alpha\beta x_{\alpha\beta}\right]$, etc., have not been evaluated. These expressions are the same as those obtained in the last section. However, it is possible to obtain expressions which are correct to second order by the simple expedient of using the values of u_i at a in Figure 4.3. In fact corresponding to equations (4.16) are equations

$$
(4.22)\begin{cases} x-x_a = 2\alpha\bar{x}_\alpha + 2\beta\bar{x}_\beta + O(h^3) \\ y-y_a = 2\alpha\bar{y}_\alpha + 2\beta\bar{y}_\beta + O(h^3) \\ u-u_a = 2\alpha\bar{u}_\alpha + 2\beta\bar{u}_\beta + O(h^3) \\ v-v_a = 2\alpha\bar{v}_\alpha + 2\beta\bar{v}_\beta + O(h^3) \\ z-z_a = 2\alpha\bar{z}_\alpha + 2\beta\bar{z}_\beta + O(h^3) \end{cases}
$$

which with equations (4.20) yield equations

$$
(4.23)\begin{cases}
x = x_a + \left[2\bar{\omega}y_d - (1+\bar{\lambda}\bar{\omega})x_d\right]/\left[1-\bar{\lambda}\bar{\omega}\right] + 0(h^3) \\
y = y_a + \left[(1+\bar{\lambda}\bar{\omega})y_d - 2\bar{\lambda}x_d\right]/\left[1-\bar{\lambda}\bar{\omega}\right] + 0(h^3) \\
u = u_a + \left[\left\{(1+\bar{\lambda}\bar{\omega})u_d + 2\bar{\lambda}v_d\right\} - 2\left\{(y-y_u)(\bar{S}-\bar{T}) - \right.\right. \\
\quad \left.\left.\bar{\lambda}(x-x_\ell)(\bar{S}+\bar{T})\right\}\right]/\left[1-\bar{\lambda}\bar{\omega}\right] + 0(h^3) \\
v = v_a + \left[\left\{2\bar{\omega}u_d + (1+\bar{\lambda}\bar{\omega})v_d\right\} - 2\left\{(x-x_\ell)(\bar{S}+\bar{T}) - \right.\right. \\
\quad \left.\left.\bar{\omega}(y-y_u)(\bar{S}-\bar{T})\right\}\right]/\left[1-\bar{\lambda}\bar{\omega}\right] + 0(h^3) \\
z = z_a + \bar{y}\bar{A}\left[-\bar{v}(x-x_a) + \bar{u}(y-y_a)\right] + 0(h^3)
\end{cases}
$$

It is important to note that the computations involved in the second order approximations are almost identical with those in the first order approximations. Accordingly, the only excuse for using the first order ones would be in the case where storage memory in a machine was too small to remember the values at the extra point a.

An alternative procedure for obtaining a second order approximation to x, y, u, v, z at b which is slightly better for the machine of small memory may be found as follows: one might expect that if he replaced $2\alpha\frac{\partial x}{\partial\alpha}$ by $x-x_\ell$, $2\alpha\frac{\partial y}{\partial\alpha}$ by $y-y_\ell$, etc., in equations (4.14)a and (4.14)c evaluating coefficients at the midpoint of ℓ and b, and similarly replaced $2\beta\frac{\partial x}{\partial\alpha}$ by $x-x_u$, etc., in equations (4.14)b and (4.14)d evaluating coefficients at the midpoint of b and u, a second order approximation would result; and this is true. To prove it, let us form the expressions

$$
(4.24)\begin{cases}
y - y_\ell - \bar{\lambda}(x - x_\ell) = 2\alpha\beta\bar{\lambda}_\beta\,\bar{x}_\alpha + 0(h^3) \\
x - x_u - \bar{\omega}(y - y_u) = 2\alpha\beta\bar{\omega}_\alpha\,\bar{y}_\beta + 0(h^3) \\
\bar{\omega}(u-u_\ell) + (v-v_\ell) + (\bar{S}+\bar{T})(x-x_\ell) = -2\alpha\beta\left[\bar{\omega}_\beta\,\bar{u}_\alpha + (\bar{S}+\bar{T})_\beta\,\bar{x}_\alpha\right] \\
\qquad + 0(h^3) \\
(u-u_u) + \bar{\lambda}(v-v_u) + (\bar{S}-\bar{T})(y-y_u) = -2\alpha\beta\left[\bar{\lambda}_\alpha\bar{v}_\beta + (\bar{S}-\bar{T})_\alpha\,\bar{y}_\beta\right] \\
\qquad + 0(h^3)
\end{cases}
$$

and replace $\bar{\lambda}$, $\bar{\omega}$, $\bar{\lambda}_\beta$, etc., using equations

$$
(4.25)\begin{cases}
\bar{\lambda} = \frac{1}{2}(\lambda_\ell + \lambda_u) + 0(h^2) \\
\bar{\omega} = \frac{1}{2}(\omega_\ell + \omega_u) + 0(h^2) \\
(\bar{S}+\bar{T}) = \frac{1}{2}\left[(S_\ell + T_\ell) + (S_u + T_u)\right] + 0(h^2) \\
\alpha\bar{x}_\alpha = \frac{1}{2}(x-x_\ell) + 0(h^2) \\
\ldots
\end{cases}
$$

$$\begin{cases} \beta \bar{y}_\beta = \frac{1}{2}(y-y_u) + O(h^2) \\ \alpha \bar{\lambda}_\alpha = \frac{1}{2}(\lambda - \lambda_\ell) + O(h^2) \end{cases}$$

etc.

As a result we find the equations

$$(4.26)\begin{cases} (y-y_\ell) - \frac{1}{2}(\lambda + \lambda_\ell)(x - x_\ell) = O(h^3) \\ (x-x_u) - \frac{1}{2}(\omega + \omega_u)(y - y_u) = O(h^3) \\ (v-v_\ell) + \frac{1}{2}(\omega + \omega_\ell)(u-u_\ell) + \frac{1}{2}\left[(S+T)+(S_\ell + T_\ell)\right] \cdot \\ \qquad (x-x_\ell) = O(h^3) \\ (u-u_u) + \frac{1}{2}(\lambda + \lambda_u)(v-v_u) + \frac{1}{2}\left[(S-T)+(S_u-T_u)\right] \cdot \\ \qquad (y-y_u) = O(h^3) \end{cases}$$

These may be solved for x, y, u, v, z.

$$(4.27)\begin{cases} x = \left[\{x_u - \frac{1}{2}(\omega+\omega_u)y_u\} + \frac{1}{2}(\omega+\omega_u)\ \{y_\ell - \frac{1}{2}(\lambda+\lambda_\ell)x_\ell\}\right] / \\ \qquad \left[1 - \frac{1}{4}(\lambda + \lambda_\ell)(\omega + \omega_u)\right] \\ y = \left[\{y_\ell - \frac{1}{2}(\lambda+\lambda_\ell)x_\ell\} + \frac{1}{2}(\lambda+\lambda_\ell)\ \{x_u - \frac{1}{2}(\omega+\omega_u)y_u\}\right] / \\ \qquad \left[1 - \frac{1}{4}(\lambda+\lambda_\ell)(\omega + \omega_u)\right] \\ u = \left[\{u_u + \frac{1}{2}(\lambda+\lambda_u)v_u - M_2(y-y_u)\} - \frac{1}{2}(\lambda+\lambda_u)\ \{\frac{1}{2}(\omega+\omega_\ell)u_\ell + \right. \\ \qquad \left. v_\ell - M_1(x-x_\ell)\}\right] / \left[1 - \frac{1}{4}(\lambda+\lambda_u)(\omega+\omega_\ell)\right] \\ v = \left[\{\frac{1}{2}(\omega+\omega_\ell)u_\ell + v_\ell - M_1(x-x_\ell)\} - \frac{1}{2}(\omega+\omega_\ell)\ \{u_u + \frac{1}{2}(\lambda+\lambda_u)v_u \right. \\ \qquad \left. - M_2(y-y_u)\}\right] / \left[1 - \frac{1}{4}(\lambda+\lambda_u)(\omega + \omega_\ell)\right] \\ z = \left[\frac{1}{2}(z_u + z_\ell)\right] + \left[\frac{1}{32}\{2(yA)+(yA)_u + (yA)_\ell\}\ \{-(2v+v_u+v_\ell)\cdot \right. \\ \qquad \left. (2x-x_u-x_\ell)+(2u+u_u+u_\ell)(2y-y_u-y_\ell)\}\right] \\ \text{where } M_1 = \frac{1}{2}\left[(S+T)+(S_\ell+T_\ell)\right] \text{ and } M_2 = \frac{1}{2}\left[(S-T)+(S_u-T_u)\right]\ ; \end{cases}$$

but these formulae must be used by iteration to yield second order approximations since λ, ω, P, Q, Ayu, Ayv must themselves be known to first order.

We turn our attention now to a third order approximation to x, y, u, v, z at a point not on a boundary. As we have suggested, we may increase the order of approximation by adjoining more points or by differentiating the differential equations or both. Because of the fact that we want our methods to apply when there are characteristics which are lines of discontinuity of derivatives of x, y, u, v, z, we prefer not to adjoin more points. The reason is that using only the points ℓ, u, a and b on the vertices of a rectangle we can always manage to have isolated characteristics of discontinuity be part of our network and thus avoid having them pass through any of the rectangles. It will then be permissible always to assume that the dependent functions are analytic inside the rectangle.

Following the general remarks above, we observe that using x, y, u, v, z as given at three arbitrary points and expanded about a fourth $\bar{P}$ we would have fifty unknowns ($\bar{x}$, $\bar{x}_\alpha$, $\bar{x}_\beta$, $\bar{x}_{\alpha\alpha}$, $\bar{x}_{\beta\beta}$, $\bar{x}_{\alpha\beta}$, $\bar{x}_{\alpha\alpha\alpha}$, etc.) to determine from the $15 + 5\,(2^{k+1}-1)$ expansions and differential equations obtained by differentiating k times. Thus k would have to be 2. The problem of differentiating the differential equations twice, obtaining 35 differential equations, and solving these 35 with 15 expansions for the fifty unknowns would appear to be quite formidable. Actually, because of the symmetry of the four points, a, b, ℓ, u, with respect to $\bar{P}$, the midpoint, the number of unknowns is so reduced that we shall only need to use five of the ten possible first derived equations.

To be precise, if we expand any function p such as x, y, u, v, z, λ, ω, etc., at a, b, ℓ, u, taking the origin at $\bar{P}$,

$$(4.28)\quad \begin{cases} p = \bar{p} + \alpha\bar{p}_\alpha + \beta p_\beta + \frac{1}{2}(\alpha^2\bar{p}_{\alpha\alpha} + \beta^2\bar{p}_{\beta\beta}) + \alpha\beta\bar{p}_{\alpha\beta} + \frac{1}{6}(\alpha^3\bar{p}_{\alpha\alpha\alpha} + \\ \quad 3\alpha\beta^2\bar{p}_{\alpha\beta\beta}) + \frac{1}{6}(3\alpha^2\beta\bar{p}_{\alpha\alpha\beta} + \beta^3\bar{p}_{\beta\beta\beta}) \\ p_\ell = \bar{p} - \alpha\bar{p}_\alpha + \beta\bar{p}_\beta + \frac{1}{2}(\alpha^2\bar{p}_{\alpha\alpha} + \beta^2\bar{p}_{\beta\beta}) - \alpha\beta\bar{p}_{\alpha\beta} - \frac{1}{6}(\alpha^3\bar{p}_{\alpha\alpha\alpha} + \\ \quad 3\alpha\beta^2\bar{p}_{\alpha\beta\beta}) + \frac{1}{6}(3\alpha^2\beta\bar{p}_{\alpha\alpha\beta} + \beta^3\bar{p}_{\beta\beta\beta}) \\ p_u = \bar{p} + \alpha\bar{p}_\alpha - \beta\bar{p}_\beta + \frac{1}{2}(\alpha^2\bar{p}_{\alpha\alpha} + \beta^2\bar{p}_{\beta\beta}) - \alpha\beta\bar{p}_{\alpha\beta} + \frac{1}{6}(\alpha^3\bar{p}_{\alpha\alpha\alpha} + \\ \quad 3\alpha\beta^2\bar{p}_{\alpha\beta\beta}) - \frac{1}{6}(3\alpha^2\beta\bar{p}_{\alpha\alpha\beta} + \beta^3\bar{p}_{\beta\beta\beta}) \\ p_a = \bar{p} - \alpha\bar{p}_\alpha - \beta\bar{p}_\beta + \frac{1}{2}(\alpha^2\bar{p}_{\alpha\alpha} + \beta^2\bar{p}_{\beta\beta}) + \alpha\beta\bar{p}_{\alpha\beta} - \frac{1}{6}(\alpha^3\bar{p}_{\alpha\alpha\alpha} + \\ \quad 3\alpha\beta^2\bar{p}_{\alpha\beta\beta}) - \frac{1}{6}(3\alpha^2\beta\bar{p}_{\alpha\alpha\beta} + \beta^3\bar{p}_{\beta\beta\beta}) , \end{cases}$$

we may form the combinations

$$(4.29)\quad p + p_a - p_\ell - p_u = 4\alpha\beta\bar{p}_{\alpha\beta} + O(h^4) ,$$

$$(4.30)\quad p_\ell + p_u = 2\bar{p} + O(h^2)$$

$$(4.31)\quad \alpha\bar{p}_\alpha = (p + p_u - p_a - p_\ell)/4 + O(h^3) ,$$

$$(4.32)\quad \beta\bar{p}_\beta = (p + p_\ell - p_a - p_u)/4 + O(h^3) ,$$

$$(4.33)\quad p - p_a = 2\alpha\bar{p}_\alpha + 2\beta\bar{p}_\beta + O(h^3) \text{ , and}$$

$$(4.34)\quad p_\ell - p_u = -2\alpha\bar{p}_\alpha + 2\beta p_\beta + O(h^3).$$

Equation (4.29) will yield third order approximations to x, y, u, v, z provided we can evaluate $\alpha\beta\,\bar{x}_{\alpha\beta}$, $\alpha\beta\,\bar{y}_{\alpha\beta}$, etc. This suggests that we differentiate equations (4.14)a), c), e) and (4.14)b), d) with respect to β and α respectively and solve for $4\alpha\beta\,\bar{x}_{\alpha\beta}$, $4\alpha\beta\,\bar{y}_{\alpha\beta}$, etc:

$$
(4.35)\quad \left\{
\begin{aligned}
4\alpha\beta\,\bar{x}_{\alpha\beta} &= \left[\bar{\omega}\; 2\beta\,\bar{\lambda}_{\beta}\; 2\alpha\,\bar{x}_{\alpha} + 2\alpha\,\bar{\omega}_{\alpha}\; 2\beta\,\bar{y}_{\beta}\right] / \; 1 - \bar{\lambda}\bar{\omega}] \\
4\alpha\beta\,\bar{y}_{\alpha\beta} &= \left[\bar{\lambda}\;\; 2\alpha\,\bar{\omega}_{\alpha} 2\beta\,\bar{y}_{\beta} + 2\beta\bar{\lambda}_{\beta}\; 2\alpha\,\bar{x}_{\alpha}\right] / \left[1 - \bar{\lambda}\,\bar{\omega}\right] \\
4\alpha\beta\,\bar{u}_{\alpha\beta} &= \left[\bar{\lambda}(\bar{S}+\bar{T})4\alpha\beta\,\bar{x}_{\alpha\beta} - (\bar{S}-\bar{T})4\alpha\beta\,\bar{y}_{\alpha\beta} + \bar{\lambda}\; 2\beta\,\bar{\omega}_{\beta} 2\alpha\bar{u}_{\alpha} - \right. \\
&\qquad \left. 2\alpha\bar{\lambda}\; 2\beta\;\bar{v}_{\beta}\right] \; / \left[1 - \bar{\lambda}\bar{\omega}\right] + \\
&\quad \left[\lambda\, 2\beta\,(\bar{S}+\bar{T})_{\beta}\, 2\alpha\bar{x}_{\alpha} - 2\alpha(\bar{S}-\bar{T})_{\alpha}\;\; 2\beta\,\bar{y}_{\beta}\right] \left[1 - \bar{\lambda}\,\bar{\omega}\right] \\
4\alpha\beta\bar{v}_{\alpha\beta} &= \left[\bar{\omega}\;\; (\bar{S}-\bar{T})4\alpha\beta\,\bar{y}_{\alpha\beta} - (\bar{S}+\bar{T})4\alpha\beta\,\bar{x}_{\alpha\beta} + \bar{\omega}\; 2\alpha\lambda_{\alpha} 2\beta\;\bar{v}_{\beta} - \right. \\
&\qquad \left. - 2\beta\bar{\omega}_{\beta} 2\alpha\bar{u}_{\alpha}\right] \; / \left[1 - \lambda\bar{\omega}\right] + \\
&\quad \left[\bar{\omega}\, 2\alpha\,(\bar{S}-\bar{T})_{\alpha}\; 2\beta\,\bar{y}_{\beta} - 2\beta\,(\bar{S}+\bar{T})_{\beta}\;\; 2\alpha\;\bar{x}_{\alpha}\right] \; / \left[1 - \bar{\lambda}\bar{\omega}\right] \\
4\alpha\beta\,\bar{z}_{\alpha\beta} &= \overline{yA}(-\bar{v}\;\; 4\alpha\beta\,\bar{x}_{\alpha\beta} + \bar{u}\;\; 4\alpha\beta\,\bar{y}_{\alpha\beta}\;) + 2\beta\,(\overline{yAu})_{\beta}\;\; 2\alpha\, y_{\alpha} - \\
&\qquad 2\beta\,(\overline{yAv})_{\beta}\;\; 2\alpha\, x\,\alpha \;\; .
\end{aligned}
\right.
$$

In order that $\alpha\beta\,\bar{x}_{\alpha\beta}$ shall be correct to third order it is necessary that $\bar{\lambda}$, $\bar{\omega}$, $\bar{P}$, $\bar{Q}$, $\overline{yAu}$, $\overline{yAv}$ be known to first order, and $\alpha\,\bar{x}_{\alpha}$, $\beta\,\bar{x}_{\beta}$, etc., and $\alpha\,\bar{\lambda}_{\alpha}$, $\beta\,\bar{\lambda}_{\beta}$, $\alpha\,\bar{P}_{\alpha}$, etc. be known to second order. The first set may be found by using equation (4.30). The second set may be found by solving equations (4.18).

$$
(4.36)\quad
\begin{aligned}
2\alpha\bar{x}_{\alpha} &= \left[\bar{\omega}\;(y_{\ell} - y_{u}) - (x_{\ell} - x_{u})\right] \; / \left[1 - \bar{\lambda}\,\bar{\omega}\right] \\
2\beta\bar{y}_{\beta} &= \;\; \left[(y_{\ell} - y_{u}) - \bar{\lambda}(x_{\ell} - x_{u})\right] / \left[1 - \bar{\lambda}\,\bar{\omega}\right] \\
2\alpha\bar{u}_{\alpha} &= -\left[(u_{\ell} - u_{u}) + \bar{\lambda}\,(v_{\ell} - v_{u}) + \bar{S}(y_{\ell} - y_{u}) - \bar{T}(y - y_{a})\right] / \\
&\qquad \left[1 - \bar{\lambda}\,\bar{\omega}\right] \\
2\beta\,\bar{v}_{\beta} &= \left[\bar{\omega}(u_{\ell} - u_{u}) + (v_{\ell} - v_{u}) + \bar{S}(x_{\ell} - x_{u}) - \bar{T}(x - x_{a})\right] / \\
&\qquad \left[1 - \bar{\lambda}\,\bar{\omega}\right]
\end{aligned}
$$

To find the third set we use equations (4.23) and (4.33) and (4.34).

Second and third order boundary processes may also be devised although we do not have a third order process yet which is elegant enough to include in this report.

To obtain a second order contour process, for example, let us call a and b two points on the contour in the $\alpha\beta$ plane, ℓ the point of intersection of vertical characteristic through a and horizontal characteristic through b, $\underline{P}$ and $\bar{P}$, the midpoints of ℓ, b and a, b respectively. We assume that x, y, u, v, z are known at ℓ and a and are desired at b. We take the origin at $\bar{P}$, expand x, y, u, v, z at ℓ and b about $\underline{P}$, and at a and b about $\bar{P}$, and form the differences and sums.

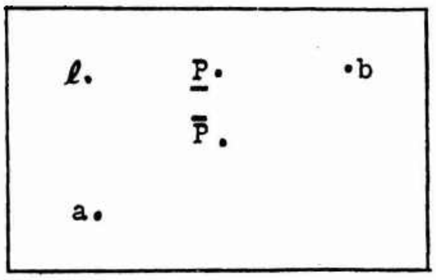


Figure 4.4

$$(4.37) \qquad p - p_{\ell} = 2\alpha \underline{p}_{\alpha} + O(h^3),$$

$$(4.38) \qquad p + p_{\ell} = 2\underline{p} + \alpha^2 \underline{p}_{\alpha\alpha} + O(h^4),$$

$$(4.39) \qquad p - p_a = 2\alpha \bar{p}_{\alpha} + 2\beta \bar{p}_{\beta} + O(h^3),$$

$$(4.40) \qquad p + p_a = 2\bar{p} + \alpha^2 \bar{p}_{\alpha\alpha} + \beta^2 \bar{p}_{\beta\beta} + 2\alpha\beta \bar{p}_{\alpha\beta} + O(h^4),$$

denoting x, y, u, v, z, λ, etc., generically by p.

Now if the map of the contour is $\beta = \tau\alpha$, the boundary conditions may be written

$$(4.41) \qquad \frac{dy}{dx} = \frac{y_{\alpha} + \tau y_{\beta}}{x_{\alpha} + \tau x_{\beta}} = f(x,y)$$

and

$$(4.42) \qquad v = u\, f(x,y) .$$

Multiplying the second member of equation (4.41) above and below by $2\beta/\tau$ or 2α we get

$$(4.43) \qquad 2\alpha y_{\alpha} + 2\beta y_{\beta} = f(x,y) \left[2\alpha x_{\alpha} + 2\beta x_{\beta} \right] .$$

Equations (4.42) and (4.43) must hold in particular at $\bar{P}$ and then we find the equations

$$(4.44) \qquad \begin{cases} y - y_a - \bar{f}(x - x_a) = O(h^3) \\ y - y_{\ell} - \lambda(x - x_{\ell}) = O(h^3) \\ \underline{\omega}(u - u_{\ell}) + (v - v_{\ell}) + (\underline{S} + \underline{T})(x - x_{\ell}) = O(h^3) \\ v - u\, f(x,y) = O(h^3) \end{cases}$$

which may be solved for second order approximation to x, y, u, v, z provided $\underline{\lambda} \neq \bar{f}$. However, $\bar{\lambda}$ would only equal $\bar{f}$ if the Mach angle were zero, i.e. the Mach number infinite. Ruling this case out, $\underline{\lambda}$ will not equal $\bar{f}$ if the grid size is small enough because of the continuity of λ. These equations are iterated to get the second order; i.e., $\underline{\lambda}$, $\bar{f}$, etc., are first taken equal to λ_{ℓ}, f_a, etc., for a first order approximation, then $\underline{\lambda}$, $\bar{f}$, etc., are found from equations (4.38) and (4.40) and used in equations (4.44) for the second order approximation.

Alternatively, in regions where extrapolation is permissible $\underline{\lambda}$, $\bar{f}$, etc., may be found to first order by extrapolation.

Section 5. Error Study

Introduction

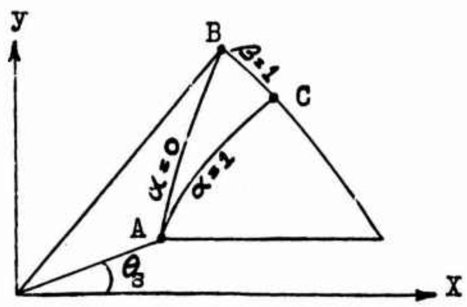
Numerical solution of differential equations involves making approximations which introduce errors. An estimate of these errors is of great importance in the analysis of the computations.

The first and thus far the only type of body studied by the methods described in the preceding sections is the cone-cylinder. A report on these computations will be published later. Before they were carried out on the ENIAC, a study of the errors involved in the numerical processes was made. It is expected that the behavior of the errors in the cone-cylinder problem typifies the behavior of errors in the calculation of flows about arbitrary pointed bodies of revolution. The method of analysis and the differential equations are the same, only the boundary conditions are specialized.

ENIAC Computations, Empirical Study

To investigate the effect of grid size and order of approximation on the computations, the flow in the expansion region for a particular cone-cylinder and Mach number was computed by the ENIAC. The case studied was one for which $\theta_s = 20^\circ$ and $M_1 = 2.12966$. The 1st order, 2nd order iterative, and 2nd order-3 point methods were used for grid sizes varying from h=1 to h=1/40.

The expansion region in the physical plane (ABC) is shown in Figure 5.1; and in the characteristic plane (AA'BC), in Figure 5.2. (h = 1 corresponds to the grid size which yields x, y, u, v at C in one step (see Figure 5.1). When h = $\frac{1}{2}$, four steps are required to obtain x, y. u, v at C; when h = 1/3, nine steps are required; etc.) We assume that the input data (x, y, u, v along the characteristics $\alpha = 0$ and $\beta = 0$) contain no error.



.Figure 5.1

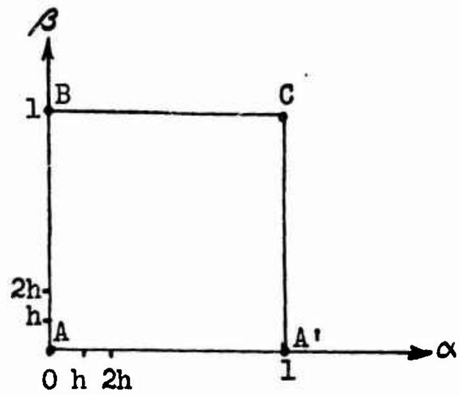


Figure 5.2

It is possible, modifying the method used by Frankl and Aleksieva[5.1] or Goursat[5.2], to obtain limits on the size of the error in the values of x, y, u, v at any point in the flow field. These limits are of considerable theoretical interest. However, in order to obtain general results it is necessary to make rather strong assumptions. The limits so obtained are much larger than necessary, and give only a poor idea of the behavior of the error as a function of grid size or order of approximation. The work of Richardson[5.3] suggests that a much more exact study of the error is possible for each specific flow problem. A natural procedure is to solve the problem for more than one grid size, h; and then at points common to several grids to fit x, y, u, v to some reasonable function of h. Assuming that these functions are valid approximations for all sufficiently small values of h, it is possible to extrapolate for the limits of x, y, u, v as h approaches zero.

The above-mentioned procedure was applied to the ENIAC computations of the cone-cylinder expansion region. The computed functions were plotted against the grid size h at the points $(1,\frac{1}{2})$, $(\frac{1}{2},\frac{1}{2})$, $(\frac{1}{2},1)$, and (1,1) in Figures 5.3, 5.4, 5.5, and 5.6 respectively for the 1st, 2nd order iterative, and 2nd order-3 point methods. The curves for the 2nd order computations were drawn with zero slope at h = 0.

The curves drawn at these four representative points should indicate the general behavior of the error in the whole region. The graphical extrapolations to h = 0 are shown in Table 5.I. Also included for comparison are the computed values for h = 1/32 by the 2nd order iterative method.

5.1 F. Frankl and P. Aleksieva, Op. Cit. ref. 1.9
5.2 Goursat, Cours d' Analyse, Par. 386
5.3 L. F. Richardson, "The deferred approach to the limit, part I - single lattice", Phil. Trans., vol. 226, 1927, p. 299

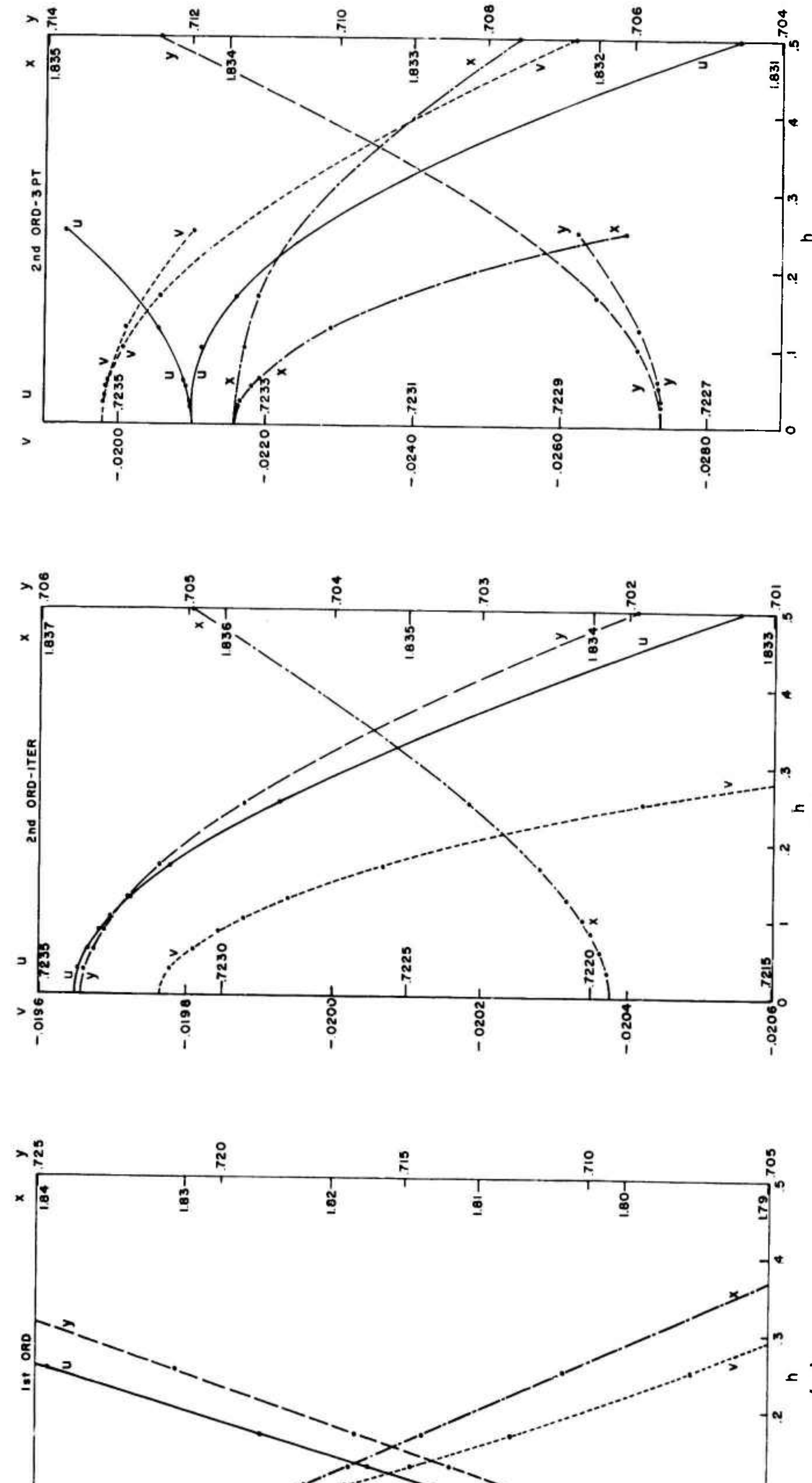


FIGURE 5.3 EFFECT OF GRID SIZE ON CALCULATIONS AT POINT $\alpha=1$, $\beta=1/2$

Table 5.I

($\alpha = 1$, $\beta = \frac{1}{2}$)	x	y	u	v
Ist ord.	1.83345	.70590	.723440	-.01976
2nd ord. iter.	1.83388	.705713	.723392	-.019765
2nd ord.-3 pt.	1.83396	.70565	.723400	-.01978
2nd ord.iter., h = 1/32	1.83391	.705693	.723393	-.019777
($\alpha = \frac{1}{2}$, $\beta = \frac{1}{2}$)	**x**	**y**	**u**	**v**
Ist ord.	1.7065	.76499	.662525	.08156
2nd ord. iter.	1.707045	.764781	.662496	.081612
2nd ord.-3 pt.	1.70736	.764710	.66248	.08161
2nd ord. iter., h=1/32	1.707056	.764761	.662483	.081609
($\alpha = \frac{1}{2}$, $\beta = 1$)	**x**	**y**	**u**	**v**
Ist ord.	3.2054	1.8086	0.66985	.04388
2nd ord. iter.	3.20525	1.80904	0.669896	.043910
2nd ord.-3 pt.	3.20550	1.80891	0.66978	.04390
2nd ord. iter., h=1/32	3.20531	1.80899	0.669783	.043918
($\alpha = 1$, $\beta = 1$)	**x**	**y**	**u**	**v**
Ist ord.	3.7152	1.5455	.70934	-.02646
2nd ord. iter.	3.71593	1.54547	.709311	-.026316
2nd ord.-3 pt.	3.7162	1.54535	.70932	-.02630
2nd ord. iter., h=1/32	3.71610	1.54535	.709290	-.026319

The agreement of the graphical extrapolations indicates that we may be assured of the accuracy of x, y, u to within 3 in the fourth figure and of v to within 1 in the third figure in all the values listed in Table 5.I. Ignoring the first order extrapolation, we find even closer agreement, which indicates that the second order iterative computation for the grid size h=1/32 is reliable to within 6 in the fifth figure for x, y, u and to within 2 in the 4th figure for v.

The graphs show that the 2nd order-3 point calculations follow two different patterns in relation to grid size. The points lie on either one of two curves, depending on whether $1/h$ is odd or even for (α =1, β =1), and on whether $1/2h$ is odd or even for the other three points.

To obtain a numerical extrapolation to h = 0 one can fit the data by the least squares method to some reasonable function of h. This was done for the computations at ($\alpha = 1$, $\beta = 1$); the functions employed are $f(h) = a + bh + ch^2$ for the 1st order values, and $f(h) = a + bh + ch^2 + dh^3$ for the 2nd order iterative and 2nd order-3 point values. Table 5.II shows the results.

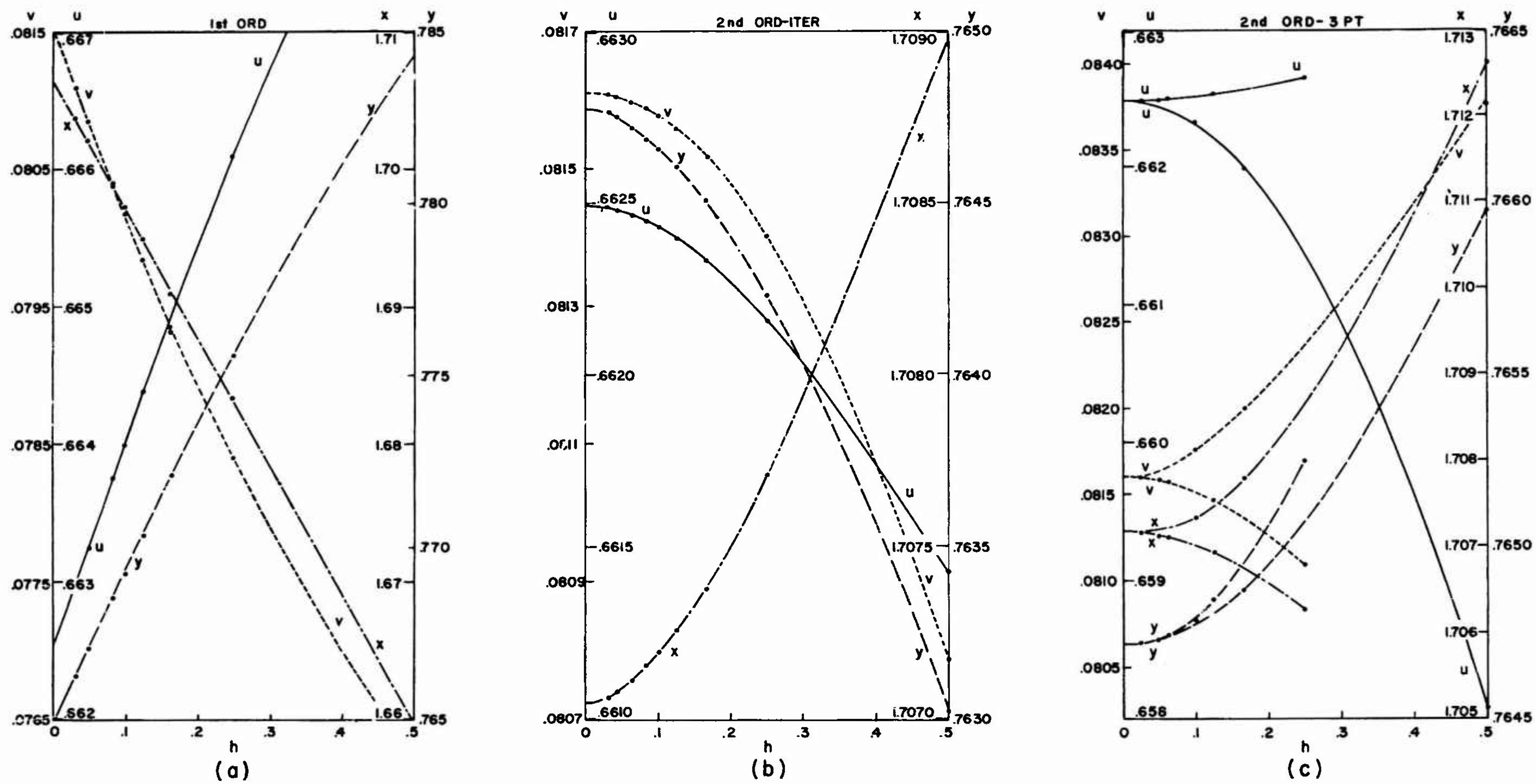


FIGURE 5.4 EFFECT OF GRID SIZE ON CALCULATIONS AT POINT $\alpha=1/2$, $\beta=1/2$

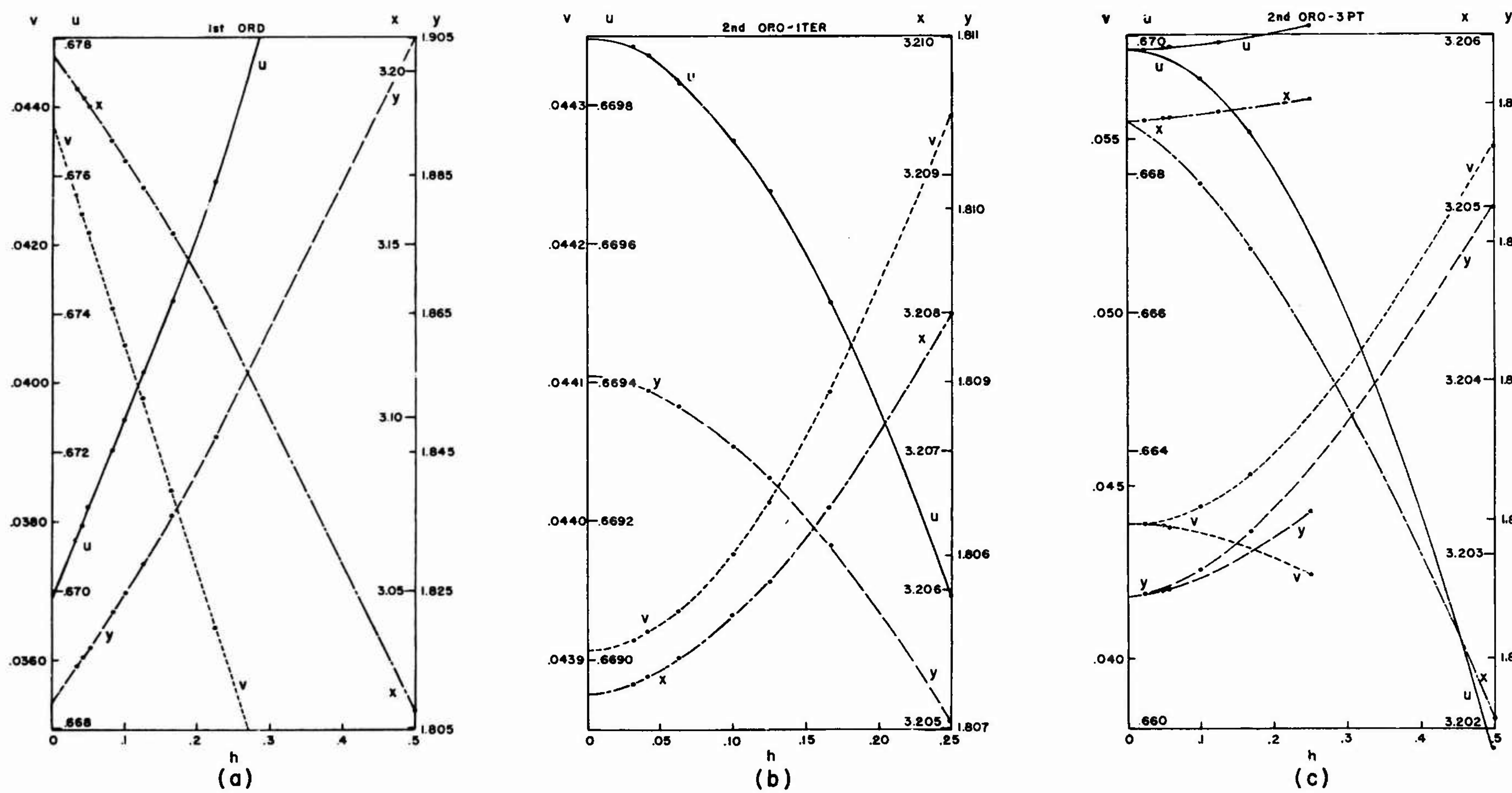


FIGURE 5.5 EFFECT OF GRID SIZE ON CALCULATIONS AT POINT $\alpha=1/2,\ \beta=1$

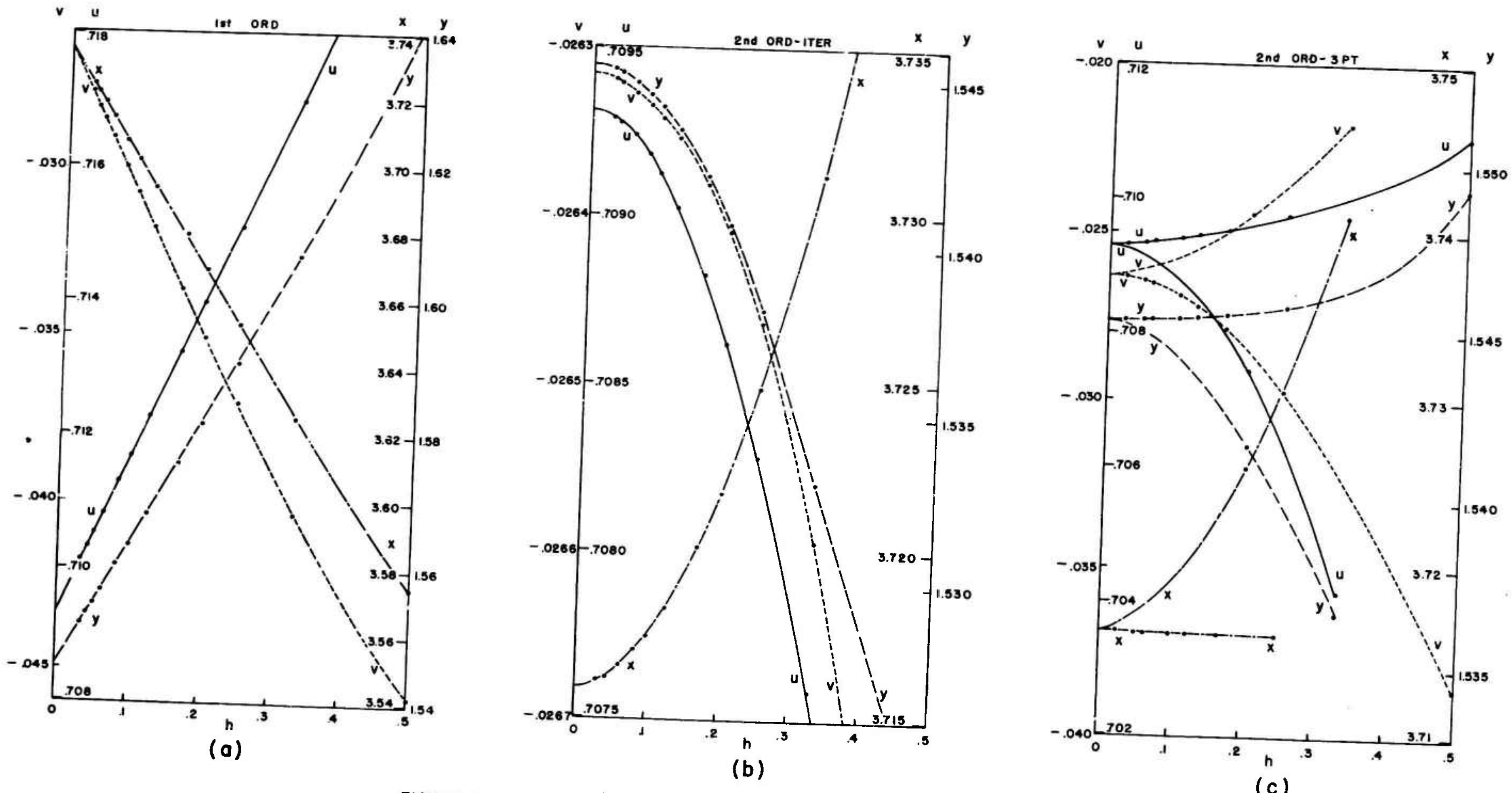


FIGURE 5.6 EFFECT OF GRID SIZE ON CALCULATIONS AT POINT $\alpha = 1$, $\beta = 1$

Table 5.II

1st order

$$x = 3.71556035 - .340695693\,h + .046337492\,h^2$$
$$y = 1.54562504 + .181446572\,h - .00554870548\,h^2$$
$$u = .70931271 + .02430564\,h - .00444836206\,h^2$$
$$v = -.026314228 - .048620732\,h + .011581095\,h^2$$

2nd order-iterative

$$x = 3.715892542 + .002385437\,h + .134694176\,h^2 - .002492252\,h^3$$
$$y = 1.545510530 - .001357115\,h - .117729376\,h^2 + .025024049\,h^3$$
$$u = .709308737 + .000075481\,h - .020277732\,h^2 + .013444669\,h^3$$
$$v = -.02630000 - .00012539676\,h - .00125911740\,h^2 - .009568309\,h^3$$

2nd order-3 point

$$x = 3.716199722 - .000608705\,h - .002697709\,h^2 + .008292560\,h^3$$
$$y = 1.545365028 + .0003562452\,h + .0040463651\,h^2 + .0051117669\,h^3$$
$$u = .709310875 - .00005547079\,h + .0096804297\,h^2 - .0077314754\,h^3$$
$$v = -.0263339776 + .0001389884\,h - .05993978635\,h^2 + .0205477726\,h^3$$

The constant terms in the polynomials are the extrapolations to $h = 0$. The range of grids used was $h = \frac{1}{4}$ to $h = 1/40$. Only even grids were used for the 2nd order-3 point data fit. The above polynomials all agree with the given data at least to within 5 in the 6th figure for x, y, u; and to within 1 in the 3rd figure for v. (The v agreement for the first order computations holds only for $h \leqq 1/20$.) Comparison shows that this numerical extrapolation to $h = 0$ agrees with the graphical extrapolation to within 3 in the 5th figure for x, y, u; and to within 1 in the 3rd figure for v.

These results suggest that the error can be moderately well represented by simpler functions; namely, by bh, for the 1st order method, and by bh^2 for the 2nd order methods. The least squares fits of the data to these functions are given in Table 5.III.

Table 5.III

1st order

$$x = 3.715007365 - .3284527749\,h$$
$$y = 1.545690071 + .1799814167\,h$$
$$u = .7093652534 + .02312716466\,h$$
$$v = -.02644997335 - .04280403863\,h$$

2nd ord. iter.

$$x = 3.716010825 + .1426135703\,h^2$$
$$y = 1.545421374 - .1164789567\,h^2$$
$$u = .7093000262 - .01673578393\,h^2$$
$$v = -.02631802511 - .0023052374\,h^2$$

Table 5.III (cont'd)

2nd ord.-3 pt.

$$
\begin{aligned}
x &= 3.716162678 - .002284617161\, h^2 \\
y &= 1.545376882 + .0065855559\, h^2 \\
u &= .7093153981 + .0075518560\, h^2 \\
v &= -.0263463811 - .0543126060\, h^2
\end{aligned}
$$

These curves differ from those in Table II at most by 5 in the 5th figure for x, y, u; and by 1 in the 3rd figure for v.

Round-Off Errors

Although the outputs of the ENIAC in this problem contain ten figures, they have less than ten significant figures, for several reasons. In order to make allowance for the wide ranges of some of the quantities encountered, certain numbers such as x and y were purposely shifted to the right on the accumulators. While this procedure insured that no numbers would exceed the capacity of the machine in the extreme cases, it meant a loss of one or two significant figures. Furthermore ENIAC multiplication and division are only correct to nine places. Thus local computations are affected by round-off errors in the sixth or seventh significant figure of x and y, and the eighth significant figure of u. As for v, since its magnitude is small (it can change sign) it may have between zero and eight significant figures locally correct. When it has none, however, it does not affect the accuracy of the other quantities.

These local errors, of the round-off variety, are in addition to the errors due to the replacement of derivatives by difference quotients. It is the principal aim of our error study to determine empirically the nature of these latter truncation errors.

Because of the above-mentioned round-off errors we cannot hope by extrapolation to zero grid size to obtain more than six significant figures.

Theoretical Study of the Truncation Error

It is natural to expect that the local errors in computing the values of x, y, u, v, at b (assuming correct values given at l, a, and u) are of order j (i.e. the error is a series in h starting with terms of order j). Then the total errors at B, made in computing x, y, u, v step-by step from boundaries CAD can be expected to be of

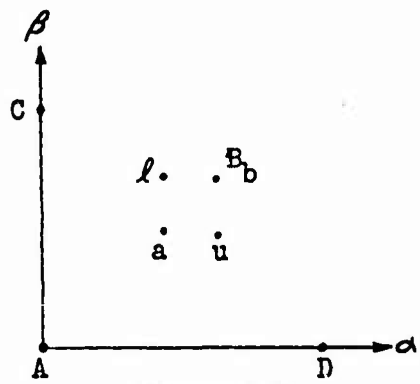


Figure 5.7

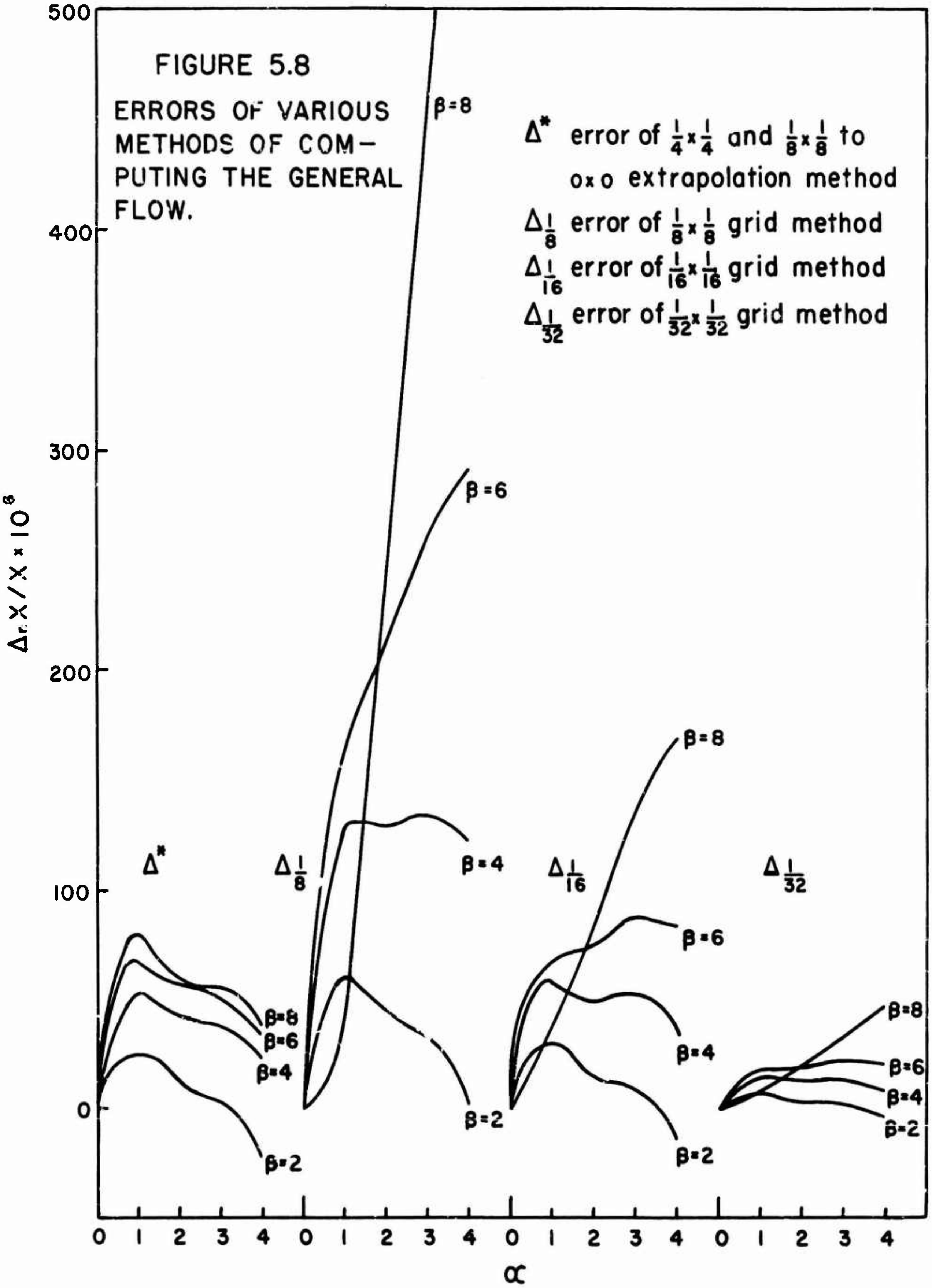


FIGURE 5.8
ERRORS OF VARIOUS METHODS OF COMPUTING THE GENERAL FLOW.
Δ* error of 1/4 × 1/4 and 1/8 × 1/8 to o × o extrapolation method
Δ 1/8 error of 1/8 × 1/8 grid method
Δ 1/16 error of 1/16 × 1/16 grid method
Δ 1/32 error of 1/32 × 1/32 grid method
β=8
β=6
β=4
β=2
Δ*
Δ 1/8
Δ 1/16
Δ 1/32
500
400
300
200
100
0
Δr X/X × 10³
0 1 2 3 4 0 1 2 3 4 0 1 2 3 4 0 1 2 3 4
α

the form:

$$(5.1) \quad x - x_L = E_x = \bar{a}_x\, h^n + \bar{b}_x\, h^{n+1} + \bar{c}_x\, h^{n+2} + \dots ,$$

$$y - y_L = E_y = \bar{a}_y\, h^n + \bar{b}_y\, h^{n+1} + \dots, \text{ etc.}$$

$$n \doteq j,$$

where x_L is the limit of x as h approaches zero, and the coefficients $\bar{a}_x$, $\bar{b}_x$, $\bar{c}_x$, etc. are functions of α and β independent of h, but differing for x, y, u, v.

If this is true and if this series converges rapidly enough for the values of h for which the flow problem has been solved, we may neglect all but the first one or two terms and solve for n, a, b and x_L (or y_L etc.). For example, if t is used generically for x, y, u or v, and if t_1, t_2, and t_3 are the values of t when h is h_1, h_2, and h_3, we should have the equations

$$(5.2) \quad t_1 = t_L + \bar{a}\, h_1^{\,n}$$

$$t_2 = t_L + \bar{a}\, h_2^{\,n}$$

$$t_3 = t_L + \bar{a}\, h_3^{\,n}$$

for t_L, a, and n to satisfy.

Eliminating $\bar{a}$ and t_L, n must satisfy the equation

$$(5.3) \quad (t_2 - t_3)(h_1^{\,n} - h_2^{\,n}) = (t_1 - t_2)(h_2^{\,n} - h_3^{\,n}).$$

In particular if $\frac{h_1}{4} = \frac{h_2}{2} = h_3$

$$(5.4) \quad n = \left[\log \frac{t_1 - t_2}{t_2 - t_3}\right] / \log 2$$

$$n = 3.32194 \log_{10} \frac{t_1 - t_2}{t_2 - t_3}$$

Having found n, $\bar{a}$ and t_L are given by

$$(5.5) \quad \bar{a} = \frac{t_1 - t_2}{h_1^{\,n} - h_2^{\,n}}$$

(5.6) $t_L = t_3 - \bar{a}\, h_3^{\,n}$

However, for some cases $\dfrac{t_1 - t_2}{t_2 - t_3}$ may be negative, and therefore the data cannot be fitted for three values of h by a function of the form

$$t = t_L + \bar{a}\, h^n.$$

This means that the term $\bar{b}\, h^{n+1}$ cannot be neglected. If the approximation

(5.7) $t = t_L + \bar{a}h^n + \bar{b}\, h^{n+1}$,

is used, the four values of h and t available are substituted

$$(5.8) \begin{cases} t_1 = t_L + \bar{a}\, h_1^{\,n} + \bar{b}\, h_1^{\,n+1} \\ t_2 = t_L + \bar{a}\, h_2^{\,n} + \bar{b}\, h_2^{\,n+1} \\ t_3 = t_L + \bar{a}\, h_3^{\,n} + \bar{b}\, h_3^{\,n+1} \\ t_4 = t_L + \bar{a}\, h_4^{\,n} + \bar{b}\, h_4^{\,n+1} \end{cases}$$

and t_L eliminated, the equations

$$(5.9) \begin{cases} t_1 - t_2 = \Delta_1 = \bar{a}\,(h_1^{\,n} - h_2^{\,n}) + \bar{b}\,(h_1^{\,n+1} - h_2^{\,n+1}) \\ t_2 - t_3 = \Delta_2 = \bar{a}\,(h_2^{\,n} - h_3^{\,n}) + \bar{b}\,(h_2^{\,n+1} - h_3^{\,n+1}) \\ t_3 - t_4 = \Delta_3 = \bar{a}\,(h_3^{\,n} - h_4^{\,n}) + \bar{b}\,(h_3^{\,n+1} - h_4^{\,n+1}) \end{cases}$$

result.

Therefore, n must satisfy the equation

$$(5.10) \begin{vmatrix} \Delta_1 & h_1^{\,n} - h_2^{\,n} & h_1^{\,n+1} - h_2^{\,n+1} \\ \Delta_2 & h_2^{\,n} - h_3^{\,n} & h_2^{\,n+1} - h_3^{\,n+1} \\ \Delta_3 & h_3^{\,n} - h_4^{\,n} & h_3^{\,n+1} - h_4^{\,n+1} \end{vmatrix} = 0$$

In particular, if $\frac{h_1}{8} = \frac{h_2}{4} = \frac{h_3}{2} = h_4$, equation (5.10) becomes

$$(5.11) \quad \begin{vmatrix} \Delta_1 & 2^{2n} & 4.2^{2n} \\ \Delta_2 & 2^n & 2.2^n \\ \Delta_3 & 1 & 1 \end{vmatrix} = 0,$$

or (5.12) $\Delta_1 - 3\Delta_2 2^n + 2\Delta_3 (2^n)^2 = 0.$

The solutions of this equation are

$$(5.13)\quad n = \left[1/\log 2\right] \left[\log\left(\frac{3\Delta_2 \pm \sqrt{9\Delta_2^2 - 8\Delta_1\Delta_3}}{4\Delta_3}\right)\right],$$

provided $9\Delta_2^2 > 8\Delta_1 \Delta_3$, and

$$\frac{3\Delta_2 \pm \sqrt{9\Delta_2^2 - 8\Delta_1\Delta_3}}{4\Delta_3} > 0 .$$

Somewhat more generally, if $\frac{h_1}{k^3} = \frac{h_2}{k^2} = \frac{h_3}{k} = h_4$, the equation to be satisfied by n is

$$(5.14)\quad \Delta_1 - (k+1)\Delta_2 k^n + k\Delta_3 (k^n)^2 = 0,$$

or

$$(5.15)\quad n = \left[1/\log k\right] \left[\log\left(\frac{(k+1)\Delta_2 \pm \sqrt{(k+1)^2\Delta_2^2 - 4k\Delta_1\Delta_3}}{2k\Delta_3}\right)\right]$$

When n has been obtained, t_L, $\bar{a}$, and $\bar{b}$ are found from three of the linear equations (5.8).

The value of equation (5.13) for hand computations is very small. h_1 must be made small enough so that it is permissible to truncate equation (5.1) at the second term. Then Δ_3 will be so small that it can be known accurately only if very many figures are carried.

In summary, if assumption (5.1) is correct, it is possible to determine n, $\bar{a}$, $\bar{b}$, etc. if the solution is carried out at enough grid sizes. If the computations are done by hand, a tiny error, whether by mistake or round-off, affects the value of n (as given by equations (5.4), (5.13), or similar formulae) so markedly that the study cannot be very valuable. In fact, it is possible for a small variation in the fifth figure of the data at small grid sizes to change the sign of the discriminant $(9\Delta_2^2 - 8\Delta_1\Delta_3)$ in equation (5.13) from positive to negative to yield a complex value of n.

Our study was made on the ENIAC, a machine which carries ten figures and rarely makes mistakes. Even these ENIAC computations, however, cannot all be relied upon to calculate satisfactory values of n, for reasons discussed above and in the section on round-off errors. Since the computations of u were found to have the most significant figures, we calculated n with them. The values of n, found by equation (5.13) from the u data at h = 1/4, 1/8, 1/16, 1/32, are listed in

Table 5.IV.

Table 5.IV

	(α =1, β =1/2)	(α=1/2,β=1/2)	(α =1/2,β=1)	(α =1,β =1)
1st ord	.99	1.0	1.0	.97
2nd ord - iter.	2.1	2.1	2.1	2.1
2nd ord - 3 pt.	1.9	2.0	2.0	2.0

From Table 5.IV we see that the order of the gross error is approximately one for the 1st order computations and approximately two for the 2nd order computations.

If assumption (5.1) is correct, as the data seem to indicate, we may make the following observations. Let us consider methods of two different orders for computing t, calling the results t_1 and t_2. We can represent the errors by

$$(5.16)\quad t_1 - t_L = \bar{a}_1 h^{n_1} (1 + \epsilon_1)$$

$$t_2 - t_L = \bar{a}_2 h^{n_2} (1 + \epsilon_2),$$ where ϵ_1 and ϵ_2 go to zero as

h goes to zero and $n_2 > n_1$. Then for any n_2 there exists an h = H, such that

$$\bar{a}\,(n_2)\, h^{n_2} < \bar{a}\,(n_1)\, h^{n_1}$$

and $h < H$. It is reasonable to expect that n(j) is an increasing function of j, the local order. Therefore, a method of any local order j gives more accurate results than methods of smaller local order for all grid sizes, h, small enough (< H).

Under the same assumptions, there would also exist some (possibly smaller) grid size H', such that the method of local order j gives results of specified accuracy with less total labor than any method of lower local order. This is true as long as the specified accuracy is as good as, or better than, the accuracy associated with grid size H'. Although this conclusion ignores the effects of round-off errors, it is probably correct even with round-off errors, provided enough significant figures are carried. The computation with higher order local approximation has fewer round-off errors (since it is carried out at larger grid size) if the specified accuracy is high enough.

On the other hand, for grid sizes larger than H' there will be lower order methods which give more accurate results for a given amount of labor. For this reason and for reasons of accuracy in extrapolating to zero grid size, the second order method was found the best for hand and machine computations.

Comparison of Extrapolation to Zero with Small Grid Computations

Having found the 2nd order iterative method most feasible for computation with the ENIAC we wish to investigate the accuracy obtainable by the process of computing at large grid sizes and extrapolating to $h = 0$. This method involves considerably less labor than computation at very small grid sizes.

Letting t be a generic symbol for x, y, u, and v, we write for t the second order function of h:

$$t(h) - t_L = ah^2 ,$$

where t_L is the desired extrapolated value. If we use two grid sizes h_1 and h_2 such that $h_2 = 1/2\, h_1$, then

$$t_L = t(h_2) + 1/3 \left[t(h_2) - t(h_1) \right] . \tag{5.17}$$

Using $h_1 = 1/16$ and $h_2 = 1/32$, the smallest available grid sizes applicable to equation (5.17), we calculate t_L throughout the expansion region, and we consider it to be the "correct" value of t. Then we form the quantity $\Delta_h = \left[t_L - t(h) \right] / t_L$ throughout the expansion region. Δ_h is the relative error in the computation at the point (α, β) resulting from the use of the finite grid size h.

We also calculate $t_L^* = t(1/8) + 1/3 \left[t(1/8) - t(1/4) \right]$ and form $\Delta^* = (t_L - t_L^*)/t_L$. It is Δ^* that we wish to compare with Δ_h for various grid sizes to see how the accuracy of extrapolation with large grids compares with that of small grid computations.

In figure 5.8 we have the relative errors in x plotted for the grid sizes 1/8, 1/16, 1/32, and for the 1/4, 1/8 extrapolation. It is evident that the 1/4, 1/8 extrapolations are not as good as the 1/32 x 1/32 computations but are appreciably better than the 1/16 x 1/16 computations. The errors in y, u, and v behave in an identical manner with those of x. The amount of labor required for grid size h is proportional to $1/h^2$. Taking the extrapolations as equivalent to computations with $h = 1/28$, the ratio of the work required is about $(4^2 + 8^2)/(28)^2$, approximately 1/10.

Similar conclusions can be drawn from the 1st order and 2nd order-3 point method computations.

R. F. Clippinger

RFClippinger

N. Gerber

N. Gerber