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SOME DEVELOPMENT OF MARKOFF'S METHOD OF RANDOM FLIGHTS, WITH THE INTENTION OF APPLICATION TO TURBULENT FLOW

by

Toyoki Koga

NOVEMBER 1965

POLYTECHNIC INSTITUTE OF BROOKLYN

DEPARTMENT
of
AEROSPACE ENGINEERING
and
APPLIED MECHANICS

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SOME DEVELOPMENT OF MARKOFF'S METHOD OF RANDOM FLIGHTS, WITH THE INTENTION OF APPLICATION TO TURBULENT FLOW[+]

by

Toyoki Koga[*]

Polytechnic Institute of Brooklyn

SUMMARY

Markoff's method of random flights is reconsidered and developed with the intention of then applying it to problems of turbulent flow. The gist is to generalize the assumption regarding the probability distribution of causal events.

1. Correlations which may exist among causal events, and the consequences appearing in the resultant events are considered. In view of the conclusion, some of the previous authors' treatments are criticized.

2. The correlation probability between two resultant events which are separated in space but caused by a common group of causal events is formulated.

3. The correlation probability between two events which are separated in time but caused by the same group of causal events evolving in time is formulated.

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LIST OF SYMBOLS

$A(\vec{p})$	defined by (2.10)
$\vec{p}^{(1)}, \vec{p}^{(2)}, \ldots, \vec{p}^{(N)}$	elementary results caused respectively by elementary causal events $\vec{q}^{(1)}, \vec{q}^{(2)}, \ldots, \vec{q}^{(N)}$
$\vec{P}: (P_1, P_2, \ldots, P_s)$	total result $\sum_1^N \vec{p}^{(\nu)}$
$\vec{q}^{(1)}, \vec{q}^{(2)}, \ldots, \vec{q}^{(N)}$	elementary causal events
$w(\vec{q}^{(1)}, \vec{q}^{(2)}, \ldots, \vec{q}^{(N)})$	probability density of the causal events
$W(\vec{P})$	probability density of $\vec{P}$
$W(\vec{P}, t; P', t')$	defined by (4.1)
Δ	defined by (2.8), (3.3), and (4.2)
Δ'	defined by (4.3)
Φ	transition probability defined in (4.1)
$\omega^{(\nu)}(\vec{q}^{(\nu)})$	probability density of causal event $\vec{q}^{(\nu)}$
ω	defined by (2.12)

SECTION I

INTRODUCTION

Markoff's method of random flights[1] is used to calculate the probability distribution of the result caused by events of which the probability of occurrence is given. The method was applied to problems of the random walk by Rayleigh[2], the statistics of electric fields due to charged particles by Holtsmark[3], and the statistics of the gravitational field due to stars by Chandrasekhar and von Neumann[4].

In those treatments, each causal event is assumed to be independent of the others. It is known that this assumption is fatal in some problems[5]. (In view of this assumption, the author believes that some of the results obtained by Holtsmark and by Chandrasekhar and von Neumann are not plausible, as will be discussed later.) As a matter of principle, this assumption is not indispensable in the method, as will be considered in Section II. It is essential to eliminate the assumption to some extent, especially when one intends to apply the method to problems of fluid mechanics and/or plasma dynamics.

Sometimes one may be interested in the correlation between two resultant events caused by a common set of causal events, as is considered in Section III. Also, it may be useful to consider the correlation between two events when one is directly caused by a certain group of events, say group A, and the other is caused by group B, which has evolved from A with a certain probability. This is considered in Section IV.

Remarks in view of the application to turbulent flow are given in Section V. There one may also find the author's answer to the possible question: why is it necessary to consider the present method which appears to be anachronistic in contrast with methods of modern turbulence theories of high mathematical sophistication?

SECTION II

CORRELATION AMONG CAUSAL EVENTS

Let us suppose that an event represented by a quantity (vector) denoted by $\vec{P}$ is the result of the sum of elementary quantities $\vec{p}^{(1)}, \vec{p}^{(2)}, \ldots\ldots, \vec{p}^{(N)}$,

$$\vec{P} = \sum_{\nu=1}^{N} \vec{p}^{(\nu)} \tag{2.1}$$

For example, $\vec{P}$ may be a momentum given to a particle or a displacement. In general, $\vec{P}$ may be a vector of s dimensions:

$$\vec{P} \equiv (P_1, P_2, \cdots, P_s). \tag{2.2}$$

It is supposed that $\vec{p}^{(\nu)}$ is a function of a set of coordinates, $q^{(\nu)} \equiv (q_1^{(\nu)}, q_2^{(\nu)}, \ldots\ldots, q_k^{(\nu)})$, which represents a causal event. The probability of occurrence of the causes between $(\vec{q}^{(1)}, \vec{q}^{(2)}, \ldots\ldots, \vec{q}^{(N)})$ and $(\vec{q}^{(1)} + d\vec{q}^{(1)}, \vec{q}^{(2)} + d\vec{q}^{(2)}, \ldots\ldots, \vec{q}^{(N)}+d\vec{q}^{(N)})$ is given by

$$\left.\begin{aligned} &w\, d^k q^{(1)} d^k q^{(2)} \cdots d^k q^{(N)} \\ &\text{where} \quad w \equiv w(\vec{q}^{(1)}, \vec{q}^{(2)}, \cdots, \vec{q}^{(N)}; t), \\ &d^k q^{(\nu)} \equiv dq_1^{(\nu)} dq_2^{(\nu)} \cdots dq_k^{(\nu)}. \end{aligned}\right\} \tag{2.3}$$

Of course, the q's may be considered as functions of time t. If there is neither sink nor source of causes, w is governed by

$$\left(\frac{\partial}{\partial t} + \sum \vec{u}^{(\nu)} \cdot \frac{\partial}{\partial \vec{q}^{(\nu)}}\right) w = 0 \tag{2.4}$$

where

$$\vec{u}^{(\nu)} = \frac{d\vec{q}^{(\nu)}}{dt}. \tag{2.5}$$

According to Markoff, we investigate the probability that

$$\vec{P} - \frac{1}{2} d\vec{P} \le \sum_{\nu=1}^{N} \vec{p}^{(\nu)} \le \vec{P} + \frac{1}{2} d\vec{P} \qquad (2.6)$$

By denoting this probability by $W(\vec{P})d^sP$, we have

$$W(\vec{P})\, d^s P = \int \cdots \int \Delta\, w \prod_{\nu=1}^{N} d^{\kappa} q^{(\nu)} \qquad (2.7)$$

where $\Delta = 1$ whenev ondition (2.6) is satisfied, and $\Delta = 0$ otherwise. Function Δ is given by Dirichlet's integral

$$\Delta = \frac{1}{\pi^s} \int \cdots \int_{\rho} \frac{\sin(\frac{1}{2} dP_1\, \rho_1)}{\rho_1} \frac{\sin(\frac{1}{2} dP_2\, \rho_2)}{\rho_2} \cdots$$

$$\times \frac{\sin(\frac{1}{2} dP_s\, \rho_s)}{\rho_s} \exp\left[i\vec{\rho} \cdot \left(\sum \vec{p}^{(\nu)} - \vec{P}\right)\right] d^s\rho$$

$$= \frac{d^s P}{(2\pi)^s} \int \cdots \int \exp\left[i\vec{\rho} \cdot \left(\sum \vec{p}^{(\nu)} - \vec{P}\right)\right] d^s\rho \qquad (2.8)$$

Hence,

$$W(\vec{P}) = \frac{1}{(2\pi)^s} \int \cdots \int \exp(-i\vec{\rho} \cdot \vec{P}) A(\vec{\rho})\, d^s\rho \qquad (2.9)$$

where

$$A(\vec{\rho}) = \int \cdots \int w \prod_{\nu=1}^{N} \exp(i\vec{\rho} \cdot \vec{p}^{(\nu)})\, d^{\kappa} q^{(\nu)}. \qquad (2.10)$$

Previous authors have traditionally assumed that

$$\left.\begin{aligned} w &= \omega^{(1)}\omega^{(2)}\cdots\omega^{(N)}, \\ \omega^{(\nu)} &= \omega^{(\nu)}(\vec{q}^{(\nu)}), \end{aligned}\right\} \tag{2.11}$$

In most cases of application, it is further assumed that

$$\omega^{(1)} \equiv \omega^{(2)} \equiv \cdots \equiv \omega^{(N)} \equiv \omega \tag{2.12}$$

and that

$$A(\vec{p}) = \left[\int\cdots\int \omega \exp(i\vec{p}\cdot\vec{p})\, d^{\kappa}q\right]^{N} \tag{2.13}$$

We have to emphasize, however, that there are many practical cases where it is not trivial to choose w given either by (2.11) or by (2.12). Probability w given by (2.3) may imply that there are certain correlations among N causal events, as for example, in such a condition as

$$w = 0$$

whene er

$$|\vec{q}^{(\nu)} - \vec{q}^{(\nu')}| \le q^{(0)}.$$

In other words, one may take into account the interaction among different events. In this sense, causal events do not occur completely at random. In the following, we consider two dynamical examples where the above consideration has some significance.

Example A. The Distribution Function of the Total Force Induced by Free Field Particles.

Let us suppose that field particles 1, 2, . . . , N exert their gravitational forces on a test body fixed at point 0. The masses of those particles are assumed to be the same. As illustrated in Fig. 1-a, the field particles are supposed to be confined in the space between two spherical walls with their centers at 0. The radii are denoted by R_1 and $R_1 + \delta R$. Here,

$$\delta R \ll R_1 \tag{2.a.1}$$

The mutual interactions among the field particles, and also the force exerted by the test body on them, are ignored because we assume that each of them has a large momentum and its motion is not significantly disturbed by those interactions. Under this circumstance, we may use relation (2.12) and

$$\omega = \frac{1}{\delta V}$$
$$\delta V = \frac{4\pi}{3}\left[(R_1 + \delta R)^3 - R_1^3\right] \tag{2.a.2}$$

$$= 4\pi R_1^2 \delta R \tag{2.a.3}$$

We write for $\vec{p}$, the force due to a field particle,

$$\vec{p} = \kappa \frac{\vec{r}}{r^3}$$
$$\vec{r} \equiv (r, \theta, \varphi) \tag{2.a.4}$$

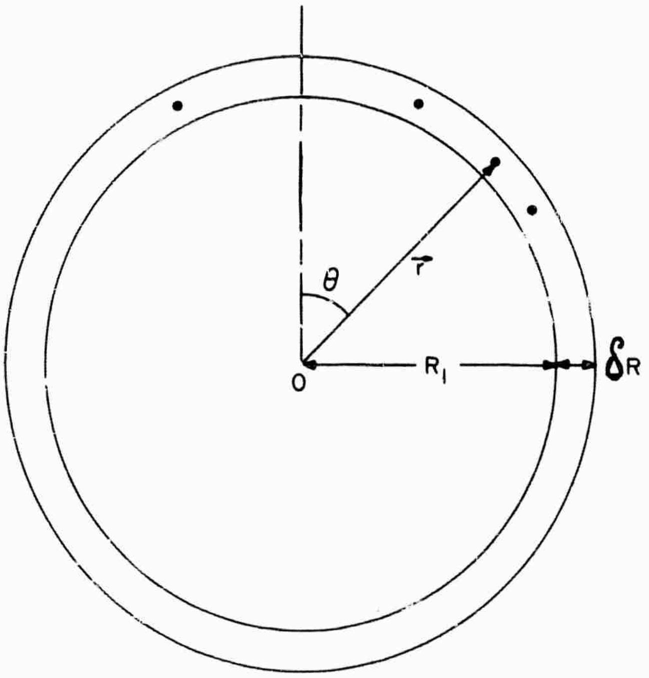

Fig. 1-a. A meridian section ($\varphi = \varphi_1$ and $\varphi_1 + \pi$) of the space between two spherical walls with their centers at 0. Since the space is not divided, a particle may exist anywhere inside the space with uniform probability.

by means of a spherical coordinate system. Writing the total force $\vec{F}$ for $\vec{P}$, Eqs. (2.9) and (2.10) yield

$$\left.\begin{aligned} W(\vec{F}) &= \frac{1}{(2\pi)^3}\iiint \exp(-i\vec{\rho}\cdot\vec{F})\, A(\vec{\rho}) \sin\theta'\, d\theta'\, d\varphi'\, d\rho, \\ A(\vec{\rho}) &= \left[\iiint_{r=R_1}^{r=R_1+\delta R} \omega \exp\left(i\vec{\rho}\cdot\frac{\kappa \vec{r}}{R_1^2}\right) r^2 \sin\theta\, d\theta\, d\varphi\, dr\right]^N \end{aligned}\right\} \qquad (2.a.5)$$

Here the domain of integration is

$$\vec{\rho} \equiv (\rho, \theta', \varphi'),$$

$$\rho: \text{ from } 0 \text{ to } \infty,$$

$$\theta': \quad " \quad 0 \quad " \quad \pi,$$

$$\varphi': \quad " \quad 0 \quad " \quad 2.$$

Since $A(\vec{\rho})$ is independent of the direction of $\vec{\rho}$, $\vec{\rho}$ is assumed to be in the direction $\theta = 0$, $\varphi = 0$ for the convenience of treating $A(\vec{\rho})$.

$$A(\vec{\rho}) = \left[\omega \int_\varphi \int_\theta \exp\left(i\rho\frac{\kappa}{R_1^2}\cos\theta\right) R_1^2\, \delta R \sin\theta\, d\theta\, d\varphi\right]^N$$

$$= \left[\omega R_1^2\, \delta R\, 4\pi \frac{R_1^2}{\rho\kappa} \sin\left(\frac{\rho\kappa}{R_1^2}\right)\right]^N$$

If we take into account $\omega R_1^2\, \delta R\, 4\pi = 1,$

then

$$A(\vec{\rho}) = \left[1 - \frac{1}{N}\left\{N - \frac{NR_1^2}{\rho\kappa}\sin\left(\frac{\rho\kappa}{R_1^2}\right)\right\}\right]^N$$

$$= \exp\left[-N + \frac{NR_1^2}{\rho\kappa}\sin\left(\frac{\rho\kappa}{R_1^2}\right)\right]$$

$$= \exp\left[-\frac{N}{3!}\left(\frac{\rho\kappa}{R_1^2}\right)^2\right] \qquad (2.a.6)$$

Since $A(\vec{\rho})$ is independent of the direction of $\vec{\rho}$, $W(\vec{F})$ is also independent of the direction $\vec{F}$.

By taking $\vec{F}$ in the direction ($\theta' = 0$, $\varphi' = 0$), we have

$$W(\vec{F}) = \frac{1}{(2\pi)^3}\iiint \exp(-i\rho F\cos\theta')\exp\left[-\frac{N}{3!}\left(\frac{\rho\kappa}{R_1^2}\right)^2\right] \times \rho^2\sin\theta'\,d\rho\,d\theta'\,d\varphi'$$

$$= \frac{1}{\pi^{3/2}}\frac{1}{\left[\frac{4N}{3!}\left(\frac{\kappa}{R_1^2}\right)^2\right]^{3/2}}\exp\left[-F^2\Big/\frac{4N}{3!}\left(\frac{\kappa}{R_1^2}\right)^2\right] \tag{2.a.7}$$

Of course, it is easily shown that

$$\iiint W(\vec{F})\,dF_x\,dF_y\,dF_z = 4\pi\int W(F)F^2dF = 1 \tag{2.a.8}$$

and that

$$\langle|F|\rangle = \int_0^\infty 4\pi W(F)F^3dF = \frac{4}{(6\pi)^{1/2}}\frac{N^{\frac{1}{2}}\kappa}{R_1^2} \tag{2.a.9}$$

$$\langle F^2\rangle = \int_0^\infty 4\pi W(F)F^4dF = N\left(\frac{\kappa}{R_1^2}\right)^2 \tag{2.a.10}$$

It is noted that $\langle F^2\rangle$ is equal to the square of the force of each single field particle multiplied by the number of the field particles.

Example B. The Distribution Function of the Total Force Induced by Restricted Field Particles.

In Example A, we assumed that each of the field particles is free in the domain between two spherical walls. In the present example, we assume that N field particles are divided into G groups confined in G separate sub-domains which are formed with additional walls set in the entire domain considered in Example A. For the sake of simplicity, the number of field particles in each group is assumed to be the same:

$$\frac{N}{G} = n \tag{2.b.1}$$

By taking G to be even, we assume that there is always a pair of sub-domains g and g' which are geometrically symmetric regarding center 0 (see Fig. 1-b).

domain	coordinates	volume
g	from θ to $\theta+\delta\theta$; " φ to $\varphi+\delta\varphi$	$R_1^2 \sin\theta\, \delta R\, \delta\theta\, \delta\varphi$
g'	" $\pi-\theta-\delta\theta$ " $\pi-\theta$; " $\varphi+\pi$ " $\varphi+\delta\varphi+\pi$	$R_1^2 \sin\theta\, \delta R\, \delta\theta\, \delta\varphi$

(2.b.2)

For $A(\vec{\rho})$, we have, instead of (2.19) or (2.a.5),

$$A(\vec{\rho}) = \prod_{g}\prod_{g'} \left[\iiint_{g} \omega_g \exp\left(i\vec{\rho}\cdot\frac{\kappa\vec{r}}{r^3}\right) r^2 \sin\theta\, dr\, d\theta\, d\varphi\right]^n \times \left[\iiint_{g'} \omega_{g'} \exp\left(i\vec{\rho}\cdot\frac{\kappa\vec{r}}{r^3}\right) r^2 \sin\theta\, dr\, d\theta\, d\varphi\right]^n \tag{2.b.3}$$

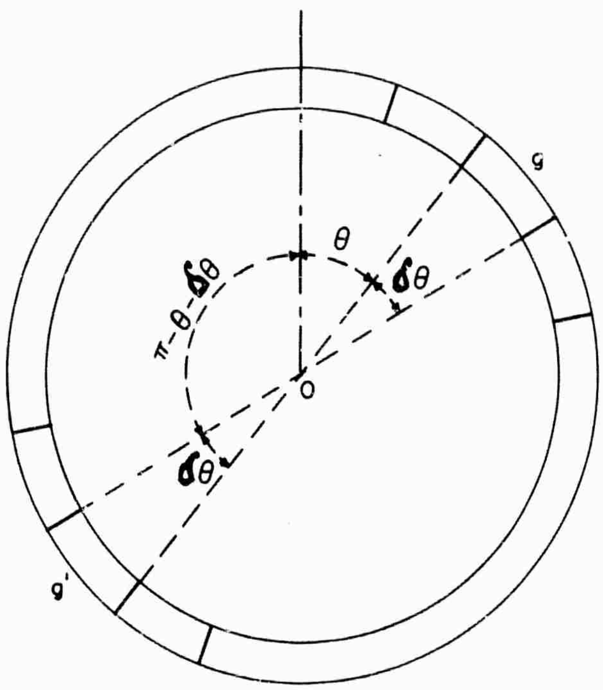

Fig. 1-b. A meridian section similar to the one illustrated in Fig. 1-a. The space between the two spheres are further divided in pairs of smaller section, g, g', etc. A particle existing in g has no probability of existence in the other part of the space.

Here,

$$\omega_g = \frac{1}{R_1^2 \sin\theta\, \delta R\, \delta\theta\, \delta\varphi} \tag{2.b.4}$$

inside domain g, but vanishes outside the domain. A similar relation exists between $\omega_{g'}$ and domain g'. It may be convenient to put

$$\sin\theta\, \delta\theta = \delta\bar{\theta} \tag{2.b.5}$$

and to consider $\delta\bar{\theta}$ invariant regarding all the sub-domains. In the present case, $A(\vec{\rho})$ is by no means independent of the direction of $\vec{\rho}$ because each sub-domain occupies a finite solid angle with respect to the center. $A(\vec{\rho})$ may fluctuate with pitches of those solid angles as the direction of $\vec{\rho}$ changes. Only at the limit $G \to \infty$, may the distribution of particles be uniform, and $A(\vec{\rho})$ become independent of the direction of $\vec{\rho}$.

In order to avoid difficulty of manipulation, we consider in a special direction ($\theta = 0$, $\varphi = 0$), and interpret the result to be valid in the limit $G \to \infty$.

Considering those conditions, we obtain

$$\begin{aligned} X_g &= \iiint_g \omega_g \exp\left(i\vec{\rho}\cdot\frac{\kappa\vec{r}}{r^3}\right) r^2 \sin\theta\, dr\, d\theta\, d\varphi \\ &\doteq \omega_g R_1^2 \delta R\, \delta\varphi \int_{\theta}^{\theta+\delta\theta} \exp\left(i\rho\frac{\kappa}{R_1^2}\cos\theta\right)\sin\theta\, d\theta \\ &\doteq \omega_g R_1^2 \delta R\, \delta\varphi\, \frac{-\exp\left[i\rho\frac{\kappa}{R_1^2}\cos(\theta+\delta\theta)\right]+\exp\left[i\rho\frac{\kappa}{R_1^2}\cos\theta\right]}{i\rho\kappa/R_1^2} \end{aligned} \tag{2.b.6}$$

$$X_{g'} = \omega_{g'} R_1^2 \delta R \delta\varphi \; \frac{-\exp\left[-i\rho \frac{\kappa}{R_1^2}\cos\theta\right] + \exp\left[-i\rho \frac{\kappa}{R_1^2}\cos(\theta+\delta\theta)\right]}{i\rho\kappa/R_1^2} \tag{2.b.7}$$

Hence,

$$\begin{aligned} X_g X_{g'} &= (\omega_g R_1^2 \delta R \delta\varphi)^2 \left[(\sin\theta\,\delta\theta)^2 - \frac{1}{4!}\left(\frac{\rho\kappa}{R_1^2}\right)^2 (\sin\theta\,\delta\theta)^4 \right. \\ &\qquad \left. + \cdots \right] \\ &= 1 - \frac{1}{4!}\left(\frac{\rho\kappa}{R_1^2}\right)^2 \overline{\delta\theta}^{\,2} + \cdots\cdots \end{aligned} \tag{2.b.8}$$

and

$$\begin{aligned} A(\vec{\rho}) &= \prod_g \prod_{g'} X_g^n X_{g'}^n \\ &= \left[1 - \frac{1}{4!}\left(\frac{\rho\kappa}{R_1^2}\right)^2 \overline{\delta\theta}^{\,2}\right]^{N/2} \\ &= \exp\left[-\frac{N}{2}\,\frac{\overline{\delta\theta}^{\,2}}{4!}\left(\frac{\rho\kappa}{R_1^2}\right)^2\right] \end{aligned} \tag{2.b.9}$$

Considering (2.9),

$$\begin{aligned} W(F) &= \frac{1}{(2\pi)^3}\iiint \exp(-i\rho F\cos\theta')\exp\left[-\frac{N}{2}\,\frac{\overline{\delta\theta}^{\,2}}{4!}\left(\frac{\rho\kappa}{R_1^2}\right)^2\right] \\ &\qquad \times \rho^2 d\rho \sin\theta' d\theta' d\varphi' \\ &= \frac{1}{\pi^{3/2}}\,\frac{\exp\left[-F^2\Big/\left(\frac{N}{12}\,\overline{\delta\theta}^{\,2}\left(\frac{\kappa}{R_1^2}\right)^2\right)\right]}{\left[\frac{4N}{2\times 4!}\,\overline{\delta\theta}^{\,2}\left(\frac{\kappa}{R_1^2}\right)^2\right]^{3/2}} \end{aligned} \tag{2.b.10}$$

In this case,

$$\langle F^2 \rangle = \frac{N}{2} \overline{\delta\theta}^{\,2} \left(\frac{k}{K_1^{\,2}} \right)^2 \tag{2.b.11}$$

instead of (2.a.10). The result is independent of $\delta\varphi$ and does not seem plausible. This defect of the result must be expected in the beginning when we assumed that $A(\vec{\rho})$ were independent of the direction of $\vec{\rho}$. As stated there, the present result is valid only in the limit $\overline{\delta\theta} \to 0$. This may be sufficient for our present purpose to demonstrate the consequence of a restriction appearing in w.

The contrast between Example A and Example B is remarkable and shows us the significance of the restriction imposed upon w. By condition (2.12), each of the causal events is completely independent of the others. The value of $\langle F^2 \rangle$ given by (2.a.10) is predictable in a simple manner: Denoting with $\vec{F}_i$, the force induced by field particle i, we have

$$\vec{F} = \sum_{i=1}^{N} \vec{F}_i$$

Hence,

$$F^2 = \sum F_i^{\,2} + 2 \sum_{i<j}\sum \vec{F}_i \cdot \vec{F}_j$$

Since $\vec{F}_i$ changes its direction as independent of the direction of $\vec{F}_j$, the second member in the right-hand side vanishes in

$$\langle F^2 \rangle = N \langle F_i^{\,2} \rangle$$

which is identical to (2.a.10).

In a previous study[5] related to multiple collisions among particles in plasmas, the author noticed the significant and particular meaning of condition (2.12), and expressed a deep doubt about the feasibility of the results obtained by Holtsmark[3] regarding charged field particles and by Chandrasekhar and von Neumann[4] regarding the gravity force due to stars. In those studies, condition (2.12) was employed. Holtsmark calculated $W(\vec{E})$, the distribution function of the electric force $\vec{E}$ exerted on an emitter by the field particles (charged particles, particles with dipole moments and particles with quadrapole moments). Although those results are finite and do not show any particular contradiction by themselves, the result in the case of charged particles may need some reconsideration. On the other hand, the case of Chandrasekhar and von Neumann is to be reconsidered more seriously. Chandrasekhar and von Neumann applied the method to the calculation of the gravity force exerted on a test star by the field stars. Their interest was not only in the total force but also in the correlation period. Hence the distribution function $W(\vec{F}, \vec{f})$ is defined in the six-dimension space of force, $\vec{F}$, and temporal change rate of force, $\vec{f}$. By an approximate formula, they defined the mean life of a force $\vec{F}$ by

$$\tau_{|F|} = |F| / (\langle |f|^2 \rangle)^{\frac{1}{2}}$$

which yields

$$\int \vec{F} \, \tau_{|F|} \, W(\vec{F}) d^3F \Big/ \int \tau_{|F|} \, W(\vec{F}) d^3F = \vec{P}$$

and

$$\int (\vec{F} \, \tau_{|F|})^2 \, W(\vec{F}) d^3F \Big/ \int \tau_{|F|} \, W(\vec{F}) d^3F = Q$$

to be finite. Here $W(\vec{F}) = \int W(\vec{F}, \vec{f}) d^3f$.

We note that the precise result Q diverges to infinity in the case of gravity force (and/or Coulomb force): So far as $\vec{P}$ and Q are concerned, the effect on the test particle exerted by the field particles which are assumed to have no mutual interaction is equivalent to the effect as calculated under the condition that only one of the field particles exerts its force on the test particle at a moment of time. In other words, the effect is equivalent to the one produced by "binary interactions" and hence the precise result of Q is easily shown to be infinite, as is well-known[5]. The convergence of the result obtained by Chandrasekhar and von Neumann seems to be due to the approximate treatment. For applying the method to problems of continuous medium (for instance, turbulent flow), the need of modifying condition (2.12) is quite obvious. This modification will be studied in a report treating turbulent flow in the near future.

SECTION III

CORRELATION BETWEEN TWO RESULTANT EVENTS SEPARATED IN SPACE

Let us suppose that two events denoted by $\vec{P}_A$ and $\vec{P}_B$ are results of a common set of causal events denoted by $\vec{q}^{(\nu)}$; $\nu = 1, 2, \ldots, N$. The probability of occurrence of those causes is given by (2.3). We consider, by analogy to (2.6), the probability that

$$\left.\begin{aligned} \vec{P}_A - \tfrac{1}{2} d\vec{P}_A &\leq \sum_{\nu=1}^{N} \vec{p}_A^{(\nu)} \leq \vec{P}_A + \tfrac{1}{2} d\vec{P}_A , \\ \vec{P}_B - \tfrac{1}{2} d\vec{P}_B &\leq \sum \vec{p}_B^{(\nu)} \leq \vec{P}_B + \tfrac{1}{2} d\vec{P}_B . \end{aligned}\right\} \tag{3.1}$$

By denoting the probability by $W(\vec{P}_A, \vec{P}_B)\, d^sP_A d^sP_B$,

$$W(\vec{P}_A, \vec{P}_B)\, d^sP_A\, d^sP_B = \int \cdots \int \Delta\, w \prod_{\nu=1}^{N} d^{\kappa} q^{(\nu)} \tag{3.2}$$

Here $\Delta = 1$ whenever condition (3.1) is satisfied, and $\Delta = 0$ otherwise. Function Δ

is given by

$$\Delta = \frac{d^s P_A \, d^s P_B}{(2\pi)^{2s}} \int \cdots \int \exp\left[i \vec{\rho}_A \cdot \left(\Sigma \vec{r}_A^{\,(\nu)} - \vec{P}_A \right) \right] d^s \rho_A$$

$$\times \int \cdots \int \exp\left[i \vec{\rho}_B \cdot \left(\Sigma \vec{r}_B^{\,(\nu)} - \vec{P}_B \right) \right] d^s \rho_B \qquad (3.3)$$

as a simple generalization of (2.8). Hence,

$$W(\vec{P}_A, \vec{P}_B) = \frac{1}{(2\pi)^{2s}} \int \cdots \int \exp\left[-i \left(\vec{\rho}_A \cdot \vec{P}_A + \vec{\rho}_B \cdot \vec{P}_B \right) \right]$$

$$\times A(\vec{\rho}_A, \vec{\rho}_B) \, d^s \rho_A \, d^s \rho_B \qquad (3.4)$$

where

$$A(\vec{\rho}_A, \vec{\rho}_B) = \int \cdots \int w \prod_{\nu=1}^{N} \exp\left[i \vec{\rho}_A \cdot \vec{r}_A^{\,(\nu)} + i \vec{\rho}_B \cdot \vec{r}_B^{\,(\nu)} \right] d^K q^{(\nu)}$$

$$(3.5)$$

The average value of, for example, $P_{A2}P_{B5}$ may be obtained as follows:

$$\langle P_{A2} P_{B5} \rangle = \int \cdots \int P_{A2} P_{B5} \, W \, d^s P_A \, d^s P_B \qquad (3.6)$$

The difference between the result in Section II and the present result is a matter of notation. If we write for (2.2)

$$\vec{P} \equiv (P_{A1}, P_{A2}, \ldots, P_{As}, P_{B1}, \ldots, P_{Bs})$$

the statement in this section may be a repetition of that in Section II without any other modification.

SECTION IV

CORRELATION BETWEEN TWO RESULTANT EVENTS SEPARATED IN TIME

We consider the probability that an event caused by a set of causal events at time t is between $\vec{P}$ and $\vec{P} + d\vec{P}$ and subsequently at time t' between $\vec{P}'$ and $\vec{P}' + d\vec{P}'$. The causal events at time t' are assumed as having evolved from the causal events at time t. The probability is given, in general, by

$$W(\vec{P}, t; \vec{P}', t')\, d^s P\, d^s P'$$

$$= \int \cdots \int \Delta\, \Delta'\, \Phi(\vec{q}, t; \vec{q}', t')\, w(\vec{q}, t) \prod_{\nu} d^k q^{(\nu)} \prod_{\nu} d^k q^{(\nu)\prime} \tag{4.1}$$

Here,

$$\Delta = 1$$

whenever

$$\vec{p} - \tfrac{1}{2} d\vec{p} \leq \sum_{\nu=1}^{N} \vec{p}^{\,(\nu)} \leq \vec{p} + \tfrac{1}{2} d\vec{p}$$

(4.2)

and

$$\Delta = 0$$

otherwise; similarly,

$$\Delta' = 1$$

whenever

$$\vec{p}' - \tfrac{1}{2} d\vec{p}' \leq \sum_{\nu=1}^{N} \vec{p}^{\,(\nu)\prime} \leq \vec{p}' + \tfrac{1}{2} d\vec{p}' \tag{4.3}$$

and

$$\Delta' = 0$$

otherwise.

$\prod_{\nu} w\, d^{K} q^{(\nu)}$ is the probability that each of the $q^{(\nu)}$'s is respectively between $\vec{q}^{(\nu)}$ and $\vec{q}^{(\nu)} + d\vec{q}^{(\nu)}$ at time t, while $\phi(\vec{q}, t; \vec{q}', t') \prod d^{K} \vec{q}^{(\nu)} \prod d^{K} \vec{q}^{(\nu)'}$ gives the probability that the causal events, being known to be between

$$\vec{q}^{(1)}, \vec{q}^{(2)}, \ldots, \vec{q}^{(N)}$$

and

$$\vec{q}^{(1)} + d\vec{q}^{(1)}, \vec{q}^{(2)} + d\vec{q}^{(2)}, \ldots, \vec{q}^{(N)} + d\vec{q}^{(N)}$$

at time t, appear at time t' between

$$\vec{q}^{(1)'}, \vec{q}^{(2)'}, \ldots, \vec{q}^{(N)'}$$

and

$$\vec{q}^{(1)'} + d\vec{q}^{(1)'}, \vec{q}^{(2)'} + d\vec{q}^{(2)'}, \ldots, \vec{q}^{(N)'} + d\vec{q}^{(N)'}.$$

Φ is called the transition probability and

$$\int \cdots \int \phi \prod_{\nu=1}^{N} d^{K} q^{(\nu)'} = 1. \tag{4.4}$$

The transition probability has often been considered in the theory of Brownian motion[4]. If the physics of the causal events is well-known, the transition probability is to be given as a consequence. In general, however, the causal events cannot be independent of the resultant events. In other words, those events make a closed ring and there is no absolute way to separate them into "causal" and "resultant". It is expected that the entire system of events would be presented by a set of integral equations, that is, by physical laws. Introducing transition probability is a conventional way to cut the closed ring of events into causal events and resultant events. In this way, physical characteristics are eliminated from the problem under consideration. In this sense, the present treatment is kinematical rather than dynamical.

Similarly as in Section II, we may write

$$\bar{W}(\vec{P}, t; \vec{P}', t') = \frac{1}{(2\pi)^{2s}} \int \cdots \int \exp(-i\vec{\rho} \cdot \vec{P}) \exp(-i\vec{\rho}' \cdot \vec{P}') \times A(\vec{\rho}, \vec{\rho}')\, d^s\rho\, d^s\rho' \tag{4.5}$$

where

$$A(\vec{\rho}, \vec{\rho}') = \int \cdots \int w \phi \prod_{\nu=1}^{N} \exp(i\vec{\rho} \cdot \vec{p}^{(\nu)})\, d^\kappa q^{(\nu)} \times \prod_{\nu=1}^{N} \exp(i\vec{\rho}' \cdot \vec{p}^{(\nu)\prime})\, d^\kappa q^{(\nu)\prime} . \tag{4.6}$$

It is expected that

$$\int \bar{W}(\vec{P}, t; \vec{P}', t')\, d^s P' = W(\vec{P}) \tag{4.7}$$

where $W(\vec{P})$ is defined in Section II. The proof is as follows: According to the definition

$$\Delta' = \frac{d^s P'}{(2\pi)^s} \int \cdots \int \exp[i\vec{\rho}' \cdot (\Sigma \vec{p}^{(\nu)\prime} - \vec{P}')]\, d^s\rho' . \tag{4.8}$$

Suppose that the entire domain of $\vec{P}'$ is divided into many small sub-domains,

- - - - - - - - - - -

Sub-domain between $\vec{P}'_{-1} - \frac{1}{2} d\vec{P}'$ and $\vec{P}'_{-1} + \frac{1}{2} d\vec{P}'$

sub-domain between $\vec{P}'_{0} - \frac{1}{2} d\vec{P}'$ and $\vec{P}'_{0} + \frac{1}{2} d\vec{P}'$ (4.9)

sub-domain between $\vec{P}'_{+1} - \frac{1}{2} d\vec{P}'$ and $\vec{P}'_{+1} + \frac{1}{2} d\vec{P}'$

- - - - - - - - - - - etc.

which cover continuously the entire domain. In each sub-domain we consider Δ' as defined previously by (4. 3). By denoting those by $\cdots \Delta'_{-1}, \Delta'_0, \Delta'_{+1}, \cdots$, we make the sum

$$\sum \Delta' = \cdots + \Delta'_{-1} + \Delta'_0 + \Delta'_{+1} + \cdots \tag{4.10}$$

Since those sub-domains cover the entire domain of P' with no overlapping, $\sum \Delta'$ is always unity regardless of the value of $\sum \vec{p}^{(\nu)\prime}$. Hence,

$$d^s\vec{P}\int_{P'} W(\vec{P},t;\vec{P}',t')\, d^sP'$$

$$= \int\cdots\int \Delta \sum\Delta' \phi(\vec{q},t;\vec{q}',t')\, w(\vec{q},t) \times \prod_\nu d^\kappa q^{(\nu)} \prod_\nu d^\kappa q^{(\nu)\prime}$$

$$= \int\cdots\int \Delta\, \phi(\vec{q},t;\vec{q}',t')\, w(\vec{q},t) \prod_\nu d^\kappa q^{(\nu)} \prod d^\kappa q^{(\nu)\prime}$$

[considering (4. 4)]

$$= \int\cdots\int \Delta\, w(\vec{q},t) \prod d^\kappa q^{(\nu)}$$

$$= \bar{W}(\vec{P})\, d^sP \tag{4.11}$$

according to the definition given in Section II.

By means of $\bar{W}(\vec{P}, t; \vec{P}'; t')$ given by (4. 5), one may obtain

$$\langle P_i P_j' \rangle = \int \cdots \int P_i P_j' W(\vec{P}, \vec{P}') d^5 P d^5 P'$$

as a function of $\tau = t' - t$. The result is expected to decrease, monotonously with τ increasing, to zero at $\tau = \tau_c$. We may call τ_c the period of correlation between P_i and P'_j.

SECTION V

REMARKS IN VIEW OF APPLICATION

The treatment presented so far is a matter of mathematics. All the possible physical implications are concealed in the distribution probability of causal events, w, and the probability of evolution of causal events named "transition probability", Φ . In a physical problem, the distinction between "causal events" and "resultant events" is often artificial. A "causal event" can be affected by "resultant events". For example, consider a segment of vortex line in a fluid. The segment, as a cause, may induce a velocity field which results in a shift of the location of another segment of vortex, as a resultant event. At the same time, the first segment may float due to the second segment. Such relations must all be included in w and Φ, and the entire closed cycle of phenomena may often be presented by a set of integro-differential equations. In this paper, the closed cycle is cut into open parts, and one of the parts is treated as if the distinction between cause and result is obvious. See the schematic illustration of the relation in Fig. 2.

As read in the title, results of the present study are expected to be applied to turbulent flow. In view of the general sophistication of current theories of turbulent flow, it seems necessary to explain briefly the motive of considering the present exotic method which, at this moment, appears to be rather anachronistic: In spite of their mathematical sophistication, most

Cause 3 = Result 2

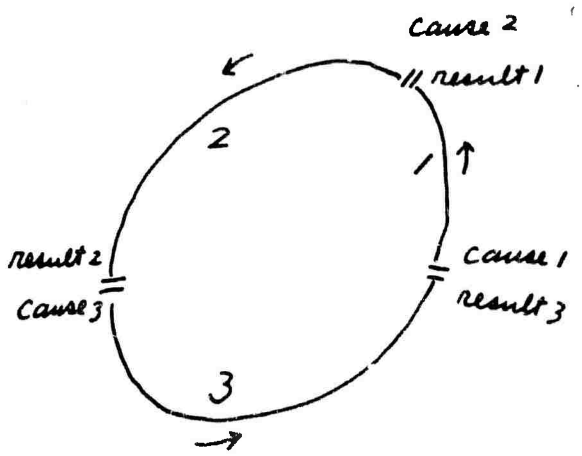

Fig. 2. The entire system of phenomenon is represented by a ring. The cause of part 1 is the result of part 3, and the result of part 1 is the cause of part 2, etc.

of the recent theories seem to bear two weak points in view of physics:

1. The significance of a vortex line, a singular line in the field, is that the strength is an integral of motion as described plainly by Helmholtz's three theorems, and turbulent flows are considered to be induced by vortex lines (of course, the strength is not permanently invariant because of viscosity stress. Here, another integral, wave motion, is ignored. Regarding wave motion, we may say the same as regarding vortex line.) By having coarse-grained directly basic differential equations of fluid dynamics, either in the Lagrangian sense or in the Eulerian sense, we have no more opportunity to introduce properly the integral of motion in a theory.

2. In order to compensate the ignorance of vortex line as an integral of motion, "Mischungsweg" was introduced by Prandtl in the Lagrangian sense and "size of eddy" by Taylor in the Eulerian sense. As confessed by those initial proposers, those lengths are simply of the sense of dimensional analysis. The structure of those lengths, in view of rational dynamics, has never been clearly explained, in spite of their frequent appearance in modern theories.

The assertion motivating the present intention is that one has to introduce singular lines of a field in advance of coarse-graining the basic equations of the field. This approach is quite analogous to the approach of Boltzmann, who considered precise dynamics of particle collisions in advance of coarse-graining the collision effect in the Boltzmann equation. The author wishes to publish some results of his study of turbulent flow in accordance with the present assertion in the near future.

SECTION VI

CONCLUSION

1. This is a purely mathematical treatment. Physical implications are represented by w and Φ.

2. It is emphasized that correlations appearing among causal events are not trivial, as illustrated by the two examples in Section II.

3. In view of (2) above, referring to the methcd of "random flights" is not necessarily proper.

SECTION VII

REFERENCES

1. Markoff, A. A.: Wahrscheinlichkeitsrechnung, Leipzig, 1912. (German translation of Russian second edition (1908); original Russian edition appeared in 1900).
2. Lord Rayleigh: Random Vibrations and Random Flights in One, Two, or Three Dimensions, Phil. Mag., 37, 321, 1919 (or Scientific papers, Vol. 6, p. 604).
3. Holtsmark, J.: Uber die Verbreiterung von Spektrallinien, Ann. d. Physik, 58, 577, 1919; also, Physik. Zeits., 20, 162, 1919 and 25, 73, 1924.
4. Chandrasekhar, S.: Stochastic Problems in Physics and Astronomy, Rev. Mod. Phys., 15, 1, 1943.
5. Koga, T.: A Kinetic Theoretical Investigation of a Fully Ionized Gas, Part II - Some Aspects of Multiple Collisions, PIBAL Report No. 863 (Polytechnic Institute of Brooklyn), January 1965.