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1. Basic Concepts and Definitions

1.1 With the advent of very complex engineering designs such
as those of high-speed computers or supersonic aircraft, it
has become increasingly important to study the relationship
between the functioning and failure of single components and
the performance of the entire system and, in particular, to
be able to make quantitative statements about the probability
that the system will perform according to specifications. It
is the aim of this paper to present some inequalities for
this probability.

1.2 We shall assume that there are only two states possible
for every component of a system, as well as for the system
itself: either it functions or it fails. When the system
consists of n components, we shall ascribe to each of them
a binary variable which will indicate its state

$$x_i = \begin{cases} 1 & \text{when the } i\text{-th } \text{ component functions} \\ 0 & \text{when the } i\text{-th } \text{ component fails} \end{cases}$$

for $i = 1, 2, \cdots, n$. Similarly we ascribe to the entire
system a binary indicator variable

$$u = \begin{cases} 1 & \text{when the system functions} \\ 0 & \text{when the system fails.} \end{cases}$$

When the design of a system is known, then the states of all n components i.e., the values of x_1, x_2, $\cdots$, x_n determine the state of the system, that is the value of u so that

$$u = \varphi (x_1, x_2, \cdots, x_n)$$

where φ is a function assuming the values 0 or 1. This function φ will be called the <u>structure</u> <u>function</u> of the system. The indicator variable x_i will sometimes be referred to as "component x_i" and φ will sometimes be called "structure φ". The n-tuple of 0's or 1's

$$(x_1, x_2, \cdots, x_n) = \underline{x}$$

will be called the <u>vector</u> <u>of</u> <u>component</u> <u>states</u> or, in short, the "state vector". It can assume any one of the 2^n values represented by the vertices of the unit-cube in n-dimensional space: $(0, 0, \cdots, 0)$, $(1, 0, \cdots, 0)$, $(1, 1, \cdots, 0)$, $\cdots$, $(1, 1, \cdots, 1)$. This set of all possible values of $\underline{x}$ will be denoted by I_n. Thus the structure function $\varphi(x_1, x_2, \cdots, x_n) = \varphi (\underline{x})$ is a binary function on I_n.

1.3 We shall furthermore assume that the state of each component is decided by chance, so that the value actually assumed by x_i is a binary random variable X_i with the probability distribution

$$\begin{aligned} \Pr \{X_i = 1\} &= p_i \\ \Pr \{X_i = 0\} &= q_i = 1 - p_i \end{aligned} \qquad i = 1, 2, \cdots, n$$

and we shall make the assumption that X_1, X_2, $\cdots$, X_n are

totally independent. The probability p_i will be called the <u>reliability of the</u> i-th component.

In the following it will always be assumed that

(1.3.1) $p_1 = p_2 = \cdots = p_n = p$

that is, that all components have the same reliability, e.g. the reliability of the least reliable component.

1.4 For a known structure function $\varphi(\underline{X})$ the value of p determines the probability that the system will function

(1.4.1) $\Pr \{\varphi(\underline{X}) = 1 \mid p\} = h_\varphi(p)$

i.e., the <u>reliability of the system</u> for given component reliability p . The function $h_\varphi(p)$ is called the <u>reliability function</u> for φ ; we will mostly denote it by $h(p)$ omitting the subscript φ .

In this paper we shall present inequalities for $h'(p)$, the derivative of the reliability function, which can be obtained when only partial information about $\varphi(\underline{x})$ is available. We shall then discuss a procedure by which such inequalities can be used to obtain some conclusions about $h(p)$.

1.5 The assumption of 1.2 is restrictive since it precludes consideration of systems whose components may function only partially and yet the systems will deliver a satisfactory performance. Similarly, the assumption of 1.3 is rather special, since often functioning or failure of different components of a system is correlated. Nevertheless, these two assumptions are a reasonable approximation to many

practical situations, and they make it possible to simplify
the theory to a manageable level.

1.6 For state vectors $\underline{x}$, $\underline{y}$ we shall use the following
notations:

$$\underline{x} \leq \underline{y} \quad \text{when} \quad x_i \leq y_i \quad \text{for} \quad i = 1, 2, \cdots, n$$

$$\underline{x} < \underline{y} \quad \text{when} \quad \underline{x} \leq \underline{y} \quad \text{and} \quad x_j < y_j \quad \text{for some} \quad j$$

$$(0_k, \underline{x}) = (x_1, x_2, \cdots, x_{k-1}, 0, x_{k+1}, \cdots, x_n)$$

$$(1_k, \underline{x}) = (x_1, x_2, \cdots, x_{k-1}, 1, x_{k+1}, \cdots, x_n)$$

$$\underline{0} = (0, 0, \cdots, 0) \ , \quad \underline{1} = (1, 1, \cdots, 1) \ .$$

A component x_k is called __essential__ for the structure $\varphi(\underline{x})$
if there exists a state vector $\underline{x}^*$ such that

$$\varphi (1_k, \underline{x}^*) \neq \varphi (0_k, \underline{x}^*)$$

If $\varphi(x)$ is any function on I_n , not necessarily a structure
function, the same definition of an essential component can be
used.

1.7 For given n there are $2^{(2^n)}$ possible different
structure functions. Among all these possible structure
functions we shall single out the class of __coherent__ structure
functions which is defined in the following, (this definition
was introduced in [1]) and which not only has some intuitive
appeal but also has been found to have a number of rather
interesting properties [2].

A structure function $\varphi(\underline{x})$ is coherent when it fulfills
the following conditions:

(1.7.1) $\qquad \phi(\underline{x}) \leq \phi(\underline{y})$ for $\underline{x} \leq \underline{y}$

(1.7.2) $\qquad \phi(\underline{0}) = 0 , \phi(\underline{1}) = 1 .$

From now on we shall assume that all structure functions considered are coherent.

1.8 A state vector $\underline{x}$ is called a __path__ for ϕ when

$$\phi(\underline{x}) = 1$$

and $\underline{x}$ is called a __cut__ for ϕ when

$$\phi(x) = 0 .$$

This terminology is analogous to that used in circuit theory.

From ϕ being coherent follows immediately that

(a) if $\underline{x}$ is a path for ϕ and $\underline{x} \leq \underline{y}$, then $\underline{y}$ is a path for ϕ ,

(b) if $\underline{x}$ is a cut for ϕ and $\underline{x} \geq \underline{y}$, then $\underline{y}$ is a cut for ϕ .

For every state vector $\underline{x} \; \varepsilon \; I_n$ we define

$$S(\underline{x}) = \sum_{i=1}^{n} x_i = \text{number of functioning components in}$$

$\underline{x}$ and call $S(\underline{x})$ the __size__ of $\underline{x}$.

For a given structure $\phi(\underline{x})$ we consider the following numbers:

(1.8.1) $A_j = $ number of paths of size j , for $j = 0, 1, 2, \dots, n$

Obviously one has

(1.8.2) $A_j \leq \binom{n}{j} , j = 0, 1, \dots, n .$

1.9 For $\phi(\underline{x})$ coherent one can prove [1] that

(1.9.1) $\dfrac{A_j}{\binom{n}{j}} \leq \dfrac{A_{j+1}}{\binom{n+1}{j}}$ for $j = 0, 1, \cdots, n - 1$

(1.9.2) $h(0) = 0, \quad h(1) = 1$

(1.9.3) $h'(p) > 0$ for $0 < p < 1$.

2. Inequalities for $h'(p)$ and grids for $h(p)$

2.1 Let us assume that, for all reliability functions $h(p)$ belonging to a certain class $\mathcal{H}$, one can prove an inequality of the form

(2.1.1) $h'(p) \geq \Psi(p,h)$, for all $0 \leq p \leq 1$, $0 \leq h \leq 1$. This means that there exists a function $\Psi(p, h)$ on the unit square $0 \leq p \leq 1$, $0 \leq h \leq 1$, such that when a curve representing $h(p) \varepsilon \mathcal{H}$ passes through a point (p, h) then the slope of that curve at that point must be at least $\Psi(p, h)$.

If the inequality is replaced by equality, then (2.1.1) becomes a differential equation

(2.1.2) $\mathcal{X}' = (p, \mathcal{X})$

which under very general assumptions on $\Psi(p, \mathcal{X})$ has a one-parametric family of solutions $\mathcal{X}_c(p)$ with the parameter c . We shall say that the family of curves representing the functions $\mathcal{X}_c(p)$ for $0 \leq p \leq 1$ forms a __grid__ __for__ __the__ __class__ $\mathcal{H}$. In view of (2.1.1), this grid has the following properties:

$1^{\underline{2}}$ Through every point (p, h) in the unit square goes exactly one grid curve $\mathcal{X}_c(p)$

$2^{\underline{2}}$ if the curve representing a reliability function $h(p) \varepsilon \mathcal{H}$

goes through a point (p, h) and $\chi_c(p)$ is the grid curve going through the same point, then $h'(p) \geq \chi_c'(p) = \psi(p, h)$. This means that if the curve $h(p) \in \mathcal{H}$ intersects any grid curve, then it intersects it from below. It should be noted, however, that there may be points in the unit square $0 \leq p \leq 1$, $0 \leq h \leq 1$ such that no curve $h(p) \in \mathcal{H}$ goes through them.

The knowledge of a grid may be utilized in various ways which will be discussed later, but the most immediate application is the following:

Assume that all one knows about a reliability function $h(p) \in \mathcal{H}$ is that for given component reliability p_0 it assumes a known value $h(p_0) = h_0$. Then there exists a parameter value c_0 such that $\chi_{c_0}(p_0) = h_0$, and from grid property 2^0 it follows that

$$h(p) \geq \chi_c(p) \quad \text{for all} \quad p \geq p_0 .$$

2.2 It is well known that for the class $\mathcal{H}_c$ consisting of reliability functions for all coherent structures the inequality

$$(2.2.1) \qquad h'(p) \geq \frac{h(p) \, [1 - h(p)]}{p(1 - p)}$$

holds for $0 \leq p \leq 1$. This inequality, obtained for two-terminal networks in [3] and generalized in [1] to all coherent systems and in [4] to the case of components with unequal reliabilities, is of the form $(2.1.1)$. The corresponding

differential equation of the form (2.1.2) is

$$(2.2.2) \quad \chi' = \frac{\chi(1 - \chi)}{p(1 - p)} \quad .$$

Its general integral is

$$(2.2.3) \quad \frac{\chi_c(p)}{1 - \chi_c(p)} = c \frac{p}{1 - p} \quad , c > 0 \ , \ 0 \le p \le 1$$

and this one-parameter family of curves forms the so-called Moore-Shannon grid. Fig. 1 indicates the shape of the curves of this family which, for c = 1 , includes the diagonal $\chi_1(p) = p$.

2.3 We shall need the following concepts defined by analogy with terms used in the theory of circuits: the length χ_ϕ of a system ϕ is the smallest number of components such that if only they function, the structure functions; the width w_ϕ of ϕ is the smallest number of components such that if only they fail, the structure fails.

According to these definitions we have

$$(2.3.1)$$

$$\phi(\underline{x}) = 0 \quad \text{for all } \underline{x} \quad \text{such that} \quad S(\underline{x}) \le \chi_0 - 1$$

$$\phi(\underline{x}) = 0 \quad \text{for some } \underline{x} \quad \text{such that} \quad \chi \le S(\underline{x}) \le n - w$$

$$\phi(\underline{x}) = 1 \quad \text{for some } \underline{x} \quad \text{such that} \quad \chi \le S(\underline{x}) \le n - w$$

$$\phi(\underline{x}) = 1 \quad \text{for all } \underline{x} \quad \text{such that} \quad n - w + 1 \le S(\underline{x}) \quad .$$

Sometimes the only information one has about a system ϕ consists of the knowledge of χ_ϕ , or w_ϕ and, possibly, A_χ or A_w . The remainder of this section will be devoted to the problem of obtaining grids when some or all of the parameters χ , w , A_χ , A_w are known.

According to (2.3.1) we compute

$$E\left\{\Phi(\underline{X})\ [S(\underline{X}) - \chi]\right\} = \sum_{j=0}^{n} \sum_{S(\underline{x})=j} \Phi(\underline{x})\ (j - \chi)\ P\{\underline{X} = \underline{x}\} =$$

$$\sum_{j=\chi}^{n-w} (j-\chi) \sum_{S(\underline{x})=j} \Phi(\underline{x})\ p^j(1-p)^{n-j} + \sum_{j=n-w+1}^{n} (j-\chi) \sum_{S(\underline{x})=j} p^j(1-p)^{n-j} =$$

$$\sum_{j=\chi}^{n-w} (j-\chi)\ A_j\ p^j(1-p)^{n-j} + \sum_{j=n-w+1}^{n} (j-\chi)\ \binom{n}{j}\ p^j(1-p)^{n-j}\ ,$$

hence, using formula (6.3) in [4] for $p_1 = p_2 = \cdots = p_n$,

we obtain

$$\text{cov}\left\{\Phi(\underline{X}), S(\underline{X})\right\} = p(1-p)\ h'(p) = E\left\{\Phi(\underline{X})\ S(\underline{X})\right\} -$$

$$E\left\{\Phi(\underline{X})\right\}\ E\left\{S(\underline{X})\right\} =$$

$$\chi\ h(p) + \sum_{j=\chi}^{n-w} (j-\chi)\ A_j\ p^j(1-p)^{n-j} +$$

$$\sum_{j=n-w+1}^{n} (j-\chi)\ \binom{n}{j}\ p^j(1-p)^{n-j} - h(p)\ np =$$

$$(\chi - np)\ h(p) + \sum_{j=\chi}^{n-w} (j-\chi)\ A_j\ p^j(1-p)^{n-j} + \sum_{j=n-w+1}^{n} (j-\chi)\ \binom{n}{j}\ p^j(1-p)^{n-j}.$$

Using the inequality

$$A_j \geq 1 \quad \text{for} \quad \chi \leq j \leq n - w$$

which follows from (2.3.1), and inequality (1.9.1), one obtains

$$A_j \geq \text{Max}\left\{A_\chi \frac{\binom{n}{j}}{\binom{n}{\chi}}, 1\right\}, \quad \text{for} \quad \chi \leq j \leq n - w\ .$$

Since

$$\frac{\binom{n}{j}}{\binom{n}{\chi}} = \frac{\chi!\ (n-\chi)!}{j!\ (n-j)!} = \frac{(n-j+1)\ (n-j+2)\ \cdots\ (n-\chi)}{(\chi+1)\ (\chi+2)\ \cdots\ j}$$

and $n-j \geq \chi$ implies $\dfrac{n-j+1}{\chi+1} > \dfrac{n-j+2}{\chi+2} > \ldots > \dfrac{n-\chi}{j} \geq 1$, hence

$$\frac{\binom{n}{j}}{\binom{n}{\chi}} > 1$$

and $n-j < \chi$ implies $\frac{n-j+1}{\chi+1} < \frac{n-j+2}{\chi+2} < \dots < \frac{n-\chi}{j} < 1$,

hence $\frac{\binom{n}{j}}{\binom{n}{\chi}} < 1$

we conclude

$$A_j > A_\chi \frac{\binom{n}{j}}{\binom{n}{\chi}} \quad \text{for} \quad j \leq n - \chi$$

$$A_j \geq 1 \quad \text{for} \quad j > n - \chi$$

and finally we obtain the inequality

$$p(1-p) \, h'(p) \geq (\chi-np) \, h(p) \, +$$

$$\frac{A_\chi}{\binom{n}{\chi}} \sum_{j=\chi}^{n-\chi} (j-\chi) \binom{n}{j} p^j (\chi-p)^{n-j} \quad +$$

(2.3.3)

$$\sum_{j=n-\chi+1}^{n-w} (j-\chi) \, p^j (1-p)^{n-j} \quad +$$

$$\sum_{j=n-w+1}^{n} (j-\chi) \binom{n}{j} p^j (1-p)^{n-j} \quad .$$

In this inequality, the first sum may be empty if $\chi > n - \chi$, and the second sum may be empty if $w > \chi - 1$. Both these sums are empty when $\chi > n - \chi$ and $w > \chi-1$ which implies $\chi + w \geq n$, so that, in view of the known inequality $1 + w \leq n + 1$, one then has $\chi + w = n$ or $\chi + w = n + 1$.

Inequality (2.3.3) is of the form (2.1.1). The corresponding differential equation of the form (2.1.2) is

$$\chi' = \frac{\chi-np}{p(1-p)} + \frac{A_\chi}{\binom{n}{\chi}} \sum_{j=\chi+1}^{n-\chi} (j-\chi) \binom{n}{j} p^{j-1}(1-p)^{n-j-1} +$$

$$\sum_{j=n-\chi+1}^{n-w} (j-1) p^{j-1}(1-p)^{n-j-1} +$$

(2.3.4)

$$\sum_{j=n-w+1}^{n} (j-\chi) \binom{n}{j} p^{j-1}(1-p)^{n-j}$$

and, again, the first or the second sum or both cam be empty.

The general solution of (2.3.4) is

$$\chi_c = cp^\chi (1-p)^{n-\chi} + A_\chi \sum_{j=\chi+1}^{n-\chi} \frac{\binom{n}{j}}{\binom{n}{\chi}} p^{j}(1-p)^{n-j} +$$

(2.3.5)

$$\sum_{j=n-\chi+1}^{n-w} p^{j}(1-p)^{n-j} + \sum_{j=n-w+1}^{n} \binom{n}{j} p^{j}(1-p)^{n-j} .$$

The family of functions (2.3.5) constitutes a grid for the class of reliability functions corresponding to coherent systems with given n, χ, w and A_χ. This class will be denoted by $\mathcal{X}_c(n, \chi, w, A_\chi)$.

If χ and w are known, but A_χ is not known, then (2.3.3) can be replaced by a (weaker) inequality by setting $A_\chi = 1$, and one obtains the grid

$$\chi_c = cp^\chi(1-p)^{n-\chi} + \sum_{j=\chi+1}^{n-\chi} \frac{\binom{n}{j}}{\binom{n}{\chi}} p^{j}(1-p)^{n-j} +$$

(2.3.6)

$$\sum_{j=n-\chi+1}^{n-w} p^{j}(1-p)^{n-j} + \sum_{j=n-w+1}^{n} \binom{n}{j} p^{j}(1-p)^{n-j}$$

for the class $\mathcal{K}_c(n, \chi\ w)$ of reliability functions corresponding to coherent systems with given n, χ, and w .

If only n and χ are known, then the resulting grid is

$$\mathcal{K}_c = cp^\chi(1-p)^{n-\chi} + \sum_{j=\chi+1}^{n-\chi} \frac{\binom{n}{j}}{\binom{n}{\chi}} p^j(1-p)^{n-j} +$$

(2.3.7)
$$\sum_{j=n-\chi+1}^{n} p^j(1-p)^{n-j}$$

When in (2.3.3) all terms on the right side are omitted, but the first, one arrives at the particularly simple grid for $\mathcal{K}_c(n,\chi)$:

(2.3.8) $\mathcal{K}_c = cp^\chi(1-p)^{n-\chi}$.

For $h(p)\ \varepsilon\ \mathcal{K}_c(n, \chi, w, A_\chi)$ one always has

$$h(p) = A_\chi p^\chi(1-p)^{n-\chi} + \sum_{j=\chi+1}^{n} A_j p^j(1-p)^{n-j} > A_\chi p^\chi(1-p)^{n-\chi}$$

so that no $h(p)$ in this class can go through points in the region $h \le A_\chi p^\chi(1-p)^{n-\chi}$.

2.4 Again using (2.3.1) one computes

$$E\left\{[1-\phi(\underline{X})]\ [n-w-S(\underline{X})]\right\} = \sum_{j=0}^{n} \sum_{S(\underline{x})=j} [1-\phi(\underline{x})]\ (n-w-j)\ P\{\underline{X}=\underline{x}\} =$$

$$\sum_{j=0}^{\chi-1} (n-w-j)\ \binom{n}{j}\ p^j(1-p)^{n-j} + \sum_{j=\chi}^{n-w-1} (n-w-j)\ A_j^*\ p^j(1-p)^{n-j}$$

where $A_j^* = \binom{n}{j} - A_j$ = number of cuts of size j . Hence

$$\text{cov}\left\{\phi(\underline{X}),\ S(\underline{X})\right\} = p(1-p)\ h'(p) = E\left\{\phi(\underline{X})\ S(\underline{X})\right\} - E\{\phi(\underline{X})\}\ E\{S(\underline{X})\} =$$

$$[1-h(p)]\ [np-(n-w)] + \sum_{j=0}^{\chi-1} (n-w-j)\ \binom{n}{j}\ p^j(1-p)^{n-j} +$$

$$\sum_{j=\chi}^{n-w-1} (n-w-j)\ A_j^*\ p^j(1-p)^{n-j}$$

and duplicating the arguments of Section 2.3 one obtains

$$p(1-p) \, h'(p) \geq [1-h(p)] \, [np-(n-w)] \, +$$

(2.4.1)

$$\sum_{j=0}^{\chi-1} (n-w-j) \binom{n}{j} p^j (1-p)^{n-j} \; +$$

$$\sum_{j=\chi}^{w-1} (n-w-j) \, p^j (1-p)^{n-j} \; +$$

$$\frac{A^*_{n-w}}{\binom{n}{w}} \sum_{j=w}^{n-w-1} (n-w-j) \binom{n}{j} p^j (1-p)^{n-j} \, ,$$

an inequality of the form (2.1.1). As was done in Section 2.3 with regard to inequality (2.3.3), we may retain all or some terms of the right side of (2.4.1), replace in each case inequality by equality, integrate the resulting differential equations, and obtain grids for the respective classes of reliability functions. We consider here explicitly only the case when all but the first term on the right side of (2.4.1) are omitted. One obtains then the simple grid

(2.4.2) $$\chi_c(p) = 1 - cp^{n-w} (1-p)^w$$

for the class $\mathcal{H}_c(n,w)$ of reliability functions for coherent systems for which n and w are known.

2.5 **The use of several grids for the same class of reliability functions.**

2.5.1 If there are several different grids for a given class $\mathcal{H}$ of reliability functions, then all these grids can be used to obtain lower bounds for an $h(p) \, \varepsilon \, \mathcal{H}$ which are better

than bounds based on any single of the grids. For example,
let us consider a class $\mathcal{H}$ for which there are two grids
$\chi_a(p)$ and $\lambda_b(p)$, with parameters a and b , respectively.
If for an $h(p) \in \mathcal{H}$ it is known that $h(p_0) = h_0$, then one
can determine a_0 and b_0 so that $\chi_{a_0}(p_0) = \lambda_{b_0}(p_0) = h_0$,
and choose that one of the curves $\chi_{a_0}(p)$, $\lambda_{b_0}(p)$ which is
steeper at p_0 . If, say, $\chi'_{a_0}(p_0) > \lambda'_{b_0}(p_0)$ then $\chi_{a_0}(p_0)$
will be used as a lower bound for $h(p)$ for $p \geq p_0$ until,
possibly, it intersects some curve of the λ-grid which is
steeper at the point of intersection, i.e., until the first
value $p_1 > p_0$ such that for some b_1 one has $\lambda_{b_1}(p_1) =$
$\chi_{a_0}(p_1)$, $\lambda'_{b_1}(p_1) > \chi'_{a_0}(p_1)$. Then $\lambda_{b_1}(p)$ can be used as a
lower bound for $p \geq p_1$, etc. Fig. 2 illustrates this
procedure.

In some cases an analytic discussion can be carried out
for the use of several grids, and a practically useful example
of such a discussion follows.

2.5.2   In sections 2.2, 2.3, and 2.4 we have seen that the
families of curves (2.3.8), (2.2.3) and (2.4.2) are grids for
the class $\mathcal{H}_c(n, \chi, w)$. For the purposes of our discussion
we rewrite the equations of these grids

(2.5.2.1) $$\chi_a(p) = ap^\chi(1-p)^{n-\chi}$$

(2.5.2.2) $$\lambda_b(p) = \frac{b \, p}{1 + (b-1)p}$$

(2.5.2.3) $$\mu_c(p) = 1 - cp^{n-w}(1-p)^w .$$

In order that the curve of each of these families passes through a given point (p_1, h_1) the corresponding parameters must assume the values

$$a_1 = \frac{h_1}{p_1^{\gamma} (1-p_1)^{n-\gamma}}$$

$$b_1 = \frac{h_1}{p_1} \cdot \frac{1-p_1}{1-h_1}$$

$$c_1 = \frac{1-h_1}{p_1^{n-w} (1-p_1)^{w}} \quad .$$

The derivatives of the three curves passing through (p_1, h_1) at that point are

$$(2.5.2.4) \qquad \chi'_{a_1}(p_1) = \frac{h_1(\gamma - np_1)}{p_1(1-p_1)}$$

$$(2.5.2.5) \qquad \lambda'_{b_1}(p_1) = \frac{h_1(1-h_1)}{p_1(1-p_1)}$$

$$(2.5.2.6) \qquad \mu'_{c_1}(p_1) = \frac{(1-h_1)(w-n+np_1)}{p_1(1-p_1)} \quad .$$

We have

$$\chi'_{a_1}(p_1) > 0 \quad \text{if and only if} \quad 0 < p_1 < \frac{\gamma}{n}$$

$$\lambda'_{b_1}(p_1) > 0 \quad \text{for all } p_1 , \quad 0 < p_1 < 1$$

$$\mu'_{c_1}(p_1) > 0 \quad \text{if and only if} \quad \frac{n-w}{n} < p_1 < 1$$

so that (2.5.2.1) is a non-trivial grid only for $0 < p < \frac{\gamma}{n}$ (2.5.2.3) only for $\frac{n-w}{n} < p_1 < 1$, while the Moore-Shannon

grid (2.5.2.2) consists of functions which increase for all p and hence is useful for $0 < p < 1$.

In view of the known inequality $\chi + w \leq n + 1$, we have $\frac{\chi}{n} \leq \frac{n-w}{n}$, except for the case when $\chi + w = n + 1$ which occurs if and only if the structure is "χ out of n " ; in this case $h(p) = \sum\limits_{j=\chi}^{n} \binom{n}{j} p^j (1-p)^{n-j}$ and is completely known. In all other cases the intervals $(0, \frac{\chi}{n})$, $(\frac{\chi}{n}, \frac{n-w}{n})$, $(\frac{n-w}{n} , 1)$ are non-overlapping, and for $0 < p < \frac{\chi}{n}$ we need to consider only grids (2.5.2.1) and (2.5.2.2), for $\frac{\chi}{n} < p < \frac{n-w}{n}$ only grid (2.5.2.2), and for $\frac{n-w}{n} < p < 1$ only grids (2.5.2.2) and (2.5.2.3). Comparing the derivatives (2.5.2.4) and (2.5.2.5) we see that the χ-grid is steeper for $0 < p < \frac{\chi}{n}$ if and only if $h > np - \chi + 1$, and comparing (2.5.2.5) and (2.5.2.6) we find that the μ-grid is steeper for $\frac{n-w}{n} < p < 1$ if and only if $h > np - n + w$. The parallel lines $h = np - \chi + 1$ and $h = np - n + w$ divide therefore the unit square in three regions, from left to right, such that the χ-grid is steepest at all points of the first region, the λ-grid in the second and the μ-grid in the third region.

2.5.3 A specific example is presented in Figure 3. For $n = 10$, $\chi = 5$, $w = 2$, the lines

(χ_1) $h = 10p - 4$

(χ_2) $h = 10p - 8$

partition the unit square in the three regions described in the preceding section. The curves of the Moore-Shannon grid (2.5.2.2), shown before in Figure 1, are reproduced in Figure 3 in solid lines and, in addition, several curves of the $\varkappa$-grid (2.5.2.1) are indicated by dotted lines, and of the μ-grid (2.5.2.3) by broken lines.

If it is known that $h(p) \in \varkappa_c$ ($n = 10$, $\gamma = 5$, $w = 2$) and that the graph of $h(p)$ goes through the point $P_1 =$ (.20, .05), then our theory tells us that for $p \geq .20$ that graph is bounded from below by a curve which goes along the $\varkappa$-curve through P_1 to its intersection with γ_1, then along the Moore-Shannon curve (in this case the diagonal) to its intersection with γ_2, then along the μ-curve. This lower bound is indicated by a heavy line.

Similarly, if a reliability function $h(p)$ of our family is known to go through $P_2 =$ (.34, .10) then for $p \geq .34$ one obtains for $h(p)$ the lower bound indicated by the heavy line beginning at P_2.

Another lower bound for $h(p)$ going through $P_3 =$ (.32, .40) is indicated by the heavy line beginning at that point.

Addendum

It should be mentioned that an improvement of the Moore-Shannon grid has been recently obtained. One can show that the following inequalities hold for all $h(p) \in \varkappa_c$

$$(-p \log p) \, h'(p) \geq -h \log h$$

$$[-(1-p) \log (1-p)] \, h'(p) \geq -(1-h) \log (1-h).$$

The corresponding grids are

(i) $\quad \chi_c(p) = p^c \quad , \, c > 0$

(ii) $\quad \chi_c(p) = 1 - (1-p)^c \, , \quad c > 0 \, .$

These grids are an improvement on (2.2.3), since (i) is steeper than the corresponding Moore-Shannon curve at every point such that $p > h$, and (ii) at every point such that $p < h$. A derivation of this new grid is being prepared for publication [5].

REFERENCES

[1]   Z. W. Birnbaum, J. D. Esary, and S. C. Saunders,
        "Multi Component Systems and Structures and Their
        Reliabilities", Technometrics, Vol. 3, No. 1, 1961,
        pp. 55-77.

[2]   J. D. Esary and A. W. Marshall, "System Structure
        and the Existence of a System Life", Technometrics,
        Vol. 6, No. 4, pp. 459-462.

[3]   E. F. Moore and C. E. Shannon, "Reliable Circuits
        Using Less Reliable Relays", Journal of the Franklin
        Institute, Vol. 262, 1956, pp. 191-208 and pp. 281-297.

[4]   J. D. Esary and F. Proschan, "Coherent Structures of
        Non-Identical Components", Technometrics, Vol. 5, 1963,
        pp. 191-209.

[5]   Z. W. Birnbaum, J. D. Esary, and A. W. Marshall,
        "Stochastic Characterization of Wear-Out for Components
        and Systems" , Boeing Scientific Research Laboratories
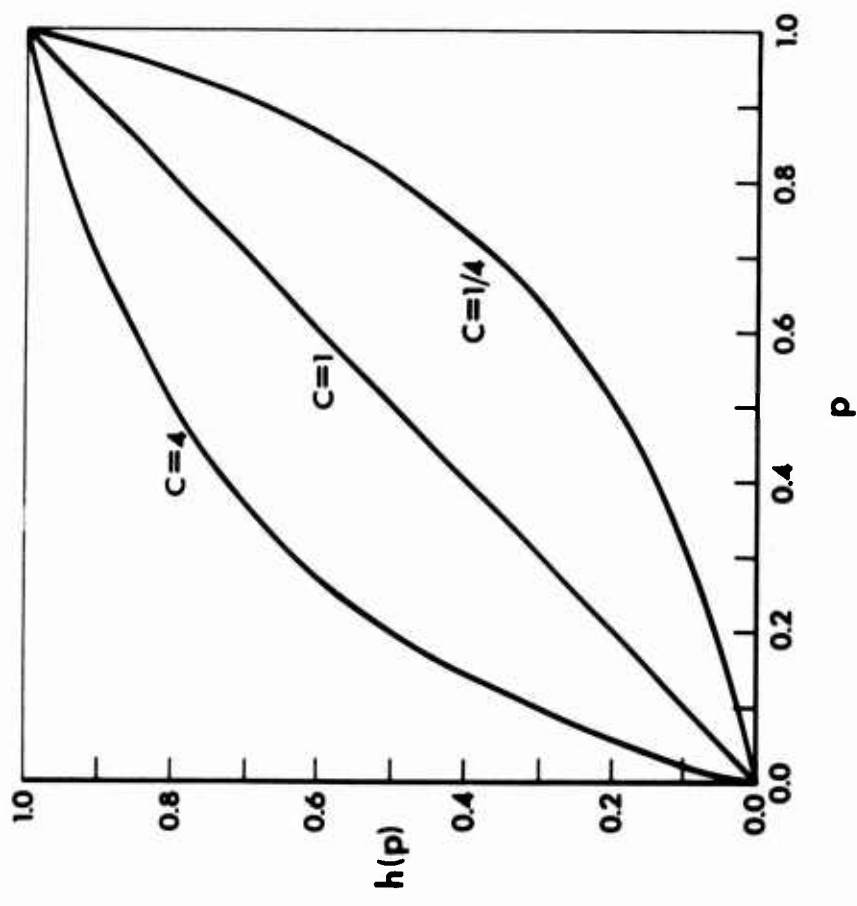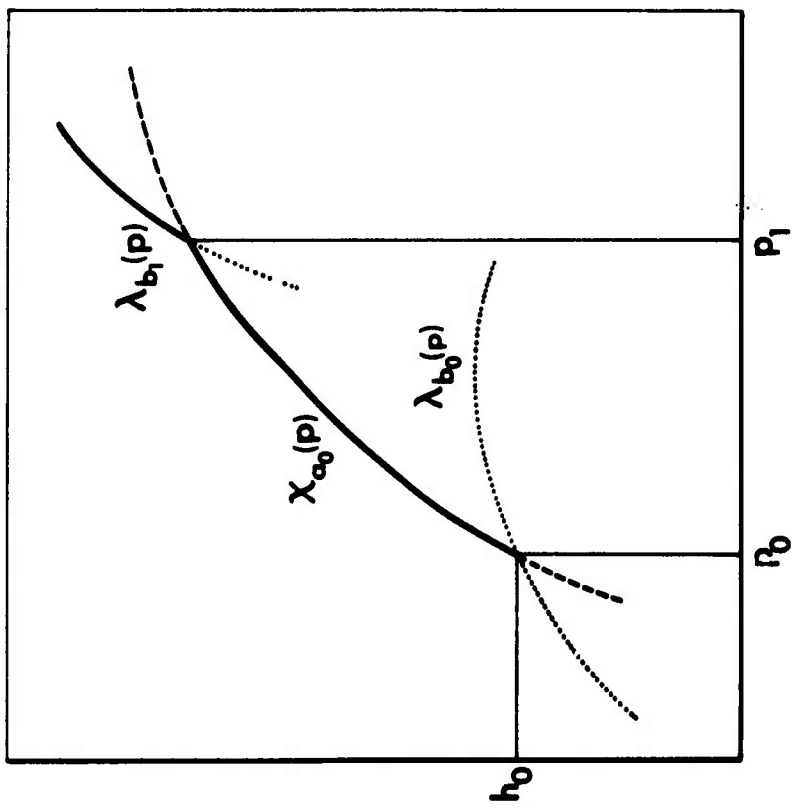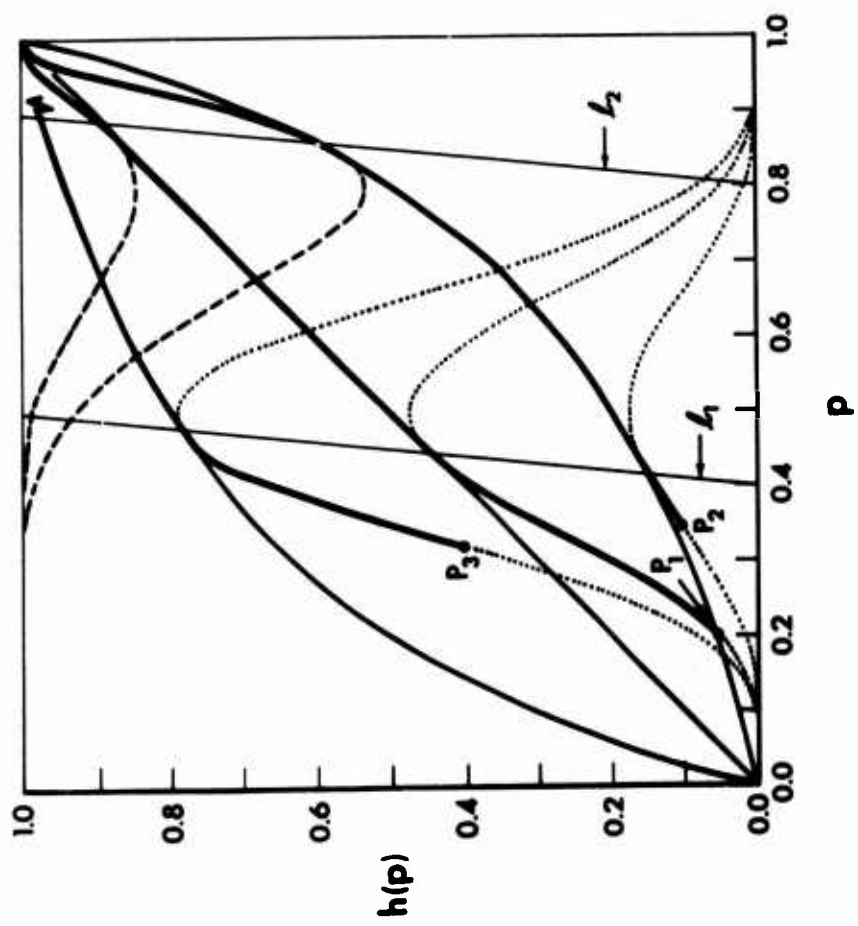   Document DI-8L-0460, in preparation.

FIGURE 1

FIGURE 2

FIGURE 3

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A method is presented for obtaining lower bounds for $h'(p)$, the derivative of the reliability function of a coherent binary structure with binary components, when only partial information about this structure is available. In particular, the case is studied when only the number of components, the length and the width of the structure are known.

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