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PART I, VOLUME 2

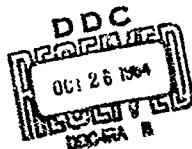
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**SIX-DEGREE-OF-FREEDOM FLIGHT PATH STUDY
GENERALIZED COMPUTER PROGRAM**

PART I, VOLUME 2 - STRUCTURAL LOADS FORMULATION

TECHNICAL DOCUMENTARY REPORT No. FDL-TDR-84-1
PART I, VOLUME 2

AUGUST 1984



**AF FLIGHT DYNAMICS LABORATORY
RESEARCH AND TECHNOLOGY DIVISION
AIR FORCE SYSTEMS COMMAND
WRIGHT-PATTERSON AIR FORCE BASE OHIO**

Project No. 1431, Task No. 1431B

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ABSTRACT

The equations of motion applicable to the Six-Degree-of-Freedom Structural Loads Program (SLP) are derived in this report. These equations are written for the determination of vehicle structural loads and response due to aerodynamic loads, loads due to control surface deflections, and environmental disturbances. Arbitrary elastic degrees of freedom (wing bending, wing torsion, body bending, etc.) and fuel slosh equations are incorporated into the overall analysis.

Newtonian flow theory is used for obtaining idealized aerodynamic pressure distributions since it is the simplest aerodynamic theory that offers sufficient generality. Accelerations, deflections, shear forces and bending moments at arbitrary stations can be computed.

This technical documentary report has been reviewed and is approved.



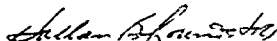
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SYMBOLS

| | |
|--------------------|---|
| A_j^{rs} | static aerodynamic terms. See Equation (195). |
| ϕ_{kvt} | given mode of vibration in degree of freedom k . |
| A_{kvt}^r | components of ϕ_{kvt} in the y coordinate system. |
| a_j^{rs} | inertia terms. See Equation (180). |
| B_j^{rs} | aerodynamic stiffness terms. See Equation (196). |
| $\dot{\theta}_j$ | the dynamically balancing rotation rate with respect to q_j^d of the vehicle relative to the vehicle axes. |
| $\dot{\theta}_j^r$ | components of $\dot{\theta}_j$ in the y coordinate system. |
| C_{jk}^r | aerodynamic damping terms. See Equation (197). |
| \bar{c}_j | the dynamically balancing translation rate with respect to q_j^d of the vehicle relative to the vehicle axes. |
| \bar{c}_j^r | components of \bar{c}_j in the y coordinate system. |
| C_{pvt} | permutation symbol. See text preceding Equation (139). |
| E | number of thrust vectoring nozzles (or "engines"). |
| e_{st} | components in the \bar{I}_T system of the \bar{J}_{st}^i vectors. |
| \bar{F} | the sum of the external forces exerted on the vehicle. |
| \bar{F}_{ih} | the external force on the h -th particle of the i -th section. |
| \bar{F}_{ihkj} | the internal force exerted on the h -th particle of the i -th section by the j -th particle of the k -th section. |
| \bar{F}_{ihkj} | the magnitude of \bar{F}_{ihkj} . |

| | |
|----------------|--|
| \bar{G} | the sum of the moments about the origin of the vehicle axes due to the external forces. |
| \bar{g} | the force per unit mass due to gravitation. |
| γ_j | the coefficient of "structural" damping associated with the j-th degree of freedom. |
| G_{rs} | products of inertia of structure and fuel about vehicle axes. |
| H'_{rsi} | the moments and the negatives of the products of inertia of section i about its own axes. See Equation (138). |
| H_{jk} | inertia coupling terms. See Equation (173). |
| H_k | modal unbalances. See Equation (150). |
| h | subscript used to denote a particle of a section. |
| \bar{h}_{ji} | the partial linear velocity with respect to q^j of the center of mass of section i relative to the vehicle axes - values obtained after dynamic balancing. |
| h_{ji}^y | components of \bar{h}_{ji} in the y coordinate system. |
| I_{rs} | moments and negatives of products of inertia of structure and fuel about vehicle axes. See Equation (145). |
| i | subscript used to denote a section of the vehicle. |
| \bar{j}_r | three unit vectors pointing respectively in the directions of the three vehicle axes y^r . See Sec. 2 and Fig. 2. |
| \bar{j}_k | three unit vectors pointing respectively in the directions of the three axes x^k of section i. See Sec. 2 and 3 and Fig. 2. |
| \bar{f}_{ki} | the partial linear velocity with respect to q^k of the center of mass of section i relative to the vehicle axes - arbitrary values given prior to dynamic balancing. |
| f_{ki}^y | components of \bar{f}_{ki} in the y coordinate system. |
| j | suffix used to denote a degree of freedom. |
| K_{jk} | components of the stiffness tensor. See Equation (90). |

| | |
|-------------|---|
| k | suffix used to denote a degree of freedom |
| I_{jk}^i | nodal inertia terms. See Equation (15) |
| L | suffix used to denote a degree of freedom. |
| M_{JK} | components of the inertia tensor. See Equation (32) |
| \bar{M} | the bending moment at a specified location. |
| M_r | components of \bar{M} in the y coordinate system. |
| m | total mass of vehicle and fuel at any instant. |
| m_i | mass of section i . |
| m_{ik} | mass of the k -th particle of section i . |
| N | the number of sections and tanks. |
| N_j | the generalized forces associated with inertia forces. See Equation (46). |
| N_j^i | nodal inertia terms. See Equation (172). |
| n | number of elastic degrees of freedom. |
| \bar{n} | a unit vector located at a certain point on the surface, perpendicular to the surface at that point, and pointing outward. |
| n^r | components of \bar{n} in the v^r coordinate system. |
| Q_j | the generalized forces associated with conservative internal forces. |
| \bar{Q}_i | position vector locating the origin of the v_i^r coordinate system with respect to the y coordinate system. See Fig. 2. |
| Q_i^r | components of \bar{Q}_i in the y coordinate system and coordinates of the center of mass of section i . |
| P_i | the number of particles in the i -th section or tank of fuel. |

| | |
|-------------------|---|
| P_j | the generalized forces associated with dissipative internal forces. |
| $D_{r,i}$ | modal moments and negatives of products of inertia of the vehicle. See Equation (53). |
| $\tilde{P}_{j,i}$ | modal moments and negatives of products of inertia of section i . See Equation (179). |
| Q_j | the generalized forces associated with external forces. See Equation (47). |
| q^i | generalized coordinate associated with the i -th degree of freedom. |
| r | suffix denoting the r -th coordinate axis in either the y or the \mathcal{V}_i system. |
| S_i | the surface of the i -th section. |
| s | suffix denoting the s -th coordinate axis in either the y or the \mathcal{V}_i system. |
| T | the kinetic energy of the vehicle and fuel. |
| T_i | the magnitude of the thrust force at the i -th nozzle. |
| t | time. Also used sometimes as a suffix in the same sense as r or s . |
| U | potential energy due to elastic deformation. |
| Y | energy dissipated thru damping. |
| \vec{V} | linear velocity of the vehicle at the origin of the vehicle axes. |
| V^r | components of \vec{V} in the y coordinate system. |
| $\vec{v}_{h,i}$ | velocity of the h -th particle of the i -th section. |
| W | the work done by the external forces. |
| w | the "piston speed" (or downwash) at a point on the surface. |
| \vec{X} | position vector of the vehicle in relation to a space-fixed frame of reference. |
| $\vec{y}_{h,i}$ | position vector of the h -th particle of the i -th section in relation to the vehicle axes. |

\bar{y}_{ih}^r components of \bar{y}_{ih} - y coordinates of the h-th particle of the i-th section.

\bar{y}_c position vector of the center of mass of the vehicle.

\bar{y}_c^i components of \bar{y}_c in the y coordinate system.

$\bar{\alpha}_{ji}$ the partial angular velocity with respect to q^j of the \bar{v}_i^j coordinates relative to the y system - values obtained after dynamic balancing.

$\bar{\alpha}_{ji}^r$ components of $\bar{\alpha}_{ji}$ in the \bar{v}_i^j coordinate system.

$\bar{\beta}_{ji}$ the partial angular velocity with respect to q^j of the \bar{v}_i^j coordinates relative to the y system - arbitrary values given prior to dynamic balancing.

$\bar{\beta}_{ji}^r$ components of $\bar{\beta}_{ji}$ in the \bar{v}_i^j coordinate system.

Γ_i^r products of inertia of section i referred to the sectional axes.

Δ_{ikhj} the distance from particle k to particle ih. See Equation (77).

Δ_{ik} inertia coupling terms. See Equation (171).

δ_{rs} the Kronecker delta
 $\delta_{rs} = 1$ when $r = s$.
 $\delta_{rs} = 0$ when $r \neq s$.

δ_j the logarithmic decrement associated with the j-th degree of freedom.

| | |
|------------------------|---|
| δ | the angle of rotation of \vec{J}_{21} and \vec{J}_{31} about J_{11} . See Sec. 9. |
| ϵ_{ijk} | inertia coupling terms. See Equation (170). |
| η_{ijk} | inertia coupling terms. See Equation (169). |
| \hat{I}_{ji}^{rs} | modal products of inertia of section i. See Equation (142). |
| λ_i | angle of swivel of nozzle (or the \vec{J}_{1i} vector) about an axis (\vec{L}_i) perpendicular to \vec{J}_1 and making an angle ϕ_1 with \vec{J}_3 . See Sec. 9. |
| μ_{ijk} | inertia coupling terms. See Equation (167). |
| ξ_j | aerodynamic modal term. See Equation (191). |
| π | ratio of circumference to diameter of a circle. |
| ρ | atmospheric density. |
| $\vec{\sigma}_{jih}$ | the partial linear velocity with respect to q^j of particle h relative to section i. |
| $\vec{\sigma}_{jih}^r$ | components of $\vec{\sigma}_{jih}$ in the \mathcal{V}_i coordinate system. |
| \vec{v}_{ih} | position vector of the h-th particle of the i-th section relative to the origin of the \mathcal{V}_i coordinate system. |
| v_{ih}^r | components of \vec{v}_{ih} in the \mathcal{V}_i system. |
| ϕ_i | angle of rotation of the axis and plane of swivel about the y^1 axis (\vec{J}_1). See Sec. 9. |
| $\vec{\Omega}$ | angular velocity of the vehicle axes. |
| Ω^r | components of $\vec{\Omega}$ in the y coordinate system. |
| ω_j | vibration frequency associated with the j-th degree of freedom. See Equation (91). |
| \boxed{KL}_j | inertia "symbols". See Equation (59). |

1. INTRODUCTION

This report includes the derivation of the equations to be used in the Structural Loads Program (SLP). This program is to be used in conjunction with the Six-Degree-of-Freedom Flight Path Computer Program (SDW), as a means to determine the vehicle structural loads and response due to aerodynamic loads, loads due to control surface deflections, and environmental disturbances (i.e., wind profiles and continuous discrete turbulence profiles). The program permits the inclusion of up to 17 elastic degrees of freedom and 40 fuel slosh modes. The elastic degrees of freedom are arbitrary, and the user may incorporate any number of modes such as body bending, wing bending, wing torsion, etc., so as to total 17. The fuel slosh modes incorporate 2 longitudinal and 2 lateral modes on each tank and the program allows one to include up to 10 tanks. It is recognized that the gross vehicle motion (large motions) influences the small motions (elastic deformations and fuel sloshing) of the vehicle, but it is assumed that these smaller motions have a negligible effect on the larger motions of the vehicle. Other basic assumptions used in this analysis are:

1. Undamped free vibration modes are used to specify the elastic deformations and fuel slosh.
2. There is no elastic or damping coupling between the degrees of freedom.
3. The aerodynamic forces can be obtained by Newtonian flow theory.
4. The fuel surface (except for the sloshing) is considered to be perpendicular to the resultant acceleration at the center of the tank.
5. The fuel slosh modes of a tank that is not vertical or horizontal can be represented by those of some hypothetical tank that is vertical or horizontal. In addition, longitudinal fuel sloshing in a horizontal cylindrical tank is represented by an analogy to a rectangular tank.
6. The effect of a rocket engine can be represented by a thrust vector, which is a simplification that assumes the center of mass flow through the nozzle to be exactly aligned with the geometric axis of the nozzle.

The complexities inherent in this type of problem are so great that certain conventions of the tensor notation are incorporated in the subsequent development in order to shorten the writing of the equations. These operations are explicitly explained as they are introduced. The analysis of the structural loads is logically developed in the following sequence:

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1. The vehicle kinematics are derived.
2. The force and moment relations are found using Newtonian mechanics.
3. The equations of motion of the elastic deformation are derived (work and energy concepts are used to check the basic formulations of Items 2 and 3).
4. The main equations to be used to determine the elastic deformations are put into terms suitable for computation.
5. The aerodynamic forces (using Newtonian flow theory) are found.
6. The analysis of the fuel slosh problem is included.
7. The thrust forces are introduced.
8. The accelerations at all locations are found.
9. The shear forces and bending moments are calculated.

The generalized forces to be used in the program are inertia forces N_j , external forces Q_j , conservative internal forces O_j and dissipative internal forces P_j . These forces are represented by Equations (46) - (49).

To clarify to some degree the subsequent analysis, the representation of the coordinate system is presented in Figure 1. The origin of the orthogonal reference frame (y^1, y^2, y^3) is represented by an arbitrary point that would be fixed in the vehicle if it remained rigid during the motion along its flight path. In conjunction with this frame of reference, are located relative coordinate systems (u^1, u^2, u^3) positioned at various points on the body to define the elastic deformations and fuel slosh motions. An absolute reference frame (X^1, X^2, X^3) is shown for generality with X the position vector connecting the origins of the reference frames. As a physical insight into the relative relations of these coordinate systems, consider the case when the vehicle center of gravity (C.G.) is the origin of the y^1, y^2, y^3 triad; then, the velocity of this point is represented by $\vec{v}_G = \frac{d}{dt} \vec{r}_{CG}$.

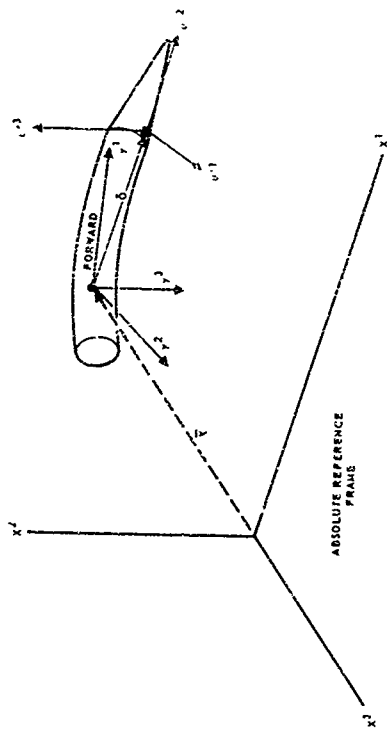


Figure 1. Coordinate Systems

c. KINEMATICS OF THE VEHICLE AND BASIC ASSUMPTIONS

In analyzing the motion of a vehicle in flight, it is convenient (perhaps necessary) to think in terms of the following types of motion: (1) the motion of the vehicle as a whole, which characterizes its "flight" and is referred to here as the gross motion of the vehicle. (2) large motions of certain parts, such as control surfaces, relative to the rest of the vehicle and large displacement of the fuel in the tanks, and (3) small elastic distortions and fuel sloshing. Types of motion (1) and (2) are determined in the basic Six-Degree-of-Freedom Flight Path Study Generalized Computer Program (SDF), and the Vehicle Physical Characteristics Subprogram (VPCS). Type (3) is to be determined in the Structural Loads Program (SLP), which, as its name indicates, is also to determine the structural loads.

It is assumed here that the relatively small elastic deformations and fuel sloshing motions have a negligible effect on the other (large) motions of the vehicle. (Some considerations associated with this assumption are investigated in following paragraphs.) It is not assumed that the large motions of the vehicle have a negligible effect on the small motions. Consequently, the large motions - types (1) and (2) - will be employed as part of the input to the Structural Loads Program.

An orthogonal right-handed triad of unit vectors $\bar{J}_1, \bar{J}_2, \bar{J}_3$ that would be fixed in the vehicle if it were perfectly rigid is introduced to provide a frame of reference (a) to represent the gross motion of the vehicle and (b) to facilitate the description of the other motions of the vehicle - types (2) and (3). Rectangular coordinates y_1, y_2, y_3 are associated respectively with the unit vectors $\bar{J}_1, \bar{J}_2, \bar{J}_3$ as shown in Figure 2. These coordinates are seen to be the components of the position vector \bar{y} , which equals $J_1 y_1 + J_2 y_2 + J_3 y_3$. The axes of these coordinates are called "vehicle" axes.

Additional orthogonal right-handed triads of unit vectors $\bar{J}'_1, \bar{J}'_2, \bar{J}'_3$ that would be fixed in the various parts of the vehicle and in the fuel in the various tanks if they were rigid are introduced as frames of reference (a) to represent the motions of the parts and the displacements of the fuel relative to the vehicle and (b) to facilitate the description of the elastic deformations and the sloshing of the fuel. Rectangular coordinates v'_1, v'_2, v'_3 are associated respectively with the unit vectors $\bar{J}'_1, \bar{J}'_2, \bar{J}'_3$ and these coordinates are the components of the vector \bar{v}' , which is the position vector in the J'_r coordinate system. The axes of these coordinates are called "section" axes. The vector \bar{O} locates the origin of the J'_r coordinate system with respect to the J_r system. Consequently, then, $\bar{v}' = \bar{O} + \bar{v}$.

The gross motion of the vehicle is that of the \bar{J}_r ($r = 1, 2, 3$) triad, which has a linear velocity \bar{V} at its origin and an angular velocity $\bar{\Omega}$. These velocities are functions of the time t , and, together with their derivatives, completely describe the gross motion of the vehicle.

Generalized coordinates are employed to specify the configuration of the vehicle and fuel relative to this frame of reference - the J_r triad. In so

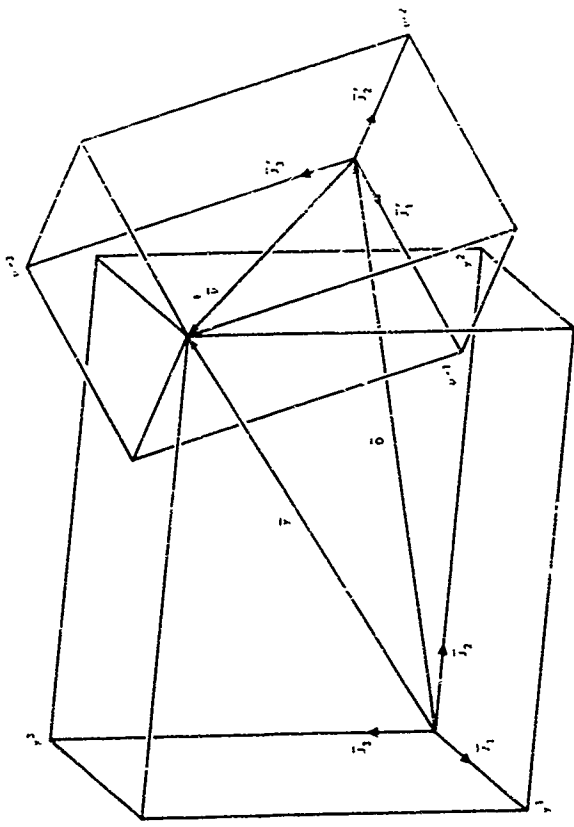


Figure 2. Vehicle and Section Coordinates

Doing, however, a distinction is made between the large motions of type (2) and the small motions - type (3). Inasmuch as the large motions are foreknown in the SIP they can be specified by means of a single generalized coordinate, for which the symbol q^r is chosen here, and which is to be equated to the time t . The use of q^r for this purpose rather than τ , even though the two are numerically equivalent, serves to distinguish the motions of type (2) from the small motion, type (1), the gross motion being represented as a function of τ but not as a function of q^r .

Uncoupled free vibration modes are used as Degrees of Freedom for specifying elastic deformations and fuel sloshing, motions of type (3), and are referred to as elastic degrees of freedom. (Good results can reasonably be expected if a sufficient number of the lower frequency modes are used.) These small motions are specified by the generalized coordinates q^1, q^2, \dots, q^n (n being the number of elastic degrees of freedom).

At this point, it is convenient to adopt the range and summation conventions of the tensor analysis as follows:

- (1) Range Convention - A coordinate suffix that occurs just once in a term is understood to represent all the integral values appropriate to its range.
- (2) Summation Convention - A coordinate suffix that occurs just twice in a term implies summation with respect to that suffix over its range.

These conventions enable us to write

$$\bar{y} = \bar{J}_r y^r \quad (= \bar{J}_1 y^1 + \bar{J}_2 y^2 + \bar{J}_3 y^3) \quad (1)$$

$$\bar{v} = \bar{J}'_r v^r \quad (= \bar{J}'_1 v^1 + \bar{J}'_2 v^2 + \bar{J}'_3 v^3) \quad (2)$$

$$\bar{\theta} = \bar{J}_r \theta^r \quad (= \bar{J}_1 \theta^1 + \bar{J}_2 \theta^2 + \bar{J}_3 \theta^3) \quad (3)$$

$$\bar{J}_r = \bar{J}_s e^s_r \quad (= \bar{J}_1 e^1_r + \bar{J}_2 e^2_r + \bar{J}_3 e^3_r) \quad (4)$$

The θ^r in equation (3) are the coordinates in the \bar{J}_r coordinate system of the origin of the \bar{J}_r system. The e^s_r in (4) are the components in the \bar{J}_s system of the \bar{J}_r vectors; for any particular choice of r and s , $e^s_r = \bar{J}'_r \cdot \bar{J}_s$, which is the cosine of the angle between \bar{J}'_r and \bar{J}_s . Equations (1), (2), and (3) illustrate the use of the summation convention; equation (4) illustrates both conventions.

Unless otherwise noted, the range of the suffix of a unit vector (\bar{J}_r or \bar{J}'_r) or of a rectangular coordinate (y^r , v^r , or θ^r) is 1, 2, 3. The range of the suffix of a generalized coordinate (q^r) will be understood to be 1, 2, ... n . Zero (as in q^0) is specifically and deliberately excluded in the use of the range and summation conventions. Thus, in specifying basic functional relations, we write -

$$\bar{J} = \bar{J}_r(t) \quad (5)$$

$$\theta^r = \theta^r(q^1, q^2) \quad (6)$$

$$e^s_r = e^s_r(q^1, q^2) \quad (7)$$

$$v^r = v^r(q^1, q^2) \quad (8)$$

Since $\vec{\omega} = \dot{\vec{\alpha}}$, we find by substitution from (1) thru (4) that

$$\begin{aligned}\vec{J} \cdot \dot{\vec{y}} &= \vec{J}_r \cdot \dot{\vec{e}}^r + I_r \dot{V}^r \\ &= \vec{J}_r \cdot \dot{\vec{e}}^r + \vec{J}_s \cdot \dot{\vec{e}}_r^s V^r \\ &= \vec{J}_r \cdot \dot{\vec{e}}^r + \vec{J}_r \cdot \dot{\vec{e}}_s^r V^s\end{aligned}\quad (9)$$

$$\dot{\vec{y}} = \dot{\vec{e}}^r + \dot{\vec{e}}_s^r V^s = \dot{y}^r(\hat{e}^r; \hat{e}^s) \quad (10)$$

The components y^r of the position vector \vec{y} are the coordinates of a particle of the vehicle, and their variation as the particle moves with respect to the \vec{J}_r frame of reference is a function of \dot{y}^r and the \dot{q}^k . Since the angular velocity of the \vec{J}_r triad is $\vec{\omega}$, and since the \vec{J}_r are functions of the time t only, their derivatives are

$$\frac{d\vec{J}_r}{dt} = \vec{\omega} \times \vec{J}_r \quad (11)$$

It is clear from (10) that $\frac{\partial \dot{\vec{y}}}{\partial \dot{t}} = 0$; therefore, from (1),

$$\frac{\partial \dot{\vec{y}}}{\partial \dot{t}} = \frac{d\dot{\vec{y}}}{dt} \dot{y}^r = \vec{\omega} \times \vec{J}_r \dot{y}^r = \vec{\omega} \times \dot{\vec{y}} \quad (12)$$

and

$$\begin{aligned}\frac{d\dot{\vec{y}}}{dt} &= \frac{\partial \dot{\vec{y}}}{\partial \dot{t}} + \frac{\partial \dot{\vec{y}}}{\partial \dot{q}^i} \dot{q}^i + \frac{\partial \dot{\vec{y}}}{\partial \dot{q}^k} \dot{q}^k \\ &= \vec{\omega} \times \dot{\vec{y}} + \frac{\partial \dot{\vec{y}}}{\partial \dot{q}^i} \dot{q}^i + \frac{\partial \dot{\vec{y}}}{\partial \dot{q}^k} \dot{q}^k,\end{aligned}\quad (13)$$

since $\dot{q}^0 = \frac{d\alpha^0}{dt} = 1$, (14)

\dot{q}^k = $\frac{d\bar{q}^k}{dt}$ and is unknown until determined in the solution of the equations of motion, which are to follow. Making use of (15) and the fact that the linear velocity of the origin of the J_r triad is \bar{V} , we find that the velocity of a particle of the vehicle is

$$\begin{aligned}\bar{v} &= \bar{V} + \frac{d\bar{y}}{dt} \\ &= \bar{V} + \bar{\omega} \times \bar{y} + \frac{\partial \bar{y}}{\partial \bar{q}^1} + \frac{\partial \bar{y}}{\partial \bar{q}^2} \dot{q}^k\end{aligned}\quad (15)$$

The acceleration of a particle can be found as follows

$$\frac{\partial}{\partial t} \left(\frac{\partial \bar{y}}{\partial \bar{q}^1} \right) = \frac{d\bar{\omega}}{dt} \times \frac{\partial \bar{y}}{\partial \bar{q}^1} = \bar{\Omega} \times \bar{\omega} \times \frac{\partial \bar{y}}{\partial \bar{q}^1} = \bar{\Omega} \times \frac{\partial \bar{y}}{\partial \bar{q}^1} \quad (16)$$

$$\frac{d}{dt} \left(\frac{\partial \bar{y}}{\partial \bar{q}^1} \right) = \bar{\Omega} \times \frac{\partial \bar{y}}{\partial \bar{q}^1} + \frac{\partial^2 \bar{y}}{\partial \bar{q}^1 \partial \bar{q}^2} \dot{q}^2 + \frac{\partial^2 \bar{y}}{\partial \bar{q}^1 \partial \bar{q}^3} \dot{q}^3 \quad (17)$$

Likewise

$$\frac{d}{dt} \left(\frac{\partial \bar{y}}{\partial \bar{q}^2} \right) = \bar{\Omega} \times \frac{\partial \bar{y}}{\partial \bar{q}^2} + \frac{\partial^2 \bar{y}}{\partial \bar{q}^2 \partial \bar{q}^1} \dot{q}^1 + \frac{\partial^2 \bar{y}}{\partial \bar{q}^2 \partial \bar{q}^3} \dot{q}^3 \quad (18)$$

The acceleration is now found by differentiation of (15) and substitution from (16), (17), and (18), with the result

$$\begin{aligned}\frac{d\bar{v}}{dt} &= \frac{d\bar{V}}{dt} + \bar{\Omega} \times \frac{d\bar{y}}{dt} + \frac{d\bar{\omega}}{dt} \times \bar{y} + \frac{d}{dt} \left(\frac{\partial \bar{y}}{\partial \bar{q}^1} \right) \dot{q}^1 + \frac{d}{dt} \left(\frac{\partial \bar{y}}{\partial \bar{q}^2} \right) \dot{q}^2 + \frac{\partial^2 \bar{y}}{\partial \bar{q}^1 \partial \bar{q}^3} \dot{q}^3 \\ &= \frac{d\bar{V}}{dt} + \bar{\Omega} \times (\bar{\Omega} \times \bar{y}) + \frac{d\bar{\Omega}}{dt} \times \bar{y} + 2 \bar{\Omega} \times \frac{\partial \bar{y}}{\partial \bar{q}^1} + \frac{\partial^2 \bar{y}}{\partial \bar{q}^1 \partial \bar{q}^2} \dot{q}^2 \\ &\quad + 2 \left(\bar{\Omega} \times \frac{\partial \bar{y}}{\partial \bar{q}^2} + \frac{\partial^2 \bar{y}}{\partial \bar{q}^2 \partial \bar{q}^1} \dot{q}^1 \right) + \frac{\partial^2 \bar{y}}{\partial \bar{q}^2 \partial \bar{q}^3} \dot{q}^3 + \frac{\partial^2 \bar{y}}{\partial \bar{q}^1 \partial \bar{q}^3} \dot{q}^3\end{aligned}\quad (19)$$

We note that

$$\begin{aligned} \frac{d\bar{\Omega}}{dt} &= \bar{J}_r \frac{d\Omega^r}{dt} + \frac{d\bar{J}_r}{dt} \Omega^r \\ &= \bar{J}_r \dot{\Omega}^r + \bar{\Omega} \times \bar{J}_r \Omega^r \\ &= \bar{J}_r \dot{\Omega}^r, \end{aligned} \tag{20}$$

because $\bar{\Omega} \times \bar{J}_r \Omega^r = \bar{\Omega} \times \bar{\Omega} = 0$.

The $\dot{\Omega}^r$, then, are the components of the angular acceleration. The linear acceleration at the origin is

$$\begin{aligned} \frac{d\bar{V}}{dt} &= \bar{J}_r \frac{dV^r}{dt} + \frac{d\bar{J}_r}{dt} V^r \\ &= \bar{J}_r \dot{V}^r + \bar{\Omega} \times \bar{V} \\ &= \bar{J}_1 (\dot{V}^1 + \Omega^2 V^3 - \Omega^3 V^2) \\ &\quad + \bar{J}_2 (V^2 + \Omega^3 V^1 - \Omega^1 V^3) \\ &\quad + \bar{J}_3 (V^3 + \Omega^1 V^2 - \Omega^2 V^1) \end{aligned} \tag{21}$$

The coefficients of the unit vectors are the components of the linear acceleration.

3. FORCES, MOMENTS, AND "DYNAMIC BALANCING"

In the application of Newton's second law of motion to the vehicle, it is necessary to have a means of identifying the particles. But the vehicle is divided into various parts (or sections) and various fuel tanks, which also need to be identified. Because of its shifty and slippery nature, the fuel cannot be regarded as part of the tank that contains it. The tank itself is treated as one or more structural sections. Subscripts are employed to identify masses or mass particles and their rectangular coordinates, a single subscript or the first of two subscripts denoting the section or the fuel contained in a certain tank, and the second subscript denoting the particle of the section or fuel. The absence of such subscripts denotes a quantity pertaining to the entire vehicle.

Thus the mass of the h -th particle of the i -th section is $m_{i,h}$ and its coordinates are $y_{i,h}^r$ and $v_{i,h}^r$. The mass of the i -th section is m_i and its "coordinates" are θ_i^r . The mass of the entire vehicle is m (without a subscript). Let P_i be the number of particles in the i -th section or tank of fuel and N be the number of sections and tanks,

$$\text{then } m_i = \sum_{h=1}^{P_i} m_{i,h} \quad (22)$$

$$\text{and } m = \sum_{i=1}^N \sum_{h=1}^{P_i} m_{i,h} = \sum_{i=1}^N m_i \quad (23)$$

Let the \bar{U}_i^r triad of the i -th section or tank of fuel be designated as the \bar{U}_i^r triad and let its origin be at the center of mass of the i -th system of particles. Then the θ_i^r are the coordinates of the center of mass of the i -th section, and

$$\sum_{h=1}^{P_i} m_{i,h} v_{i,h}^r = 0 \quad (24)$$

Also, let y_c^r be the coordinates of the center of mass of the vehicle. Then, with the aid of (10), (22), and (24) it is found that

$$\begin{aligned} m y_c^r &= \sum_{i=1}^N \sum_{h=1}^{P_i} m_{i,h} y_{i,h}^r \\ &= \sum_{i=1}^N \sum_{h=1}^{P_i} m_{i,h} (\theta_i^r + e_{3i}^r v_{i,h}^r) \\ &= \sum_{i=1}^N m_i \theta_i^r \end{aligned} \quad (25)$$

The angular momentum of the vehicle about the origin

$$\vec{L} = \sum_{h=1}^N \vec{r}_h \times m_h \dot{\vec{y}}_h$$

is the position vector of the center of mass of the vehicle

With the aid of Newton's second law of motion, it can be shown that the sum of the forces exerted on all the particles of the vehicle equals the sum of the external forces only. Let this be designated by \vec{F} , then Newton's second law of motion (and (19)),

$$\begin{aligned} \vec{F} &= \sum_{h=1}^N \sum_{k=1}^k m_{h,k} \frac{d\vec{v}_{h,k}}{dt} \\ &= \sum_{h=1}^N \sum_{k=1}^k m_{h,k} \left[\frac{d\vec{y}}{dt} + \bar{\Omega} \times (\bar{\Omega} \times \vec{y}_{h,k}) + \frac{d\bar{\Omega}}{dt} \times \vec{y}_{h,k} \right. \\ &\quad \left. + 2\bar{\Omega} \times \frac{\partial \vec{y}_{h,k}}{\partial t} + \frac{\partial^2 \vec{y}_{h,k}}{\partial t^2} + 2\left(\bar{\Omega} \times \frac{\partial \vec{y}_{h,k}}{\partial t} + \frac{\partial \vec{y}_{h,k}}{\partial t} \times \bar{\Omega}\right) \dot{\vec{q}} \right. \\ &\quad \left. + \frac{\partial^2 \vec{y}_{h,k}}{\partial t^2} \cdot \dot{\vec{q}} \cdot \dot{\vec{q}} + \frac{\partial \vec{y}_{h,k}}{\partial t} \cdot \ddot{\vec{q}} \right] \end{aligned} \quad (27)$$

Likewise, the sum of the moments about the origin of the J_h triad due to all the forces is equal to the sum of the moments due to the external forces only. Let this be designated by \vec{G} , then, by Newton's second law of motion and (19),

$$\begin{aligned} \vec{G} &= \sum_{h=1}^N \sum_{k=1}^k \vec{y}_h \times m_{h,k} \left(\frac{d\vec{v}_{h,k}}{dt} \right) \\ &= \sum_{h=1}^N \sum_{k=1}^k m_{h,k} \vec{y}_h \times \left[\frac{d\vec{y}}{dt} + \bar{\Omega} \times (\bar{\Omega} \times \vec{y}_{h,k}) + \frac{d\bar{\Omega}}{dt} \times \vec{y}_{h,k} \right. \\ &\quad \left. + 2\bar{\Omega} \times \frac{\partial \vec{y}_{h,k}}{\partial t} + \frac{\partial^2 \vec{y}_{h,k}}{\partial t^2} + 2\left(\bar{\Omega} \times \frac{\partial \vec{y}_{h,k}}{\partial t} + \frac{\partial \vec{y}_{h,k}}{\partial t} \times \bar{\Omega}\right) \dot{\vec{q}} \right. \\ &\quad \left. + \frac{\partial^2 \vec{y}_{h,k}}{\partial t^2} \cdot \dot{\vec{q}} \cdot \dot{\vec{q}} + \frac{\partial \vec{y}_{h,k}}{\partial t} \cdot \ddot{\vec{q}} \right] \end{aligned} \quad (28)$$

Inasmuch as the J_p triad provides a frame of reference to represent the gross motion of the vehicle, its linear and angular velocities and accelerations $V, \bar{\Omega}, d\bar{V}/dt$, and $d\bar{\Omega}/dt$ are those of the vehicle as a whole. It would be strictly proper to require these velocities and accelerations to satisfy (27) and (28); but it has been assumed that the elastic deformations and fuel sloshing, motions of type (3), have a negligible effect on the large motions of the vehicle; therefore, $\bar{V}, d\bar{V}/dt$, and $d\bar{\Omega}/dt$ are regarded as not being functions of the generalized coordinates q^k or their derivatives \dot{q}^k and \ddot{q}^k . The fact that \bar{P} and \bar{C} may be significantly affected by the elastic deformation of aerodynamic surfaces is arbitrarily disregarded here, and the portions of (27) and (28) involving \dot{q}^k and \ddot{q}^k are simply ignored in the process of determining the gross motion of the vehicle. This leaves us, for the determination of the gross motion,

$$\begin{aligned} \bar{F} = & \sum_{i=1}^N \sum_{h=1}^P m_{i,h} \left[\frac{d\bar{V}}{dt} + \bar{\Omega} \times (\bar{\Omega} \times \bar{y}_{i,h}) + \frac{d\bar{\Omega}}{dt} \times \bar{y}_{i,h} \right. \\ & \left. + 2 \bar{\Omega} \times \frac{\partial \bar{y}_{i,h}}{\partial q^k} + \frac{\partial^2 \bar{y}_{i,h}}{\partial q^k \partial q^l} \right] \end{aligned} \quad (29)$$

$$\begin{aligned} \bar{C} = & \sum_{i=1}^N \sum_{h=1}^P m_{i,h} \bar{y}_{i,h} \times \left[\frac{d\bar{V}}{dt} + \bar{\Omega} \times (\bar{\Omega} \times \bar{y}_{i,h}) + \frac{d\bar{\Omega}}{dt} \times \bar{y}_{i,h} \right. \\ & \left. + 2 \bar{\Omega} \times \frac{\partial \bar{y}_{i,h}}{\partial q^k} + \frac{\partial^2 \bar{y}_{i,h}}{\partial q^k \partial q^l} \right] \end{aligned} \quad (30)$$

with none of these terms being regarded as functions of the q^k .

Equation (29) contains the summations $\sum_{i=1}^N \sum_{h=1}^P m_{i,h} \bar{y}_{i,h}$ and $\sum_{i=1}^N \sum_{h=1}^P m_{i,h} \frac{\partial^2 \bar{y}_{i,h}}{\partial q^k \partial q^l}$. If the terms of (29) are not to be functions of

the q^k , the partial derivatives of these summations with respect to the q^k should be equal to zero. Furthermore, if these same partial derivatives are equal to zero, the terms of (27) involving the \dot{q}^k and the \ddot{q}^k will vanish, because they contain these partial derivatives as factors. In fact, it is sufficient for this purpose for

to be zero, because $\sum_{i=1}^N \sum_{h=1}^P m_{i,h} \frac{\partial^2 \bar{y}_{i,h}}{\partial q^k \partial q^l}$

$$\frac{\partial}{\partial q^k} \sum_{i=1}^N \sum_{h=1}^P m_{i,h} \frac{\partial \bar{y}_{i,h}}{\partial q^l} = \frac{\partial}{\partial q^k} \sum_{i=1}^N \sum_{h=1}^P m_{i,h} \frac{\partial \bar{y}_{i,h}}{\partial q^l} = 0$$

$$\sum_{i=1}^N \sum_{h=1}^P m_{i,h} \frac{\partial^2 \bar{y}_{i,h}}{\partial q^k \partial q^l} = 0.$$

Reference to Equation (26) sheds a little more light on this problem:

$$\sum_{i=1}^N \sum_{h=1}^P m_{i,h} \frac{\partial \bar{y}_{i,h}}{\partial q^k} = \sum_{i=1}^N m_i \frac{\partial \bar{y}_i}{\partial q^k} = m \frac{\partial \bar{y}}{\partial q^k} \quad (31)$$

From the physical viewpoint it is clear that \bar{y}_k will not change such as a result of elastic deformation, but that fact does not preclude the possibility of its changing rapidly; therefore, it is rash to assume arbitrarily that (31) will equal zero. Rather, it is desirable to impose its being zero as a condition to be satisfied by the elastic degrees of freedom.

The components of the inertia tensor (or elements of tr. inertia matrix) of the vehicle are given by the formula (Reference 7, Equation 2-12)

$$M_{jr} = \sum_{i=1}^N \sum_{h=1}^6 m_{ih} \frac{\partial \bar{y}_i}{\partial q^k} \cdot \frac{\partial \bar{y}_h}{\partial q^k} \quad (32)$$

It is known that $M_{jr} = 0$ when the j -th degree of freedom is a form of motion in which the vehicle behaves as a rigid body, having a translation rate \bar{c}_j and a rotation rate \bar{b}_j relative to the J_r triad, and the k -th degree of freedom is a normal free-free mode of vibration. When such is the case,

$$\frac{\partial \bar{y}_{ih}}{\partial q^k} = \bar{c}_j \bar{b}_j \times \bar{y}_{ih} \quad (33)$$

and from (32)

$$\begin{aligned} M_{jr} &= \sum_{i=1}^N \sum_{h=1}^6 m_{ih} (\bar{c}_j \bar{b}_j \times \bar{y}_{ih}) \cdot \frac{\partial \bar{y}_{ih}}{\partial q^k} \\ &= \bar{c}_j \cdot \sum_{i=1}^N \sum_{h=1}^6 m_{ih} \frac{\partial \bar{y}_{ih}}{\partial q^k} + \bar{b}_j \cdot \sum_{i=1}^N \sum_{h=1}^6 m_{ih} \bar{y}_{ih} \times \frac{\partial \bar{y}_{ih}}{\partial q^k} \\ &= 0 \end{aligned} \quad (34)$$

Now \bar{c}_j and \bar{b}_j , being any translation and rotation rates of the vehicle, are arbitrary; therefore,

$$\sum_{i=1}^N \sum_{h=1}^6 m_{ih} \frac{\partial \bar{y}_{ih}}{\partial q^k} = 0 \quad (35)$$

and

$$\sum_{i=1}^N \sum_{h=1}^6 m_{ih} \bar{y}_{ih} \times \frac{\partial \bar{y}_{ih}}{\partial q^k} = 0 \quad (36)$$

when the k -th degree of freedom is a normal free-free mode of vibration.

Thus, if the elastic degrees of freedom satisfy the condition that they are normal free-free modes of vibration, Equation (35) is satisfied, the terms in (27) involving \bar{Q}^k and \bar{Q}^l which (eliminating the difference between (27) and (29)), and the terms on the right side of (29) are not functions of the q^k . However, the use of normal free-free modes does not preclude more than this. Equation (36) directly eliminates one term of (28), and differentiation of (36) leads to the elimination of other terms, as follows:

$$\begin{aligned} & \frac{\partial}{\partial q^k} \sum_{h=1}^n \sum_{h=1}^n m_{,h} \bar{y}_{,h} \times \frac{\partial \bar{y}_{,h}}{\partial q^k} \\ &= \sum_{h=1}^n \sum_{h=1}^n m_{,h} \bar{y}_{,h} \times \frac{\partial^2 \bar{y}_{,h}}{\partial q^k \partial q^k} + \sum_{h=1}^n \sum_{h=1}^n \bar{y}_{,h} \times \frac{\partial^2 \bar{y}_{,h}}{\partial q^k \partial q^k} \\ &= 0, \text{ or} \end{aligned} \quad (37)$$

$$\begin{aligned} & \sum_{h=1}^n \sum_{h=1}^n m_{,h} \bar{y}_{,h} \times \frac{\partial^2 \bar{y}_{,h}}{\partial q^k \partial q^k} = \sum_{h=1}^n \sum_{h=1}^n m_{,h} \frac{\partial \bar{y}_{,h}}{\partial q^k} \times \frac{\partial \bar{y}_{,h}}{\partial q^k} \\ &= 0 \end{aligned} \quad (38)$$

because interchanging the superscripts k and l does not affect the left side of (38) whereas it reverses the sign of the right side, and only zero equals its opposite. Furthermore, the superscript l could be replaced by 0 in the two equations above; thus, (28) is reduced to

$$\begin{aligned} \bar{C} &= \sum_{h=1}^n \sum_{h=1}^n m_{,h} \bar{y}_{,h} \times \left[\frac{d^2 \bar{y}}{dt^2} + \bar{\Omega} \times (\bar{\Omega} \times \bar{y}_{,h}) + \frac{d\bar{\Omega}}{dt} \times \bar{y}_{,h} \right. \\ & \left. + 2 \bar{\Omega} \times \frac{\partial \bar{y}_{,h}}{\partial q^k} + \frac{\partial^2 \bar{y}_{,h}}{\partial q^k \partial q^k} + 2 \bar{\Omega} \times \frac{\partial \bar{y}_{,h}}{\partial q^k} \right] \end{aligned} \quad (39)$$

which could be used for whatever value it might have in solving for the \bar{q}^k , it being recognized that the terms of this equation are functions of the q^k , in contrast to the use of equations (29) and (30).

The results thus accomplished by the use of normal free-free modes of vibration can also be brought about by a "dynamic balancing" of each degree of freedom individually. In order to do this, let

$$\frac{\partial \bar{y}_{,h}}{\partial q^k} = \bar{a}_{k,h} + \bar{b}_{k,h} \bar{y}_{,h} + c_k \quad (40)$$

the $\bar{a}_{k,ih}$ being given (not necessarily free-free) modes of vibration, and the \bar{b}_k and \bar{c}_k being as defined in connection with (33) except that, instead of being arbitrary, they are now unknowns to be determined in such a way that (35) and (36) will be satisfied. Substitution from (40) into (35) results in

$$\begin{aligned} & \sum_{i=1}^N \sum_{h=1}^P m_{ih} (\bar{a}_{k,ih} + \bar{b}_k \times \bar{y}_{ih} + \bar{c}_k) \\ &= \sum_{i=1}^N \sum_{h=1}^P m_{ih} \bar{a}_{k,ih} + m \bar{b}_k \times \bar{y}_c + m \bar{c}_k = 0, \end{aligned} \quad (41)$$

and substitution into (36) results in

$$\begin{aligned} & \sum_{i=1}^N \sum_{h=1}^P m_{ih} \bar{y}_{ih} \times (\bar{a}_{k,ih} + \bar{b}_k \times \bar{y}_{ih} + \bar{c}_k) \\ &= \sum_{i=1}^N \sum_{h=1}^P m_{ih} \bar{y}_{ih} \times (\bar{a}_{k,ih} + \bar{b}_k \times \bar{y}_{ih}) + m \bar{y}_c \times \bar{c}_k \\ &= 0 \end{aligned} \quad (42)$$

Now let us eliminate \bar{c}_k by forming the vector product of \bar{y}_c with (41) and subtracting it from (42). This results in

$$\begin{aligned} & \sum_{i=1}^N \sum_{h=1}^P m_{ih} \bar{y}_{ih} \times (\bar{a}_{k,ih} + \bar{b}_k \times \bar{y}_{ih}) \\ & - \bar{y}_c \times \sum_{i=1}^N \sum_{h=1}^P m_{ih} \bar{a}_{k,ih} - m \bar{y}_c \times (\bar{b}_k \times \bar{y}_c) = 0, \end{aligned} \quad (43)$$

which can be solved for the \bar{b}_k . Once the \bar{b}_k are obtained, (41) can be used to obtain the \bar{c}_k . When the \bar{b}_k and \bar{c}_k are obtained in this manner, the use of (40) results in $\frac{\partial \bar{y}_{ih}}{\partial t}$ that satisfy (35) and (36). These may be called

"dynamically balanced" modes. They have the practical advantage of being much more easily obtained than the normal free-free modes.

4. EQUATIONS OF MOTION FOR THE ELASTIC DEFORMATIONS

In the preceding section, the influence of the internal forces and the distribution of the aerodynamic pressures over the surface of the vehicle were deliberately disregarded. In this section, it will be necessary to give them full consideration, because their effect on the elastic deformations cannot be disregarded and because the purpose of this section is to deduce equations of motion for the determination of the elastic deformations in the various degrees of freedom. For the sake of suitable notation, let \bar{F}_{ih} denote the external force on the h -th particle of the i -th section, and let \bar{F}_{ikhj} represent the internal force exerted on the h -th particle of the i -th section by the j -th particle of the k -th section. By Newton's second law of motion, then, the total force exerted against the h -th particle of the i -th section is

$$M_{ih} \frac{d\bar{v}_{ih}}{dt} = \bar{F}_{ih} + \sum_{k=1}^N \sum_{j=1}^{P_k} \bar{F}_{ikhj} \quad (44)$$

The equations of motion in terms of generalized forces are obtained from (44) by forming the scalar product of $\frac{\partial \bar{v}_{ih}}{\partial \dot{q}_j}$ with each term and summing over h and i . Thus

$$\begin{aligned} \sum_{i=1}^N \sum_{h=1}^{P_i} M_{ih} \frac{\partial \bar{v}_{ih}}{\partial \dot{q}_j} \cdot \frac{d\bar{v}_{ih}}{dt} &= \sum_{i=1}^N \sum_{h=1}^{P_i} \frac{\partial \bar{v}_{ih}}{\partial \dot{q}_j} \cdot \bar{F}_{ih} \\ &+ \sum_{i=1}^N \sum_{h=1}^{P_i} \sum_{k=1}^N \sum_{j=1}^{P_k} \frac{\partial \bar{v}_{ih}}{\partial \dot{q}_j} \cdot \bar{F}_{ikhj} \end{aligned} \quad (45)$$

For convenience, the generalized forces are separated into four types and designated as follows:

1. Those associated with inertia forces are

$$N_j = \sum_{i=1}^N \sum_{h=1}^{P_i} M_{ih} \frac{\partial \bar{v}_{ih}}{\partial \dot{q}_j} \cdot \frac{d\bar{v}_{ih}}{dt} \quad (46)$$

2. Those associated with external forces are

$$Q_j = \sum_{i=1}^N \sum_{h=1}^{P_i} \frac{\partial \bar{v}_{ih}}{\partial \dot{q}_j} \cdot \bar{F}_{ih} \quad (47)$$

3. Those associated with conservative internal forces are O_j .
4. Those associated with dissipative internal forces are P_j .

Since the \bar{F}_{ikhj} denote the internal forces, we may let

$$Q_j + P_j = - \sum_{i=1}^M \sum_{k=1}^R m_{i,k} \sum_{j=1}^R \frac{\partial \bar{y}_{i,k}}{\partial q^j} \cdot \bar{F}_{i,k} \dot{q}^j \quad (48)$$

and substitution from these last three equations into (45) leads to

$$L_j - \bar{Q}_j + P_j = Q_j \quad (49)$$

Substitution from (49) into (46) results in

$$\begin{aligned} N_j = & \sum_{i=1}^M \sum_{k=1}^R m_{i,k} \frac{\partial \bar{F}_{i,k}}{\partial q^j} \cdot \left[\frac{d\bar{V}}{dt} + \bar{\Omega} \times (\bar{\Omega} \times \bar{y}_{i,k}) + \frac{d\bar{\Omega}}{dt} \times \bar{y}_{i,k} \right. \\ & + 2\bar{\Omega} \times \frac{\partial \bar{y}_{i,k}}{\partial q^j} + \frac{\partial^2 \bar{y}_{i,k}}{\partial q^j \partial q^j} + 2(\bar{\Omega} \times \frac{\partial \bar{y}_{i,k}}{\partial q^j} + \frac{\partial^2 \bar{y}_{i,k}}{\partial q^j \partial q^j}) \dot{q}^k \\ & \left. + \frac{\partial^2 \bar{y}_{i,k}}{\partial q^k \partial q^j} \dot{q}^k \dot{q}^l + \frac{\partial^2 \bar{y}_{i,k}}{\partial q^k \partial q^k} \dot{q}^k \right] \quad (50) \end{aligned}$$

If we make use of (32), (35), (36), (38), and some new symbols in an examination of the individual terms of (50), we obtain a simpler and more practical expression for N_j , as follows:

$$\sum_{i=1}^M \sum_{k=1}^R m_{i,k} \frac{\partial \bar{y}_{i,k}}{\partial q^j} \cdot \frac{d\bar{V}}{dt} = \frac{d\bar{V}}{dt} \cdot \sum_{i=1}^M \sum_{k=1}^R m_{i,k} \frac{\partial \bar{y}_{i,k}}{\partial q^j} = 0 \quad (51)$$

$$\begin{aligned} & \sum_{i=1}^M \sum_{k=1}^R m_{i,k} \frac{\partial \bar{y}_{i,k}}{\partial q^j} \cdot \bar{\Omega} \times (\bar{\Omega} \times \bar{y}_{i,k}) \\ & = \sum_{i=1}^M \sum_{k=1}^R m_{i,k} [(\bar{\Omega} \cdot \frac{\partial \bar{y}_{i,k}}{\partial q^j}) (\bar{\Omega} \cdot \bar{y}_{i,k}) - (\frac{\partial \bar{y}_{i,k}}{\partial q^j}) \cdot \bar{y}_{i,k}] (\bar{\Omega} \cdot \bar{\Omega}) \\ & = \sum_{i=1}^M \sum_{k=1}^R m_{i,k} [\bar{\Omega}^T \bar{y}_{i,k} \bar{\Omega}^T \frac{\partial \bar{y}_{i,k}}{\partial q^j} - \bar{\Omega}^T \bar{\Omega}^T \bar{y}_{i,k} \frac{\partial \bar{y}_{i,k}}{\partial q^j}] \\ & = \bar{\Omega}^T \bar{\Omega}^T \sum_{i=1}^M \sum_{k=1}^R m_{i,k} (\bar{y}_{i,k} \frac{\partial \bar{y}_{i,k}}{\partial q^j} - \delta_{rs} \bar{y}_{i,k} \frac{\partial \bar{y}_{i,k}}{\partial q^j}) = -\bar{\Omega}^T \bar{\Omega}^T P_{rs} \quad (52) \end{aligned}$$

where

$$P_{rs} = \sum_{i=1}^M \sum_{k=1}^R m_{i,k} (\delta_{rs} \bar{y}_{i,k} \frac{\partial \bar{y}_{i,k}}{\partial q^j} - \bar{y}_{i,k} \frac{\partial \bar{y}_{i,k}}{\partial q^j}) \quad (53)$$

$$\begin{aligned} & \sum_{i=1}^M \sum_{k=1}^R m_{i,k} \frac{\partial \bar{y}_{i,k}}{\partial q^j} \cdot \frac{d\bar{\Omega}}{dt} \times \bar{y}_{i,k} = \frac{d\bar{\Omega}}{dt} \cdot \sum_{i=1}^M \sum_{k=1}^R m_{i,k} \bar{y}_{i,k} \times \frac{\partial \bar{y}_{i,k}}{\partial q^j} \\ & = 0 \quad (54) \end{aligned}$$

$$\sum_{i=1}^n \sum_{k=1}^R m_{i,k} \frac{\partial \bar{T}_i}{\partial \dot{q}^k} \cdot \bar{\Omega} \times \frac{\partial \bar{T}_i}{\partial \dot{q}^k} \dot{q}^k = \bar{\Omega} \cdot \sum_{i=1}^n \sum_{k=1}^R m_{i,k} \frac{\partial \bar{T}_i}{\partial \dot{q}^k} \dot{q}^k \times \frac{\partial \bar{T}_i}{\partial \dot{q}^k} \dot{q}^k = 0 \quad (55)$$

$$\sum_{i=1}^n \sum_{k=1}^R m_{i,k} \frac{\partial \bar{T}_i}{\partial \dot{q}^k} \dot{q}^k \cdot \frac{\partial \bar{T}_i}{\partial \dot{q}^k} \dot{q}^k = \boxed{00, j}$$

$$\sum_{i=1}^n \sum_{k=1}^R m_{i,k} \frac{\partial \bar{T}_i}{\partial \dot{q}^k} \dot{q}^k \cdot \bar{\Omega} \times \frac{\partial \bar{T}_i}{\partial \dot{q}^k} \dot{q}^k = \bar{\Omega} \cdot \sum_{i=1}^n \sum_{k=1}^R m_{i,k} \frac{\partial \bar{T}_i}{\partial \dot{q}^k} \dot{q}^k \times \frac{\partial \bar{T}_i}{\partial \dot{q}^k} \dot{q}^k = 0$$

$$\sum_{i=1}^n \sum_{k=1}^R m_{i,k} \frac{\partial \bar{T}_i}{\partial \dot{q}^k} \dot{q}^k \cdot \frac{\partial^2 \bar{T}_i}{\partial \dot{q}^k \partial \dot{q}^k} = \boxed{0K, j} \quad (57)$$

$$\sum_{i=1}^n \sum_{k=1}^R m_{i,k} \frac{\partial \bar{T}_i}{\partial \dot{q}^k} \dot{q}^k \cdot \frac{\partial \bar{T}_i}{\partial \dot{q}^k \partial \dot{q}^k} = \boxed{K, j} \quad (58)$$

$$\sum_{i=1}^n \sum_{k=1}^R m_{i,k} \frac{\partial \bar{T}_i}{\partial \dot{q}^k} \dot{q}^k \cdot \frac{\partial \bar{T}_i}{\partial \dot{q}^k} = M_{jk} \quad (59)$$

Substitution from (51) through (60) into (50) results in

$$N_j = -\Omega^r \Omega^s P_{r,sj} + \boxed{00, j} + 2 \boxed{0K, j} \dot{q}^k + \boxed{K, j} \dot{q}^k \dot{q}^k + M_{jk} \ddot{q}^k \quad (61)$$

It is also possible to use the familiar Lagrangean expression for N_j in terms of the kinetic energy T . To show that this is so, we note from (15) that

$$\frac{\partial \bar{T}_i}{\partial \dot{q}^k} = \frac{\partial \bar{T}_i}{\partial \dot{q}^k} \quad (62)$$

and from (15) and (18) that

$$\frac{\partial \bar{T}_i}{\partial \dot{q}^k} \dot{q}^k = \bar{\Omega} \times \frac{\partial \bar{T}_i}{\partial \dot{q}^k} \dot{q}^k + \frac{\partial^2 \bar{T}_i}{\partial \dot{q}^k \partial \dot{q}^k} \dot{q}^k + \frac{\partial \bar{T}_i}{\partial \dot{q}^k \partial \dot{q}^k} \dot{q}^k \dot{q}^k = \frac{dT_i}{dt} \left(\frac{\partial \bar{T}_i}{\partial \dot{q}^k} \dot{q}^k \right) \quad (63)$$

The kinetic energy is given by the well known formula

$$T = \frac{1}{2} \sum_{i=1}^n \sum_{k=1}^R m_{i,k} \bar{T}_{i,k} \cdot \bar{T}_{i,k} \quad (64)$$

whence, with the aid of (62) and (63),

$$\frac{\partial T}{\partial \dot{q}^k} = \sum_{i=1}^n \sum_{k=1}^R m_{i,k} \bar{T}_{i,k} \cdot \frac{\partial \bar{T}_{i,k}}{\partial \dot{q}^k} \dot{q}^k$$

$$= \sum_i^N \sum_{k=1}^P m_{i,k} \bar{r}_{i,k} \frac{d\bar{v}_{i,k}}{dq_i^k}, \quad (5)$$

$$\frac{d}{dt} \left(\frac{\partial I}{\partial \dot{q}_i^k} \right) = \sum_{k=1}^P m_{i,k} \left[\frac{d\bar{v}_{i,k}}{dt} \frac{\partial \bar{v}_{i,k}}{\partial \dot{q}_i^k} + \bar{v}_{i,k} \cdot \frac{d}{dt} \left(\frac{\partial \bar{v}_{i,k}}{\partial \dot{q}_i^k} \right) \right], \quad (6)$$

and

$$\begin{aligned} \frac{\partial I}{\partial \dot{q}_i^k} &= \sum_{k=1}^P m_{i,k} \bar{v}_{i,k} \cdot \frac{\partial \bar{v}_{i,k}}{\partial \dot{q}_i^k} \\ &= \sum_{k=1}^P m_{i,k} \bar{v}_{i,k} \cdot \frac{d}{dt} \left(\frac{\partial \bar{v}_{i,k}}{\partial \dot{q}_i^k} \right). \end{aligned} \quad (67)$$

Subtraction of (67) from (66) results in the Lagrangian expression

$$\frac{d}{dt} \left(\frac{\partial I}{\partial \dot{q}_i^k} \right) - \frac{\partial I}{\partial q_i^k} = \sum_{k=1}^P m_{i,k} \frac{d\bar{v}_{i,k}}{dt} \cdot \frac{\partial \bar{v}_{i,k}}{dt} = N_j \quad (68)$$

by the defining equation (46). The use of this expression to obtain (61) leads to the interesting discovery that

$$\Omega^r \Omega^s P_{r,sj} = \frac{1}{2} \Omega^r \Omega^s \frac{\partial I}{\partial q_i^k} r_s \quad (69)$$

or that $\frac{\partial I}{\partial \dot{q}_i^k} = P_{r,sj} + P_{s,rj}$, (70)

where $I_{r,s} = \sum_{k=1}^P m_{i,k} (\delta_{rs} y_{i,k}^r y_{i,k}^s - y_{i,k}^r y_{i,k}^s)$ (71)
 = moment and negatives of products of inertia of structure and fuel about vehicle axes.

It is also interesting and useful to observe from (60), (62), (65), and the fact that $\frac{\partial y_{i,k}^r}{\partial q_i^k}$ is not a function of the q^k that

$$M_{ijk} = \frac{\partial^2 I}{\partial q_i^k \partial q_j^k} \quad (72)$$

This is especially useful in treating the inertia effects of fuel slosh.

Recalling the definition of $\bar{F}_{i,k} A_{kj}$, we know by Newton's third law of motion that

$$\bar{F}_{i,k} A_{kj} = -\bar{F}_{kj} A_{ji}, \quad (73)$$

and that $\bar{F}_{i,k} A_{kj}$ is parallel to $\bar{y}_{i,k} - \bar{y}_{kj}$,

$$\text{so that } \bar{F}_{ik} \times (\bar{y}_{ik} - \bar{y}_{kj}) = 0 \quad (74)$$

Substitution from (73) into (48) results in

$$\begin{aligned} Q_j + P_j &= \sum_{i=1}^N \sum_{k=1}^N \sum_{j=1}^N \frac{\partial \bar{F}_{ik}}{\partial q_j} \cdot \bar{F}_{kjl} \\ &= \sum_{i=1}^N \sum_{k=1}^N \sum_{j=1}^N \frac{\partial \bar{F}_{ik}}{\partial q_j} \cdot \bar{F}_{iklj} \end{aligned} \quad (75)$$

If (45) is added to (75) and the sum divided by 2, the result is

$$Q_j + P_j = -\frac{1}{2} \sum_{i=1}^N \sum_{k=1}^N \sum_{j=1}^N \left[\frac{\partial}{\partial q_i} (\bar{y}_{ik} - \bar{y}_{kj}) \right] \cdot \bar{F}_{iklj} \quad (76)$$

$$\text{Let } |\bar{y}_{ik} - \bar{y}_{kj}| = \Delta_{iklj}, \quad (77)$$

which is the distance from particle ik to particle lj ; and

$$\text{let } |\bar{F}_{iklj}| = F_{iklj}, \quad (78)$$

positive when it tends to increase Δ_{iklj} and negative when it tends to decrease Δ_{iklj} .

Then, since F_{iklj} is parallel to $\bar{y}_{ik} - \bar{y}_{kj}$, it can be shown that (76) is equivalent to

$$Q_j + P_j = -\frac{1}{2} \sum_{i=1}^N \sum_{k=1}^N \sum_{j=1}^N F_{iklj} \frac{\partial}{\partial q_i} \Delta_{iklj}. \quad (79)$$

Let U be the potential energy due to elastic deformation, and let V be the energy dissipated through damping. These both represent work done in overcoming internal forces; therefore,

$$\begin{aligned} \frac{d}{dt}(U+V) &= -\sum_{i=1}^N \sum_{k=1}^N \sum_{j=1}^N \bar{F}_{iklj} \cdot \bar{V}_{ik} \\ &= \sum_{i=1}^N \sum_{k=1}^N \sum_{j=1}^N \bar{F}_{iklj} \cdot \bar{V}_{ik} \\ &= \sum_{i=1}^N \sum_{k=1}^N \sum_{j=1}^N \bar{F}_{iklj} \cdot \bar{V}_{kj} \\ &= -\frac{1}{2} \sum_{i=1}^N \sum_{k=1}^N \sum_{j=1}^N \bar{F}_{iklj} \cdot (\bar{V}_{ik} - \bar{V}_{kj}) \\ &= -\frac{1}{2} \sum_{i=1}^N \sum_{k=1}^N \sum_{j=1}^N \bar{F}_{iklj} \cdot [\bar{\Omega} \times (\bar{y}_{ik} - \bar{y}_{kj}) \\ &\quad + \frac{\partial}{\partial q_i} (\bar{y}_{ik} - \bar{y}_{kj}) + \frac{\partial}{\partial q_i} (\bar{y}_{ik} - \bar{y}_{kj}) \dot{q}^i] \end{aligned} \quad (80)$$

because of (15). The term

$$\bar{F}_{ikl_j} \bar{\Omega} \times (\bar{y}_{ik} - \bar{y}_{lj}) = -\bar{\Omega} \bar{F}_{ikl_j} \times (\bar{y}_{ik} - \bar{y}_{lj}) = 0 \quad (81)$$

because of (14); therefore, because of (76),

$$\frac{d}{dt}(U+V) = O_0 + P_0 + (O_j + P_j)\dot{q}^j, \quad (82)$$

where $O_0 + P_0 =$

$$\begin{aligned} & -\frac{1}{2} \sum_{i=1}^N \sum_{k=1}^N \sum_{l=1}^N \bar{F}_{ikl_j} \cdot \frac{d}{dq^j} (\bar{y}_{ik} - \bar{y}_{lj}) \\ & = -\frac{1}{2} \sum_{i=1}^N \sum_{k=1}^N \sum_{l=1}^N \bar{F}_{ikl_j} \frac{d}{dq^j} \Delta_{iklj}. \end{aligned} \quad (83)$$

Since the O_j are associated with conservative internal forces and the P_j are associated with dissipative internal forces, it is clear from the definitions of U and V and from (82) that

$$\frac{dU}{dt} = O_0 + O_j \dot{q}^j \quad (84)$$

and

$$\frac{dV}{dt} = P_0 + P_j \dot{q}^j. \quad (85)$$

Now U is a function of q^j and the \dot{q}^j but not of their time derivatives; therefore,

$$\frac{dU}{dt} = \frac{\partial U}{\partial q^0} + \frac{\partial U}{\partial \dot{q}^j} \dot{q}^j. \quad (86)$$

From (84) and (86), we see that

$$O_0 = \frac{\partial U}{\partial q^0} \quad \text{and} \quad O_j = \frac{\partial U}{\partial \dot{q}^j}. \quad (87)$$

This can be extremely helpful in the computation of the O_j .

If the \bar{F}_{ikl_j} are only the dissipative internal forces, equation (79) may be directly useful for the calculation of the P_j . In using equation (47) to compute the Q_j , it is not necessary to include the force due to gravity because such a force can be represented as $m_i \bar{g}$, in which case

$$\sum_{i=1}^n \sum_{k=1}^n m_{i,k} \bar{q}_j \cdot \frac{\partial \bar{v}_i}{\partial q_j} = \bar{q}_j \cdot \sum_{i=1}^n \sum_{k=1}^n m_{i,k} \frac{\partial \bar{v}_i}{\partial q_j} = 0 \quad (88)$$

when (35) is satisfied.

We now introduce certain assumptions here and proceed to some further treatment of the C_j and P_j . First, let us assume that U is a minimum when the q_j are equal zero, then

$$0_j = \frac{\partial U}{\partial q_j} = 0 \quad \text{when the } q_j = 0; \quad (89)$$

and, as a close and convenient approximation (Reference 7, Equation 4-44)

$$0_j = K_{jk} q^k, \quad (90)$$

where $K_{jk} = \frac{\partial^2 U}{\partial q^j \partial q^k}$ evaluated for the $q^j = 0$.

Inasmuch as undamped free vibration modes are used as degrees of freedom for specifying elastic deformations and fuel sloshing, there is a frequency ω_j associated with the j -th degree of freedom for all the values of j . For the first degree of freedom,

$$\omega_1 = \sqrt{\frac{K_{11}}{M_{11}}} \quad (91)$$

$$\text{or } K_{11} = (\omega_1)^2 M_{11}. \quad (92)$$

Likewise,

$$K_{22} = (\omega_2)^2 M_{22}, \quad (93)$$

$$K_{33} = (\omega_3)^2 M_{33}$$

and so forth for all the degrees of freedom. It is now further assumed, and this must be carefully noted, that the degrees of freedom will be so chosen that there will be no elastic coupling, that is, so that

$$K_{jk} = 0 \quad \text{when } j \neq k. \quad (94)$$

(Working the degrees of freedom in this fashion is a common practice in the analysis of flutter stability.) A general expression for the k_{jk} is

$$k_{jk} = (\omega_j)^2 M_{jj} \delta_j^L \delta_k^L, \quad (95)$$

which, substitution into (90) yields

$$Q_j = (\omega_j)^2 \sum_{i,j} M_{ij} q^i. \quad (96)$$

These equations (91) through (96) are based on the mathematical relations expressing the vibratory motion of the system in $\sin \omega t$ for the given degrees of freedom at a time. A further pursuit of this line of thought, linked with the association of a coefficient of "structural" damping g_j with each degree of freedom leads to a simple formula, analogous to (95), for k_j . This is

$$P_j = g_j^2 \omega_j M_{jj} \dot{q}^j. \quad (97)$$

The determination of g_j^2 from the logarithmic decrement δ_j^L is simple, as follows:

$$g_j^2 = \frac{2\delta_j^L}{\sqrt{4\pi^2 + (\delta_j^L)^2}} \quad (98)$$

$$\cong \delta_j^L / \pi \quad \text{when } \delta_j^L \text{ is small.}$$

5. WORK AND ENERGY RELATIONS

While it adds nothing to the present formulation of the equations of motion, it is a valuable check on the basic formulation to investigate the work and energy relations. Using (15), (27), (28), (46), and (64) given by

$$\begin{aligned} \frac{dT}{dt} &= \sum_{i=1}^n \sum_{k=1}^3 m_{ik} \bar{v}_{ik} \cdot \frac{d\bar{v}_{ik}}{dt} \\ &= \sum_{i=1}^n \sum_{k=1}^3 m_{ik} \frac{d\bar{v}_{ik}}{dt} \cdot (\bar{V} + \bar{\Omega} \times \bar{y}_{ik} + \frac{\partial \bar{y}_{ik}}{\partial t} + \frac{\partial \bar{y}_{ik}}{\partial q^j} \dot{q}^j) \\ &= \bar{V} \cdot \bar{F} + \bar{\Omega} \cdot \bar{G} + N_0 + N_j \dot{q}^j. \end{aligned} \quad (99)$$

$\frac{dU}{dt}$ and $\frac{dV}{dt}$ are given by (84) and (85), and, if W is work done by the external forces,

$$\begin{aligned} \frac{dW}{dt} &= \sum_{i=1}^n \sum_{k=1}^3 \bar{F}_{ik} \cdot \bar{v}_{ik} \\ &= \sum_{i=1}^n \sum_{k=1}^3 \bar{F}_{ik} \cdot (\bar{V} + \bar{\Omega} \times \bar{y}_{ik} + \frac{\partial \bar{y}_{ik}}{\partial t} + \frac{\partial \bar{y}_{ik}}{\partial q^j} \dot{q}^j) \\ &= \bar{V} \cdot \bar{F} + \bar{\Omega} \cdot \bar{G} + Q_0 + Q_j \dot{q}^j, \end{aligned} \quad (100)$$

use having been made of (47).

By substituting the suffix 0 for j in Equations (45) thru (49), we find that

$$N_0 + O_0 + P_0 = Q_0. \quad (101)$$

Because of this and (49)

$$N_0 + O_0 + P_0 + (N_j + O_j + P_j) \dot{q}^j = Q_0 + Q_j \dot{q}^j, \quad (102)$$

where

$$\begin{aligned} \bar{V} \cdot \bar{F} + \bar{\Omega} \cdot \bar{G} + N_0 + Q_0 + P_0 + (N_1 + Q_1 + P_1) \dot{q}^1 \\ = \bar{V} \cdot \bar{F} + \bar{\Omega} \cdot \bar{G} + \dot{w}_0 + Q_1 \dot{q}^1. \end{aligned} \quad (103)$$

Substitution into (103) from (79), (84), (85), and (100) results in

$$\frac{dT}{dt} + \frac{dU}{dt} + \frac{dV}{dt} = \frac{dw}{dt}, \quad (104)$$

$$T + U + V = W + \text{CONSTANT}, \quad (105)$$

which simply states the fact that the work done by the external forces must be either stored in the form of kinetic or potential energy or dissipated.

6. PRACTICAL EXPRESSION OF THE INERTIAL FORMULAE

Not all of the foregoing equations are needed in the computational phase of investigating the elastic deformations of a vehicle in flight, but those that are essential for this purpose (in Sections 3 and 4), so far having been only rather abstractly expressed, need to be presented in terms that are suitable for practical use. Among these essential equations are (53), (56), (58), (59), and (60), and they are full of partial derivatives of y_{ik} (or its components) with respect to the generalized coordinates. For the purpose of computation, these partial derivatives need to be expressed in detail. For convenience, the subscripts i and k are temporarily dropped, and the basic notions of equations (1) thru (10) are developed and extended.

Let us introduce the vectors \bar{h}_0 , \bar{h}_k , $\bar{\alpha}_0$, and $\bar{\alpha}_k$ having to do with the linear and angular velocities of the J' coordinate system relative to the J system and defined as follows:

$$\bar{h}_0 = \frac{\partial \bar{C}}{\partial q^0} = \bar{J}_r \frac{\partial \bar{C}^r}{\partial q^0} = \bar{J}_r h_0^r \quad (106)$$

$$\bar{h}_k = \frac{\partial \bar{C}}{\partial q^k} = \bar{J}_r \frac{\partial \bar{C}^r}{\partial q^k} = \bar{J}_r h_k^r \quad (107)$$

$$\bar{\alpha}_0 = \bar{J}'_1 \left(\bar{J}'_3 \cdot \frac{\partial \bar{J}'_2}{\partial q^0} \right) + \bar{J}'_2 \left(\bar{J}'_1 \cdot \frac{\partial \bar{J}'_3}{\partial q^0} \right) + \bar{J}'_3 \left(\bar{J}'_2 \cdot \frac{\partial \bar{J}'_1}{\partial q^0} \right) \quad (108)$$

$$\bar{\alpha}_k = \bar{J}'_1 \left(\bar{J}'_3 \cdot \frac{\partial \bar{J}'_2}{\partial q^k} \right) + \bar{J}'_2 \left(\bar{J}'_1 \cdot \frac{\partial \bar{J}'_3}{\partial q^k} \right) + \bar{J}'_3 \left(\bar{J}'_2 \cdot \frac{\partial \bar{J}'_1}{\partial q^k} \right) \quad (109)$$

\bar{h}_0 and \bar{h}_k are the partial linear velocities with respect to q^0 and q^k of the origin of the J' coordinate system relative to the J system. $\bar{\alpha}_0$ and $\bar{\alpha}_k$ are the partial angular velocities with respect to q^0 and q^k of the J' coordinate system relative to the J system.

There should be no particular difficulty in regard to the linear velocities, but some discussion of the angular velocities is definitely needed. First, let us note that

$$\begin{aligned} \bar{J}'_r \cdot \bar{J}'_s &= \delta_{rs} = 1 \text{ when } r=s \\ &= 0 \text{ when } r \neq s \end{aligned}$$

Then

$$\frac{\partial}{\partial q^0} (\bar{J}'_r \cdot \bar{J}'_s) = \bar{J}'_r \cdot \frac{\partial \bar{J}'_s}{\partial q^0} + \bar{J}'_s \cdot \frac{\partial \bar{J}'_r}{\partial q^0} = 0. \quad (110)$$

Also, let us denote the components in the \bar{J}_r system of $\bar{\alpha}_0$ and $\bar{\alpha}_k$ respectively as α'_0 and α'_k . Then, from (103) and (110),

$$\left. \begin{aligned} \alpha'_0 &= \bar{J}_3 \cdot \frac{\partial \bar{J}_2}{\partial q_0} = -\bar{J}_2 \cdot \frac{\partial \bar{J}_1}{\partial q_0} \\ \alpha'_2 &= \bar{J}_1 \cdot \frac{\partial \bar{J}_3}{\partial q_0} = -\bar{J}_3 \cdot \frac{\partial \bar{J}_1}{\partial q_0} \\ \alpha'_3 &= \bar{J}_2 \cdot \frac{\partial \bar{J}_1}{\partial q_0} = -\bar{J}_1 \cdot \frac{\partial \bar{J}_2}{\partial q_0} ; \end{aligned} \right\} \quad (111)$$

and likewise, from (109),

$$\left. \begin{aligned} \alpha'_k &= \bar{J}_3 \cdot \frac{\partial \bar{J}_2}{\partial q_k} = -\bar{J}_2 \cdot \frac{\partial \bar{J}_1}{\partial q_k} \\ \alpha'_2 &= \bar{J}_1 \cdot \frac{\partial \bar{J}_3}{\partial q_k} = -\bar{J}_3 \cdot \frac{\partial \bar{J}_1}{\partial q_k} \\ \alpha'_3 &= \bar{J}_2 \cdot \frac{\partial \bar{J}_1}{\partial q_k} = -\bar{J}_1 \cdot \frac{\partial \bar{J}_2}{\partial q_k} \end{aligned} \right\} \quad (112)$$

These equations enable us to write

$$\bar{\alpha}_0 = \bar{J}_r \alpha'_0 \quad \text{and} \quad \bar{\alpha}_k = \bar{J}_r \alpha'_k \quad (113)$$

and to show that

$$\bar{\alpha}_0 \times \bar{J}_r = \frac{\partial \bar{J}_r}{\partial q_0} \quad \text{and} \quad \bar{\alpha}_k \times \bar{J}_r = \frac{\partial \bar{J}_r}{\partial q_k} \quad (114)$$

In further anticipation of terms to arise in the equations of motion, we derive from (111), (112), (113), and (114) the following relations:

$$\begin{aligned} \frac{\partial \alpha'_0}{\partial q_k} - \frac{\partial \alpha'_k}{\partial q_0} &= \frac{\partial \bar{J}_3}{\partial q_k} \cdot \frac{\partial \bar{J}_2}{\partial q_0} - \frac{\partial \bar{J}_2}{\partial q_0} \cdot \frac{\partial \bar{J}_1}{\partial q_k} \\ &= \alpha'^2_0 \alpha'^3_k - \alpha'^2_0 \alpha'^2_k . \end{aligned} \quad (115)$$

By symmetry, this can be extended and generalized to

$$\bar{J}_r \left(\frac{\partial \alpha'_0}{\partial q_k} - \frac{\partial \alpha'_k}{\partial q_0} \right) = \bar{\alpha}_0 \times \bar{\alpha}_k . \quad (116)$$

From (113) and (114), it is readily found that

$$\begin{aligned} \frac{\partial \bar{\alpha}_0}{\partial q^0} &= \bar{J}'_r \frac{\partial \alpha_0^r}{\partial q^0} + \frac{\partial \bar{J}'_r}{\partial q^0} \alpha_0^r \\ &= \bar{J}'_r \frac{\partial \alpha_0^r}{\partial q^0} + \bar{\alpha}_k \times \bar{\alpha}_0. \end{aligned} \quad (117)$$

Elimination of $\bar{\alpha}_0 \times \bar{\alpha}_k$ between (116) and (117) results in

$$\frac{\partial \bar{\alpha}_0}{\partial q^0} = \bar{J}'_r \frac{\partial \alpha_0^r}{\partial q^0} \quad (118)$$

Likewise, $\frac{\partial \bar{\alpha}_k}{\partial q^0} = \bar{J}'_r \frac{\partial \alpha_k^r}{\partial q^0}$, (119)

and substitution from (118) and (119) into (116) results in

$$\frac{\partial \bar{\alpha}_k}{\partial q^0} - \frac{\partial \bar{\alpha}_0}{\partial q^0} = \bar{\alpha}_0 \times \bar{\alpha}_k. \quad (120)$$

The derivation of (115) thru (120) was of such generality that, in any of them, the suffix zero could be replaced by a letter. In the physical realm, this means that the relations expressed by these equations are applicable between degrees of freedom involving only small motions (type (3)) as well as between the large motions of type (2) and the motions of type (3).

In accordance with (8), $\frac{\partial v^r}{\partial q^0} = 0$, but $\frac{\partial v^r}{\partial q^k}$ may or may not be zero. Let us introduce

$$\sigma_k^r = \frac{\partial v^r}{\partial q^k} \quad (121)$$

Then, since

$$\begin{aligned} \bar{v} &= \bar{J}'_r v^r, \\ \frac{\partial \bar{v}}{\partial q^0} &= \frac{\partial \bar{J}'_r}{\partial q^0} v^r = \bar{\alpha}_0 \times \bar{J}'_r v^r \\ &= \bar{\alpha}_0 \times \bar{v}, \end{aligned} \quad (122)$$

$$\begin{aligned} \text{and } \frac{\partial \bar{v}}{\partial q^k} &= \bar{J}'_r \frac{\partial v^r}{\partial q^k} + \frac{\partial \bar{J}'_r}{\partial q^k} v^r \\ &= \bar{J}'_r \sigma_k^r + \bar{\alpha}_k \times \bar{J}'_r v^r \\ &= \bar{\sigma}_k + \bar{\alpha}_k \times \bar{v}. \end{aligned} \quad (123)$$

(121), (122), (123), and (123) facilitate the writing of the following

$$\frac{\partial \bar{y}}{\partial q^0} = \frac{\partial \bar{x}_0}{\partial q^0} + \frac{\partial \bar{v}}{\partial q^0} \\ = \bar{v}_0 + \bar{\alpha}_0 \times \bar{v}, \quad (124)$$

$$\frac{\partial \bar{y}}{\partial q^k} = \frac{\partial \bar{x}_k}{\partial q^k} + \frac{\partial \bar{v}}{\partial q^k} \\ = \bar{h}_k + \bar{\alpha}_k \times \bar{v} \quad (125)$$

Let us introduce the two assumptions

$$\frac{\partial h_k^i}{\partial q^j} = 0 \quad \text{and} \quad \frac{\partial \alpha_k^i}{\partial q^j} = 0 \quad (126)$$

and find expressions for the second partial derivatives of \bar{y} with respect to the generalized coordinates. It can be found without too much trouble that, under these assumptions,

$$\frac{\partial^2 \bar{y}}{\partial q^0 \partial q^0} = \frac{\partial \bar{h}_0}{\partial q^0} + \bar{\alpha}_0 \times (\bar{\alpha}_0 \times \bar{v}) + \frac{\partial \bar{\alpha}_0}{\partial q^0} \times \bar{v}, \quad (127)$$

$$\frac{\partial^2 \bar{y}}{\partial q^0 \partial q^k} = \bar{\alpha}_0 \times (\bar{\alpha}_k + \bar{\alpha}_k \times \bar{v}) + \frac{\partial \bar{h}_0}{\partial q^k} + \frac{\partial \bar{\alpha}_0}{\partial q^k} \times \bar{v}, \quad (128)$$

$$\frac{\partial^2 \bar{y}}{\partial q^k \partial q^l} = \bar{\alpha}_k \times \bar{\alpha}_l + \bar{\alpha}_l \times \bar{\alpha}_k + \bar{\alpha}_k \times (\bar{\alpha}_l \times \bar{v}) + \frac{\partial \bar{\alpha}_k}{\partial q^l} \times \bar{v}. \quad (129)$$

The first important question that arises now is, how do we equate () with (125)? In answer, let us observe that (125) is the general form of expression for $\partial \bar{y} / \partial q^k$; that the \bar{q}_{kik} are given as possible or tentative values of $\partial^2 \bar{y} / \partial q^k \partial q^k$, and that, therefore, the \bar{q}_{kik} will be given in the same form as (125). Thus, let the given linear velocity of the center of mass of section i be \bar{f}_{ki} , and let its given angular velocity be $\bar{\beta}_{ki}$. Then

$$\bar{q}_{kik} = \bar{f}_{ki} + \bar{\sigma}_{kik} + \bar{\beta}_{ki} \times \bar{v}_{ik}; \quad (130)$$

and substitution from this into (40) results in

$$\frac{\partial^2 \bar{y}}{\partial q^k \partial q^k} = \bar{f}_{ki} + \bar{\sigma}_{kik} + \bar{\beta}_{ki} \times \bar{v}_{ik} + \bar{h}_k \times \bar{f}_{ik} + \bar{v}_k \\ = \bar{f}_{ki} + \bar{v}_k + \bar{h}_k \times (\bar{v}_k + \bar{v}_{ik}) + \bar{\sigma}_{kik} + \bar{\beta}_{ki} \times \bar{v}_{ik} \\ = \bar{h}_{ki} + \bar{\sigma}_{kik} + \bar{\alpha}_k \times \bar{v}_{ik} \quad (131)$$

$$\bar{h}_{ki} = \bar{f}_{ki} + \bar{c}_{ki} + \bar{b}_{ki} \times \theta. \quad (121)$$

$$\text{and } \bar{\sigma}_{ki} = \bar{\beta}_{ki} + \bar{b}_{ki} \quad (123)$$

Now we consider the practical problem of solving (43) for the \bar{h}_{ki} . The use of $\bar{\sigma}_{ki} + \bar{v}_{ki}$ for \bar{y}_{ki} and of Eq. (130), the first term in (43) becomes

$$\begin{aligned} & \sum_{k=1}^M \sum_{i=1}^P m_{ik} \bar{y}_{ik} \times \bar{a}_{ki} \\ &= \sum_{k=1}^M \sum_{i=1}^P m_{ik} (\bar{\sigma}_{ki} + \bar{v}_{ki}) \times (\bar{f}_{ki} + \bar{\sigma}_{ki} : \bar{\beta}_{ki} \times \bar{v}_{ki}) \\ &= \sum_{k=1}^M \left[m_{ik} \bar{\sigma}_{ki} \times \bar{f}_{ki} + \bar{\sigma}_{ki} \times \sum_{i=1}^P m_{ik} \bar{\sigma}_{ki} \right. \\ & \quad + \bar{\sigma}_{ki} \times (\bar{\beta}_{ki} \times \sum_{i=1}^P m_{ik} \bar{v}_{ki}) - \bar{f}_{ki} \times \sum_{i=1}^P m_{ik} \bar{v}_{ki} \\ & \quad \left. + \sum_{i=1}^P m_{ik} \bar{v}_{ki} \times (\bar{\sigma}_{ki} + \bar{\beta}_{ki} \times \bar{v}_{ki}) \right]. \quad (134) \end{aligned}$$

From (2), (24), and (121), the following results are obtained

$$\sum_{k=1}^P m_{ik} \bar{v}_{ki} = \bar{f}_{ki} \sum_{k=1}^P m_{ik} \bar{v}_{ki}^T = 0, \quad (135)$$

$$\begin{aligned} \sum_{k=1}^P m_{ik} \bar{\sigma}_{ki} &= \bar{f}_{ki} \sum_{k=1}^P m_{ik} \bar{\sigma}_{ki}^T \\ &= \bar{f}_{ki} \frac{\partial}{\partial q^k} \sum_{k=1}^P m_{ik} \bar{v}_{ki}^T = 0. \quad (136) \end{aligned}$$

This eliminates three terms of (134), and its final term is transformed as follows

$$\begin{aligned} & \sum_{k=1}^P m_{ik} \bar{v}_{ki} \times (\bar{\beta}_{ki} \times \bar{v}_{ki}) \\ &= \sum_{k=1}^P m_{ik} [\bar{\beta}_{ki} (\bar{v}_{ki} \cdot \bar{v}_{ki}) - \bar{v}_{ki} (\bar{\beta}_{ki} \cdot \bar{v}_{ki})] \end{aligned}$$

$$\begin{aligned}
&= \bar{J}_r \beta_{ki}^{rs} \sum_{h=1}^{P_i} m_{ih} (\delta_{rs} v_{ih}^t v_{ih}^t - v_{ih}^r v_{ih}^s) \\
&= \bar{J}_r \beta_{ki}^{rs} H'_{rsi}, \tag{137}
\end{aligned}$$

$$\text{where } H'_{rsi} = \sum_{h=1}^{P_i} m_{ih} (\delta_{rs} v_{ih}^t v_{ih}^t - v_{ih}^r v_{ih}^s). \tag{138}$$

The H'_{rsi} are easily seen to be the moments and the negatives of the products of inertia of section i about its own axes.

For convenience in treating the remaining terms of (134), we introduce the permutation symbol $C_{rst} = 0$ if two suffixes are the same
 $= 1$ if (rst) is an even permutation of (123)
 $= -1$ if (rst) is an odd permutation of (123) ,
the even permutations of (123) being (123) , (231) , and (312) , and the odd permutations being (321) , (213) , and (132) . Thus

$$\bar{J}_r \times \bar{J}_s = C_{rst} \bar{J}_t, \tag{139}$$

$$\sum_{i=1}^N m_i \bar{\theta}_i \times \bar{r}_{ki} = C_{rst} \bar{J}_t \sum_{i=1}^N m_i \theta_i^r \bar{r}_{ki}^s, \tag{140}$$

$$\text{and } \sum_{h=1}^{P_i} m_{ih} \bar{v}_{ih} \times \bar{\sigma}_{kih} = C_{rst} \bar{J}_t \Lambda_{ki}^{rs}, \tag{141}$$

$$\text{where } \Lambda_{ki}^{rs} = \sum_{h=1}^{P_i} m_{ih} v_{ih}^r \sigma_{kih}^s. \tag{142}$$

Substitution from (135) thru (141) into (134) results in

$$\begin{aligned}
\sum_{i=1}^N \sum_{h=1}^{P_i} m_{ih} \bar{y}_{ih} \times \bar{\alpha}_{kih} &= \sum_{i=1}^N (C_{rst} \bar{J}_t m_i \theta_i^r \bar{r}_{ki}^s \\
&\quad + C_{rst} \bar{J}_t \Lambda_{ki}^{rs} + \bar{J}_r \beta_{ki}^{rs} H'_{rsi}) \\
&= \bar{J}_r \sum_{i=1}^N (C_{rst} m_i \theta_i^r \bar{r}_{ki}^s + C_{stv} e_{si}^r \Lambda_{ki}^{tv} \\
&\quad + e_{si}^r H'_{rst} \beta_{ki}^{st}). \tag{143}
\end{aligned}$$

The second term of (4) is

$$\begin{aligned}
 & \sum_{i=1}^N \sum_{k=1}^P m_{ik} \bar{y}_{ik} \times (\bar{x}_k \times \bar{y}_{ik}) \\
 & - \sum_{i=1}^N \sum_{k=1}^P m_{ik} [\bar{x}_k (\bar{y}_{ik} \cdot \bar{y}_{ik}) - \bar{y}_{ik} (\bar{x}_k \cdot \bar{y}_{ik})] \\
 & = \bar{J}_r \mathcal{L}_k^s \sum_{i=1}^N \sum_{k=1}^P m_{ik} (\delta_{rs} y_{ik}^s y_{ik}^s - y_{ik}^r y_{ik}^s) \\
 & = \bar{J}_r \mathcal{L}_k^s I_{rs}, \tag{14a}
 \end{aligned}$$

where

$$\begin{aligned}
 I_{rs} &= \sum_{i=1}^N \sum_{k=1}^P m_{ik} (\delta_{rs} y_{ik}^s - y_{ik}^r y_{ik}^s) \\
 &= \sum_{i=1}^N \sum_{k=1}^P m_{ik} (\delta_{rs} (\theta_i^s + e_{ii}^s v_{ik}^s) (\theta_i^s + e_{ii}^s v_{ik}^s) \\
 &\quad - (\theta_i^s + e_{ii}^s v_{ik}^s) (\theta_i^s + e_{ii}^s v_{ik}^s)) \\
 &= \sum_{i=1}^N \sum_{k=1}^P m_{ik} (\delta_{rs} (\theta_i^s \theta_i^s + v_{ik}^s v_{ik}^s) - (\theta_i^s \theta_i^s + e_{ii}^s e_{ii}^s v_{ik}^s v_{ik}^s)) \\
 &= \sum_{i=1}^N [m_i (\delta_{rs} \theta_i^s \theta_i^s - \theta_i^s \theta_i^s) \\
 &\quad + e_{ii}^s e_{ii}^s \sum_{k=1}^P m_{ik} (\delta_{rs} v_{ik}^s v_{ik}^s - v_{ik}^s v_{ik}^s)] \\
 &= \sum_{i=1}^N [m_i (\delta_{rs} \theta_i^s \theta_i^s - \theta_i^s \theta_i^s) + e_{ii}^s e_{ii}^s H_{rs}^i]. \tag{14b}
 \end{aligned}$$

The I_{rs} are the moments and the negatives of the products of inertia of the vehicle about its axes.

The third term of (4) contains

$$\begin{aligned}
 \sum_{i=1}^N \sum_{k=1}^P m_{ik} \bar{a}_{kik} &= \sum_{i=1}^N \sum_{k=1}^P m_{ik} (\bar{f}_{kic} + \bar{\sigma}_{kik} + \bar{\beta}_{kic} \times \bar{v}_{ik}) \\
 &= \sum_{i=1}^N m_i \bar{f}_{kic} \\
 &= \bar{J}_r \sum_{i=1}^N m_i \bar{f}_{kic}, \tag{14c}
 \end{aligned}$$

and so that

$$\begin{aligned} m \bar{y}_c \times (\bar{L}_K \times \bar{y}_c) &= m [\bar{L}_K (\bar{y}_c \cdot \bar{y}_c) - \bar{y}_c (\bar{L}_K \cdot \bar{y}_c)] \\ &= m \bar{L}_K^c (\delta_{rs} y_c^r y_c^s - y_c^r y_c^s). \end{aligned} \quad (147)$$

Substitution from (143) thru (147) into (43) results in

$$\begin{aligned} &\bar{J}_r \sum_{i=1}^n (C_{rst} m_i \theta_i^r j_{ki}^s + C_{stu} e_{si}^r \Lambda_{ki}^{-tu} + e_{si}^r H_{stc}^i \beta_{ki}^{-t}) \\ &+ \bar{J}_r \beta_{ki}^s \bar{I}_{rs} - \bar{y}_c \times \bar{J}_c \sum_{i=1}^n m_i j_{ki}^r \\ &- \bar{J}_r \beta_{ki}^s m (\delta_{rs} y_c^r y_c^s - y_c^r y_c^s) = 0. \end{aligned} \quad (148)$$

Noting that $\bar{y}_c \times \bar{J}_c = C_{rst} \bar{J}_r y_c^s$, we can put this in the form

$$\begin{aligned} [\bar{I}_{rs} - m (\delta_{rs} y_c^r y_c^s - y_c^r y_c^s)] \beta_{ki}^s &= C_{rst} y_c^s \sum_{i=1}^n m_i j_{ki}^r \\ &- \sum_{i=1}^n (C_{rst} m_i \theta_i^r j_{ki}^s + C_{stu} e_{si}^r \Lambda_{ki}^{-tu} + e_{si}^r H_{stc}^i \beta_{ki}^{-t}). \end{aligned} \quad (149)$$

$$\text{Let } H_{ki}^s = \sum_{i=1}^n m_i j_{ki}^r \quad (150)$$

$$\text{and } L_{ki}^s = \sum_{i=1}^n m_i \theta_i^r j_{ki}^s; \quad (151)$$

then (149) becomes

$$\begin{aligned} [\bar{I}_{rs} - m (\delta_{rs} y_c^r y_c^s - y_c^r y_c^s)] \beta_{ki}^s &= C_{rst} y_c^s H_{ki}^r \\ &- C_{rst} L_{ki}^s - \sum_{i=1}^n (C_{stu} e_{si}^r \Lambda_{ki}^{-tu} + e_{si}^r H_{stc}^i \beta_{ki}^{-t}), \end{aligned} \quad (152)$$

which can be solved for the β_{ki}^s by familiar techniques. With the aid of (41), (146), (150), and the fact that $\bar{L}_K \times \bar{y}_c = \bar{J}_r C_{rst} \beta_{ki}^s y_c^r$, it is easily seen that

$$C_K^r = C_{rst} y_c^s \beta_{ki}^s - \frac{1}{m} H_K^r. \quad (153)$$

$$\begin{aligned}
\bar{J}_r \alpha_{kik}^r &= \bar{J}_r f_{ki}^r + \bar{J}_{s_i} \bar{J}_{k_i} \sigma_{ki}^s + \bar{J}_{t_i} \times \bar{J}_{u_i} B_{ki}^t v_{ik}^u \\
&= \bar{J}_r (f_{ki}^r + e_{s_i}^r \bar{J}_{k_i} \sigma_{ki}^s) + \bar{J}_{s_i} C_{seu} B_{ki}^t v_{ik}^u \\
&= \bar{J}_r [f_{ki}^r + e_{s_i}^r (\sigma_{ki}^s + C_{seu} B_{ki}^t v_{ik}^u)], \\
\text{or} \\
\alpha_{kik}^r &= f_{ki}^r + e_{s_i}^r (\sigma_{ki}^s + C_{seu} B_{ki}^t v_{ik}^u). \quad (154)
\end{aligned}$$

Similarly (40) is transformed to

$$\begin{aligned}
\bar{J}_r \frac{\partial y_{ik}^r}{\partial q_k^r} &= \bar{J}_r \alpha_{kik}^r + \bar{J}_s \times \bar{J}_t C_{rst} B_{ki}^s y_{ik}^t + \bar{J}_r C_k^r \\
&= \bar{J}_r (\alpha_{kik}^r + C_{rst} B_{ki}^s y_{ik}^t + C_k^r) \\
\text{or} \\
\frac{\partial y_{ik}^r}{\partial q_k^r} &= C_k^r + C_{rst} B_{ki}^s y_{ik}^t + f_{ki}^r \\
&\quad + e_{s_i}^r (\sigma_{ki}^s + C_{seu} B_{ki}^t v_{ik}^u). \quad (155)
\end{aligned}$$

We now turn our attention to the evaluation of the P_{rsj} , defined in (53).

$$\begin{aligned}
\sum_{i=1}^M \sum_{k=1}^{P_i} m_{i,k} y_{ik}^r \frac{\partial y_{ik}^r}{\partial q_k^r} &= \sum_{i=1}^M \sum_{k=1}^{P_i} m_{i,k} y_{ik}^r [C_k^r + C_{seu} B_{ki}^t y_{ik}^t \\
&\quad + f_{ki}^r + e_{s_i}^r (\sigma_{ki}^s + C_{seu} B_{ki}^t v_{ik}^u)]. \quad (156)
\end{aligned}$$

where $\Gamma_{\rho\sigma} = \Gamma_{\sigma\rho}$ and $\Gamma_{\rho\rho} = 1$.

$$\sum_{i=1}^N \sum_{k=1}^P m_{ik} y_{ik}^r C_j^s = m_{ij}^r C_j^s, \quad (15)$$

$$\sum_{i=1}^N \sum_{k=1}^P m_{ik} y_{ik}^r C_{\rho\sigma} b_{ij}^s y_{ik}^r = C_{\rho\sigma} b_{ij}^s G_{\rho\sigma}, \quad (15B)$$

$$\begin{aligned} \text{trace } G_{\rho\sigma} &= \sum_{i=1}^N \sum_{k=1}^P m_{ik} y_{ik}^r y_{ik}^r \\ &= \delta_{\rho\sigma} I_{\rho\rho} / 2 = I_{\rho\rho}; \end{aligned} \quad (159)$$

$$\sum_{i=1}^N \sum_{k=1}^P m_{ik} y_{ik}^r j_{ji}^s = \sum_{i=1}^N m_{ij}^r \theta_i^s j_{ji}^s; \quad (160)$$

$$\begin{aligned} &\sum_{i=1}^N \sum_{k=1}^P m_{ik} y_{ik}^r e_{ii}^s \sigma_{jik}^t \\ &= \sum_{i=1}^N \sum_{k=1}^P m_{ik} (\theta_i^r + e_{ii}^r v_{ik}^r) e_{ii}^s \sigma_{jik}^t \\ &= \sum_{i=1}^N e_{ii}^r e_{ii}^s \sum_{k=1}^P m_{ik} v_{ik}^r \sigma_{jik}^t \\ &= \sum_{i=1}^N e_{ii}^r e_{ii}^s \Lambda_{ji}^{st}; \end{aligned} \quad (161)$$

$$\begin{aligned} &\sum_{i=1}^N \sum_{k=1}^P m_{ik} y_{ik}^r e_{ii}^s C_{\rho\sigma} b_{ij}^s v_{ik}^r \\ &= C_{\rho\sigma} \sum_{i=1}^N e_{ii}^s b_{ij}^s \sum_{k=1}^P m_{ik} (\theta_i^r + e_{ii}^r v_{ik}^r) v_{ik}^r \\ &= C_{\rho\sigma} \sum_{i=1}^N e_{ii}^s b_{ij}^s e_{\rho i}^r \Gamma_{\rho\sigma} v_i, \end{aligned} \quad (162)$$

$$\begin{aligned} \Gamma_{\rho\sigma}^{\nu} &= \sum_{k=1}^n m_{ik} v_{ik}^{\rho} v_{ik}^{\sigma} \\ &- \delta_{\rho\sigma} v \text{Hgg} / 2 - H'_{\rho\nu} c \end{aligned} \quad (152)$$

Substitution from (51) thru (62) into (152) results in:

$$\begin{aligned} \sum_{i=1}^n \sum_{k=1}^n m_{ik} \cdot y_{ik}^{\rho} \frac{\partial y_{ik}^{\sigma}}{\partial q^j} &= m y_c C_j^{\rho} + C_{c2i} b_j^{\rho} G_{ru} \\ &+ \sum_{i=1}^n [m_i \sigma_i^{\rho} \beta_{ji}^{\rho} + e_{i2}^{\rho} e_{i1}^{\rho} \Lambda_{ji}^{\rho} + C_{iuv} e_{i1}^{\rho} e_{i2}^{\rho} \beta_{ji}^{\rho} \Gamma_{\rho\nu}^{\rho}]. \end{aligned} \quad (16)$$

From this, setting S equal to r (and summing over r),

$$\begin{aligned} \sum_{i=1}^n \sum_{k=1}^n m_{ik} y_{ik}^{\rho} \frac{\partial y_{ik}^{\sigma}}{\partial q^j} &= \sum_{i=1}^n \sum_{k=1}^n m_{ik} y_{ik}^{\rho} \frac{\partial y_{ik}^{\sigma}}{\partial q^j} \\ &= m y_c C_j^{\rho} + C_{ru} b_j^{\rho} G_{ru} \\ &+ \sum_{i=1}^n [m_i \sigma_i^{\rho} \beta_{ji}^{\rho} + e_{i2}^{\rho} e_{i1}^{\rho} \Lambda_{ji}^{\rho} + C_{iuv} e_{i1}^{\rho} e_{i2}^{\rho} \beta_{ji}^{\rho} \Gamma_{\rho\nu}^{\rho}] \\ &= m y_c C_j^{\rho} + \sum_{i=1}^n (m_i \sigma_i^{\rho} \beta_{ji}^{\rho} + \Lambda_{ji}^{\rho}), \end{aligned} \quad (165)$$

since $G_{ru} = G_{ur}$, $e_{i2}^{\rho} e_{i1}^{\rho} = \delta_{ur}$, and

$$\Gamma_{\rho\nu}^{\rho} = \Gamma_{\nu\rho}^{\rho}.$$

P_{rsj} is now obtained by substitution from (164) and (165) into (53).

$$\begin{aligned}
P_{rsj} &= \delta_{rs} [m_j y_c^t C_j^t + \sum_{i=1}^n m_i \sigma_i^t j_{ci}^t + \Lambda_{ji}^{tt}] - m_j y_c^t C_j^t \\
&- C_{rsu} b_j^t \theta_{ru} - \sum_{i=1}^n (m_i \sigma_i^t j_{ci}^t + e_{ci}^t e_{ci}^t \Lambda_{ji}^{tt}) \\
&\quad \cdot C_{eur} e_{ci}^t e_{pi}^t \beta_{ji}^{tt} / \gamma_{pvi}^t \\
&= m(\delta_{rs} y_c^t C_j^t - y_c^t C_j^t) - C_{rsu} b_j^t \theta_{ru} \\
&+ \sum_{i=1}^n [m_i \sigma_i^t j_{ci}^t - \sigma_i^t j_{ci}^t] + \delta_{rs} \Lambda_{ji}^{tt} - e_{ci}^t e_{ci}^t \Lambda_{ji}^{tt} \\
&\quad - C_{eur} e_{pi}^t e_{ci}^t \beta_{ji}^{tt} / \gamma_{pvi}^t \\
&= m(\delta_{rs} y_c^t C_j^t - y_c^t C_j^t) - C_{rsu} b_j^t \theta_{ru} \\
&+ \delta_{rs} L_j^{tt} - L_j^{tt} + \sum_{i=1}^n [e_{ci}^t e_{ci}^t (\delta_{ru} \Lambda_{ji}^{tt} - \Lambda_{ji}^{tt}) \\
&\quad - C_{eur} e_{pi}^t e_{ci}^t \beta_{ji}^{tt} / \gamma_{pvi}^t] \tag{166}
\end{aligned}$$

$$\text{Let } \mu_{jk} = \sum_{i=1}^n \sum_{k=1}^B m_{ik} \sigma_{ji}^t \sigma_{ki}^t, \tag{167}$$

and substitute from (155) into (60).

Then, after simplification and making use of (22), (23), (25), (135), (136), (138), (142), (145), (150), (151), and (159), the result is obtained that

$$\begin{aligned}
h_{ijk} &= m [C_j^t C_k^t + C_{rst} (C_j^t L_k^{tt} + C_k^t L_j^{tt}) y_c^t] \\
&+ \mu_{jk} + b_j^t L_k^{tt} I_{rs} + C_j^t H_k^t + C_k^t H_j^t \\
&+ C_{rst} (b_j^t L_k^{tt} + b_k^t L_j^{tt}) + \sum_{i=1}^n m_i j_{ci}^t j_{ci}^t
\end{aligned}$$

$$\begin{aligned}
& + C_{rst} (b_j^r \sum_{k_1}^n e_{r_1}^+ \Lambda_{k_1}^{st} + b_{k_1}^u \sum_{r_1}^n e_{r_1}^+ \Lambda_{j_1}^{st}) \\
& - C_{rst} \sum_{k_1}^n (\beta_{j_1}^r \Lambda_{k_1}^{st} + \beta_{k_1}^r \Lambda_{j_1}^{st}) \\
& + \omega_j^r \sum_{k_1}^n e_{s_1}^r \beta_{r_1}^t H_{s_1}^+ + b_{k_1}^r \sum_{s_1}^n e_{s_1}^r \beta_{r_1}^t b_{s_1}^+ \\
& + \sum_{k_1}^n \beta_{j_1}^r \beta_{k_1}^s H_{r_1}^+ \quad (168)
\end{aligned}$$

This equation is confirmed by writing out an expression for the kinetic energy T and making use of (72).

To provide more compact notation, let

$$D_{jk} = \sum_{s_1}^n m_{s_1} \theta_{j_1}^r \theta_{k_1}^r, \quad (169)$$

$$\Lambda_k^r = C_{stu} \sum_{s_1}^n e_{s_1}^r \Lambda_{k_1}^{tu}, \quad (170)$$

$$\Delta_{jk} = C_{rst} \sum_{k_1}^n (\beta_{j_1}^r \Lambda_{k_1}^{st} + \beta_{k_1}^r \Lambda_{j_1}^{st}), \quad (171)$$

$$N_k^r = \sum_{s_1}^n e_{s_1}^r \beta_{k_1}^s H_{s_1}^+, \quad \text{and} \quad (172)$$

$$H_{jk} = \sum_{s_1}^n \beta_{j_1}^r \beta_{k_1}^s H_{r_1}^+ \quad (173)$$

$$\begin{aligned}
\text{Then } M_{jk} = & m [C_j^r C_k^r + C_{rst} (c_j^r b_k^s + c_k^r b_j^s) y_c^t] \\
& + b_j^r b_k^s I_{rst} + C_j^r H_k^+ + C_k^r H_j^+ \\
& + b_j^r (C_{rst} L_k^{st} + \Lambda_k^r + N_k^r) \\
& + b_k^r (C_{rst} L_j^{st} + \Lambda_j^r + N_j^r) \\
& + D_{jk} + \Delta_{jk} + H_{jk} + \omega_{jk}. \quad (174)
\end{aligned}$$

If the subscript i in (120) is replaced by L and the result applied to (129), it can be demonstrated that interchanging K and L does not change the value of (129), which is as it should be. This "symmetry" of (129) depends on the retention of the last term; but $\frac{\partial \bar{\alpha}_k}{\partial q^L}$, which is a factor in the last term, is difficult to obtain and likely to be small; therefore, a means of dropping it out without destroying the symmetry of the equation is sought. This is accomplished by a simple averaging, as follows:

$$\frac{\partial^2 \bar{y}}{\partial q^k \partial q^L} = \frac{\partial^2 \bar{y}}{\partial q^L \partial q^k} = \frac{1}{2} \left(\frac{\partial^2 \bar{y}}{\partial q^k \partial q^L} + \frac{\partial^2 \bar{y}}{\partial q^L \partial q^k} \right) \quad (175)$$

$$\cong \bar{\alpha}_K \times \bar{\sigma}_L + \bar{\alpha}_L \times \bar{\sigma}_K + \frac{1}{2} [\bar{\alpha}_K \times (\bar{\alpha}_L \times \bar{v}) + \bar{\alpha}_L \times (\bar{\alpha}_K \times \bar{v})]$$

$$= \bar{\alpha}_K \times \bar{\sigma}_L + \bar{\alpha}_L \times \bar{\sigma}_K - (\bar{\alpha}_K \cdot \bar{\alpha}_L) \bar{v} + \frac{1}{2} [(\bar{\alpha}_K \cdot \bar{v}) \bar{\alpha}_L + (\bar{\alpha}_L \cdot \bar{v}) \bar{\alpha}_K]$$

Employing (125) and (175) in (59) results in

$$\boxed{KLj} \cong \sum_{i=1}^N \sum_{k=1}^{P_i} M_{ik} (\bar{h}_{ji} + \bar{\sigma}_{jik} + \bar{\alpha}_{ji} \times \bar{v}_{ik}) \cdot \{ \bar{\alpha}_{ki} \times \bar{\sigma}_{Li} + \bar{\alpha}_{Li} \times \bar{\sigma}_{ki} - (\bar{\alpha}_{ki} \cdot \bar{\alpha}_{Li}) \bar{v}_{ik} + \frac{1}{2} [(\bar{\alpha}_{ki} \cdot \bar{v}_{ik}) \bar{\alpha}_{Li} + (\bar{\alpha}_{Li} \cdot \bar{v}_{ik}) \bar{\alpha}_{ki}] \} \quad (176)$$

If this is now expanded, the result is

$$\boxed{KLj} \cong \sum_{i=1}^N [C_{rot} (\alpha_{ki}^{-t} S_{Lji}^{-st} + \alpha_{Li}^{-t} S_{kji}^{-st}) + \alpha_{ki}^{-t} \alpha_{Li}^{-s} Q_{ji}^{-rs} + (\alpha_{ki}^{-t} p_{Li}^{-rs} + \alpha_{Li}^{-t} p_{ki}^{-rs}) \alpha_{ji}^{-s}] \quad (177)$$

where
$$S_{jki}^{-rs} = \sum_{k=1}^{P_i} M_{ik} \sigma_{jik}^r \sigma_{ik}^s \quad (178)$$

$$p_{ji}^{-rs} = \sum_{k=1}^{P_i} M_{ik} (\delta_{rs} v_{ik}^t \sigma_{jik}^t - v_{ik}^r \sigma_{jk}^s) = \delta_{rs} \Lambda_{ji}^{-tt} - \Lambda_{ji}^{-rs} \quad (179)$$

$$\text{and } a_{ji}^{-rs} = \frac{1}{2} [\alpha_{ji}^{rt} (C_{stu} \Gamma_{rui} + C_{rtu} \Gamma_{oui}) - p_{ji}^{-rs} - p_{ji}^{-sr}] \quad (180)$$

If we note that $S_{jki}^{-rs} = 0$ when j and/or k equal zero and that $p_{ji}^{-rs} = 0$ when $j = 0$, then it follows from (177) that

$$\boxed{OK, j} \equiv \sum_{i=1}^N (C_{rst} \alpha_{oi}^{-r} S_{kji}^{-st} + \alpha_{oi}^{-t} p_{ki}^{-rs} \alpha_{ji}^{-s} + \alpha_{ki}^{-r} \alpha_{oi}^{-s} a_{ji}^{-rs}) \quad (181)$$

$$\text{and } \boxed{OQj} \equiv \sum_{i=1}^N \alpha_{oi}^{-r} \alpha_{oi}^{-s} a_{ji}^{-rs} \quad (182)$$

7. PRACTICAL EXPRESSION OF THE AERODYNAMIC FORMULAS

The aerodynamic forces, being external, are accounted for by the use of Equation (47). In this use of (47), however, only those particles lying on the surface of the vehicle will be involved. In the present formulation, we have recourse to the simplest available aerodynamic theory that offers sufficient generality, namely, Newtonian flow theory. Let \bar{n} be a unit vector located at a certain point on the surface, perpendicular to the surface at that point, and pointing outward. The velocity of that point is \bar{v} , as given in (15). Letting ρ be the atmospheric density, then, according to Newtonian flow theory, the aerodynamic force per unit area (\bar{F}) at the given point on the surface of the vehicle is as follows:

1. When $\bar{n} \cdot \bar{v} \leq 0$, $\bar{F} = 0$. (183)

2. When $\bar{n} \cdot \bar{v} > 0$, $\bar{F} = -\bar{n} \rho (\bar{n} \cdot \bar{v})^2$. (184)

The scalar $\bar{n} \cdot \bar{v}$ may be called the "piston speed" of the given point (or the downwash at that point) and is symbolized by w ; thus, when $w > 0$,
 $\bar{F} = -\bar{n} \rho w^2$. (185)

For the purpose of evaluating w , it is noted that the first and last terms of \bar{v} as given in (15) are, under normal conditions, much more significant than the two middle terms. Dropping these less significant terms results in

$$w = \bar{n} \cdot (\bar{V} + \frac{\partial \bar{V}}{\partial q^k} \dot{q}^k). \quad (186)$$

A considerable practical advantage can be realized if w^2 in (185) is replaced by a linear approximation (or expansion) about the elastically undeformed configuration. Regarding w as a function of q^k and \dot{q}^k for this purpose, and using the subscript 0 to denote the undeformed configuration, we obtain

$$w^2 = w_0^2 + 2 w_0 \left(\frac{\partial w_0}{\partial q_0^k} q^k + \frac{\partial w_0}{\partial \dot{q}_0^k} \dot{q}^k \right). \quad (187)$$

Now $w_0 = \bar{V} \cdot \bar{n}$,
 $\frac{\partial w_0}{\partial q_0^k} = \bar{V} \cdot \frac{\partial \bar{n}}{\partial q_0^k}$,
 $\frac{\partial w_0}{\partial \dot{q}_0^k} = \bar{n} \cdot \frac{\partial \bar{V}}{\partial \dot{q}_0^k}$;

and, therefore,

$$w^2 = (\bar{V} \cdot \bar{n})^2 + 2 \bar{V} \cdot \bar{n} \left(\bar{V} \cdot \frac{\partial \bar{n}}{\partial q_0^k} q^k + \bar{n} \cdot \frac{\partial \bar{V}}{\partial \dot{q}_0^k} \dot{q}^k \right). \quad (189)$$

In applying (47) to the calculation of the generalized air dynamic forces, we are impelled to replace the summation over K with an integration over the surface of the i -th section by virtue of the fact that only points on the surface are involved. Let S_i be the surface the i -th section then

$$\begin{aligned}
 (18) \quad & - \sum_{i=1}^N \int_{S_i} \frac{\partial \vec{F}}{\partial q_i} \cdot \vec{F} \, ds \\
 & = - \sum_{i=1}^N \int_{S_i} \frac{\partial \vec{F}}{\partial q_i} \cdot \vec{n} \, e \, w^2 \, ds \\
 & = - \sum_{i=1}^N \int_{S_i} \xi_j \left[(\vec{V} \cdot \vec{n})^2 \right. \\
 & \quad \left. + 2\vec{V} \cdot \vec{n} \left(\vec{V} \frac{\partial \vec{n}}{\partial q_i} \cdot \vec{q}^i + \xi_k \dot{q}^k \right) \right] ds, \quad (190)
 \end{aligned}$$

where $\xi_j = \vec{n} \cdot \frac{\partial \vec{F}}{\partial q_j}$. (191)

The following development of formulas serves to make this more practical for numerical computations

$$\vec{n} = \vec{J}'_r \, n^{r'} \quad (192)$$

$$\begin{aligned}
 \frac{\partial \vec{n}}{\partial q^k} &= \frac{\partial \vec{J}'_r}{\partial q^k} \, n^{r'} + \vec{J}'_r \, \frac{\partial n^{r'}}{\partial q^k} \\
 &= \vec{c}_k \times \vec{n} + \vec{J}'_r \, \frac{\partial n^{r'}}{\partial q^k} \\
 &= \vec{J}'_r \left(c_{rsk} \, c^s_k \, n^{r'} + \frac{\partial n^{r'}}{\partial q^k} \right). \quad (193)
 \end{aligned}$$

Inserting this into (120) and transforming it in other ways leads to

$$\begin{aligned}
 Q_j &= -e \sum_{i=1}^N \xi_j [v^r v^s (\bar{J}_r \cdot \bar{J}_{ti}) (\bar{J}_s \cdot \bar{J}_{ui}) n^{-t} n^{-u} \\
 &\quad + 2v^r v^s (\bar{J}_r \cdot \bar{J}_{ti}) (\bar{J}_s \cdot \bar{J}_{ui}) n^{-t} (c_{uv} \alpha_{ki}^{-v} n^{-x} + \frac{\partial n^{-u}}{\partial q^k}) q^k \\
 &\quad + 2v^r (\bar{J}_r \cdot \bar{J}_{ti}) n^{-t} \xi_k \dot{q}^k] ds \\
 &= -e v^r v^s \sum_{i=1}^N e_{ti}^r e_{ui}^s \int_{s_i} \xi_j n^{-t} n^{-u} ds \\
 &\quad - 2e v^r v^s \sum_{i=1}^N e_{ti}^r e_{ui}^s (c_{uv} \alpha_{ki}^{-v} \int_{s_i} n^{-t} n^{-x} \xi_j ds \\
 &\quad \quad + \int_{s_i} \xi_j n^{-t} \frac{\partial n^{-u}}{\partial q^k} ds) q^k \\
 &\quad - 2e v^r \sum_{i=1}^N e_{ti}^r \int_{s_i} n^{-t} \xi_j \xi_k ds \dot{q}^k \\
 &= -e (v^r v^s A_{jk}^{rs} + 2v^r v^s B_{jk}^{rs} q^k + 2v^r C_{jk}^r \dot{q}^k), \quad (194)
 \end{aligned}$$

where $A_{jk}^{rs} = \sum_{i=1}^N e_{ti}^r e_{ui}^s \int_{s_i} \xi_j n^{-t} n^{-u} ds,$ (195)

$$\begin{aligned}
 B_{jk}^{rs} &= \sum_{i=1}^N e_{ti}^r e_{ui}^s (c_{uv} \alpha_{ki}^{-v} \int_{s_i} \xi_j n^{-t} n^{-x} ds \\
 &\quad + \int_{s_i} \xi_j n^{-t} \frac{\partial n^{-u}}{\partial q^k} ds), \quad (196)
 \end{aligned}$$

and $C_{jk}^r = \sum_{i=1}^N e_{ti}^r \int_{s_i} \xi_j \xi_k n^{-t} ds.$ (197)

Substitution from (125) into (191) results in

$$\begin{aligned}
 \xi_j &= \bar{h} \cdot (\bar{h}_j + \bar{\sigma}_j + \bar{\alpha}_j \times \bar{v}) \\
 &= n^{\circ} (e_{\circ j}^{\circ} h_j^{\circ} + \sigma_j^{\circ} + c_{\circ j}^{\circ} \alpha_j^{\circ} v^{\circ}). \quad (198)
 \end{aligned}$$

It is assumed (or introduced as a limitation on the applicability of this formulation) that all thrust forces are directed by thrust "vectoring" (that is, tilting) nozzles and that all such nozzles are symmetric with respect to the axis (or line) of thrust. Regarding each thrust vectoring nozzle as a rigid but movable structural section identifiable by a subscript i , we place therein a triad of unit vectors \bar{J}_{ij} with \bar{J}_{ij1} pointing in the direction of the thrust and coinciding with the line of thrust. Because of the symmetry of the nozzle, its center of mass will lie on the thrust axis and the usual condition that the origin of the \bar{J}_{ij} triad be at the center of mass of the section can be complied with.

At the center of mass of a section, $\frac{\partial \bar{J}_{ij1}}{\partial q^k} = \frac{\partial \bar{J}_{ij2}}{\partial q^k} = \bar{h}_{jk}$. Representation of the thrust force at the i -th nozzle as $\bar{J}_{ij1} T_i$ and substitution into (47) results in the following expression for the generalized forces associated with the thrust forces:

$$\begin{aligned} Q_j &= \sum_{i=1}^E \bar{h}_{ji} \cdot \bar{J}_{ij1} T_i \\ &= \sum_{i=1}^E \bar{J}_{ij1} \cdot \bar{J}_{ij1} \bar{h}_{ji} T_i \\ &= \sum_{i=1}^E e_{ij1}^T \bar{h}_{ji} T_i, \end{aligned} \tag{199}$$

E being the number of thrust vectoring nozzles (or "engines").

As has already been indicated in (7), the e_{ij1}^T are functions of q^0 and the q^k . The correct inclusion of the dependence of the e_{ij1}^T on the q^k would be the best procedure and would enable the program to reveal the interaction between thrust and elastic deformation even to the point of detecting instabilities if any existed. However, doing this would impose considerable additional difficulty and go beyond the scope of the program; therefore, the dependence of the e_{ij1}^T on the q^k will be disregarded here. On the other hand, their dependence on q^0 must be and is included as shown in the following section.

9. DIRECTION COSINES OF MOVABLE STRUCTURAL SECTION

In addition to thrust vectoring nozzles, there are such movable structural sections as control surfaces of various types. For the sake of simplicity it is assumed that the large motions of all control surfaces consist of nothing more than a rotation about a fixed axis. It is convenient to place the \bar{J}'_{1i} vector of such a section parallel to, but not necessarily on, this axis of rotation. Doing this makes it possible to employ Eulerian angles to define the orientation of both thrust vectoring nozzles and control surfaces.

These angles are shown in Figure 3 and defined as follows.

ϕ_i = angle of rotation of the plane and axis of swivel about the y^i axis (\bar{J}_1).

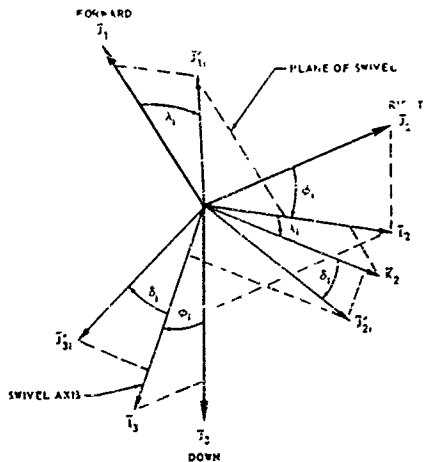
λ_i = angle of swivel of nozzle (or the \bar{J}'_{1i} vector) about an axis (\bar{J}_3) perpendicular to \bar{J}_1 and making an angle ϕ_1 with \bar{J}_3 .

δ_i = angle of rotation of \bar{J}'_{2i} and \bar{J}'_{3i} about \bar{J}'_{1i} .

By familiar processes of vector analysis and classical mechanics, it is known that the direction cosines relating the \bar{J}'_{ri} vectors to the \bar{J}_s vectors are the following e_{ri}^s :

$$\left. \begin{aligned}
 e_{1i}^1 &= \cos \lambda_i \\
 e_{2i}^1 &= -\sin \lambda_i \cos \delta_i \\
 e_{3i}^1 &= \sin \lambda_i \sin \delta_i \\
 e_{1i}^2 &= \cos \phi_i \sin \lambda_i \\
 e_{2i}^2 &= \cos \phi_i \cos \lambda_i \cos \delta_i - \sin \phi_i \sin \delta_i \\
 e_{3i}^2 &= -\cos \phi_i \cos \lambda_i \sin \delta_i - \sin \phi_i \cos \delta_i \\
 e_{1i}^3 &= \sin \phi_i \sin \lambda_i \\
 e_{2i}^3 &= \sin \phi_i \cos \lambda_i \cos \delta_i + \cos \phi_i \sin \delta_i \\
 e_{3i}^3 &= -\sin \phi_i \cos \lambda_i \sin \delta_i + \cos \phi_i \cos \delta_i
 \end{aligned} \right\} \quad (200)$$

It is understood that ϕ_i , λ_i , and δ_i are functions of q^o as determined by auto-pilot or flight programmer commands.



NOTE:

- a) I_2 AND I_3 ARE 1 . ROTATE ABOUT AND ARE 1 TO I_1
- b) I_{11} AND K_2 ARE 1 . ROTATE ABOUT AND ARE 1 TO I_3
- c) I_{21} AND I_{31} ARE 1 . ROTATE ABOUT AND ARE 1 TO I_{11}

Figure 3. Orientation Angles of Movable Sections

10. FORMULAS FOR THE STRUCTURAL LOADS

The shear force \bar{S} at a specified location on the vehicle is the negative of the sum of the internal forces exerted by all the particles of the vehicle on the particles located on one side of the chosen shear plane. For the sake of economy, the side of the shear plane selected for this purpose will be the side on which the smaller number of particles is found. This will usually be the side away from (or lying outboard of) the center of mass. Let the absence of specific designation as to which particles and sections are included in a summation be understood to mean summation over the particles on the chosen side of the shear plane. Then, with the aid of (44),

$$\begin{aligned} \bar{S} &= - \sum_i \sum_k \sum_{h=1}^N \sum_{j=1}^{P_i} \bar{F}_{ikhj} = - \sum_i \sum_k (\bar{F}_{ik} - m_{ik} \frac{d\bar{v}_{ik}}{dt}) \\ &= \sum_i \sum_k \bar{F}_{ik} - \sum_i \sum_k m_{ik} \frac{d\bar{v}_{ik}}{dt} \end{aligned} \quad (201)$$

Except for the number of particles included in the summation, the last term of (201) is the same as the right side of (27). A practical symmetric expression for $\frac{d^2 \bar{y}}{dt^2}$ is given in (175). This can be used in (27), which in turn is to be used in (201).

The bending moment \bar{M} at the specified location is the negative of the sum of the moments about a point \bar{y} in the shear plane due to the internal forces exerted by all the particles of the vehicle on the particles located on one side of the shear plane. In like manner to that employed in determining \bar{S} , it is found that

$$\begin{aligned} \bar{M} &= - \sum_i \sum_k (\bar{y}_{ik} - \bar{y}) \times \sum_{h=1}^N \sum_{j=1}^{P_i} \bar{F}_{ikhj} \\ &= \sum_i \sum_k (\bar{y}_{ik} - \bar{y}) \times (\bar{F}_{ik} - m_{ik} \frac{d\bar{v}_{ik}}{dt}) \\ &= \sum_i \sum_k \bar{y}_{ik} \times \bar{F}_{ik} - \sum_i \sum_k m_{ik} \bar{y}_{ik} \times \frac{d\bar{v}_{ik}}{dt} \\ &\quad - \bar{y} \times \bar{S} \end{aligned} \quad (202)$$

Except for the extent of the summation, the next to the last term of (20) is just the right side of (20), which can be used to expand this term for practical use.

$$\bar{S} = \bar{J}_r S_r \quad \text{and} \quad \bar{M} = \bar{J}_r M_r . \quad (203)$$

The actual numerical quantities to be computed are the components S_r of \bar{S} and M_r of \bar{M} . It is assumed for the purposes of this program that the selected shear plane will be perpendicular to one of the \bar{J}_r vectors. The choice of the shear plane will affect the interpretation of the results S_r and M_r ($r=1, 2, 3$). For example, if the shear plane is perpendicular to \bar{J}_3 , then S_3 is the component of the shear force in the direction of the y^3 axis, S_1 is the shear force in the direction of the y^1 axis, S_2 is the normal force (being perpendicular to the shear plane), M_3 is the bending moment about an axis parallel to y^3 , M_1 is the bending moment about an axis parallel to y^1 , and M_2 is the torque.

It is a prerequisite to (27) that the sum of the internal forces exerted by and on all the particles of the vehicle equals zero. It is likewise prerequisite to (28) that the sum of the moments about any point in the vehicle due to the internal forces exerted by and on all the particles of the vehicle equals zero. These facts are deduced from Newton's third law of motion; and it follows from these and the definitions of \bar{S} and \bar{M} , leading to (201) and (202), that determining \bar{S} and \bar{M} by summing over the opposite side of the shear plane should change their signs but not their magnitudes.

It has been noted in Section 3 that (27) and (28) are not satisfied in the SLP because of the assumption that the elastic deformations and fuel sloshing motions have a negligible effect on the large motions of the vehicle and on \bar{F} and \bar{G} . Furthermore, the aerodynamic theory employed here (Newtonian flow) is different from that employed in the SDF program. This difference between the two programs further jeopardizes the agreement between them as to \bar{F} and \bar{G} and, hence, the satisfaction of (27) and (28) in the SLP; therefore, it cannot be expected that summing over the opposite side of the shear plane will satisfy the theoretical requirement of changing only the signs of \bar{S} and \bar{M} . This represents a failure to satisfy Newton's third law of motion and may prove to be a serious defect in the program.

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ENCLISION OF FUEL SLOSHING EFFECTS

The basic formulations for fuel sloshing as obtained from available literature are presented in Appendix I. Here the concern is how to incorporate the effects of fuel sloshing in the Structural Loads Program.

Within each tank, there are two directions of sloshing, designated somewhat loosely as longitudinal and lateral. We choose the letter u as an indicator of the sloshing direction, longitudinal sloshing being indicated by letting $u=1$, and lateral sloshing being indicated by letting $u=2$.

For each sloshing direction, there are two possible sloshing modes. The letter s is the mode indicator, the first mode being indicated by letting $s=1$, and the second mode being indicated by letting $s=2$.

Since there are two sloshing directions and two possible modes for each direction, there are four possible sloshing degrees of freedom for each tank. The number of the tank is designated by the letter i , and the degree of freedom is designated by k . The following formula is used to determine k in terms of i , u , and s .

$$k = 4(i-1) + u + 2(s-1). \quad (204)$$

The following tabulation illustrates these relations.

| | 1 | | | | 2 | | | | 3 | | | |
|---|---|---|---|---|---|---|---|---|---|----|----|----|
| S | | 1 | | 2 | | 1 | | 2 | | 1 | | 2 |
| u | 1 | 2 | 1 | 2 | 1 | 2 | 1 | 2 | 1 | 2 | 1 | 2 |
| K | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |

The program allows for a maximum of ten tanks; therefore, the largest possible value of k for a fuel sloshing mode is 40. The number of the elastic degrees of freedom for structural deformation, control surface rotation, and so forth, starts with 41 and proceeds to a maximum of 57, giving a possibility of 17 "structural" degrees of freedom.

The greatest problem that arises in connection with the effects of fuel sloshing in the Structural Loads Program is the computation of the terms $M_{F_{xi}}$ (138), ΔW_{xi} (142), and S_{jxi} (179).

In Appendix I, formulas are given for the effective moments of inertia of the fuel about tank axes for rectangular and cylindrical tanks. The equivalence between these and the $M_{F_{xi}}$ is as follows, the subscript F denoting fuel:

$$\left. \begin{aligned} H_{Fz} &= J_{Fz1} - I_{Fyz} \\ H_{Fz1} &= J_{Fz1} - I_{Fyz} \\ H'_{Fz1} &= J_{Fz1} - I_{Fyz} \\ &= 0 \text{ when } r \neq 5 \end{aligned} \right\} \text{ for a rectangular tank.} \quad (207)$$

$$\left. \begin{aligned} H'_{Fz} &= J'_{Fz1} - 0 \\ H'_{Fz1} &= J'_{Fz1} - I_{Fyz} \\ H'_{Fz1} &= J'_{Fz1} - I_{Fyz} I_{yz} / I_{yy} \\ H'_{Fz1} &= 0 \text{ when } r \neq 5 \end{aligned} \right\} \text{ for a horizontal cylindrical tank.} \quad (208)$$

$$\left. \begin{aligned} H'_{Fz1} &= J'_{Fz1} = m_{z1} \\ H'_{Fz2} &= J'_{Fz2} = m_{z2} \\ H'_{Fz3} &= J'_{Fz3} = 0 \\ H'_{Fz1} &= 0 \text{ when } r \neq 5 \end{aligned} \right\} \text{ for a vertical cylindrical tank.} \quad (209)$$

All $H'_{Fz1} = 0$ for a spherical tank. (208)

In rectangular tank and for longitudinal sloshing in horizontal cylindrical tanks, a spring-mass mechanical analogy is used. Each mass in this analogy has motion in one, and only one, degree of freedom; therefore, it can be identified by the subscript n , in accordance with equation (204). Likewise, the location of m_n is given by the coordinates x_n, y_n , and z_n . In the case of longitudinal oscillations, $x_n = q^n$ and $y_n = 0$; for lateral oscillations, $x_n = 0$ and $y_n = q^n$; in either case, z_n is simply z_n , a constant.

For the purpose of evaluating the Δ_{z1}^{rs} and the S_{z1}^{rs} , it is necessary to relate the masses and coordinates just discussed with those appearing in (162) and (178). An inspection of (204) quickly discloses that the particular mass particle within tank i is identified by

$$i = u + 2(s-1), \quad (209)$$

so that

$$k = 4(i-1) + 1 \quad (210)$$

With this relation between the subscripts i, k , and k established, it is clear that

$$m_{i,k} = m_u \quad (211)$$

$$U_{i,k} = \begin{cases} x_u = q^u & \text{when } u=1 \\ y_u = q^u & \text{when } u=2 \end{cases} \quad (212)$$

$$U_k^2 = \begin{cases} y_k & \text{when } u=1 \\ q^k & \text{when } u \geq 2 \end{cases} \quad (215)$$

$$U_k^3 = z_k = \text{constant} \quad (216)$$

From (171), $\sigma_{k,h}^r = \partial U_{k,h}^r / \partial q^h$ (215)

Making proper applications now results in the formulas

$$\sigma_{k,h}^1 = \partial x_k / \partial q^h = \begin{cases} 1 & \text{when } u=1 \\ 0 & \text{when } u \geq 2 \end{cases} \quad (217)$$

$$\sigma_{k,h}^2 = \partial y_k / \partial q^h = \begin{cases} 0 & \text{when } u=1 \\ 1 & \text{when } u \geq 2 \end{cases} \quad (218)$$

$$\sigma_{k,h}^3 = \partial z_k / \partial q^h = 0 \quad (219)$$

Summarizing (212) thru (219) results in

$$U_{k,h}^r = \begin{cases} q^k & \text{when } r=u \\ 0 & \text{when } r \neq u \end{cases} \quad (220)$$

$$= z_k \text{ when } r=3 \quad (221)$$

$$\sigma_{k,h}^{r,t} = \begin{cases} 1 & \text{when } t=u \\ 0 & \text{otherwise} \end{cases} \quad (222)$$

Substitution from (211), (219), (220), and (221) into (162) and letting the unknown $q^k = 0$ for this purpose results in

$$\Lambda_{k,h}^{r,t} = \begin{cases} m_k z_k & \text{when } r=3 \text{ and } t=u \\ 0 & \text{otherwise} \end{cases} \quad (223)$$

Similar substitution into (178) results in

$$S_{j,k}^{r,s} = \begin{cases} m_k & \text{when } r,s = u \\ & \text{and } j=k \\ 0 & \text{otherwise} \end{cases} \quad (224)$$

The reader is reminded that (222) and (223) are applicable only to rectangular tanks and to longitudinal sloshing in horizontal cylindrical tanks. For other tanks, the $\Lambda_{k,h}^{r,t}$ are assumed to be non-existent, and the $S_{j,k}^{r,s}$ are more or less circumvented by arriving at the $U_{j,k}$ (which equal $\sum_{r=1}^3 S_{j,k}^{r,s}$) by another process.

The formulation presented in Appendix I for lateral oscillations in horizontal cylindrical tanks, for vertical cylindrical tanks, and for spherical tanks lead to expressions for the kinetic and potential energies of fuel sloshing rather than a spring-mass mechanical analogy. Once these expressions for the kinetic energy are extended to account for the other motions of the vehicle as well as the sloshing of the fuel, they can be used as in (72) to obtain expressions for the contribution of the fuel to the M_{ik} .

To lateral sloshing in a horizontal cylindrical tank we affix the subscript i to T , M , ρ , and α and make the following substitutions in the final equation given in Appendix I for the kinetic energy

$$\left. \begin{aligned} \dot{q}_i^2 &= v_i'^2 \\ l &= l_{ni} \\ A_{s+1} &= A_{si} \\ B_{s+1} &= B_{si} \\ \left[\frac{v_i'^2}{\lambda_{s+1}} \right]^2 &= \lambda_{si} \end{aligned} \right\} \quad (204)$$

In addition to this, we introduce a transformation of coordinates. Let

$$p = 4(s-1) + u, \quad (225)$$

then (204) can be expressed

$$k = \dot{p} + 2(s-1) \quad (226)$$

and we let

$$q^{s+2} = \frac{R_{ni}}{\lambda_{si}} q^{p+2(s-1)} \quad (227)$$

This results in

$$\begin{aligned} T_i &= \frac{1}{2} N_{ni} (v_i'^2)^2 + \rho \alpha_i l_{ni} R_{ni}^2 \sum_{s=1}^{\infty} \frac{A_{si}}{\lambda_{si}} (\dot{q}^{p+2(s-1)})^2 \\ &\quad + 2 \rho_i \alpha_i^2 l_{ni} R_{ni}^2 v_i'^2 \sum_{s=1}^{\infty} B_{si} q^{p+2(s-1)} \end{aligned} \quad (228)$$

Differentiation of T_i results in

$$\begin{aligned} \frac{\partial T_i}{\partial q^{p+2(s-1)}} &= 2 \rho_i \alpha_i^2 l_{ni} R_{ni}^2 (A_{si}/\lambda_{si}) \dot{q}^{p+2(s-1)} \\ &\quad + 2 \rho_i \alpha_i^2 l_{ni} R_{ni}^2 v_i'^2 B_{si} \end{aligned} \quad (229)$$

$$\frac{\partial^2 T_1}{\partial q^2 \partial \alpha^2} = 2 \rho_1 \alpha_1 \dot{b}_{n1} R_{u1}^2 A_{s1} / \lambda_{s1} \quad (121)$$

$$\frac{\partial^2 T_1}{\partial \alpha \partial q^2} = 2 \rho_1 \alpha_1^2 \dot{b}_{n1} R_{u1}^2 B_1 \dot{h}'_{j1} h''_{j1} \quad (122)$$

j denoting a "structural" (that is, non-fuel signal) degree of freedom, and \dot{h}'_{j1} , h''_{j1} being demonstrably equal to $\frac{\partial}{\partial q} \dot{h}_{j1}$ by (62) (120), - recognition that here $\dot{\alpha}_{n1}$ and \dot{b}_{n1} equal $\dot{z} = \dot{c}$.

From (230), for the case in which j and k denote lateral sloshing in a horizontal cylindrical tank, we define

$$u_{jk} = 2 \rho_1 \alpha_1 \dot{b}_{n1} R_{u1}^2 A_{s1} / \lambda_{s1} \quad \text{when } j = k = j + \alpha (s-1) \\ = 0 \quad \text{when } j \neq k \quad (232)$$

From (231), for the case in which k denotes lateral sloshing in a horizontal cylindrical tank, we define

$$m'_{k1} = 2 \rho_1 \alpha_1^2 \dot{b}_{n1} R_{u1}^2 B_{s1} \quad (233)$$

For sloshing in a spherical tank, we affix the subscript s to T , M_s , ρ , α , and R and make the following substitutions in the final equation given in Appendix I for the kinetic energy:

$$\dot{q}^1 = v_1 r^1 \quad (r = 1, \dots, 2) \\ C_{s+3} = C_s \\ D_{s+3} = D_s \\ [\sqrt{\lambda_{s+3}}]^2 = \lambda'_{s1} \quad (234)$$

In addition to this, we use (225) and (226) again and introduce the following transformation of coordinates:

$$q^{s+3} = \frac{R_1}{\lambda'_{s1}} q^{s+\alpha(s-1)} \quad (235)$$

This results in

$$T_1 = \frac{1}{2} M_{r1} (v_1 r^1)^2 + \frac{1}{2} \pi \rho_1 \alpha_1^2 R_1^3 \sum_{s=1}^{s-1} \frac{C_{s1}}{\lambda'_{s1}} (q^{s+\alpha(s-1)})^2 \\ + \pi \rho_1 \alpha_1^2 R_1^3 v_1 r^1 \sum_{s=1}^{s-1} D_{s1} q^{s+\alpha(s-1)} \quad (236)$$

Differentiation of T_i results in

$$\frac{\partial T_i}{\partial \dot{q}^{j+p+2(s-1)}} = \pi \rho_i a_i^2 R_i^3 \frac{C_{si}}{\lambda'_{si}} \dot{q}^{j+p+2(s-1)} + \pi \rho_i a_i^3 R_i^3 U_i^T D_{si} \quad (237)$$

$$\frac{\partial^2 T_i}{\partial \dot{q}^{j+p+2(s-1)} \partial \dot{q}^{k+p+2(s-1)}} = \pi \rho_i a_i^2 R_i^3 C_{si} / \lambda'_{si} \quad (238)$$

$$\frac{\partial^2 T_i}{\partial \dot{q}^j \partial \dot{q}^{k+p+2(s-1)}} = \pi \rho_i a_i^3 R_i^3 D_{si} (l_{ri}^c h_{ji}^-) \quad (239)$$

j denoting a structural degree of freedom.

From (238), for the case in which j and k denote sloshing in a spherical tank, we define

$$\begin{aligned} u_{jk} &= \pi \rho_i a_i^2 R_i^3 C_{si} / \lambda'_{si} \\ &\quad \text{when } j = k = p + 2(s-1) \\ &= 0 \quad \text{when } j \neq k \end{aligned} \quad (240)$$

From (239), for the case in which k denotes sloshing in a spherical tank, we define

$$m'_k = \pi \rho_i a_i^3 R_i^3 D_{si} \quad (241)$$

Making use of the m'_k from either (233) or (241), we compute for a spherical tank or for lateral sloshing in a horizontal cylindrical tank

$$\phi_{jk} = l_{ri}^c (h_{ji}^t m'_k + h_{ki}^t m_j) \quad (242)$$

The equivalence of this to (231) or (239) should be noted.

For sloshing in a vertical cylindrical tank, we note from the given equation for T in Appendix I that

$$\frac{\partial^2 T}{\partial \dot{q}^i \partial \dot{q}^j} = m_{ij} \quad (243)$$

Now q^i and q^j need to be related to the coordinates for measurement of the structural deflections and the sloshing of the fluid in order to determine expressions for the M_{jk} . For this purpose, we employ Equation (2-16) from Reference (7). Putting this equation into the terms that are appropriate to the present purpose results in

$$M_{jk} = \sum_{p=1}^n \frac{\partial q^k}{\partial q^j} \frac{\partial q^j}{\partial q^p} \quad (244)$$

In determining the partial derivatives of (244), we distinguish between "structural" and "fuel slosh" degrees of freedom, as before. Thus, i.e. structural degrees of freedom (assuming $u=2$, that is, that the motion is in the (ξ, η) plane).

$$\left. \begin{aligned} \frac{\partial q^k}{\partial q^j} &= \delta_{jk}^r, \quad \frac{\partial q^k}{\partial q^j} = \alpha_{jk}^r, \\ \frac{\partial q^k}{\partial q^j} &= 1 \text{ when } j = k \text{ replaced by } j \\ \frac{\partial q^k}{\partial q^j} &= 0 \end{aligned} \right\} \quad (245)$$

For fuel sloshing degrees of freedom

$$\left. \begin{aligned} \frac{\partial q^k}{\partial q^{j+2(s-1)}} &= 0, \quad \frac{\partial q^k}{\partial q^{j+2(s-1)}} = 0 \\ \frac{\partial q^k}{\partial q^{j+2(s-1)}} &= 0, \\ \frac{\partial q^k}{\partial q^{j+2(s-1)}} &= 1 \text{ when } s+3 \text{ is replaced by } j+2(s-1) \end{aligned} \right\} \quad (246)$$

The following substitutions are also made:

$$\left. \begin{aligned} m_{11} &= M_{F1}, \quad m_{13} = u_{1k1}^r, \quad m_{1, s+3} = u_{1k}^r, \quad j+2(s-1), \\ m_{22} &= J_{F11}^r, \quad m_{23} = u_{2k1}^r, \quad m_{2, s+3} = u_{2k}^r, \quad j+2(s-1); \\ m_{33} &= u_{3k1}^r, \quad m_{2, s+3} = u_{2k}^r, \quad j+2(s-1); \\ m_{s+3, s+3} &= u_{s+3k}^r, \quad j+2(s-1) \end{aligned} \right\} \quad (247)$$

Finally, substitution from (245), (246), and (247) into (244) results in

$$\begin{aligned} M_{jk} &= M_{F1}^r \delta_{21, j1}^r \delta_{21}^r h_{k1}^r + u_{1k1}^r \delta_{21}^r h_{j1}^r \\ &\quad + J_{F11}^r \alpha_{j1}^r \alpha_{k1}^r + u_{2k1}^r \alpha_{j1}^r + u_{1k1}^r \delta_{21}^r h_{j1}^r \\ &\quad + u_{2k1}^r \alpha_{j1}^r + u_{1k1}^r \delta_{21}^r h_{j1}^r \end{aligned} \quad (248)$$

$$\begin{aligned} M_{j, j+2(s-1)} &= u_{1k}^r, \quad j+2(s-1) \delta_{21}^r h_{j1}^r + u_{2k}^r, \quad j+2(s-1) \alpha_{j1}^r \\ &\quad + u_{j1}^r, \quad j+2(s-1) = \phi_{j, k1} \end{aligned} \quad (249)$$

$$M_{j+2}(s) p_{j+2}(s) = u_{j+2}(s) p_{j+2}(s) + u_{j+1} \\ = 0 \text{ when } j \neq 1 \quad (250)$$

For all modes, in spherical and vertical cylindrical tanks, and lateral sloshing in horizontal tanks, there is no dynamic balancing.

2. POINT OF ROTATION AND DYNAMIC BALANCING OF CERTAIN TYPE FOR FLEXIBLE SECTIONS

Within every section is a point p (with sectional position vector \bar{p} and vehicle position vector \bar{x}_1) which may or may not coincide with the center of mass of the section. Since \bar{O}_1 is the vehicle position vector of the center of mass of section 1,

$$\bar{x}_1 = \bar{O}_1 + \bar{p}_1 \quad (251)$$

The point p_1 is fixed in the physical material of the section and moves with it when and if it moves. Thus, it may correspond with a particle of the section.

If the section is "movable", that is, has motion of type (2), then \bar{p}_1 is a point of rotation of the section - a point in the section that does not move relative to the vehicle but about which the section rotates in a type (2) motion. If the section rotates about a fixed axis, \bar{p}_1 lies somewhere on this axis.

As for sections of type (3), the unbalanced motion of \bar{p}_1 relative to the vehicle coordinates is given prior to that of any other point in the section. If the degree of freedom deforms the section, and if the section is movable, \bar{p}_1 does not move relative to the vehicle in that degree of freedom. On the other hand, if the section is "fixed", or if the degree of freedom does not deform the section, \bar{p}_1 may move relative to the vehicle coordinates in that degree of freedom.

From (130), (135), (136), and (22),

$$\sum_{k=1}^{P_1} m_{1,k} \bar{a}_{1,k} = \sum_{k=1}^{P_1} m_{1,k} \bar{f}_{1,k} = m_1 \bar{f}_{1,1} \quad (252)$$

$$\bar{f}_{1,1} = \frac{1}{m_1} \sum_{k=1}^{P_1} m_{1,k} \bar{a}_{1,k} \quad (253)$$

$$\bar{O}_{1,k} = \bar{O}_{1,k} - \bar{f}_{1,1} = \bar{O}_{1,k} + \bar{U}_{1,k} \quad (254)$$

Here the $\bar{O}_{1,k}$ and the $\bar{O}_{1,k}$ are given arbitrarily, and the $\bar{f}_{1,1}$ and $\bar{U}_{1,k}$ are determined from them. This is necessary for the satisfaction of (136) when the degree of freedom (k) deforms the section (1).

When $\bar{U}_{1,k} = \bar{p}_1$, then $\bar{U}_{1,1} = \bar{x}_1$, $\bar{a}_{1,k} = \bar{a}_{1,k}$, $\bar{O}_{1,k} = \bar{J}_{1,k} \frac{\partial \bar{f}_{1,1}}{\partial q_k} = \bar{e}_{1,k}$, and we introduce

$$\bar{x}_{1,k} = \bar{a}_{1,k} \cdot \bar{e}_{1,k} = \bar{f}_{1,1} + \bar{B}_{1,k} \cdot \bar{q}_k \quad (255)$$

When the degree of freedom (n) does not deform the section (1), then $\bar{\sigma}_{n,h} = 0$, $\bar{\rho}_{n,h} = 0$, (136) is still satisfied, and $\bar{x}_{n,h} = \bar{a}_{n,h}$.

When the section is movable and is deformed by the degree of freedom, determine the $\alpha'_{n,h}$ and the $\beta'_{n,h}$. (The $\beta'_{n,h}$ are arbitrary.) Then compute and submit

$$f'_{n,h} = \bar{m}_h \sum_{k=1}^n m_k \alpha'_{n,h,k} \quad (256)$$

$$\sigma'_{n,h} = \alpha'_{n,h} - f'_{n,h} - C_{n,h} \beta'_{n,h} \quad (257)$$

The $\alpha'_{n,h}$ in this case. This fact effects the location of $\bar{a}_{n,h}$, the point of rotation. If the section is fixed and deformed, $\bar{a}_{n,h}$ be arbitrarily located and the $\alpha'_{n,h}$ are unimportant.

When the section is movable and is deflected but not deformed by the degree of freedom, determine and submit $\bar{x}'_{n,h}$ and $\beta'_{n,h}$ (without primes).

The symbols, data to be submitted, computations and equations used by the SLP are given in Appendix II.

13. REFERENCES

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APPENDIX I

BASIC FORMULATIONS FOR FUEL SLOSHING

INTRODUCTION

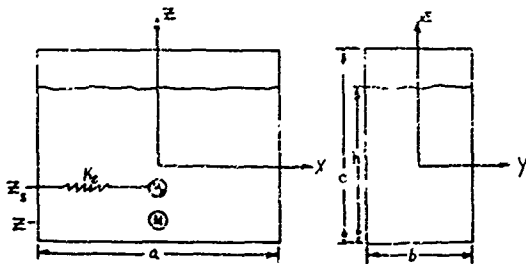
For sufficiently small amplitudes of motion, the dynamic effects of the sloshing of fuel in a partially filled tank have been analyzed in terms of the natural modes and frequencies of the small, free-surface oscillations of the fuel. Formulations for the solution of this problem are presented for rectangular, cylindrical, and spherical tanks. In Part I of this Appendix, natural modes and frequencies are presented for rectangular and cylindrical tanks in either a vertical or a horizontal position. In Part II, an approximate procedure is established for dealing with rectangular and cylindrical tanks that are neither vertical nor horizontal. Part III has to do with the treatment of the bending mode shape of an equivalent vertical tank. Part IV of this Appendix concerns the inclusion of fuel damping in the SLP.

PART I

The formulas presented in this part for rectangular and cylindrical tanks apply only to vertical or horizontal positions. Those presented for spherical tanks do not need to be qualified as to position.

In order to simplify the problem to a point where convenient, explicit solutions could be obtained in most cases, a number of assumptions were made concerning the nature of the fuel, the motions of the fuel and the shape of the tank. The fuel was assumed to be non-viscous and incompressible and all tank motions, except those normal to the mean free surface of the fuel, were restricted to small accelerations and perturbations. Although the non-viscous assumption has been made, a damping factor will be included in the final SLP Structural Loads Program to account for the fuel viscosity and the use of baffles. It should also be noted that, in the final program, provisions are made for assuming any combination of rectangular, cylindrical or spherical tanks for multiple tank vehicles.

1. Rectangular tank - In the case of a rectangular tank, a spring mass mechanical model is used to describe the motions below are for a system consisting of a fixed mass M and a set of undamped spring-masses m_s , so constrained as to move only parallel to the bottom of the tank and, in the case of the horizontal rectangular tank, parallel to the XZ -plane. The origin of the axes is located at the center of gravity of the undisturbed fuel with the fixed mass M located Z_s and the spring-masses at Z_s constrained by springs with stiffness K_s for the s th mode. Shown below is the horizontal rectangular tank undergoing longitudinal oscillations, or oscillations in the XZ -plane. The motion in the XZ -plane is assumed to be the same regardless of the Y location. The following equations have been developed by Graben in Reference (1).



Definitions:

- a = tank length parallel to X-axis
- b = tank width parallel to Y-axis
- c = tank height parallel to Z-axis
- h = fuel height parallel to vertical axis
- $\rho a b h$ = total fuel mass
- s = fuel node index = 1, 2, 3, ... ∞ . ∞ denotes the number of nodes selected for use.
- ρ = fuel density
- g = acceleration of tank normal to menul free surface of fuel
- w_F = $g h$ = total fuel weight
- r_1 = h/b = tank aspect ratio
- I_{xy} = moment of inertia about Y-axis if the fuel were solidified
- I_{fy} = effective moment of inertia about the Y-axis
- ω_s = frequency of the s^{th} mode of free surface oscillation.

Equations:

$$\omega_s = \left[g(2s-1) \frac{\pi}{2} \text{TANH}[(2s-1)\pi r_1] \right]^{1/2}$$

$$m_s = M_F \frac{8 \text{TANH}^2[(2s-1)\pi r_1]}{\pi^2 (2s-1)^2 r_1}$$

$$K_s = \frac{8 W_F \text{TANH}^2[(2s-1)\pi r_1]}{h \pi^2 (2s-1)^2}$$

$$M = M_F - M_r \sum_{s=1}^{\infty} \frac{8 \text{TANH}[(2s-1)\pi r_1]}{\pi^3 (2s-1)^3 r_1}$$

$$Z_c = \frac{D}{2} - \frac{8 \text{TANH}[(2s-1)\pi] L_1}{(2s-1)^3 \frac{\pi}{2} r_1}$$

$$\bar{Z} = -\frac{1}{M} \sum_{s=1}^{\infty} m_s \bar{Z}_s$$

$$I_{xx} = I_{sy} \left\{ 1 - \frac{4}{1+r_1^2} + \frac{768}{r_1(1+r_1^2)\pi^3} \sum_{s=1}^{\infty} \frac{\text{TANH}[(2s-1)\pi] L_1^2}{(2s-1)^5} \right\}$$

$$I_{sy} = M_F \frac{a^2 + h^2}{12}$$

In the case of a horizontal rectangular tank undergoing lateral oscillations in the XZ -plane the definitions and equations are unchanged except that the moments of inertia are now I_{rx} and I_{sx} and the tank aspect ratio now becomes $r_2 = b/a$ with

$$\omega_s = [g(2s-1) \frac{\pi}{b} \text{TANH}[(2s-1)\pi r_2]]^{1/2} \text{ and } I_{sx} = \frac{b^2 + h^2}{12} M_F$$

If the tank is now rotated so that the X -axis is vertical with the fuel oscillating parallel to the XZ -plane, the value of $\omega_s = gcbh$ and the tank aspect ratio becomes $r_3 = b/c$. The moments of inertia are still taken about the y -axis and the equations for Z_s and Z are the same except they become Z_s and Z distances and use r_3 instead of r_2 . The equations for ω_s and I_{sy} become:

$$\omega_s = [g(2s-1) \frac{\pi}{c} \text{TANH}[(2s-1)\pi r_3]]^{1/2}$$

$$I_{sy} = M_F \frac{c^2 + h^2}{12}$$

With the X -axis vertical but with the oscillations parallel to the XY -plane, the tank aspect ratio $r_4 = b/b$, the moments of inertia are I_{yz} and I_{zz} and the equations for ω_s and I_{zz} become:

$$\omega_s = [g(2s-1) \frac{\pi}{b} \text{TANH}[(2s-1)\pi r_4]]^{1/2}$$

$$I_{zz} = M_F \frac{b^2 + h^2}{12}$$

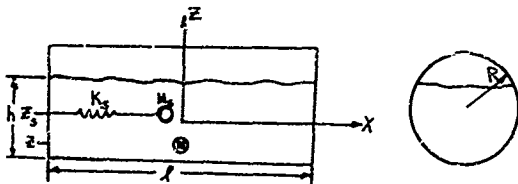
This completes the specification of the equations for the spring-mass analogy for rectangular tanks in a horizontal or vertical orientation. Therefore, the angle the X -axis makes with the horizontal determines which set of equations more accurately approximates the situation.

2. Cylindrical Tank - The formulations for the cylindrical tank are not nearly as straight forward as were those for the rectangular tanks. Three different methods have been used to define the fuel motion for the different

tank orientations. An extensive literature survey indicated that for the case of longitudinal oscillations in a horizontal cylindrical tank the same approach could be used as in the case of rectangular tanks although no development of the equations could be found. Reference (2) suggested that the natural frequencies for the horizontal cylindrical tank are:

$$\omega_s = \left[\frac{g \pi^2}{\lambda} \text{TANH} \left(\frac{s \pi h}{\lambda} \right) \right]^{1/2} \quad \text{where } s = 1, 2, 3, \dots \omega_s$$

Comparing this equation with the corresponding frequency equation for rectangular tanks indicates that the cylindrical tank aspect ratio is $r = h/\lambda$. Making like comparisons the following development is suggested:



Definitions:

- l = tank length parallel to X-axis
- R = tank radius
- h = fuel height parallel to vertical axis
- M_s = total fuel mass
- s = 1, 2, 3, ... etc.
- r = h/λ = tank aspect ratio

Equations:

$$\omega_s = \left[\frac{g \pi^2}{\lambda} \text{TANH} (s \pi r) \right]^{1/2}$$

$$M_s = M_F \frac{R \text{TANH} (s \pi r)}{\pi^2 s^2 r}$$

$$K_s = \frac{g M_s \text{TANH}^2 (s \pi r)}{h \pi^2 s^2 r}$$

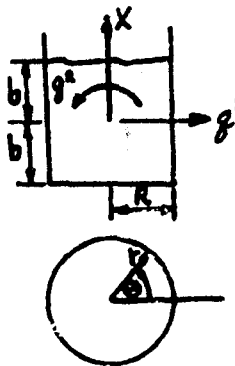
$$M = M_F - M_F \int_0^{\pi} \frac{R \text{TANH} (s \pi r)}{\pi^2 s^2 r}$$

$$Z_s = \frac{h}{2} - \frac{h \text{TANH}(\frac{1}{2} S \pi r)}{\frac{1}{2} S \pi r}$$

$$\bar{Z} = -\frac{1}{M} \sum_{s=1}^{\infty} m_s Z_s$$

$$I_{FY} = I_{cy} \left\{ 1 - \frac{4}{1+r^2} + \frac{768}{r(1+r^2)^3} \pi^6 \sum_{s=1}^{\infty} \frac{\text{TANH}(\frac{1}{2} S \pi r)}{S^3} \right\}$$

The second method to be used on the cylindrical tank follows the formulations of J. W. Miles found in Reference (3) for an upright circular cylinder. In this analysis the potential and kinetic energy expressions are derived with allowances made for tank flexibility. First the potential energy (U) and the kinetic energy (T) expressions are stated and then the potential energy coefficients (k_{ij}) and the inertia coefficients (u_{ij}) are defined.



Definitions:

- $i, j = s = 1, 2, 3, \dots, \infty$
- $q^i(t)$ = generalized coordinates
- $q_1(t)$ = a translation along $\theta = 0$
- $q_2(t)$ = a rotation about the centroidal axis $\theta = \frac{\pi}{2}$
- $q_3(t) f(x)$ = a simple bending displacement along $\theta = 0$
- $q_{s+3} \psi_{s+3}(r, \theta)$ = sloshing displacements
- $f(x)$ = bending mode shape of tank
- $f'(x) = df(x)/dx$
- $\psi_{s+3}(r, \theta)$ = s^{th} mode shape of fuel
- $M = 2\pi \rho R^2 b$ = total mass of fuel

Definitions (Continued)

- s = index indicating fuel slosh modes
 - b = one half fluid h. height
 - r = tank radius
 - g = acceleration of tank along X-axis
 - β_s = sth zero of the first derivative of the Bessel Function of the first order and the first kind.
- ($\beta_1 = 1.84119, \beta_2 = 5.33144, \beta_3 = 8.53631, \beta_4 = 11.79600$)

$$c_{ij}^1 = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Equations

$$U = \frac{1}{2} \sum_i \sum_j k_{ij} g^i g^j$$

Reference (3) defines the potential energy coefficients, k_{ij} as shown below:

$$k_{11} = k_{12} = k_{21} = k_{13} = k_{31} = k_{22} = k_{23} = k_{32} = k_{11,5+3}$$

$$= k_{5+3,1} = 0$$

$$k_{33} = Mg \left\{ \frac{b^2}{2\beta_3^2} [f''(b) - f''(-b)] + \frac{1}{2\pi} \int_b^b x f''(x) dx \right. \\ \left. - \frac{1}{2} \int_{x_0}^b f(x) dx + \frac{1}{2} \int_{-b}^{x_0} f''(x) dx \right\}$$

$$k_{5+3,5+3} = \frac{1}{4} \frac{Mg^2}{b\beta_3^2} (\beta_3^2 - 1)$$

$$k_{2,5+3} = k_{5+3,2} = \frac{1}{2} \frac{Mg^2}{b\beta_3^2}$$

$$k_{3,3+3} = k_{3+3,3} = -\frac{1}{2} \frac{Mg^2}{b\beta_3^2} f'(b)$$

$$T = \frac{1}{2} \sum_i \sum_j m_{ij} \dot{g}^i \dot{g}^j$$

where the coefficients, $m_{s,3}$, are defined as follows.

$$m_{11} = M$$

$$m_{12} = m_{13} = 0$$

$$m_{13} = m_{21} = M \left\{ \bar{r}_0 - \frac{R^2}{\beta b} [f'(a) - f'(-b)] \right\}$$

$$m_{1,5+3} = m_{5+3,1} = \frac{MR}{2\beta b^2}$$

$$m_{2,2} = M \left(\frac{b^2}{3} - \frac{3R^2}{4} \right) + \frac{MR^2}{\beta} \sum_{s=1}^{\infty} \frac{\text{TANH} \left(\frac{2s b}{\beta} \right)}{\beta_s^2 (\beta_s^2 - 1)}$$

$$m_{23} = m_{3,2} = +M \left\{ \frac{1}{2b} \int_{-b}^b x f(x) dx + \frac{Rb}{17^2} \sum_{s=1}^{\infty} \frac{\Psi_{2s-1} F_{2s-1}}{(2s-1)^2} \right. \\ \left. + [f'(b) + f'(-b)] \left[\frac{b^2}{3} - \frac{R^2}{\beta} + \frac{2Rb^2}{17^2} \sum_{s=1}^{\infty} \frac{\Psi_{2s-1}}{(2s-1)^2} \right] \right\}$$

$$m_{2,5+3} = m_{5+3,2} = \frac{MR}{2\beta b^2} \left[\frac{2R \text{TANH} \left(\frac{\beta b}{R} \right)}{\beta_s b} - 1 \right]$$

$$m_{3,3} = MF_0^2 + \frac{M}{2} \sum_{s=1}^{\infty} \Psi_s F_s^2 - \frac{MR^2}{b} \sum_{s=1}^{\infty} \frac{f'(b) Y_s(b) - f'(-b) Y_s(-b)}{\beta_s^2 - 1} \\ + \frac{F_0 MR^2}{\beta b} [f'(-b) - f'(b)] + \frac{2f'_0}{17^2} \sum_{s=1}^{\infty} \frac{1 - \Psi_{2s}}{5^{2s}} [(-)^{s+1} f'(b) - f'(-b)] F_s \\ + \frac{1}{b} \sum_{s=1}^{\infty} \frac{\dots 2Ab}{\beta_s^2 (\beta_s^2 - 1)} \left\{ [f^2(b) + f'^2(-b)] \cosh \left(\frac{2\beta_s b}{R} \right) \right. \\ \left. - 2f'(b)f'(-b) \right\}$$

$$m_{3,5+3} = m_{5+3,3} = \frac{MR}{2\beta} Y_5(b) - \frac{MR^2}{2\beta b^2} \left[\text{Csch} \left(\frac{2\beta b}{R} \right) \right] [f(b) \cosh \left(\frac{2\beta b}{R} \right) - f'(-b)]$$

$$m_{s,s+3} = \frac{MR}{2bA_s^2} (\beta_s^2 - 1) \coth \left(\frac{2\beta_s b}{R} \right)$$

$$\text{where } A_s' = \frac{S\pi R}{2b}$$

$$\Psi(0) = 1.0$$

$$\Psi_s = \Psi(A_s') = 1 - \left[\frac{I_1(A_s')}{I_1(A_s')} \right]$$

I_1 and I_2 are certain Bessel functions or certain forms of functions of some related kind.

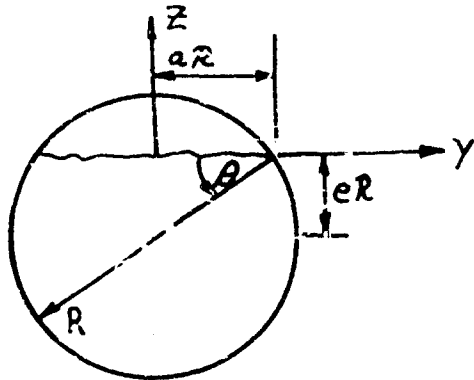
$$F_s = \left(\frac{R}{2b} \right) \int_{-b}^b f(y) \cos \left[\sin \left(\frac{y+b}{2b} \right) \right] dy \quad \text{when } S > 0.$$

$$F_0 = \frac{1}{2b} \int_{-b}^b f(x) dx$$

$$Y_s(b) = \int_{\pi/2}^{\infty} (-1)^p \left[\beta_s^2 + \left(\frac{S\pi R}{2b} \right)^2 \right]^{-1} F_p$$

These equations have been used in the experimental analysis of Reference (4). In this report, resonant bending frequencies and mode shapes were determined experimentally and were shown to be generally in agreement with the theoretical predictions. The differences were attributed primarily to variations of actual mode shapes from those assumed in the theory.

The third method to be used on cylindrical tanks was formulated by B. Duzinsky in Reference (5). This is also the method to be used for spherical tanks. In this report an integrodifferential approach is used and the method of solution developed for the first three fuel slosh modes, which as indicated in the literature is a sufficient number of modes for most practical problems. The tank orientation under consideration is a horizontal cylindrical tank undergoing lateral oscillations. In this case the generalized coordinate denoting motion along the Y-axis is q and again the fuel sloshing generalized coordinates are q^{s+3} . The dimensions of the q^{s+3} in this development, however, are (length)² rather than length as in the case of the previous generalized coordinates. The slosh height, \mathcal{P}^{s+3} , at the side of the tank can be expressed as a function of the q^{s+3} by the following relation, $\mathcal{P}^{s+3} = q^{s+3} \omega^{s+2} \epsilon$. It should be noted that in this analysis tank bending is ignored and that with the non-viscous assumption rotation of this cylindrical tank and rotation of the spherical tank need not be considered.



Definitions:

- q^1 = a translation along the Y-axis
- R = tank radius
- e = fuel height parameter = -1.0 for empty tank
= 0 for half full tank
= +1.0 for full tank
- θ = $\sin^{-1} e$
- a = $\cos \theta$
- λ_{s+3} = frequency parameter = $\frac{R}{C} \omega^{2s+3}$
- s = fuel slosh mode index = 1, 2, 3, ... ∞
- l = tank length
- ρ = fuel density
- \ddot{C} = tank acceleration along X-axis
- M_F = total mass of fuel

Equations:

$$U = \frac{\rho a R l}{g} \sum_{s=1}^{\infty} \omega_{s+3}^4 A_{s+3} (q^{s+3})^2$$

$$U = \frac{\rho a g l}{R} \sum_{s=1}^{\infty} [\sqrt{\lambda_{s+3}}]^4 A_{s+3} (q^{s+3})^2$$

$$T = \frac{1}{2} M_F (\dot{q}^1)^2 + \frac{\rho a R l}{g} \sum_{s=1}^{\infty} \omega_{s+3}^2 A_{s+3} (\dot{q}^{s+3})^2 + \frac{2 \rho l}{g} (a R)^2 \dot{q}^1 \sum_{s=1}^{\infty} \omega_{s+3}^2 B_{s+3} \dot{q}^{s+3}$$

$$T = \frac{1}{2} M_F (\dot{g}')^2 + \rho a \times \sum_{s=1}^{\infty} [\sqrt{\lambda_{s+3}}]^2 A_{s+3} (\dot{g}^{s+3})^2 \\ + 2 \rho a^2 R \dot{g}' \sum_{s=1}^{\infty} [\sqrt{\lambda_{s+3}}]^2 \bar{D}_{s+3} \dot{g}^{s+3}$$

The non-dimensional modal parameters A_{s+3} and \bar{D}_{s+3} along with $\sqrt{\lambda_{s+3}}$ are presented in Figures (4), (5), and (6) respectively for the first three odd slash modes as functions of the full height parameter e . It should be noted that for values of e near 1.0 to $e = 1.0$, the curves in Figures (6) and (9) tend to infinity as they approach $e = 1.0$. The above type of solution near $e = 1.0$, the curves have been used to intersect $e = 1.0$ to provide for an approximate but finite solution for the full tank. For this reason the solution of the equations for the nearly full to full cylindrical and spherical tanks must be used with caution.

3. Spherical Tanks - Following the same method and definitions as used above, the solutions for the spherical tank may be obtained.

$$\text{Equations } U = \frac{\pi \rho l}{2g} (aR)^2 \sum_{s=1}^{\infty} \omega_{s+3}^2 C_{s+3} (g^{s+3})^2 \\ U = \frac{1}{2} \pi \rho g a^2 \sum_{s=1}^{\infty} [\sqrt{\lambda_{s+3}}]^4 C_{s+3} (g^{s+3})^2 \\ T = \frac{1}{2} M_F (\dot{g}')^2 + \frac{\pi \rho l}{2g} (aR)^2 \sum_{s=1}^{\infty} \omega_{s+3}^2 C_{s+3} (\dot{g}^{s+3})^2 \\ + \frac{\pi \rho l}{g} (aR)^2 \dot{g}' \sum_{s=1}^{\infty} \omega_{s+3}^2 D_{s+3} \dot{g}^{s+3} \\ T = \frac{1}{2} M_F (\dot{g}')^2 + \frac{1}{2} \pi \rho a^2 R \sum_{s=1}^{\infty} [\sqrt{\lambda_{s+3}}]^2 C_{s+3} (\dot{g}^{s+3})^2 \\ + \pi \rho a^2 R^2 \dot{g}' \sum_{s=1}^{\infty} [\sqrt{\lambda_{s+3}}]^2 D_{s+3} \dot{g}^{s+3}$$

As before, the values of the non-dimensional modal parameters C_{s+3} and D_{s+3} along with $\sqrt{\lambda_{s+3}}$ are plotted versus e in Figures (7), (8), and (9) respectively. As discussed previously, the solutions of the equations are only approximate solutions as the full condition is approached.

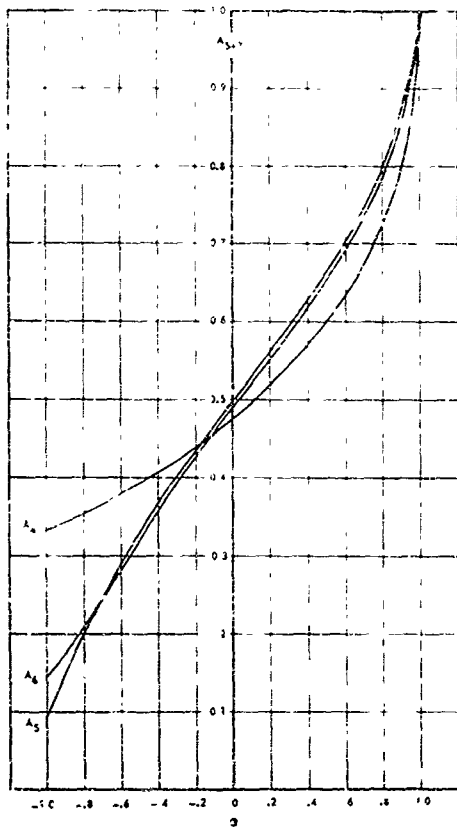


Figure 4. Variation of $A_{\xi_{42}}$ with Fuel Height Parameter, ξ ,
Cylindrical Tank

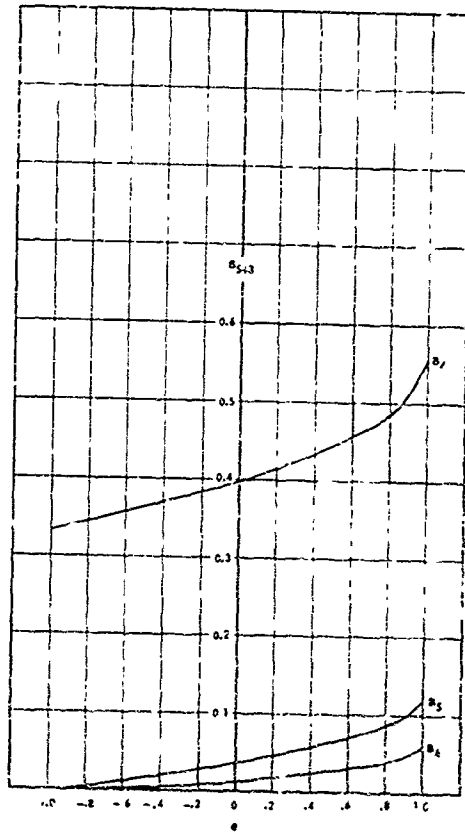


Figure 5 Variation of B_{S+3} with Fuel Height Parameter, α , Cylindrical Tank

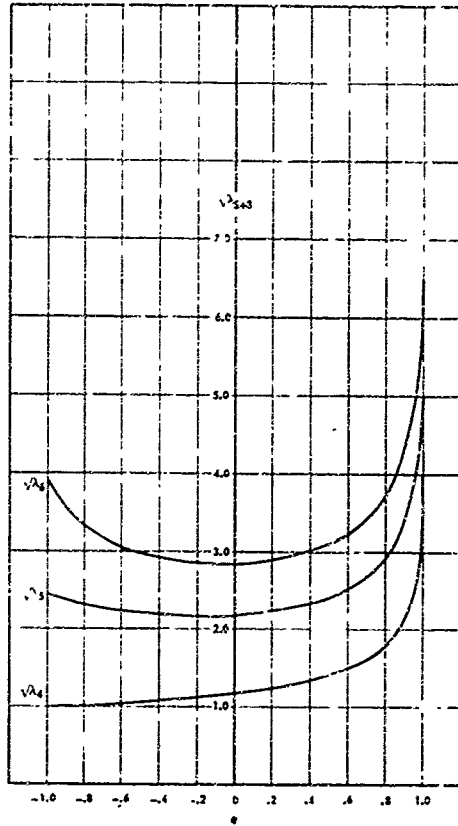


Figure 6. Variation of Cylindrical Frequency Parameter with Fuel Height Parameter

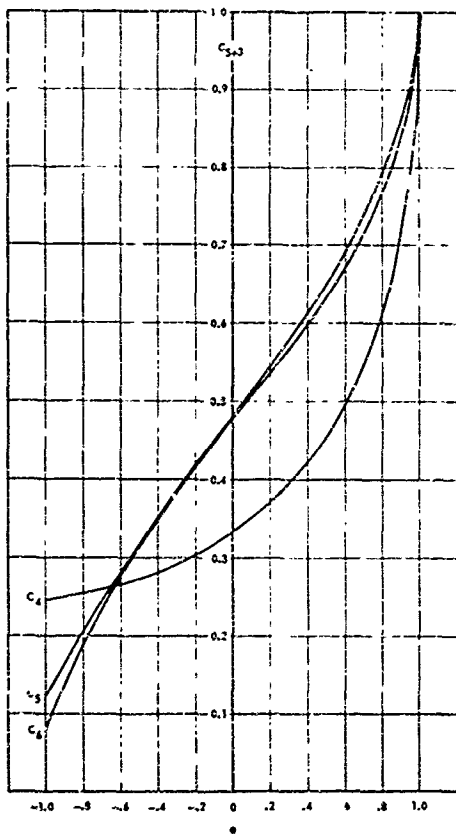


Figure 7. Variation of C_{S+3} with Fuel Height Parameter, e , Spherical Tank

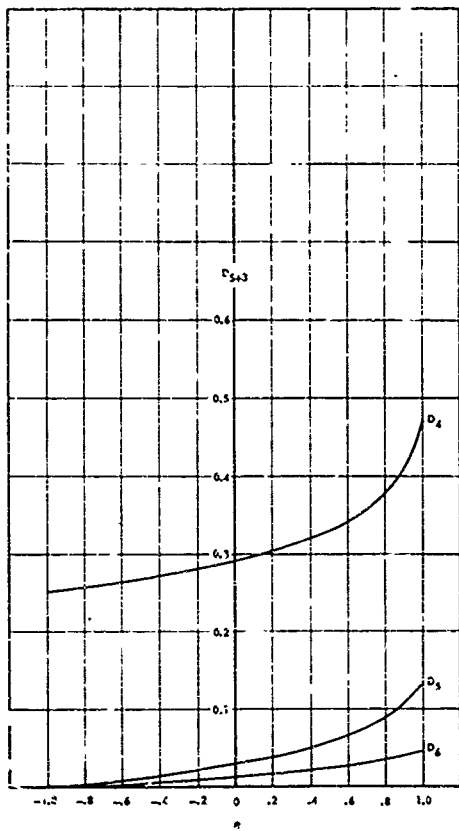


Figure 8. Variation of D_{3+3} with Fuel Height Parameter, e ,
Spherical Tank

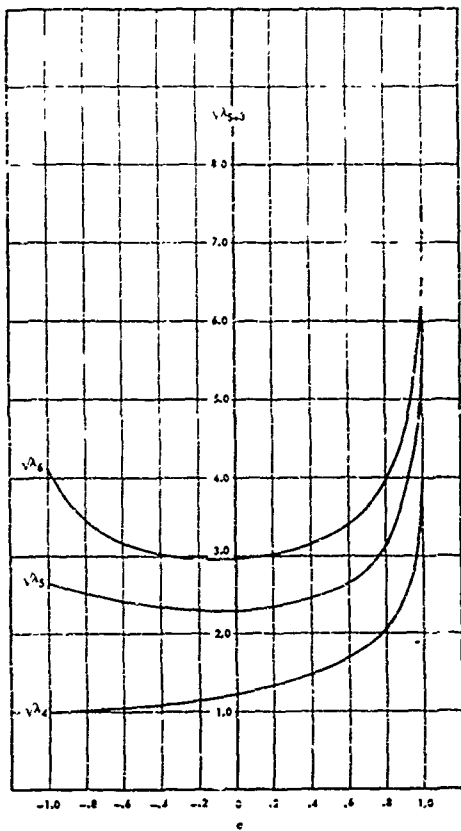


Figure 9. Variation of Spherical Frequency Parameter with Fuel Height Parameter

* 1968 V. K. S. Chandra, Mysore, India

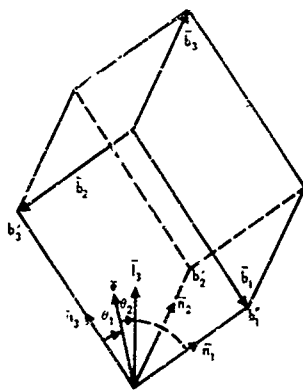
PART II

The computations called for in this part are carried out in the Vehicle Physical Characteristics Subprogram (VPSS), which runs with the basic SIF program rather than the SLP.

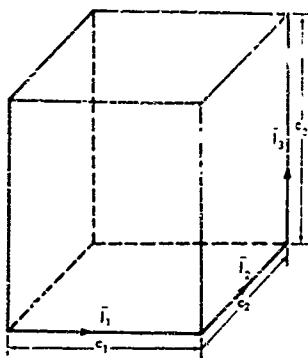
The free surface of the fuel is parallel to the horizon only when the tank has no lateral or longitudinal acceleration. It is anticipated, however, that such will not usually be the case. The assumption now being made concerning the fuel orientation is that the free surface is always perpendicular to the "resultant tank acceleration," defined as the actual acceleration at the tank center due to the gross motion of the vehicle plus the force per unit mass due to gravity. The pertinent angles for the tank orientation, therefore, are not angles usually defined as the tank or vehicle pitch, roll, and yaw angles; they are the angles between the body axis system and the resultant acceleration. This means that at any instant of time the resultant tank acceleration must first be found and then the free surface of the fuel set perpendicular to it. Since the fuel slosh equations presented in Part I are valid only vertical or horizontal axes, the tank walls must be set perpendicular or parallel to the free surface. As this is done, the real tank dimensions in the body axis system are replaced by those of a different but "equivalent" tank of the same volume. This equivalent tank is, therefore, a tank whose dimensions and orientation are a function of the angles the real tank makes with the resultant tank acceleration. As the real tank for example pitches from 0° to 90° , the equivalent tank concept provides a continuous transition to classify the tank as being either vertical or horizontal. The tank geometrical corner was chosen as being common to both the real tank and the equivalent tank.

The equivalent tank concept is by no means an exact representation but does give an approximation of the real situation. One very significant parameter to fuel sloshing is the length of the free surface. The equivalent tank concept permits the free surface length to increase or decrease as it does in the real situation, but only approximates the actual free surface length. This concept also simplifies the computation of the moments and products of inertia and the C.G. of the fuel, as the fuel changes its gross position in the tank due to the gross motion of the vehicle.

Consider now the problem of obtaining the equivalent rectangular tank dimensions and then the moments of inertia and C.G. of the fuel in the equivalent tank as if the fuel were solidified. As shown in Figure 10, λ_3 is the unit



REAL TANK



EQUIVALENT TANK

Figure 10. Real and Equivalent Rectangular Tanks

acceleration vector of the tank. One corner of the real tank is chosen as the origin of a right handed triad having unit vectors \bar{n}_1 , \bar{n}_2 and \bar{n}_3 pointing along adjacent edges of the tank and chosen so that the angle between \bar{l}_3 and \bar{n}_1 is not less than the angle between \bar{l}_3 and either \bar{n}_2 or \bar{n}_3 . The vertex for the common origin of these vectors is chosen so that these angles are not greater than $\pi/2$. The unit vector \bar{e} is defined as a unit vector perpendicular to \bar{n}_1 , lying in the plane of \bar{l}_3 and \bar{n}_1 , and making an acute angle with \bar{l}_3 . The vectors \bar{l}_1 , \bar{l}_2 and \bar{l}_3 are the vectors defining the real tank size and orientation. The magnitudes of these vectors (not necessarily respectively) l_1 , l_2 and l_3 are the lengths of the sides of the tank in the direction of the unit vectors \bar{n}_1 , \bar{n}_2 and \bar{n}_3 respectively, as shown in Figure 10. The unit vector \bar{e} and the angles θ_1 and θ_2 can be defined as:

$$\bar{e} = \frac{\bar{l}_3 - (\bar{l}_3 \cdot \bar{n}_1) \bar{n}_1}{\sqrt{1 - (\bar{l}_3 \cdot \bar{n}_1)^2}}$$

$$\theta_2 = \text{ARC COS} (\bar{l}_3 \cdot \bar{e})$$

$$= \text{ARC COS} \sqrt{1 - (\bar{l}_3 \cdot \bar{n}_1)^2}$$

$$\theta_1 = \text{ARC COS} (\bar{n}_3 \cdot \bar{e})$$

$$= \text{ARC COS} [\bar{l}_3 \cdot \bar{n}_3 \text{ SEC } \theta_2]$$

where θ_1 and θ_2 must be positive acute angles.

The dimensions of the equivalent rectangular tank can now be obtained. Referring to Figure 10, the equivalent tank may be thought of as the tank obtained by taking the real tank, with \bar{l}_3 coincident with one of its edges, and then adjusting the real tank dimensions to the equivalent tank dimensions as the tank is rotated first through θ_1 , and then through θ_2 . This then replaces the real tank, which is actually in the position described by θ_1 and θ_2 but with its sides not perpendicular to the free surface, by an equivalent tank with the same volume and approximately the same free surface length with its sides perpendicular to the free surface. Defined below are the equivalent tank dimensions C_1 , C_2 , and C_3 in terms of the real tank dimensions l_1 , l_2 and l_3 . The intermediate tank dimension C' is defined as the length of C_3 after the tank has been rotated through θ_1 , but not through θ_2 .

$$C' = \sqrt{L_1 L_2} \tan[\theta_1 + (1 - \frac{g_0}{g}) \text{ARC TAN } \sqrt{L_1/L_2}]$$

$$C_1 = \sqrt{L_1 C'} \cot[\theta_2 + (1 - \frac{g_0}{g}) \text{ARCTAN } \sqrt{C'/L_1}]$$

$$C_2 = \sqrt{L_2 L_1} \cot[\theta_1 + (1 - \frac{g_0}{g}) \text{ARC TAN } \sqrt{L_2/L_1}]$$

$$C_3 = \sqrt{L_1 C'} \tan[\theta_2 + (1 - \frac{g_0}{g}) \text{ARC TAN } \sqrt{C'/L_1}]$$

The unit vectors giving the direction of \bar{r}_1 and \bar{r}_2 are.

$$\bar{r}_1 = \frac{\bar{m}_1 - (\bar{r}_2 \cdot \bar{m}_1) \bar{r}_2}{\sqrt{1 - (\bar{r}_2 \cdot \bar{m}_1)^2}}$$

$$\bar{r}_2 = \frac{\bar{r}_2 \times \bar{m}_1}{\sqrt{1 - (\bar{r}_2 \cdot \bar{m}_1)^2}}$$

Equations for the fuel moments of inertia, J_1 , J_2 and J_3 , as if the fuel were solidified, taken about the fuel C.G., and the C.G. location \bar{z}_F of the fuel, measured from the equivalent tank center along the \bar{r}_3 axis are shown below. The total mass of fuel in the tank is M_F and the fuel height along \bar{r}_3 is h .

$$J_1 = \frac{1}{12} M_F [(c_2)^2 + h^2]$$

$$J_2 = \frac{1}{12} M_F [(c_1)^2 + h^2]$$

$$J_3 = \frac{1}{12} M_F [(c_1)^2 + (c_2)^2]$$

$$\bar{z}_F = -\frac{1}{2} \bar{r}_3 (c_2 - h)$$

$$h = M_F / \rho C_2$$

An approach similar to that used for the rectangular tank is presented for the cylindrical tank. There are two major differences between the equivalent rectangular and cylindrical tanks. The cross section of the equivalent tank, taken perpendicular to the resultant acceleration, is always rectangular for the equivalent rectangular tank. This cross section for the equivalent cylindrical tank may be rectangular or circular depending on the angle between the

resultant acceleration and the resultant length vector. If this cross section is circular the equivalent tank is considered as being vertical, and if rectangular, the equivalent tank is considered as being horizontal. The second difference between the equivalent rectangular and cylindrical tank concerns the angles used to specify their orientation. For the equivalent rectangular tank two angles were needed. For the equivalent cylindrical tank, the angle between the resultant acceleration and the tank length vector \vec{L} is the only angle needed to specify the tank orientation.

As shown in Figure 11, \vec{L} is the cylindrical tank length vector, \vec{L}_3 is the resultant acceleration unit vector and θ is the angle between them. Defining \vec{L}_1 as the tank length unit vector then

$$\vec{L} = \vec{L}_1 / h$$

$$u = \vec{L}_3 \cdot \vec{L}$$

$$\theta = \text{ARCCOS } |u|$$

If $|u|$ is greater than $1/\sqrt{2}$ the equivalent tank is vertical. Defining L and R as the real tank length and radius respectively, and L_v and R_v as the equivalent tank length and radius respectively, then the relation between them is:

$$L_v = \sqrt{2LR} \text{ TAN} [\theta + (1 - \frac{\theta}{\pi}) \text{ARC TAN } \sqrt{L/2R}]$$

$$R_v = \sqrt{\pi L / L_v}$$

$$h = M_F / \pi \rho R_v^2$$

Equations for the moments of inertia of the fuel, as if the fuel were solidified, taken about the fuel C.G., J_1 , J_2 & J_3 , and the C.G. location \bar{Z}_F of the fuel measured along L_3 from the tank center are shown below.

$$J_1 = J_2 = \frac{1}{12} \pi \rho R_v^4 h (3R_v^2 + h^2)$$

$$J_3 = \frac{1}{2} \pi \rho R_v^4 h$$

$$\bar{Z}_F = -\frac{1}{2} \bar{L}_3 (L_v - h)$$

If $u^2 = 1.0$, the components of the unit vectors \vec{L}_1 and \vec{L}_2 giving the new directions are:

$$L_1^1 = -\frac{L_3^1 L_3^2}{\sqrt{1 - (L_3^1)^2}} \quad L_1^2 = \sqrt{1 - (L_3^1)^2} \quad L_1^3 = -\frac{L_3^1 L_3^3}{\sqrt{1 - (L_3^1)^2}}$$

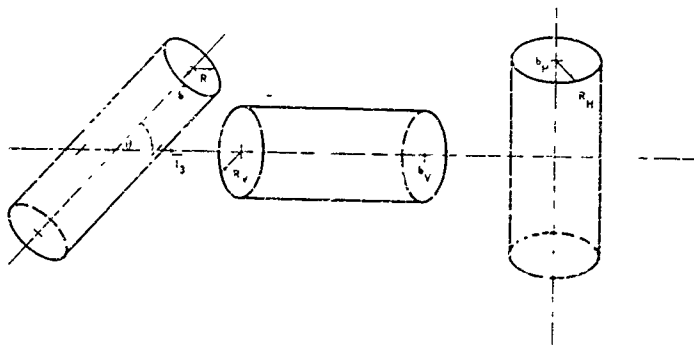


Figure 11. Resultant Acceleration and Cylindrical Tank

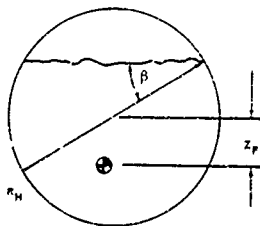


Figure 12. Horizontal Cylindrical Tank and Spherical Tank

$$l_2 = -\frac{l_3^2}{\sqrt{1-(l_3^2)^2}} \quad l_2^2 = 0 \quad l_3 = \frac{l_3^1}{\sqrt{1-(l_3^1)^2}}$$

If $u < 1.0$, the components of \bar{l}_1 and \bar{l}_2 are

$$l_1 = \frac{l_1^1 - u l_3^1}{\sqrt{1-(u)^2}} \quad l_1^2 = \frac{l_1^2 - u l_3^2}{\sqrt{1-(u)^2}} \quad l_3 = \frac{l_3^1 - u l_1^1}{\sqrt{1-(u)^2}}$$

$$l_1^1 = \frac{l_3^1 l_3^2 - l_1^2 R^2}{\sqrt{1-(u)^2}} \quad l_1^2 = \frac{l_3^2 l_1^1 - l_1^2 R^2}{\sqrt{1-(u)^2}} \quad l_3 = \frac{l_3^1 l_3^2 - l_1^2 R^2}{\sqrt{1-(u)^2}}$$

If $|u|$ is less than or equal to $\frac{1}{2}$ the equivalent cylindrical tank is horizontal. Defining L_e and $R_H \rightarrow$ the equivalent horizontal tank length and radius respectively, and expressing this as functions of the real tank dimensions L and R gives

$$L_e = \sqrt{2}LR \cot\left[\theta + \left(1 - \frac{\theta}{\pi}\right) \text{ARCTAN} \sqrt{L/2R}\right]$$

$$R_H = \sqrt{R^2 L / L_e}$$

In order to obtain the expressions for the moments of inertia and C.G. of the horizontal tank, the angle β , shown in Figure 1, must be found. The equation relating β to the fuel mass M_f is

$$M_f = \rho L_e R_H^2 \left(\frac{\pi}{2} + \beta + \sin \beta \cos \beta\right)$$

$$\left[(M_f / \rho L_e R_H^2) - \frac{\pi}{2}\right] = \beta + \sin \beta \cos \beta$$

Now let $C_H = [(M_f / \rho L_e R_H^2) - \frac{\pi}{2}]$, Newton's method can then be used to find β . Defining $f(\beta)$ and $f'(\beta)$ as shown below:

$$f(\beta) = \beta + \sin \beta \cos \beta - C_H$$

$$f'(\beta) = 2 \cos^2 \beta$$

Let β_0 be the initial estimate of β and calculate β

$$\beta_0 = \frac{1}{2} C_H$$

$$\beta_1 = \beta_0 - \frac{f(\beta_0)}{f'(\beta_0)}$$

The quantities $\left| \frac{f(\beta)}{f'(\beta)} \right|$ and $|f(\beta)|$ can then be tested to determine if they are both less than (1×10^{-7}) , the arbitrarily chosen degree of accuracy. If this is true then $\beta = \beta_1$. If the desired degree of accuracy has not been obtained, β_1 is used as the next estimate of β , and a β_2 must be calculated. The test is made again to determine if the desired value of β has been obtained, and if not, the iteration must be continued until the desired conditions are satisfied. The fuel moments of inertia, C.G. and height can then be obtained as functions of β .

$$h = R_H (1 + \sin \beta)$$

$$Z_F = - \frac{2 R_H \cos^3 \beta}{3 \left[\frac{\pi}{2} + \beta + \sin \beta \cos \beta \right]}$$

$$J_1 = M_F \left[\frac{1}{2} R_H^2 + Z_F \left(\frac{1}{2} R_H \sin \beta - Z_F \right) \right]$$

$$J_2 = M_F \left[\frac{1}{4} R_H^2 + \frac{1}{12} L_H^2 - Z_F \left(Z_F - \frac{3}{2} R_H \sin \beta \right) \right]$$

$$J_3 = M_F \left[\frac{1}{4} R_H^2 + \frac{1}{12} L_H^2 - \frac{1}{4} R_H Z_F \sin \beta \right]$$

$$\bar{Z}_F = Z_F \bar{J}_3$$

The spherical tank dimensions do not need adjustment because for any tank orientation, the free surface length will remain unchanged. The orientation of the free surface within the tank will, however, change positions in the tank. The angle β for the spherical tank is defined in Figure 12, and the same iteration method as described previously must be used to solve for β . The following equations must be used for this iteration.

$$M_F = \pi R^3 \left(\frac{2}{3} + \sin \beta - \frac{1}{3} \sin^3 \beta \right)$$

$$\left[\frac{M_F}{\pi R^3} - \frac{2}{3} \right] = \sin \beta - \frac{1}{3} \sin^3 \beta$$

$$C = \left[\frac{M_F}{\pi R^3} - \frac{2}{3} \right]$$

$$f(\beta) = \sin \beta - \frac{1}{3} \sin^3 \beta - C$$

$$f'(\beta) = \cos^3 \beta$$

$$\beta_c = C$$

Using the value of β obtained from the iteration, the following equations for the moment of inertia, fuel height, and C.G. can be solved.

$$z_F = - \frac{3R \cos^3 \beta}{4(2 + 3 \sin \beta - \sin^3 \beta)}$$

$$h = R(1 + \sin \beta)$$

$$J_1 = J_2 = \frac{1}{8} M_F (2R^2 + 3z_F R \sin \beta - 5z_F^2)$$

$$J_3 = \frac{8}{3} M_F (R^2 - z_F R \sin \beta)$$

$$\bar{z}_F = z_F \bar{l}_3$$

The components of \bar{l}_1 and \bar{l}_2 giving the new directions are:

$$l_1^1 = - \frac{l_2^1 l_2^2}{\sqrt{1 - (l_2^3)^2}} \quad l_1^2 = \sqrt{1 - (l_2^3)^2} \quad l_1^3 = - \frac{l_2^3 l_2^2}{\sqrt{1 - (l_2^3)^2}}$$

$$l_2^1 = - \frac{l_2^2}{\sqrt{1 - (l_2^3)^2}} \quad l_2^2 = 0 \quad l_2^3 = \frac{l_2^3}{\sqrt{1 - (l_2^3)^2}}$$

PART III

Inasmuch as the length of an "equivalent" vertical cylindrical tank is most likely to be different from that of the real tank, and since the bending mode shape $f(x)$ is given for the length of the real tank but must be applied along the length of the equivalent tank, it is necessary to find some way of adapting the use of $f(x)$ to a changing tank length.

To accomplish this, a new argument X_i is introduced which does not vary with the tank length, and f is given as $f(X_i)$. Separate considerations must be given to the two cases $u_i > 1/\sqrt{2}$ and $u_i < 1/\sqrt{2}$, where u_i equals $CCS \ominus$; and for a vertical "equivalent" tank $|u_i| > 1/\sqrt{2}$. X_i is defined as distance along the axis measured from the center of cylindrical tank i , nondimensionalized with respect to the tank length. In use with the real tank, X_i is positive in the direction so chosen in connection with the submission of data to the VPC82. In use with a vertical equivalent tank, X_i is positive in the direction of the "resultant acceleration" if $u_i > 1/\sqrt{2}$, and positive in the direction opposite to the "resultant acceleration" if $u_i < -1/\sqrt{2}$.

For the submission of data, we note that

b_i is the length of the real tank, that $b_w X_i$ is actual distance measured in feet from the center of the real tank, that $-\frac{1}{2} \leq X_i \leq \frac{1}{2}$, and that

$$f_i' = \frac{df_i}{dx_i} = \frac{df_i}{dX_i} \frac{dX_i}{dx_i} = \frac{df_i}{dX_i} / b_i.$$

We next consider the mathematical relations connected with the equivalent tank.

When $u_i > 1/\sqrt{2}$, these are as shown below:

$$X_i = b_{wi} X_i' + (b_{wi} - h_i) / 2$$

$$\frac{X_i}{h_i} = \frac{b_{wi}}{h_i} \left(\frac{1}{2} + X_i' \right) - \frac{1}{2}$$

$$dX_i = b_{wi} dX_i'$$

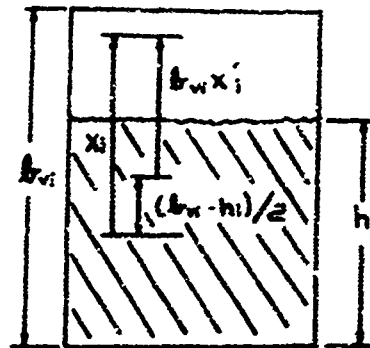
$$X_i' = \frac{X_i - (b_{wi} - h_i) / 2}{b_{wi}}$$

$$= \frac{X_i}{b_{wi}} - \left(1 - \frac{h_i}{b_{wi}} \right) / 2$$

$$= \frac{h_i}{b_{wi}} \left(\frac{1}{2} + \frac{X_i}{h_i} \right) - \frac{1}{2}$$

When $X_i = -\frac{h_i}{2}$, $X_i' = -\frac{1}{2}$

When $X_i = \frac{h_i}{2}$, $X_i' = \frac{h_i}{b_{wi}} - \frac{1}{2}$.



"Equivalent Tank"

$$F_{oi}' = \frac{b_{vi}}{h_i} \int_{-\frac{1}{2}}^{\frac{b_{vi}}{b_{vi}} - \frac{1}{2}} f_i dx_i$$

$$F_{si}' = -\frac{b_{vi}}{h_i} \int_{-\frac{1}{2}}^{\frac{b_{vi}}{b_{vi}} - \frac{1}{2}} f_i \cos \left[\pi \frac{b_{vi}}{h_i} \left(\frac{1}{2} - x_i' \right) \right] dx_i$$

$$f_{vi}' = f_i' \left(-\frac{1}{2} \right)$$

$$f_{vi}' = f_i' \left(\frac{h_i}{b_{vi}} - \frac{1}{2} \right)$$

$$G_i = \frac{b_{vi}}{h_i} \int_{-\frac{1}{2}}^{\frac{h_i}{b_{vi}} - \frac{1}{2}} x_i' f_i dx_i$$

When $u_i = -\frac{1}{2}$, the pertinent relations are as follows:

$$x_i = (b_{vi} - h_i)/2 - b_{vi} x_i'$$

$$dx_i = -b_{vi} dx_i'$$

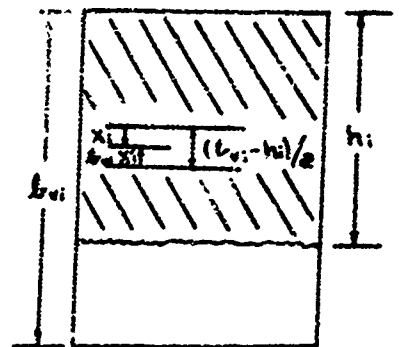
$$\frac{x_i}{h_i} = \frac{b_{vi}}{h_i} \left(\frac{1}{2} - x_i' \right) - \frac{1}{2}$$

$$x_i' = \frac{(b_{vi} - h_i)/2 - x_i}{b_{vi}}$$

$$= \frac{1}{2} - \frac{h_i}{b_{vi}} \left(\frac{1}{2} + \frac{x_i}{h_i} \right)$$

$$\text{When } x_i = -\frac{h_i}{2}, x_i' = \frac{1}{2}$$

$$\text{When } x_i = \frac{h_i}{2}, x_i' = \frac{1}{2} - \frac{h_i}{b_{vi}}$$



"Equivalent Tank"

$$F_{oi}' = -\frac{b_{vi}}{h_i} \int_{\frac{1}{2}}^{\frac{1}{2} - \frac{h_i}{b_{vi}}} f_i dx_i = \frac{b_{vi}}{h_i} \int_{\frac{1}{2} - \frac{h_i}{b_{vi}}}^{\frac{1}{2}} f_i dx_i$$

$$F'_{si} = -2 \frac{b_{vi}}{h_i} \int_{\frac{1}{2} - \frac{h_i}{b_{vi}}}^{\frac{1}{2}} f_i \cos \left[\pi \frac{b_{vi}}{h_i} \left(\frac{1}{2} - x'_i \right) \right] dx'_i$$

$$= 2 \frac{b_{vi}}{h_i} \int_{\frac{1}{2} - \frac{h_i}{b_{vi}}}^{\frac{1}{2}} f_i \cos \left[-\pi \frac{b_{vi}}{h_i} \left(\frac{1}{2} - x'_i \right) \right] dx'_i$$

$$f'_{bi} = f'_i \left(\frac{1}{2} \right)$$

$$f'_{ti} = f'_i \left(\frac{1}{2} - \frac{h_i}{b_{vi}} \right)$$

$$G_i = -\frac{b_{vi}}{h_i} \int_{\frac{1}{2} - \frac{h_i}{b_{vi}}}^{\frac{1}{2}} x'_i f_i dx'_i$$

PART IV

As indicated in equations (97) and (98), provisions have been made to include damping in the SLP. Knowledge of what numerical values to use for ζ_j (or δ_j) is important to a careful investigation of fuel sloshing in a vehicle in flight. An extensive literature search found that very little fuel damping data exists except for upright cylindrical tanks. Reference (6) did present the equation below, which can be used to obtain the logarithmic decrement δ_j for an upright cylindrical tank as a function of the kinematic viscosity ν , the fuel height h , the acceleration due to gravity g , and the tank radius R :

$$\delta = \frac{3.23\sqrt{\nu} [1 + 2(1 - h/R) \operatorname{csch}(3.63h/R)]}{[R^2 g \tanh(1.84h/R)]^{1/4}}$$

This equation is for a tank with no baffles. Most of the other references found were for upright cylindrical tanks with various baffling configurations.

Because of the scarcity of data on fuel damping, no equations such as the one just given (which is of limited applicability) are employed in the SLP. Rather, it is left to the user to determine in his own way constant values of ζ_j for submittal as input to the program. As long as the submitted values of ζ_j are greater than zero, they will at least prevent the infinite continuation of whatever fuel slosh modes are excited by the motion of the vehicle.

APPENDIX II

SYMBOLS, DATA TO BE SUBMITTED,
DEFINITIONS AND EQUATIONS USED
IN THE STRUCTURAL LOADS PROGRAM

Symbols

- A^r components in the y coordinate system of the linear acceleration of the vehicle at the origin of the vehicle axes.
SIAR7T
- $A_{Ei,h}^r$ components in the y coordinate system of the linear acceleration of the h-th particle of the i-th section due to elastic deformation.
ADAERT
- $A_{Ri,h}^r$ components in the y coordinate system of the linear acceleration of the h-th particle of the i-th section due to rigid motion.
ADARHT
- A_j^{rs} static aerodynamic terms.
- A_{si} nondimensional fuel slosh modal parameters for lateral motion of horizontal cylindrical tanks.
- A'_{si} a quantity used with vertical cylindrical tanks.
TAAPS
- A_i^{rst} sectional aerodynamic shear force terms for rigid vehicle, referred to vehicle axes.
SAAPT
- A_i^{rst} sectional aerodynamic bending moment terms for rigid vehicle, referred to vehicle axes.
SAAPT
- $a_{ih}^r \quad A_{Ei,h}^r + A_{Ri,h}^r$ components in the y coordinate system of the linear acceleration of the h-th particle of the i-th section.
ADAHT

a_{jk}^{rs} inertia terms.
SFLAPS

a_{ki}^{r} components in the v_i^r coordinate system of the given mode of vibration in degree of freedom k before balancing.

a_{ki}^{r} components in the v_i^r system of the given partial linear velocity with respect to q^k of the point of rotation of movable section i relative to the vehicle before balancing.

α_{si} modal functions of cylindrical tank aspect ratios.
TAAR

α_{usi} modal functions of rectangular tank aspect ratios.
TAAR

B_{si} nondimensional fuel slosh modal parameters for lateral motion of horizontal cylindrical tanks.

B_{jk}^{rs} aerodynamic stiffness terms.

B_{ki}^{rst} sectional aerodynamic shear force terms, referred to vehicle axial.
SABPIT

B_{ki}^{rst} sectional aerodynamic bending moment terms, referred to vehicle axes.
SABPPT

l_{wi} lengths of equivalent horizontal cylindrical tanks. Same as in VPOS.
TABHTT

- l_{vi} lengths of equivalent vertical cylindrical tanks. Same as in VPC.
- TABVTE
- \dot{q}_k^r components of the dynamically balancing rotational rate with respect to q^N of the vehicle relative to the vehicle axes.
- SEBPTT - STRUCTURE
TABPTT - TANKS
- C_{si} nondimensional fuel slosh modal parameters for lateral motion of spherical tanks.
- C_{JK}^r aerodynamic damping terms.
- C_{ki}^{rs} sectional aerodynamic shear force terms referred to vehicle axes.
- SACEPT
- C_{ki}^{rfs} sectional aerodynamic bending moment terms referred to vehicle axes.
- SACEPT
- C_{li}, C_{li} lengths of the "horizontal" edges of the (equivalent) rectangular tanks. Same as in VPC.
- TACBTE
- \dot{q}_k^r components of the dynamically balancing translation rate with respect to q^N of the vehicle relative to the vehicle axes.
- SECTKT - STRUCTURE
TACTKT - TANKS
- D_{si} nondimensional fuel slosh modal parameters for lateral motion of spherical tanks.

D_{rs}^r moments and negatives of products of inertia of structure and fuel about axes thru the vehicle center of mass and parallel to the vehicle axes.
DDEKTS

D_{ih}^r deflections of the h -th particle of the i -th section due to elastic deformation.
ADDEHT

D_{rsK} modal products of inertia of part of the vehicle.
DDEKS

d_{jk}^r dynamic balancing term.
SFEJK

E number of thrust vectoring nozzles (or "engines").
NDEHG

E'_k modal inertia term.
SFEKTS

e base of natural log system.

e_{si}^r components in the \bar{J}_V system of the \bar{J}'_{sk} vector. Same as in \bar{J}_{CS} .
TAESRT

F_{rs} moments and negatives of products of inertia of fuel about vehicle axes.
TAFRTS

F_{si}^r certain integrals connected with fuel slosh in vertical cylindrical tanks.
TAFPS

f_i bending mode shapes for cylindrical tanks.
FT&ELR

f_i slopes of bending mode shapes for cylindrical tanks.
FT&ELR

f'_{ki} f'_{ki} at the bottom of the fluid in tank i.
TAFPB

f'_{ki} f'_{ki} at the top (or surface) of the fluid in tank i.
TAFPT

G_{rs} products of inertia of structure and fuel about vehicle axes.
SFGRSS

G_{rs} products of inertia for part of the vehicle.
SIGLRS

G_i, G_i', G_i'', G_i''' integrals connected with fuel slosh in vertical cylindrical tanks.
TAGI, TAGPI, TAGPP, TAG3P

g_i the magnitude of the "resultant acceleration" at the center of tank i. Same as in VPCS.
TAGITT

g_a components of force per unit mass due to gravity. Same as in VPCS.
SOGRAP

g_j the coefficient of "structural" damping associated with the j-th degree of freedom.
AEGPJ

H_{jk} inertia coupling terms.

- H_k^r modal unbalances.
SFATKS

- h_i height of fuel in "equivalent" tank i. Same as in VPCS.
FAAIFT

- h_{ji}^r components of the partial linear velocity with respect to q^r of the center of mass of section i relative to the vehicle axes -- value obtained after dynamic balancing.
TAHKI
SFHKI

- HF_k^r modal unbalances for fuel.
TAHFIS

- HF_{jk} inertia coupling terms for fuel.
TAHFJS

- HS_{ik} inertia coupling terms for structure.
SESSJS

- HS_k^r modal unbalances for structure.
SEHSTS

- H'_{sti} moments and negatives of products of inertia for part of a section.
SIHHIS

- I_{rs} moments and negatives of products of inertia of structure and fuel about vehicle axes.
SFIRSS

- I'_{Fri} moments of inertia of fuel in "equivalent" tanks about axes parallel to vehicle axes.

- II_{rs} moments and negatives of products of inertia for part of the vehicle.
SIIIRS

J_{ri} moments of inertia of section i about section axes. Same as in VPCS.
TAJR7S

J_{Fri} moments of inertia of fuel as if it were solid "equivalent" tanks about tank axes. Same as in VPCS.
TAF7S

J'_{Fri} effective moments of inertia of fuel about tank axes.
TAJF7S

j_{ki}^r components of the partial linear velocity with respect to q^k of the center of mass of section i relative to the vehicle axes -- arbitrary values given prior to dynamic balancing.
TAJTI - TANKS
SEJKI - STRUCTURE

K_{ri} products of inertia of section i referred to sectional axes. Same as in VPCS.
TAKR7S

L_{k}^{rs}
 L_k modal inertia terms.
SFL7S

l_{ui}^r components of orthogonal unit vectors giving directions of acceleration oriented axes, l_{1i} and l_{2i} being parallel to the surface of the fuel in tank i , and l_{3i} being perpendicular to the surface of the fuel. Same as in VPCS.
TALSRT

L_{k}^{st}
 L_k modal inertia terms for fuel.
TALF7S

L_{k}^{st}
 L_k modal inertia terms for structure.
SEL7S

M_{Fi} total masses of fuel in tanks. Same as in VPCS.
TAMF7S

M_{JK} components of the inertia tensor.
 SPMJKS

M^r bending moments at a specified location on flexible vehicle without wind (components in the y-z coordinate system).
 SPM7TT

M_{θ}^r bending moments of flexible vehicle with wind.
 SPM7TT

M_{R1}^r bending moments on rigid vehicle without wind and without thrust forces.
 SPMR1T

M_{R2}^r bending moments on rigid vehicle without wind but with thrust forces.
 SPMR2T

$M_{R\theta}^r$ bending moments on rigid vehicle with wind and with thrust forces.
 SPMR\theta T

$MA_{E\theta}^r$ aerodynamic bending moments about the origin due to elastic deformation, without wind.
 SAMAE\theta T

$MA_{R\theta}^r$ aerodynamic bending moments about the origin due to rigid motion, without wind.
 SAMAR\theta T

$MA_{E\theta}^r$ aerodynamic bending moments about the origin due to elastic deformation, with wind.
 SAME\theta T

$MA_{R\theta}^r$ aerodynamic bending moments about the origin due to rigid motion, with wind.
 SAMR\theta T

M_G^r bending moments about the origin due to gravity.
SGMRT

M_i^{EG} inertial bending moments about the origin due to elastic deformation.
SIMIET

M_i^{R0} inertial bending moments about the origin due to rigid motion.
SIMIRT

M_T^{R0} bending moments about the origin due to the thrust forces of the engines.
STMTRT

m total mass of vehicle and fuel at any instant. Same as in VPCS.
AMASS

m_i mass of structural section i . Same as in VPCS.
TAMI7S

m_{ih} mass of the h -th particle of section i .
SEMIH

m_{ki} effective fuel slosh masses in tank i .

m' mass of part of the vehicle and fuel.
SIMP7S

m'_i mass of part of section i .
SIMPIS

m_{ki}^r fuel mass inertia coupling terms for spherical and horizontal cylindrical tanks.
 TAMPKS

N_i number of aerodynamic parts (or surfaces) in section i.
 NANI

N_j^r total inertia terms.

N_{Ei}^r sectional aerodynamic bending moment terms resulting from elastic deformation, without wind.
 SANLIP

N_{Ri}^r sectional aerodynamic bending moment terms resulting from rigid motion, without wind.
 SANRIP

$N_{E&I}^r$ sectional aerodynamic bending moment terms resulting from elastic deformation, with wind.
 SANEEP

$N_{R&I}^r$ sectional aerodynamic bending moment terms resulting from rigid motion, with wind.
 SANREIP

N_{Fk}^r modal inertia terms for fuel.
 TAMPKS

N_{Sk}^r modal inertia terms for structure.
 SEISKS

n number of elastic degrees of freedom.
 NOST.

n_i^r

components in the V^i coordinate system of a unit vector at point A on the surface of section i , perpendicular to the surface and pointing outward.

AENPR

P_i

number of particles (or masses) in section i

NOPI

P_{Fri}

products of inertia of fuel in tank i referred to axes parallel to vehicle axes.

P_{Tsj}

modal moments and negatives of products of inertia of vehicle and fuel.

SEPJTS - STRUCTURE

TAPJTS - TANKS

p_i^r

sectional coordinates of the point of rotation of movable section i . Same as in VPCS.

SEPPR

p_{ji}^{rs}

modal moments and negatives of products of inertia of section i .

P_{rsk}

modal moments and negatives of products of inertia of part of the vehicle.

SIPPKS

Q_j

the generalized forces associated with thrust forces.

THQTJT

q_j

generalized coordinate associated with the j -th degree of freedom.

PH.77T

q_{ki}^{rs}

modal moments and negatives of products of inertia of part of section i .

SI. PKS

- R_i radius of spherical tank i . Same as in VPCS.
TAR1PT
- R_{Hi} radius of equivalent horizontal cylindrical tank i .
Same as in VPCS.
TARH7E
- R_{Vi} radius of equivalent vertical cylindrical tank i .
Same as in VPCS.
TARV7E
- R_k^r modal products of inertia of part of the vehicle referred to
vehicle axes.
SIRKRS
- R_{ji}^{rs} sectional aerodynamic force terms for rigid vehicle.
AERTUT
- R_i^{rso} sectional aerodynamic shear force terms for rigid vehicle.
SARPTT
- P_i^{rstu} sectional aerodynamic bending moment terms for rigid vehicle.
SARPLT, SARP2T, SARP3T
- R_k^r a term of R_k^r .
SIRKRS
- r_{1i}, r_{2i} aspect ratios of "equivalent" tanks. Same as in VPCS.
TARL
- r_{3i} a quotient of aspect ratios of rectangular tank i .
TARTL

- S Area of section. Units: in².
TAGR73
- S_{rs} Moments and products of inertia of structure of entire vehicle about origin in FPS.
TAGR73
- S_{ix} Area of the i -th section of the x -th section.
ANSIHT
- S^r Shear forces at a specified location on flexible vehicle without wind (components in the r coordinate system).
SPS77P
- S^r_v Shear forces on flexible vehicle with wind.
SPS87P
- S^r_E Shear forces due to elastic deformation without wind.
SPS87P
- $S^r_{E_v}$ Shear forces due to elastic deformation with wind.
SPS8BP
- S^r_{R1} Shear forces on rigid vehicle without wind and without thrust force.
SPSR1P
- S^r_{R2} Shear forces on rigid vehicle without wind but with thrust forces.
SPSR2P
- $S^r_{R_v}$ Shear forces on rigid vehicle with wind.
SPSR8P

S_{JKL}^{rs} sectional aerodynamic stiffness terms.
ANSTUT

S_A^r aerodynamic shear forces due to elastic deformation, without wind.
SASRAP

$S_{A_r}^r$ aerodynamic shear forces due to rigid motion, without wind.
SASRAP

$S_{A_{E\&}}^r$ aerodynamic shear forces due to elastic deformation, with wind.
SASREP

$S_{A_{R\&}}^r$ aerodynamic shear forces due to rigid motion, with wind.
SASREP

$S_{A_{Ei}}^r$ elastic contribution of section i to the aerodynamic shear forces, without wind.
SASAEP

$S_{A_{Ri}}^r$ rigid contribution of section i to the aerodynamic shear forces, without wind.
SASARP

$S_{A_{E\&i}}^r$ elastic contribution of section i to the aerodynamic shear forces, with wind.
SASEEP

$S_{A_{R\&i}}^r$ rigid contribution of section i to the aerodynamic shear forces, with wind.
SASREP

S_G^r shear forces due to gravity.
SGSRP

$S I_E^r$ inertial shear forces due to elastic deformation.
SISIEP

$S I_R^r$ inertial shear forces due to rigid motion.
SISIRP

$S J_R^r$ shear forces due to the thrust forces of the engines.
STSTRP

T number of tanks. Same as in VPCS.
NOCAN

T_{xi} components in the y coordinate system of the thrust force at the
i-th nozzle.
EMXZP(1)

T_{yi} components in the y coordinate system of the thrust force at the
i-th nozzle.
EMXZP(2)

T_{zi} components in the y coordinate system of the thrust force at the
i-th nozzle.
EMXZP(3)

T_{jki}^r sectional aerodynamic damping terms.
ASTIJT

T_{ki}^{rs} sectional aerodynamic shear force terms.
SATPIT

T_{ki}^{rst} sectional aerodynamic bending moment terms.
SATPPT

t tire.

U_{rs} aerodynamic terms.
SAUPRT

U_{ki}^{rst} sectional aerodynamic shear force terms.
SAUPII

U_{ki}^{rstu} sectional aerodynamic bending moment terms.
SAUPII, SAUP2I, SAUP3I

u_i cosine of angle between resultant acceleration and axis of cylindrical tank i . Same as in VPCS.
TAUS

V^r components in the y coordinate system of the linear velocity of the vehicle at the origin of the vehicle axes.
AEVRT

V_a^r components of the velocity of the wind.
AEVRAT

V_b^r $V^r - V_a^r$.
AEVRET

V_c^r components of the vehicle velocity at the center of mass.
AEVRCT

\dot{V}_c^r dV_c^r/dt
SIVDCF

U_{Eih}^r components of the velocity of particle h of section i relative to the vehicle axes.
ADVERT

W_{KL}^r inertia coupling terms for part of the vehicle - error to vehicle axes.
SI/KLS

W_{KL}^s one term of W_{KL}^r .

W_i number of fuel slosh modes in each direction for tank i . (≤ 2)
NØ-PI

$W_{E;h}$ the "piston speed" (or downwash) at the h -th surface of the i -th section.
AØ-PIHT

$W_{E;h}^r$ one term of $A_{E;h}^r$.
ADWØHT

X_i distance along the axis measured from the center of cylindrical tank i , nondimensionalized with respect to the tank length. In use with the real tank, X_i is positive in the direction so chosen in the VPCS data to be submitted, number 5. In use with a vertical equivalent tank, X_i is positive in the direction of the "resultant acceleration" if u_i is positive, and positive in the direction opposite to the "resultant acceleration" if u_i is negative. ($u_i = \cos \Theta_i$ and in the case of a vertical "equivalent" tank ($u_i > 1/\sqrt{2}$).

X_i^r coordinates of geometric center of tank i or of the point of rotation of movable section i . Same as in VPCS.

$X_{KI}^r = \sum_{k=1}^2 e_{ki}^r X_{ki}^r$
TANK - TANK
SEKX - STRUCTURE

$$\chi_{ki}^r \quad \ddot{a}_{ki}^r - p_{ki}^r = \gamma_{ki}^r + \sum \sum \frac{1}{2} \frac{3}{2} C_{rsL} B_{ki}^{rs} \dot{\phi}_i^{rs}$$

χ_{EiA}^r one term of A_{EiA}^r .
ADXEMT

Y^r static unbalances of part of the vehicle referred to vehicle axes.
SIIR7S

Y_i^r static unbalances of part of section i , referred to vehicle axes.
SIYRIS

Y_x^r dynamic unbalances of part of the vehicle referred to vehicle axes.
SIYKKS

Y_{KL}^r inertia coupling terms for part of the vehicle referred to vehicle axes.
SIYKLS

Y_{si}^i, Y_{si}'' certain summations connected with fuel slosh in vertical cylindrical tanks.
TAYPI, TAYPP

y_{iA}^r coordinates of the h -th particle of the i -th section in the y coordinate system.
ADYIHT, SAYHET

y_i^{rs} products of inertia of part of section i , referred partly to sectional axes and partly to vehicle axes.
SIYYIS

y_{ki}^{rs} modal products of inertia of part of section i , referred partly to sectional axes and partly to vehicle axes.
SIYKKS

Z_{ki} Z_{usi} distance in tank i from fuel center of mass to spring
mass, S , positive up.
TAZITP

\bar{z}_i distance from geometric center to center of fuel mass,
positive upward, for equivalent tank i . Same as in VPCS
TAZBAR

\bar{z}_i^r coordinates of the center of mass of section i . Same as in
VPCS.
SEZSI

\bar{z}_{fi}^r coordinates of center of mass of fuel in tank i . Same as in
VPCS.
TAZFRT

\bar{z}_c^r coordinates of the center of mass of the vehicle. Same as in
VPCS.
ZCFBT

ω_{ji}^r components of partial angular velocity with respect to q^j
of the \bar{J}_{ri} coordinates relative to the \bar{J}_r system - values
obtained after dynamic balancing.
SFAPR
TAAPR

β_i fuel height angle for spherical and horizontal cylindrical tanks;
that is, the angle between the free surface and a line from the
center of the tank to the intersection of the free surface with
the wall of the tank. Same as in VPCS.
TABRTR

β_{ki}^r components in the \bar{J}_r system of the partial angular velocity
with respect to q^k of the \bar{J}_{ri} coordinates relative to the \bar{J}_r
system -- arbitrary values given prior to dynamic balancing.
SEBKI - STRUCTURE
TABKI - TANKS

B_{ji}^r components in the \bar{J}_r system of the partial angular velocity with respect to q^j of the \bar{J}_{r_i} coordinates relative to the \bar{J}_r system -- arbitrary values given prior to dynamic balancing.
TABRK - TANKS
SEBRK - STRUCTURES

Γ_{rsi} products of inertia of section i , referred to the sectional axes.
TACGFS - TANKS
SECGFS - STRUCTURES

Π_{rsi} products of inertia of part of section i , referred to the sectional axes.
SICGFS

$\gamma_1, \gamma_2, \gamma_3$ constants obtained from Bessel functions and used with vertical cylindrical tanks.
TAGAM

Δ_{jk} inertia coupling terms.

Δ_{JK}^F inertia coupling terms for fuel.
TADFJS

Δ_{JK}^S inertia coupling terms for structure.
SEGGJS

Δt time increment used in the numerical integration.

E_j^r modal inertia terms.
SPGPSS

S_{jki}^{rs} inertia coupling terms.
SIZZFS

η_{jk} inertia coupling terms.

Ω_{jk}^F

inertia coupling terms for fuel.
TATEFS

 Ω_{jk}^S

inertia coupling terms for structure.
SETASS

 Θ_j^{rs}

modal inertia terms for fuel.
TATFJS

 Θ_j^{rs}

modal inertia terms for structure.
SETSJS

 Λ_k^r

modal inertia terms.

 Λ_{ki}^{rs}

modal products of inertia of section i.
TAFLAM - TANKS
SECLAM - STRUCTURE

 Λ_{ki}^{rs}

modal products of inertia of part of section i.
SICLLS

 Λ_k^r

modal inertia terms for fuel.
TACLFS

 Λ_k^r

modal inertia terms for structure.
SECLSS

 λ_{si}

frequency parameters for lateral motion of horizontal cylindrical tanks.

 λ'_{si}

frequency parameters for lateral motion of spherical tanks.

μ_{JK} inertia coupling terms for fuel and structure.
TAMUJS - TANKS
SEMUTS - STRUCTURE

μ_{JK} fuel slosh inertia terms for vertical cylindrical tanks.
TAMUFS, TAUPPS, TARJFS, TAMUPO, TAMUJS

ξ_{jih} aerodynamic modal term.
AEXIJT

π ratio of circumference to diameter of a circle.
PI

ρ the atmospheric density.
EMRHPS

ρ_i density of fuel in tank i (used with spherical and horizontal cylindrical tanks. Same as in VPCJ).
TARHPS

ρ_{ki}^{ir} components in the v_i system of the given partial linear velocity with respect to q^k of the point of rotation of movable section i relative to the section.

$\sum_j F_j^{rs}$ modal inertia terms for fuel.
TACSF5

$\sum_j S_j^{rs}$ modal inertia terms for structure.
SECSS5

σ_{kih}^{ir} $\partial v_{ih}^{ir} / \partial q^k$
SESPSH, AESIPT

T_{jih}^{ir} $\partial n_{ih}^{ir} / \partial q^j$
AETAPT

γ_i^r

static unbalance of part of a section, referred to sectional axes.
SIGAPS

 \bar{u}_i^r

coordinates in the \bar{J}_r^r system of particle i .
SEVPRH, AEPRT, SIVEGT

 Φ_{jk}

fuel slosh inertia terms.
SEPJKS STRUCTURE
TAPJKS - TANKS

 Φ_{kLiA}^r

kinematic modal coupling term such that

$$\Phi_{kLiA}^r + \Phi_{LkLiA}^r = \partial T_{kiA}^r / \partial q^L$$

ADPKHT

 ψ_{si}

a function used in connection with vertical cylindrical tanks.
TAPSI

 ψ_{ki}^r

dynamic unbalance of part of a section, referred to sectional axes.
SIPSI

 Ω^r

components in the y coordinate system of the angular velocity of the vehicle axes.
ADPGR

 $\dot{\Omega}^r$

components in the y coordinate system of the angular acceleration of the vehicle axes.
SIPGR

 ω_{ki}

fuel slosh frequency in the k-th degree of freedom.
TAPKI

 ω_j

vibration frequency associated with the j-th degree of freedom.
SEMJ

γ^c
k.h

$\delta y^i / \delta q^k$
ADCKT

KI, J

inertia "symbols."
983 75

Date to be Submitted

1. n (= number of elastic degrees of freedom).
2. For all tanks, w_i ($0 \leq w_i \leq 2$);
 $\alpha_{ki}^r, \beta_{ki}^r$ for $k > 40$, that is, for structural k's.
3. For all cylindrical tanks,
 $\left. \begin{array}{l} f_{ki}^r \text{ versus } X_i^r \\ f_{ki}^r \text{ versus } X_i^r \end{array} \right\} \left(-\frac{1}{2} \leq X_i^r \leq \frac{1}{2} \right)$ for $k > 40$.
4. For structure, including tanks but not fuel, ρ_i^r, P_i
 - a. If $P_i = 1$, then X_{ki}^r and β_{ki}^r are given.
($m_{ih}, v_{ih}^r, \sigma_{kih}^r$ are not given).
 - b. If $P_i > 1$, then m_{ih}, v_{ih}^r and σ_{kih}^r are given ($h = 1, 2, \dots, P_i$;
 $r = 1, 2, 3; k = 41, 42, \dots$).
If $\sigma_{kih}^r = 0$, then X_{ki}^r and β_{ki}^r are given.
If $\sigma_{kih}^r \neq 0$, then f_{ki}^r and β_{ki}^r are given.
5. For aerodynamic parts of structural sections,
 $n_{ih}^r, \sigma_{jih}^r, v_{ih}^r, S_{ih}, T_{jih}^r, N_i (\leq 10)$
6. For all degrees of freedom, g_j .
7. For structural vibration modes, ω_j .
8. For the computation of structural loads, designations of points in the structure, and, with each point, associated sections, tanks, and engines (that is, thrust vectoring nozzles). Points are designated by giving the numbers of sections and the sectional coordinates U_{ij} of the points. With each section, there must also be an indication of which particles and which aerodynamic parts will be included in the summations. For some sections, all particles and aerodynamic parts will be used; for such sections, the user should so indicate, because this results in simplification of some formulas. Submit values of g .
9. For the computation of accelerations and deflections, designation of points in the vehicle. Points are designated by giving the numbers i of sections or tanks and h of particles within sections for which $P_i = 1$. If i designates a tank or a section for which $P_i = 1$, h is not given.

2. Required from the basic SDF Program

1. M_r for all tanks.
2. $e, v_e^r, v_a^r, \Omega^r, \dot{\Omega}^r, \dot{V}_c^r, (r=1, 2, 3)$
3. E (= number of engines. Same as in VPCS.)
4. $T_{x_i}, T_{y_i}, T_{z_i}, (i=1, 2, \dots, E)$
These are functions of time.
5. g_a^r

Let Required from the VPS Subprogram

1. For structure, including tanks but not fuel,
 S_{rs}
 m_i, z_i, e_i, J_i, K_i ($r, s = 1, 2, 3, i = 1, 2, \dots, S$)
for the entire vehicle (structure, tanks, and fuel).
 S, T
2. g_i
 λ_{ui} ($u = 1, 2, 3$)
3. For all tanks,
 z_i, \bar{z}_i
4. Information as to whether or not each equivalent tank is rectangular, horizontal cylindrical, vertical cylindrical, or spherical.
5. For rectangular tanks,
 $r_{1i}, r_{2i}, c_{1i}, c_{2i}, h_i, J_{Fi}$
6. For horizontal cylindrical tanks,
 $l_{xi}, r_{xi}, \beta_i, R_{xi}, e_i, h_i, J_{Fi}$
7. For vertical cylindrical tanks,
 $l_{vi}, r_{vi}, u_i, R_{vi}, h_i$
8. For spherical tanks,
 β_i, R_i, e_i

$$r_{0i} = r_{1i} \quad \text{if } r_{1i} = r_{2i}, r_{3i} = 1$$

$$r_{0i} < r_{1i} \quad \text{if } r_{2i} = r_{1i}/r_{3i}$$

$$r_{0i} > r_{1i} \quad \text{if } r_{3i} = r_{2i}/r_{1i}$$

$$ar_{usi} = (2s-1)\pi r_{ui} \quad (u=1,2,3; s=1,2, \dots, \omega_i)$$

$$\sinh ar_{usi} = \frac{1}{2} (e^{ar_{usi}} - e^{-ar_{usi}})$$

$$\cosh ar_{usi} = \frac{1}{2} (e^{ar_{usi}} + e^{-ar_{usi}})$$

$$\tanh ar_{usi} = \frac{\sinh ar_{usi}}{\cosh ar_{usi}}$$

$$\tanh \frac{ar_{ui}}{2} = \frac{\sinh ar_{usi}}{1 + \cosh ar_{usi}}$$

$$\omega_{ki} = \omega_{usi} = \sqrt{g_i (2s-1) \frac{\pi}{c_{ui}} \tanh ar_{usi}} \quad (u=1,2)$$

$$m_{ki} = m_{usi} = M_{Fi} \frac{8 \tanh ar_{usi}}{(2s-1)^2 \pi^2 r_{ui}} \quad (u=1,2)$$

$$Z_{ki} = Z_{usi} = \frac{h_i}{2} \left[1 - \frac{4 \tanh \frac{ar_{usi}}{2}}{ar_{usi}} \right] \quad (u=1,2)$$

$$J'_{F1i} = J_{F1i} \left\{ 1 - \frac{4}{1+(r_{2i})^2} + \frac{768}{r_{2i}[1+(r_{2i})^2]\pi^2} \sum_{s=1}^{\omega_i} \frac{\tanh \frac{ar_{2i}}{2}}{(2s-1)^2} \right\}$$

$$J'_{F2i} = J_{F2i} \left\{ 1 - \frac{4}{1+(r_{1i})^2} + \frac{768}{r_{1i}[1+(r_{1i})^2]\pi^2} \sum_{s=1}^{\omega_i} \frac{\tanh \frac{ar_{1i}}{2}}{(2s-1)^2} \right\}$$

$$J_{F3i} = J_{F3i} \left\{ 1 - \frac{4}{1 + (r_{3i})^2} + \frac{768}{r_{3i} [1 + (r_{3i})^2] \pi^6} \sum_{n=1}^{\infty} \frac{\tanh \frac{n \pi z_i}{2}}{(2n-1)^5} \right\}$$

$$S_{uffic} = 4(s-1) + 2 + 2(s-1) \quad (s=1, 2)$$

$$\Delta_{ki}^{rt} = m_{ki} Z_{ki} \quad \text{when } r=3 \text{ and } t=u$$

$$= 0 \quad \text{otherwise } (r, t = 1, 2, 3)$$

$$u_{jk} = m_{ki} \quad \text{when } j=k$$

$$= 0 \quad \text{when } j \neq k$$

2. For horizontal cylindrical tanks,

$$a_{r_{si}} = s \pi r_{ii} \quad (s = 1, 2, \dots, w_i)$$

$$\sinh a_{r_{si}} = \frac{1}{2} (e^{a_{r_{si}}} - e^{-a_{r_{si}}})$$

$$\cosh a_{r_{si}} = \frac{1}{2} (e^{a_{r_{si}}} + e^{-a_{r_{si}}})$$

$$\tanh a_{r_{si}} = \frac{\sinh a_{r_{si}}}{\cosh a_{r_{si}}}$$

$A_{s1}, B_{s1},$ and $\sqrt{\lambda_{s1}}$ versus $\sin \beta_1$.

| $\sin \beta_1$ | A_{11} | A_{21} | A_{31} | B_{11} | B_{21} | B_{31} | $\sqrt{\lambda_{11}}$ | $\sqrt{\lambda_{21}}$ | $\sqrt{\lambda_{31}}$ |
|----------------|----------|----------|----------|----------|----------|----------|-----------------------|-----------------------|-----------------------|
| .50 | .333 | .088 | .145 | .353 | 0 | 0 | 1.0000 | 2.4495 | 3.8730 |
| .55 | .343 | .139 | .175 | .338 | .002 | 0 | 1.0100 | 2.3800 | 3.5400 |
| .60 | .355 | .192 | .207 | .344 | .005 | .001 | 1.0222 | 2.3155 | 3.2939 |
| .70 | .367 | .240 | .242 | .350 | .009 | .002 | 1.0400 | 2.2750 | 3.1400 |
| .80 | .380 | .285 | .279 | .356 | .013 | .003 | 1.0550 | 2.2394 | 3.0216 |
| .90 | .394 | .325 | .314 | .362 | .017 | .004 | 1.0650 | 2.2000 | 2.9400 |
| .95 | .408 | .362 | .352 | .368 | .022 | .005 | 1.0793 | 2.1771 | 2.8862 |
| 1.00 | .438 | .430 | .420 | .382 | .028 | .010 | 1.1176 | 2.1564 | 2.8266 |
| .00 | .475 | .502 | .486 | .397 | .040 | .014 | 1.1362 | 2.1679 | 2.8213 |
| .10 | .520 | .560 | .551 | .413 | .048 | .019 | 1.1500 | 2.2158 | 2.8688 |
| .20 | .544 | .592 | .584 | .422 | .053 | .022 | 1.1700 | 2.2500 | 2.9200 |
| .30 | .544 | .592 | .584 | .422 | .053 | .022 | 1.1700 | 2.2500 | 2.9200 |
| .40 | .570 | .623 | .618 | .432 | .059 | .026 | 1.3198 | 2.3108 | 2.9816 |
| .50 | .602 | .659 | .652 | .443 | .065 | .029 | 1.3800 | 2.3800 | 3.0800 |
| .60 | .635 | .698 | .688 | .453 | .071 | .032 | 1.4594 | 2.4240 | 3.2062 |
| .70 | .674 | .741 | .731 | .467 | .078 | .035 | 1.5700 | 2.6400 | 3.4000 |
| .75 | .698 | .767 | .755 | .474 | .082 | .037 | 1.6500 | 2.7500 | 3.5300 |
| .80 | .728 | .791 | .783 | .481 | .087 | .040 | 1.7435 | 2.9017 | 3.7202 |
| .85 | .762 | .826 | .816 | .493 | .093 | .044 | 1.8900 | 3.0800 | 3.9500 |
| .90 | .808 | .867 | .858 | .508 | .101 | .047 | 2.1300 | 3.4300 | 4.3300 |
| .95 | .875 | .919 | .919 | .528 | .110 | .053 | 2.4800 | 4.0000 | 4.9800 |
| 1.00 | 1.000 | 1.000 | 1.000 | .558 | .121 | .058 | 3.5000 | 5.5000 | 7.0000 |

$$\tanh \frac{ar_{2i}}{2} = \frac{\sinh ar_{2i}}{1 + \cosh ar_{2i}}$$

$$J_{2i} = W_{2i} = \sqrt{\frac{g_i S_i \pi}{L_{2i}}} \tanh \frac{ar_{2i}}{2}$$

$$m_{ki} = m'_{1si} = M_{Fi} \frac{B \tanh \frac{ar_{2i}}{2}}{\pi S_i - r_{ii}}$$

$$Z_{ki} = Z_{1si} = \frac{h_i}{2} \left(i - \frac{4 \tanh \frac{ar_{2i}}{2}}{ar_{2i}} \right)$$

$$J'_{Fii} = 0.$$

$$J'_{F2i} = J_{F2i} \left\{ 1 - \frac{4}{1 + (r_{ii})^2} + \frac{768}{r_{ii} [1 + (r_{ii})^2]} \pi \epsilon \sum_{s=1}^{4S} \frac{\tanh \frac{ar_{2i}}{2}}{S^3} \right\}$$

$$J'_{F3i} = J_{F3i} \left\{ 1 - \frac{4}{1 + (r_{ii})^2} + \frac{768}{r_{ii} [1 + (r_{ii})^2]} \pi \epsilon \sum_{s=1}^{4S} \frac{\tanh \frac{ar_{2i}}{2}}{S^3} \right\}$$

$$m_{ki} = m_{2si} = \frac{2 \rho_i b_{wi} (R_{wi})^2 A_{2i} \cos \beta_i}{\lambda_{2i}} \quad (s=1, 2, \dots, w_i)$$

$$w_{ki} = w_{2si} = \sqrt{g_i / R_{wi}} \sqrt{\lambda_{2i}}$$

$$m'_{ki} = m'_{2si} = 2 \rho_i b_{wi} (R_{wi})^2 B_{2i} \cos^2 \beta_i$$

$$r_{2i} = 1 + \sin \beta_i$$

If $r_{ii} = 0$, then m_{ki} , J'_{Fii} , m'_{ki} , and $r_{2i} = 0$.

$$\text{Suffix } k = 4(1-1) + u + 2(s-1) \quad (u=1,2)$$

$$\Lambda_k^{r0} = \gamma_u Z_{ku} \text{ when } r=3 \text{ and } r+u=1$$

$$= 0 \text{ otherwise (i.e. } s=1,2,3)$$

$$\mu_{jk} = m_{ku} \text{ when } j=k$$

$$= 0 \text{ when } j \neq k.$$

3. For vertical cylindrical tanks,

$$\gamma_1 = 1.04119, \gamma_2 = 5.33144, \gamma_3 = 2.53631$$

$$\alpha r_{si} = \gamma_s r_{ii} \quad (s=1,2,3)$$

$$\sinh \alpha r_{si} = \frac{1}{2} (e^{\alpha r_{si}} - e^{-\alpha r_{si}})$$

$$\cosh \alpha r_{si} = \frac{1}{2} (e^{\alpha r_{si}} + e^{-\alpha r_{si}})$$

$$\operatorname{csch} \alpha r_{si} = \frac{1}{\sinh \alpha r_{si}}$$

$$\tanh \alpha r_{si} = \frac{\sinh \alpha r_{si}}{\cosh \alpha r_{si}}$$

$$\tanh \frac{\alpha r_{si}}{2} = \frac{\sinh \alpha r_{si}}{1 + \cosh \alpha r_{si}}$$

$$J'_{F11} = J'_{F21} = M_{F1} \left[\frac{(h_1)^2}{12} - \frac{3(R_{v1})^2}{4} + \frac{16(R_{v1})^2}{r_{11}} \sum_{s=1}^3 \frac{\tanh \frac{\alpha h_1}{2}}{\delta_s^3 (Y_s^2 \cdot)} \right]$$

$$J'_{F31} = 0, \quad J'_{F11} = J'_{F21} = 0.$$

$$J'_{F31} = 0$$

$$A'_{s1} = \frac{8\pi}{r_{11}}$$

$\Psi_{s1} = \psi(A'_{s1})$ according to the following table.

If $r_{11} = 0, \Psi_{s1} = 0$

Ψ_{s1} versus A'_{s1}

| A'_{s1} | Ψ_{s1} |
|-----------|-------------|
| 0 | 1.0 |
| .25 | .982 |
| .41 | .956 |
| 1.00 | .791 |
| 1.46 | .661 |
| 1.88 | .569 |
| 2.18 | .506 |
| 2.49 | .444 |
| 3.00 | .372 |
| 3.77 | .290 |
| 4.00 | .268 |
| 4.45 | .233 |
| 5.00 | .203 |
| 5.79 | .180 |
| 6.82 | .152 |
| 8.60 | .122 |
| 10.00 | .101 |

If $A'_{s1} > 10, \Psi_{s1} = 0$

When $u_i > i/\sqrt{2}$

$$F'_{ki} = \frac{b_{vi}}{h_i} \int_{-\frac{1}{2}}^{\frac{h_i}{b_{vi}} - \frac{1}{2}} f'_{ki} dx'_i \quad (k > 40)$$

$$F'_{kbi} = 2 \frac{b_{vi}}{h_i} \int_{-\frac{1}{2}}^{\frac{h_i}{b_{vi}} - \frac{1}{2}} f'_{ki} \cos \left[\pi \frac{b_{vi}}{h_i} \left(\frac{1}{2} + x'_i \right) \right] dx'_i$$

$$f'_{kbi} = f'_{ki} \text{ evaluated for } x'_i = -\frac{1}{2}$$

$$f'_{kvi} = f'_{ki} \text{ evaluated for } x'_i = \frac{h_i}{b_{vi}} - \frac{1}{2}$$

$$G_{ki} = \frac{b_{vi}}{h_i} \int_{-\frac{1}{2}}^{\frac{h_i}{b_{vi}} - \frac{1}{2}} x'_i f'_{ki} dx'_i$$

$$G'_{ki} = b_{vi} \int_{-\frac{1}{2}}^{\frac{h_i}{b_{vi}} - \frac{1}{2}} (f'_{ki})^2 dx'_i$$

$$G''_{ki} = b_{vi} \int_{-\frac{1}{2}}^{\frac{h_i}{b_{vi}} - \frac{1}{2}} (f'_{ki})^2 dx'_i$$

$$G'''_{ki} = b_{vi} \int_{-\frac{1}{2}}^{\frac{h_i}{b_{vi}} - \frac{1}{2}} x'_i (f'_{ki})^2 dx'_i$$

When $u_i < -1/\sqrt{2}$

$$f_{ki} = \frac{2}{h_i} \int_{\frac{1}{2} - \frac{h_i}{b_{vi}}}^{\frac{1}{2}} f_{ki} dx_i$$

$$F'_{ksi} = 2 \frac{b_{vi}}{h_i} \int_{\frac{1}{2} - \frac{h_i}{b_{vi}}}^{\frac{1}{2}} f_{ki} \cos \left[s\pi \frac{b_{vi}}{h_i} \left(\frac{1}{2} - x_i \right) \right] dx_i$$

$$f'_{ksi} = f'_{ki} \text{ evaluated for } x_i = \frac{1}{2}$$

$$f''_{ksi} = f''_{ki} \text{ evaluated for } x_i = \frac{1}{2} - \frac{h_i}{b_{vi}}$$

$$G_{ki} = -\frac{b_{vi}}{h_i} \int_{\frac{1}{2} - \frac{h_i}{b_{vi}}}^{\frac{1}{2}} x_i f_{ki} dx_i$$

$$G'_{ki} = b_{vi} \int_0^{\frac{1}{2}} (f'_{ki})^2 dx_i$$

$$G''_{ki} = b_{vi} \int_{\frac{1}{2} - \frac{h_i}{b_{vi}}}^{\frac{1}{2}} (f'_{ki})^2 dx_i$$

$$G'''_{ki} = -b_{vi} \int_{\frac{1}{2} - \frac{h_i}{b_{vi}}}^{\frac{1}{2}} x_i (f'_{ki})^2 dx_i$$

$$Y_{ks} = \sum_{p=0}^s \left\{ (-1)^p F_{kps} / \left[(\gamma_s)^2 + \left(\frac{R_{vi}}{r_{ii}} \right)^2 \right] \right\}$$

$$Y_{ks} = \sum_{p=0}^s \left\{ F'_{kps} / \left[(\gamma_s)^2 + \left(\frac{R_{vi}}{r_{ii}} \right)^2 \right] \right\}$$

$$u''_{k1} = M_{Fi} \left[F'_{ks1} - \frac{R_{vi}}{4r_{ii}} (f'_{ks1} - f'_{kb1}) \right]$$

$$u''_{1,s+3,1} = \frac{M_{Fi}}{\gamma_s^2 r_{ii}}$$

$$u''_{2ki} = M_{Fi} \left[\frac{b_{vi} - h_i}{2} F'_{ks1} + b_{vi} G_{ki} + \frac{4h_i}{\pi^2} \sum_{s=1}^2 \frac{\psi_{2s-1,1} F'_{ks2-1,1}}{(2s-1)^2} \right. \\ \left. + (f'_{kbi} + f'_{kb1}) \left(\frac{h_i^2}{12} - \frac{\gamma_{vi}}{8} + \frac{8h_i^2}{\pi^4} \sum_{s=1}^2 \frac{\psi_{2s-1,1}}{(2s-1)^4} \right) \right]$$

$$u''_{2,s+3,1} = \frac{M_{Fi} R_{vi}}{\gamma_s^2} \left[\frac{2 \tanh \frac{\alpha h_i}{2}}{\alpha r_{ii}} - \frac{1}{2} \right]$$

$$u'_{kxi} = M_{Fi} \left\{ (F'_{ks1})^2 + \frac{1}{2} \sum_{s=1}^2 \psi_{2s} (F'_{ks1})^2 - \frac{2R_{vi}}{r_{ii}} \sum_{s=1}^2 \frac{f'_{kvi} Y'_{ksi} - f'_{kvi} Y'_{ks1}}{\gamma_s^2 - 1} \right. \\ \left. + \frac{R_{vi}}{4r_{ii}} (f'_{kbi} - f'_{kvi}) + \frac{h_i}{\pi^2} \sum_{s=1}^2 F'_{ks1} \frac{1 - \psi_{2s}}{s^2} \left[(-1)^{s+1} f'_{kvi} + f'_{kbi} \right] \right. \\ \left. + \frac{2R_{vi}^2}{r_{ii}} \sum_{s=1}^2 \frac{\operatorname{csch} \alpha h_i}{\gamma_s^2 (\gamma_s^2 - 1)} \left[(f'_{kvi})^2 + (f'_{kbi})^2 \right] \cosh \alpha h_i - 2f'_{kvi} f'_{kbi} \right\}$$

$$u_{k, s+3, i} = \frac{M_{Fi}}{r_{ii}} \left\{ \gamma'_{k+1} - \frac{R_{Vi}}{\delta_s^3} (\operatorname{csch} a_{s, i}) \right\} \left[f'_{k+1} \cosh(a_{s, i} - f'_{k+1}) \right]$$

$$u'_{0+3, s+3, i} = \frac{M_{Fi}}{2 r_{ii} \delta_s^3} (\delta_s^2 - 1) / \tanh a_{s, i}$$

$$u''_{s+3, i} = \frac{\delta_s \delta_s}{R_{Vi}} \tanh a_{s, i}$$

If $r_{ii} = 0$, $M_{Fi} = 0$, the u'_{mn} equal zero.

$$\Delta_{ki}^{inc} = 0.$$

$$u_{jk} = u'_{0+3, s+3, i} \quad \text{when } j = k = 4(i-1) + u + 2(s-1)$$

$$= 0 \quad \text{when } j \neq k$$

$$u'_{jk} = 0 \quad \text{when both } j \text{ and } k > 40 \text{ but } j \neq k.$$

4. For spherical tanks,

C_{si} , D_{si} , and $\sqrt{\lambda'_{si}}$ versus $\sin \beta_i$

| $\sin \beta_i$ | C_{1i} | C_{2i} | C_{3i} | D_{1i} | D_{2i} | D_{3i} | $\sqrt{\lambda'_{1i}}$ | $\sqrt{\lambda'_{2i}}$ | $\sqrt{\lambda'_{3i}}$ |
|----------------|----------|----------|----------|----------|----------|----------|------------------------|------------------------|------------------------|
| -1.00 | .245 | .122 | .080 | .251 | 0 | 0 | 1.0000 | 2.6500 | 4.1200 |
| -.90 | .250 | .164 | .134 | .253 | .002 | .001 | 1.0149 | 2.5787 | 3.7855 |
| -.80 | .255 | .204 | .184 | .256 | .003 | .002 | 1.0304 | 2.5080 | 3.4900 |
| -.70 | .260 | .242 | .220 | .259 | .006 | .003 | 1.0550 | 2.4600 | 3.2700 |
| -.60 | .268 | .276 | .272 | .262 | .011 | .004 | 1.0790 | 2.3937 | 3.1575 |
| -.50 | .273 | .307 | .302 | .266 | .012 | .005 | 1.0997 | 2.3585 | 3.0750 |
| -.40 | .281 | .350 | .350 | .270 | .016 | .006 | 1.1225 | 2.3238 | 2.9983 |
| -.20 | .305 | .415 | .418 | .278 | .023 | .011 | 1.1790 | 2.2913 | 2.9274 |
| 0 | .335 | .475 | .485 | .290 | .032 | .015 | 1.2490 | 2.2956 | 2.9138 |
| .20 | .370 | .535 | .545 | .301 | .039 | .018 | 1.3379 | 2.3452 | 2.9648 |
| .30 | .392 | .565 | .578 | .309 | .044 | .020 | 1.4000 | 2.3973 | 3.0259 |
| .40 | .420 | .595 | .612 | .318 | .050 | .022 | 1.4629 | 2.4495 | 3.0871 |
| .50 | .453 | .632 | .650 | .328 | .058 | .024 | 1.5800 | 2.5600 | 3.2200 |
| .60 | .492 | .672 | .692 | .338 | .066 | .027 | 1.6643 | 2.6571 | 3.3377 |
| .70 | .543 | .718 | .740 | .353 | .074 | .030 | 1.8200 | 2.8200 | 3.5900 |
| .75 | .576 | .742 | .768 | .363 | .081 | .032 | 1.9200 | 2.9300 | 3.7300 |
| .80 | .620 | .770 | .794 | .373 | .087 | .033 | 2.0784 | 3.1321 | 3.9281 |
| .85 | .671 | .804 | .826 | .388 | .094 | .037 | 2.1800 | 3.3700 | 4.1900 |
| .90 | .732 | .847 | .868 | .407 | .104 | .040 | 2.4200 | 3.7700 | 4.6400 |
| .95 | .818 | .907 | .922 | .433 | .117 | .043 | 2.7700 | 4.4300 | 5.3000 |
| 1.00 | 1.000 | 1.000 | 1.000 | .470 | .134 | .047 | 4.0000 | 6.0000 | 7.5000 |

$$m_{si} = m_{usi} = \pi \rho_i R_i^2 \frac{C_{si}}{\lambda'_{si}} \cos^2 \beta_i, \quad (i=1, 2, \dots, n)$$

$$\omega_{si} = \omega_{usi} = \sqrt{g/R_i} \sqrt{\lambda'_{si}}$$

$$m'_{ki} = m'_{usi} = \pi \rho_i R_i^3 D_{si} \cos^3 \beta_i$$

$$J'_{F11} - J'_{F21} = J'_{F31} = 0$$

If $\beta_i = \pm \frac{\pi}{2}$, then

m_{ki} , \dot{m}_{ki} , and m'_{ki} equal zero.

$$\Lambda'_{ki} = 0$$

$$\Lambda_{jk} = m_{ki} \text{ when } j = k = \pm(i-1) + u + e(s-1)$$

$$= 0 \text{ where } j \neq k.$$

5. For all tanks,

$$\begin{bmatrix} I_{F11} & -P_{F21} & -P_{F21} \\ -P_{F31} & I_{F21} & -P_{F11} \\ -P_{F21} & -P_{F11} & I_{F31} \end{bmatrix} =$$

$$\begin{bmatrix} l_{11}^1 & l_{21}^1 & l_{31}^1 \\ l_{11}^2 & l_{21}^2 & l_{31}^2 \\ l_{11}^3 & l_{21}^3 & l_{31}^3 \end{bmatrix} \begin{bmatrix} J'_{F11} & 0 & 0 \\ 0 & J'_{F21} & 0 \\ 0 & 0 & J'_{F31} \end{bmatrix} \begin{bmatrix} l_{11}^1 & l_{11}^2 & l_{11}^3 \\ l_{21}^1 & l_{21}^2 & l_{21}^3 \\ l_{31}^1 & l_{31}^2 & l_{31}^3 \end{bmatrix}$$

$$F_{11} = \sum_{i=1}^T [M_{F_i} (\dot{z}_{F_i}^2 \dot{z}_{F_i}^2 + \dot{z}_{F_i}^3 \dot{z}_{F_i}^3) + I_{F_{11}}]$$

$$F_{12} = - \sum_{i=1}^T (M_{F_i} \dot{z}_{F_i}^1 \dot{z}_{F_i}^2 + P_{F_{21}})$$

$$F_{13} = - \sum_{i=1}^T (M_{F_i} \dot{z}_{F_i}^1 \dot{z}_{F_i}^3 + P_{F_{21}})$$

$$F_{21} = F_{12}$$

$$F_{22} = \sum_{i=1}^T [M_{F_i} (\dot{z}_{F_i}^2 \dot{z}_{F_i}^2 + \dot{z}_{F_i}^1 \dot{z}_{F_i}^1) + I_{F_{22}}]$$

$$F_{23} = - \sum_{i=1}^T (M_{F_i} \dot{z}_{F_i}^2 \dot{z}_{F_i}^3 + P_{F_{11}})$$

$$F_{31} = F_{13}, \quad F_{32} = F_{23}$$

$$F_{33} = \sum_{i=1}^T [M_{F_i} (\dot{z}_{F_i}^3 \dot{z}_{F_i}^3 + \dot{z}_{F_i}^1 \dot{z}_{F_i}^1) + I_{F_{33}}]$$

$$\Gamma'_{F_{11}} = (J'_{F_{21}} + J'_{F_{31}} - J'_{F_{11}}) / 2$$

$$\Gamma'_{F_{22}} = (J'_{F_{31}} + J'_{F_{11}} - J'_{F_{21}}) / 2$$

$$\Gamma'_{F_{33}} = (J'_{F_{11}} + J'_{F_{21}} - J'_{F_{31}}) / 2$$

$$H_{F_i}^0 = \sum_{i=1}^T M_{F_i} \dot{z}_{F_i}^0$$

$$LF_k^s = \sum_{i=1}^T M_{Fi} \bar{z}_{Fi}^s f_k^s$$

$$B_{k,i} = \sum_{j=1}^3 \lambda_{ji} B_{k,i}$$

$$f_{k,i}^r = X_{k,i}^r + (l_{1,i}^r B_{k,i}'^2 - l_{2,i}^r B_{k,i}'^1) \bar{z}$$

$$\{\Lambda F_k\} = \sum_{i=1}^T \begin{bmatrix} l_{1,i}^1 & l_{2,i}^1 & l_{3,i}^1 \\ l_{1,i}^2 & l_{2,i}^2 & l_{3,i}^2 \\ l_{1,i}^3 & l_{2,i}^3 & l_{3,i}^3 \end{bmatrix} \begin{Bmatrix} -\Lambda_{k,i}^{3,2} \\ \Lambda_{k,i}^{3,1} \\ 0 \end{Bmatrix}$$

$$\{NF_k\} = \sum_{i=1}^T \begin{bmatrix} l_{1,i}^1 & l_{2,i}^1 & l_{3,i}^1 \\ l_{1,i}^2 & l_{2,i}^2 & l_{3,i}^2 \\ l_{1,i}^3 & l_{2,i}^3 & l_{3,i}^3 \end{bmatrix} \begin{bmatrix} J_{F1i} & 0 & 0 \\ 0 & J_{F2i} & 0 \\ 0 & 0 & J_{F3i} \end{bmatrix} \begin{Bmatrix} B_{k,i}'^1 \\ B_{k,i}'^2 \\ B_{k,i}'^3 \end{Bmatrix}$$

$$\Sigma F_j = \sum_{i=1}^T \begin{bmatrix} l_{1,i}^1 & l_{2,i}^1 & l_{3,i}^1 \\ l_{1,i}^2 & l_{2,i}^2 & l_{3,i}^2 \\ l_{1,i}^3 & l_{2,i}^3 & l_{3,i}^3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\Lambda_{j,i}^{3,1} & \Lambda_{j,i}^{3,2} & 0 \end{bmatrix} \begin{bmatrix} l_{1,i}^1 & l_{1,i}^2 & l_{1,i}^3 \\ l_{2,i}^1 & l_{2,i}^2 & l_{2,i}^3 \\ l_{3,i}^1 & l_{3,i}^2 & l_{3,i}^3 \end{bmatrix}$$

$$\Theta F_j = \sum_{i=1}^T \begin{bmatrix} l_{1,i}^1 & l_{2,i}^1 & l_{3,i}^1 \\ l_{1,i}^2 & l_{2,i}^2 & l_{3,i}^2 \\ l_{1,i}^3 & l_{2,i}^3 & l_{3,i}^3 \end{bmatrix} \begin{bmatrix} 0 & \beta_{j,i}^{3,1} \Gamma_{F1i} & -\beta_{j,i}^{3,2} \Gamma_{F2i} \\ -\beta_{j,i}^{3,1} \Gamma_{F2i} & 0 & \beta_{j,i}^{3,2} \Gamma_{F3i} \\ \beta_{j,i}^{3,1} \Gamma_{F3i} & -\beta_{j,i}^{3,2} \Gamma_{F3i} & 0 \end{bmatrix} \begin{bmatrix} l_{1,i}^1 & l_{1,i}^2 & l_{1,i}^3 \\ l_{2,i}^1 & l_{2,i}^2 & l_{2,i}^3 \\ l_{3,i}^1 & l_{3,i}^2 & l_{3,i}^3 \end{bmatrix}$$

$$\Delta F_{jk} = -\beta_{j1}^{11} \Lambda_{k1}^{10} + \beta_{j1}^{12} \Lambda_{k1}^{11} \\ - \beta_{j1}^{11} \Lambda_{k1}^{12} + \beta_{j1}^{12} \Lambda_{k1}^{13} \quad (j, k = 1, 2, 3)$$

$$D F_{jk} = \sum_{i=1}^T M_{Fi} \sum_{r=1}^3 f_{ji}^r f_{ki}^r \quad (j, k = 1, 2, 3)$$

$$H F_{jk} = \sum_{i=1}^T [\beta_{j1}^{11} \beta_{j1}^{12} \beta_{j1}^{13}] \begin{bmatrix} J'_{F_{1i}} & 0 & 0 \\ 0 & J'_{F_{2i}} & 0 \\ 0 & 0 & J'_{F_{3i}} \end{bmatrix} \begin{Bmatrix} \beta_{k1}^{11} \\ \beta_{k1}^{12} \\ \beta_{k1}^{13} \end{Bmatrix}$$

6. For structure, including tanks but not fluid,

$$\text{If } P_i > 1, \Delta_{ki}^{irs} = \sum_{h=1}^{P_i} m_{ih} v_{ih}^r \sigma_{kih}^s$$

$$(r, s = 1, 2, 3, k = 1, 2, \dots, n; i = 1, 2, \dots, \infty)$$

$$\text{If } P_i = 1, \Delta_{ki}^{irs} = 0.$$

$$\Gamma_{111} = (J_{21} + J_{31} - J_{11})/2$$

$$\Gamma_{221} = (J_{31} + J_{11} - J_{21})/2$$

$$\Gamma_{31} = (J_{11} + J_{21} - J_{31})/2$$

If $P > 1$ or if $\sigma_{k1}^r \neq 0$

$$\beta_{ki}^{r,s} = \sum_{s=1}^3 e_{si}^r \beta_{ki}^s$$

$$f_{ki}^r = \lambda_{ki}^r - \begin{vmatrix} e_{1i}^r & B_{ki}^{r1} & f_{ki}^{r1} \\ e_{2i}^r & B_{ki}^{r2} & f_{ki}^{r2} \\ e_{3i}^r & B_{ki}^{r3} & f_{ki}^{r3} \end{vmatrix}$$

If $P_1 > 1$ and if $\sigma_{ki1}^r \neq 0$,

$$f_{ki}^r = \sum_{s=1}^3 e_{si}^r f_{ki}^{rs}$$

$$HS_k^{ss} = \sum_{i=1}^3 m_i f_{ki}^{ss}$$

$$LS_k^{ss} = \sum_{i=1}^3 m_i g_{ki}^s f_{ki}^{ss}$$

$$\{ \Lambda_{S_k} \} = \sum_{i=1}^3 \begin{bmatrix} e_{1i}^1 & e_{2i}^1 & e_{3i}^1 \\ e_{1i}^2 & e_{2i}^2 & e_{3i}^2 \\ e_{1i}^3 & e_{2i}^3 & e_{3i}^3 \end{bmatrix} \left\{ \begin{array}{l} \Lambda_{ki}^{r2} - \Lambda_{ki}^{r3} \\ \Lambda_{ki}^{r1} - \Lambda_{ki}^{r3} \\ \Lambda_{ki}^{r2} - \Lambda_{ki}^{r1} \end{array} \right\}$$

$$\{N_{\xi_k}\} = \sum_{i=1}^3 \begin{bmatrix} e_{1i}^1 & e_{2i}^1 & e_{3i}^1 \\ e_{1i}^2 & e_{2i}^2 & e_{3i}^2 \\ e_{1i}^3 & e_{2i}^3 & e_{3i}^3 \end{bmatrix} \begin{bmatrix} J_{1i} & -K_{3i} & K_{2i} \\ -K_{3i} & J_{2i} & -K_{1i} \\ -K_{2i} & -K_{1i} & J_{3i} \end{bmatrix} \begin{Bmatrix} \beta_{k1}^{1i} \\ \beta_{k1}^{2i} \\ \beta_{k1}^{3i} \end{Bmatrix}$$

$$\sum S_i = \sum_{i=1}^3 \begin{bmatrix} e_{1i}^1 & e_{2i}^1 & e_{3i}^1 \\ e_{1i}^2 & e_{2i}^2 & e_{3i}^2 \\ e_{1i}^3 & e_{2i}^3 & e_{3i}^3 \end{bmatrix} \begin{bmatrix} (\Lambda_{ji}^{22} + \Lambda_{ji}^{33}) & -\Lambda_{ji}^{12} & -\Lambda_{ji}^{13} \\ -\Lambda_{ji}^{21} & (\Lambda_{ji}^{33} + \Lambda_{ji}^{11}) & -\Lambda_{ji}^{13} \\ -\Lambda_{ji}^{31} & -\Lambda_{ji}^{22} & (\Lambda_{ji}^{11} + \Lambda_{ji}^{22}) \end{bmatrix} \begin{Bmatrix} e_{1i}^1 & e_{2i}^2 & e_{3i}^3 \\ e_{2i}^1 & e_{2i}^2 & e_{2i}^3 \\ e_{3i}^1 & e_{3i}^2 & e_{3i}^3 \end{Bmatrix}$$

$$\Theta S_j = \sum_{i=1}^3 \begin{bmatrix} e_{1i}^1 & e_{2i}^1 & e_{3i}^1 \\ e_{1i}^2 & e_{2i}^2 & e_{3i}^2 \\ e_{1i}^3 & e_{2i}^3 & e_{3i}^3 \end{bmatrix}$$

$$\begin{bmatrix} (\beta_{ji}^{12} K_{2i} - \beta_{ji}^{13} K_{3i}) & (\beta_{ji}^{13} \Gamma_{1ii} - \beta_{ji}^{11} K_{2i}) & (\beta_{ji}^{11} K_{3i} - \beta_{ji}^{12} \Gamma_{1ii}) \\ (\beta_{ji}^{12} K_{1i} - \beta_{ji}^{13} \Gamma_{22i}) & (\beta_{ji}^{13} K_{3i} - \beta_{ji}^{11} K_{1i}) & (\beta_{ji}^{11} \Gamma_{22i} - \beta_{ji}^{12} K_{3i}) \\ (\beta_{ji}^{12} \Gamma_{33i} - \beta_{ji}^{13} K_{1i}) & (\beta_{ji}^{13} K_{2i} - \beta_{ji}^{11} \Gamma_{33i}) & (\beta_{ji}^{11} K_{1i} - \beta_{ji}^{12} K_{2i}) \end{bmatrix}$$

$$\begin{bmatrix} e_{1i}^1 & e_{2i}^1 & e_{3i}^1 \\ e_{1i}^2 & e_{2i}^2 & e_{3i}^2 \\ e_{1i}^3 & e_{2i}^3 & e_{3i}^3 \end{bmatrix}$$

$$\Phi_{j\alpha} = 0.$$

$$\Delta S_{jk} = \sum_{i=1}^3 \left[\beta_{ji}^{i1} (\Lambda_{ki}^{i23} - \Lambda_{ki}^{i32}) + \beta_{ji}^{i2} (\Lambda_{ki}^{i31} - \Lambda_{ki}^{i13}) + \beta_{ji}^{i3} (\Lambda_{ki}^{i12} - \Lambda_{ki}^{i21}) \right]$$

$$a^{i1} (\Lambda_{ki}^{i23} - \Lambda_{ki}^{i32}) + a^{i2} (\Lambda_{ki}^{i31} - \Lambda_{ki}^{i13}) + a^{i3} (\Lambda_{ki}^{i12} - \Lambda_{ki}^{i21})$$

$$\eta S_{jk} = \sum_{i=1}^3 m_i \sum_{r=1}^3 f_{ji}^r f_{ki}^r$$

$$HS_{jk} = \sum_{i=1}^3 \left[\beta_{ji}^{i1} \beta_{ki}^{i1} \beta_{ji}^{i2} \beta_{ki}^{i2} \beta_{ji}^{i3} \beta_{ki}^{i3} \right] \begin{bmatrix} J_{11} & -K_{31} & -K_{21} \\ -K_{31} & J_{21} & -K_{11} \\ -K_{21} & -K_{11} & J_{31} \end{bmatrix} \begin{Bmatrix} \beta_{ki}^{i1} \\ \beta_{ki}^{i2} \\ \beta_{ki}^{i3} \end{Bmatrix}$$

$$\mu_{jk} = \sum_{i=1}^3 \sum_{h=1}^{p_i} m_{i,h} \sum_{r=1}^3 \sigma_{ji,h}^{ir} \sigma_{ki,h}^{ir}$$

$$S_{jk}^{ir} = \sum_{h=1}^{p_i} m_{i,h} \sigma_{ji,h}^{ir} \sigma_{ki,h}^{ir}$$

7. For structure and tanks, including fluid,

$$I_{rs} = S_{rs} + F_{rs} \quad (r, s = 1, 2, 3)$$

$$G_{rs} = \delta_{rs} \sum_{\alpha=1}^3 I_{\alpha\alpha} / 2 - I_{rs},$$

$$\delta_{rs} = 1 \quad \text{when } r=s$$

$$= 0 \quad \text{when } r \neq s$$

$$D = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix} - m \begin{bmatrix} (\tilde{z}_c^1 \tilde{z}_c^2 + \tilde{z}_c^3 \tilde{z}_c^4) & -\tilde{z}_c^1 \tilde{z}_c^2 & -\tilde{z}_c^3 \tilde{z}_c^4 \\ -\tilde{z}_c^2 \tilde{z}_c^1 & (\tilde{z}_c^3 \tilde{z}_c^4 + \tilde{z}_c^5 \tilde{z}_c^6) & -\tilde{z}_c^3 \tilde{z}_c^4 \\ -\tilde{z}_c^4 \tilde{z}_c^3 & -\tilde{z}_c^5 \tilde{z}_c^6 & (\tilde{z}_c^7 \tilde{z}_c^8 + \tilde{z}_c^9 \tilde{z}_c^{10}) \end{bmatrix}$$

$$\{b_k\} = \begin{Bmatrix} b_k^1 \\ b_k^2 \\ b_k^3 \end{Bmatrix} \quad (k = 1, 2, \dots, n)$$

$$\{\Lambda_k\} = \{\Delta F_k\} + \{\Delta S_k\}$$

$$\{N_k\} = \{NF_k\} + \{NS_k\}$$

$$H_k^0 = \sum_{i=1}^6 m_i \tilde{z}_i^0 f_{ki}^0 + \sum_{i=1}^7 M_{pi} \tilde{z}_i^0 f_{ki}^0 = HF_k^0 + HS_k^0$$

$$L_k^{sc} = \sum_{i=1}^6 m_i \tilde{z}_i^s f_{ki}^s + \sum_{i=1}^7 M_{pi} \tilde{z}_i^s f_{ki}^s = LF_k^{sc} + LS_k^{sc}$$

$$\{E_k\} = \{\Lambda_k\} + \{N_k\} + \begin{Bmatrix} L_k^{sc} - L_k^i \\ L_k^{sc} - L_k^i \\ L_k^{sc} - L_k^i \end{Bmatrix}$$

$$\{E_k\} = \begin{Bmatrix} \tilde{z}_c^1 H_k^2 - \tilde{z}_c^2 H_k^1 \\ \tilde{z}_c^3 H_k^1 - \tilde{z}_c^4 H_k^2 \\ \tilde{z}_c^5 H_k^2 - \tilde{z}_c^6 H_k^1 \end{Bmatrix} - \{E_k\}$$

$$\{b_k\} = D^{-1} \{E_k\}$$

$$\begin{pmatrix} C_n^1 \\ C_n^2 \\ \vdots \end{pmatrix} = \begin{pmatrix} \partial_c^2 b_n^2 - \partial_c^2 b_n^2 \\ \partial_c^3 b_n^1 - \partial_c^1 b_n^3 \\ \partial_c^1 b_n^2 - \partial_c^2 b_n^1 \end{pmatrix} = \frac{1}{m} \begin{pmatrix} H_n^1 \\ H_n^2 \\ H_n^3 \end{pmatrix}$$

$$\begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} = m \begin{bmatrix} (\partial_c^2 C_j^2 + \partial_c^1 C_j^3) & \partial_c^1 C_j^2 & -\partial_c^1 C_j^3 \\ -\partial_c^2 C_j^1 & (\partial_c^3 C_j^2 + \partial_c^1 C_j^1) & -\partial_c^2 C_j^3 \\ -\partial_c^3 C_j^1 & -\partial_c^2 C_j^2 & (\partial_c^1 C_j^1 + \partial_c^2 C_j^3) \end{bmatrix}$$

$$- \begin{bmatrix} (b_j^1 G_{12} - b_j^2 G_{21}) & (b_j^2 G_{11} - b_j^1 G_{12}) & (b_j^1 G_{22} - b_j^2 G_{21}) \\ (b_j^2 G_{23} - b_j^3 G_{32}) & (b_j^3 G_{21} - b_j^1 G_{22}) & (b_j^1 G_{32} - b_j^2 G_{31}) \\ (b_j^3 G_{31} - b_j^1 G_{32}) & (b_j^1 G_{31} - b_j^2 G_{32}) & (b_j^2 G_{32} - b_j^3 G_{31}) \end{bmatrix}$$

$$+ \begin{bmatrix} (L_j^{22} + L_j^{33}) & -L_j^{12} & -L_j^{13} \\ -L_j^{11} & (L_j^{33} + L_j^{11}) & -L_j^{23} \\ -L_j^{11} & -L_j^{22} & (L_j^{11} + L_j^{22}) \end{bmatrix} + \Sigma F_j + \Sigma S_j - \Theta F_j - \Theta S_j \quad (j = 1, 2, \dots, n)$$

$$\Delta_{jk} = \Delta F_{jk} + \Delta S_{jk}$$

$$\eta_{jk} = \eta F_{jk} + \eta S_{jk}$$

$$H_{jk} = H F_{jk} + H S_{jk}$$

$$\text{For fuel } \alpha_{ki}^r = B_{ki}^r + \sum_{r=1}^3 l_{r,i}^s b_{r,k}^s$$

$$h_{ki}^1 = c_{ki}^1 + f_{ki}^1 + b_{r_1}^2 z_{r_1}^3 - b_{r_2}^3 z_{r_2}^2$$

$$h_{ki}^2 = c_{ki}^2 + f_{ki}^2 + b_{r_2}^3 z_{r_2}^1 - b_{r_3}^1 z_{r_3}^2$$

$$h_{ki}^3 = c_{ki}^3 + f_{ki}^3 + b_{r_3}^1 z_{r_3}^2 - b_{r_1}^2 z_{r_1}^1$$

For a spherical tank or for lateral sloshing ($u=2$)
in a horizontal cylindrical tank,

$$\phi_{jk} = \sum_{r=1}^3 l_{2r}^r (h_{j1}^r m'_{kr} + h_{k1}^r m'_{j1})$$

if j or k denotes a fuel slosh mode (j or $k \leq 40$).

$\phi_{jk} = 0$ if neither j nor k denotes a fuel

slosh mode (j and $k > 40$).

For vertical cylindrical tanks, assuming $u=2$,

$$\Phi_{jk} = \sum_{r=1}^3 [u'_{jr} + \sum_{r=1}^3 l_{r1} (h'_{j1} u'_{rk} + h''_{r1} u''_{j1}) \\ + \alpha'_{j1} u''_{2k} + \alpha''_{k1} u''_{j2}]$$

for rectangular tanks and for longitudinal

sloshing in horizontal cylindrical tanks, $\Phi_{jk} = 0$.

$$d'_{jk} = c'_j b_k^3 - c'_j b_k^2 + c''_k b_j^3 - c''_k b_j^2$$

$$d^2_{jk} = c^3_j b_k^1 - c^3_j b_k^2 + c^2_k b_j^1 - c^2_k b_j^3$$

$$d^3_{jk} = c^1_j b_k^2 - c^2_j b_k^1 + c^1_k b_j^2 - c^2_k b_j^1$$

For structure and tanks, including fuel.

$$M_{jk} = m \sum_{r=1}^3 (c'_j c''_k + z_c^r d'_{jk}) + \sum_{r=1}^3 (c^r_j H''_k + c''_k H^r_j + b^r_j \epsilon''_k + b''_k \epsilon^r_j)$$

$$+ \eta_{jk} + \Delta_{jk} + H_{jk} + u_{jk} + \Phi_{jk} \\ + [b^1_j \ b^2_j \ b^3_j] \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix} \begin{Bmatrix} b^1_k \\ b^2_k \\ b^3_k \end{Bmatrix}$$

For fuel,

$$a_{ji}^{12} = a_{ji}^{21} = \frac{1}{2} \alpha_{ji}^{13} (J'_{F21} - J'_{F11})$$

$$a_{ji}^{13} = a_{ji}^{31} = \frac{1}{2} \left[\alpha_{ji}^{12} (J'_{F11} - J'_{F31}) + \Lambda_{ji}^{121} \right]$$

$$a_{ji}^{123} = a_{ji}^{32} = \frac{1}{2} \left[\alpha_{ji}^{11} (J'_{F31} - J'_{F21}) + \Lambda_{ji}^{123} \right]$$

$$[KL, j] F = \sum_{i=1}^T \left[\alpha_{ki}^{11} \quad \alpha_{ki}^{12} \quad \alpha_{ki}^{13} \right] \begin{bmatrix} 0 & a_{ji}^{12} & a_{ji}^{13} \\ a_{ji}^{21} & 0 & a_{ji}^{23} \\ a_{ji}^{31} & a_{ji}^{32} & 0 \end{bmatrix} \begin{bmatrix} \alpha_{L1}^{11} \\ \alpha_{L1}^{12} \\ \alpha_{L1}^{13} \end{bmatrix}$$

$$- \sum_{i=1}^T \left[(\alpha_{ki}^{13} \Lambda_{L1}^{131} + \alpha_{L1}^{13} \Lambda_{ki}^{131}) \alpha_{ji}^{11} \right]$$

$$+ (\alpha_{ki}^{12} \Lambda_{L1}^{122} + \alpha_{L1}^{12} \Lambda_{ki}^{122}) \alpha_{ji}^{12}$$

For structure, $\alpha_{ji}^{1r} = \beta_{ji}^{1r} + \sum_{s=1}^3 e_{r1}^s b_j^s$

$$a_{ji}^{11} = \alpha_{ji}^{12} K_{21} - \alpha_{ji}^{13} K_{31} - \Lambda_{ji}^{122} - \Lambda_{ji}^{133}$$

$$a_{ji}^{12} = \frac{1}{2} \left[\alpha_{ji}^{13} (J_{21} - J_{11}) + \alpha_{ji}^{12} K_{11} - \alpha_{ji}^{11} K_{21} - \Lambda_{ji}^{112} + \Lambda_{ji}^{121} \right]$$

$$a_{ji}^{11} = \frac{1}{2} [\alpha_{ji}^{12} (J_{11} - J_{31}) + \alpha_{ji}^{11} K_{21} - \alpha_{ji}^{13} K_{11} + \Lambda_{ji}^{11} - \Lambda_{ji}^{131}]$$

$$a_{ji}^{12} = \alpha_{ji}^{13} K_{31} - \alpha_{ji}^{11} K_{11} - \Lambda_{ji}^{133} - \Lambda_{ji}^{11}$$

$$a_{ji}^{13} = \frac{1}{2} [\alpha_{ji}^{11} (J_{31} - J_{21}) + \alpha_{ji}^{13} K_{21} - \alpha_{ji}^{12} K_{31} + \Lambda_{ji}^{123} - \Lambda_{ji}^{132}]$$

$$a_{ji}^{133} = \alpha_{ji}^{11} K_{11} - \alpha_{ji}^{12} K_{21} - \Lambda_{ji}^{111} - \Lambda_{ji}^{122}$$

$$a_{ji}^{21} = a_{ji}^{12} ; a_{ji}^{21} = a_{ji}^{13} ; a_{ji}^{22} = a_{ji}^{133}$$

$$\begin{aligned} \boxed{KL, J} S &= \sum_{i=1}^3 [\alpha_{ki}^{11} (\xi_{Lji}^{123} - \xi_{Lji}^{132}) + \alpha_{li}^{11} (\xi_{Kji}^{123} - \xi_{Kji}^{132}) \\ &+ \alpha_{ki}^{12} (\xi_{Lji}^{111} - \xi_{Lji}^{113}) + \alpha_{li}^{12} (\xi_{Kji}^{131} - \xi_{Kji}^{113}) \\ &+ \alpha_{ki}^{13} (\xi_{Lji}^{112} - \xi_{Lji}^{121}) + \alpha_{li}^{13} (\xi_{Kji}^{112} - \xi_{Kji}^{121})] \\ &+ \sum_{i=1}^3 [\alpha_{ki}^{11} \alpha_{ni}^{12} \alpha_{ki}^{13}] \begin{bmatrix} a_{ji}^{11} & a_{ji}^{12} & a_{ji}^{13} \\ a_{ji}^{21} & a_{ji}^{22} & a_{ji}^{23} \\ a_{ji}^{31} & a_{ji}^{32} & a_{ji}^{33} \end{bmatrix} \begin{Bmatrix} \alpha_{li}^{11} \\ \alpha_{li}^{12} \\ \alpha_{li}^{13} \end{Bmatrix} \\ &+ \sum_{i=1}^3 [\alpha_{ki}^{11} \alpha_{ki}^{12} \alpha_{ni}^{13}] \begin{bmatrix} (\Lambda_{li}^{12} - \Lambda_{li}^{23}) & -\Lambda_{li}^{12} & -\Lambda_{li}^{13} \\ -\Lambda_{li}^{11} & (\Lambda_{li}^{11} - \Lambda_{li}^{11}) & -\Lambda_{li}^{23} \\ -\Lambda_{li}^{21} & -\Lambda_{li}^{32} & (\Lambda_{li}^{11} - \Lambda_{li}^{11}) \end{bmatrix} \begin{Bmatrix} \alpha_{ji}^{11} \\ \alpha_{ji}^{12} \\ \alpha_{ji}^{13} \end{Bmatrix} \end{aligned}$$

(This equation continues on the next page)

$$+ \sum_{i=1}^3 [\alpha_{i1} \alpha_{i2} \alpha_{i3}] \begin{bmatrix} (\Lambda_{k1}^{22} + \Lambda_{k1}^{33}) & -\Lambda_{k1}^{12} & -\Lambda_{k1}^{13} \\ -\Lambda_{k1}^{21} & (\Lambda_{k1}^{33} + \Lambda_{k1}^{11}) & -\Lambda_{k1}^{23} \\ -\Lambda_{k1}^{31} & -\Lambda_{k1}^{32} & (\Lambda_{k1}^{11} + \Lambda_{k1}^{22}) \end{bmatrix} \begin{Bmatrix} \alpha_{j1}^1 \\ \alpha_{j1}^2 \\ \alpha_{j1}^3 \end{Bmatrix}$$

$$[K_{LJ}] = [K_{LJ}]_F + [K_{LJ}]_S.$$

8 For aerodynamic parts of structural sections,

using data submitted under this same heading,

$$\left. \begin{aligned} h_{ki}^1 &= c_k^1 + j_{ki}^1 + b_k^2 z_i^3 - b_k^3 z_i^2 \\ h_{ki}^2 &= c_k^2 + j_{ki}^2 + b_k^3 z_i^1 - b_k^1 z_i^3 \\ h_{ki}^3 &= c_k^3 + j_{ki}^3 + b_k^1 z_i^2 - b_k^2 z_i^1 \end{aligned} \right\} \text{for structural} \\ \text{sections.}$$

$$\begin{aligned} \xi_{jih} &= n_{jih} \left(\sum_{r=1}^3 e_{jih}^r h_{ji}^r + \sigma_{jih}^1 + \alpha_{ji}^2 v_{ih}^3 - \alpha_{ji}^3 v_{ih}^2 \right) \\ &+ n_{jih}^{12} \left(\sum_{r=1}^3 e_{jih}^r h_{ji}^r + \sigma_{jih}^{12} + \alpha_{ji}^3 v_{ih}^1 - \alpha_{ji}^1 v_{ih}^3 \right) \\ &+ n_{jih}^{13} \left(\sum_{r=1}^3 e_{jih}^r h_{ji}^r + \sigma_{jih}^{13} + \alpha_{ji}^1 v_{ih}^2 - \alpha_{ji}^2 v_{ih}^1 \right) \end{aligned}$$

$$w_{ik} = [V_1^1 \ V_2^2 \ V_3^3] \begin{bmatrix} e_{1i} & e_{2i} & e_{3i} \\ e_{1j}^2 & e_{2j}^2 & e_{3j}^2 \\ e_{1i}^3 & e_{2i}^3 & e_{3i}^3 \end{bmatrix} \begin{Bmatrix} n_{1i}^1 \\ n_{1j}^2 \\ n_{1i}^3 \end{Bmatrix}$$

$$V_1^r = V^r - V_a^r, \quad V_2^r = V_c^r - \Omega^2 g_c^r = -\Omega^2 g_c^r,$$

$$V^2 = V_c^2 - \Omega^2 g_c^2 + \Omega^1 g_c^2, \quad V^3 = V_c^3 - \Omega^1 g_c^3 - \Omega^2 g_c^3$$

In the three following summations, include

only the terms for which $w_{ik} > 0$:

$$\left. \begin{aligned} 1. R_{ji}^{tu} &= \sum_{k=1}^N \delta_{jik} n_{ik}^t n_{ik}^u s_{ik} \\ 2. S_{jki}^{tu} &= \sum_{k=1}^N \delta_{jia} n_{ik}^t T_{kik}^u s_{ik} \\ 3. T_{jki}^u &= \sum_{k=1}^N \delta_{jik} \delta_{kik} n_{ik}^t s_{ik} \end{aligned} \right\} (t, u = 1, 2, 3)$$

$$[A_j^{rs}] = \sum_{i=1}^N \begin{bmatrix} e_{1i}^1 & e_{2i}^1 & e_{3i}^1 \\ e_{1i}^2 & e_{2i}^2 & e_{3i}^2 \\ e_{1i}^3 & e_{2i}^3 & e_{3i}^3 \end{bmatrix} \begin{bmatrix} R_{ji}^{11} & R_{ji}^{12} & R_{ji}^{13} \\ R_{ji}^{21} & R_{ji}^{22} & R_{ji}^{23} \\ R_{ji}^{31} & R_{ji}^{32} & R_{ji}^{33} \end{bmatrix} \begin{bmatrix} e_{1i}^1 & e_{2i}^1 & e_{3i}^1 \\ e_{1i}^2 & e_{2i}^2 & e_{3i}^2 \\ e_{1i}^3 & e_{2i}^3 & e_{3i}^3 \end{bmatrix}$$

$$J_{J_1}^{t1} = S_{J_1}^{t1} + \alpha_{K_1}^{12} R_{J_1}^{t3} - \alpha_{K_1}^{13} R_{J_1}^{t2}$$

$$J_{J_2}^{t2} = S_{J_2}^{t2} + \alpha_{K_1}^{13} R_{J_1}^{t1} - \alpha_{K_1}^{11} R_{J_1}^{t3}$$

$$J_{J_3}^{t3} = S_{J_3}^{t3} + \alpha_{K_1}^{11} R_{J_1}^{t2} - \alpha_{K_1}^{12} R_{J_1}^{t1}$$

$$[B_{JK}^{rs}] = \sum_{i=1}^3 \begin{bmatrix} e_{1i}^r & e_{2i}^r & e_{3i}^r \\ e_{1i}^s & e_{2i}^s & e_{3i}^s \\ e_{1i}^t & e_{2i}^t & e_{3i}^t \end{bmatrix} \begin{bmatrix} U_{JK}^{11} & U_{JK}^{12} & U_{JK}^{13} \\ U_{JK}^{21} & U_{JK}^{22} & U_{JK}^{23} \\ U_{JK}^{31} & U_{JK}^{32} & U_{JK}^{33} \end{bmatrix} \begin{bmatrix} e_{1i}^1 & e_{2i}^2 & e_{3i}^3 \\ e_{1i}^2 & e_{2i}^3 & e_{3i}^1 \\ e_{1i}^3 & e_{2i}^1 & e_{3i}^2 \end{bmatrix}$$

$$C_{JK}^r = \sum_{i=1}^3 \sum_{t=1}^3 e_{oi}^r T_{iK}^t$$

9. For the engines,

$$\mathcal{Q}_j = \sum_{i=1}^3 (h_{ji}^1 T_{xi} + h_{ji}^2 T_{yi} + h_{ji}^3 T_{zi})$$

10 Equations of Motion.

$$\begin{aligned} \sum_{K=1}^3 M_{JK} \ddot{q}^K &= \sum_{F=1}^3 \sum_{S=1}^3 \Omega^r \Omega^s P_{rSj} + \mathcal{Q}_j \\ &- e \sum_{F=1}^3 \sum_{S=1}^3 V_F^r V_S^s A_{rSj} - (\omega_j)^2 M_{Sj} q^S \\ &- 2e \sum_{K=1}^3 \sum_{L=1}^3 \sum_{S=1}^3 V_F^r V_S^s B_{JK}^{rS} q^K - g_j^i \omega_j M_{ij} q^j \\ &- 2e \sum_{K=1}^3 \sum_{L=1}^3 V_F^r C_{JK}^s q^K - \sum_{K=1}^3 \sum_{L=1}^3 \boxed{K(L)j} q^K q^L \end{aligned}$$

Possibly, $q^j(t+\Delta t) = q^j(t) + \dot{q}^j(t) \Delta t$

$$q^j(t+\Delta t) = q^j(t) + \dot{q}^j(t) \Delta t$$

If a better routine is available, use it.

ii For the computation of structural loads due to aerodynamic forces, including only the terms called for in data submittal 8 (p. 113) and for which $\alpha_{r,h} > 0$,

$$R_i^{jqtu} = \sum_h n_{ih}^{jq} n_{ih}^{it} n_{ih}^{iu} S_{ih} \quad (q, t, u = 1, 2, 3)$$

$$U_{ki}^{jqt1} = \alpha_{ki}^{j2} R_i^{jqt3} - \alpha_{ki}^{j3} R_i^{jqt2} + \sum_h n_{ih}^{jq} n_{ih}^{it} T_{kih}^{j1} S_{ih}$$

$$U_{ki}^{jqt2} = \alpha_{ki}^{j2} R_i^{jqt1} - \alpha_{ki}^{j1} R_i^{jqt3} + \sum_h n_{ih}^{jq} n_{ih}^{it} T_{kih}^{j2} S_{ih}$$

$$U_{ki}^{jqt3} = \alpha_{ki}^{j1} R_i^{jqt2} - \alpha_{ki}^{j2} R_i^{jqt1} + \sum_h n_{ih}^{jq} n_{ih}^{it} T_{kih}^{j3} S_{ih}$$

$$T_{ki}^{jqt} = \sum_h n_{ih}^{jq} n_{ih}^{it} S_{kih} S_{ih}$$

$$A_i^{jprs} = \sum_{q=1}^3 \sum_{t=1}^3 \sum_{u=1}^3 e_{qi}^p e_{ki}^r e_{ui}^s R_i^{jqtu}$$

$$B_{ki}^{jprs} = \sum_{q=1}^3 \sum_{t=1}^3 \sum_{u=1}^3 e_{qi}^p e_{ki}^r e_{ui}^s U_{ki}^{jqtu}$$

$$C_{ki}^{jpr} = \sum_{q=1}^3 \sum_{t=1}^3 e_{qi}^p e_{ki}^r T_{ki}^{jqt}$$

$$SA_{R_i}^r = -e \sum_{s=1}^3 \sum_{t=1}^3 V^s V^t A_i^{rst} \quad (r=1,2,3)$$

$$SA_{E_i}^r = -2e \sum_{k=1}^n \sum_{s=1}^3 V^s \left(\sum_{t=1}^3 V^t B_{ki}^{rst} q^k + C_{ki}^{rs} q^k \right)$$

$$SA_{R_{k,i}}^r = -e \sum_{s=1}^3 \sum_{t=1}^3 V^s V^t A_i^{rst}$$

$$SA_{E_{k,i}}^r = -2e \sum_{k=1}^n \sum_{s=1}^3 V^s \left(\sum_{t=1}^3 V^t B_{ki}^{rst} q^k + C_{ki}^{rs} q^k \right)$$

$$R_i^{rstu} = \sum_h v_{ih}^r n_{ih}^s n_{ih}^t n_{ih}^u S_{ih}$$

$$U_{ki}^{rst1} = \alpha_{ki}^{12} R_i^{rst3} - \alpha_{ki}^{13} R_i^{rst2} + \sum_h v_{ih}^r n_{ih}^s n_{ih}^t T_{kih}^1 S_{ih}$$

$$U_{ki}^{rst2} = \alpha_{ki}^{13} R_i^{rst1} - \alpha_{ki}^{11} R_i^{rst3} + \sum_h v_{ih}^r n_{ih}^s n_{ih}^t T_{kih}^2 S_{ih}$$

$$U_{ki}^{rst3} = \alpha_{ki}^{11} R_i^{rst2} - \alpha_{ki}^{12} R_i^{rst1} + \sum_h v_{ih}^r n_{ih}^s n_{ih}^t T_{kih}^3 S_{ih}$$

$$T_{ki}^{rst} = \sum_h v_{ih}^r n_{ih}^s n_{ih}^t S_{k,i,h} S_{i,h}$$

$$A_i^{rst} = \sum_{t=1}^3 \sum_{u=1}^3 e_{ti}^r e_{ui}^s \left[e_{ii}^n (R_i^{rstu} - R_i^{srtu}) + e_{ii}^n (R_i^{rsut} - R_i^{rstu}) + e_{ii}^n (R_i^{ruts} - R_i^{rstu}) \right]$$

$$B_{ki}^{rst} = \sum_{t=1}^3 \sum_{u=1}^3 e_{ti}^r e_{ui}^s \left[e_{ii}^n (U_{ki}^{rstu} - U_{ki}^{srtu}) + e_{ii}^n (U_{ki}^{rsut} - U_{ki}^{rstu}) + e_{ii}^n (U_{ki}^{ruts} - U_{ki}^{rstu}) \right]$$

$$C_{ki}^{nr} = \sum_c^3 e_{ci}^r [e_{ii}^n (T_{ki}^{n23t} - T_{ki}^{n32t}) \\ + e_{2i}^n (T_{ki}^{n31t} - T_{ki}^{n13t}) + e_{3i}^n (T_{ki}^{n2t} - T_{ki}^{n2it})]$$

$$N_{Ri}^r = -\rho \sum_{s=1}^3 \sum_{c=1}^3 v^s v^c A_i^{rst}$$

$$N_{Ei}^r = -2\rho \sum_{k=1}^n \sum_{s=1}^3 v^s \left(\sum_{c=1}^3 v^c B_{ki}^{rst} + C_{ki}^{rs} q^k \right)$$

$$N_{Rki}^r = -\rho \sum_{s=1}^3 \sum_{c=1}^3 v_s^s v_c^c A_i^{rst}$$

$$N_{Eki}^r = -2\rho \sum_{k=1}^n \sum_{s=1}^3 v_k^s \left(\sum_{c=1}^3 v_c^c B_{ki}^{rst} + C_{ki}^{rs} q^k \right)$$

$$SA_R^r = \sum_i SA_{Ri}^r$$

$$SA_E^r = \sum_i SA_{Ei}^r$$

$$SA_{kl}^r = \sum_i SA_{Rki}^r$$

$$SA_{Ekl}^r = \sum_i SA_{Eki}^r$$

$$MA_{R0}^1 = \sum_i (z_i^2 SA_{Ri}^3 - z_i^3 SA_{Ri}^2 + N_{Ri}^1)$$

$$MA_{R0}^2 = \sum_i (z_i^3 SA_{Ri}^1 - z_i^1 SA_{Ri}^3 + N_{Ri}^2)$$

$$MA_{R0}^3 = \sum_i (z_i^1 SA_{Ri}^2 - z_i^2 SA_{Ri}^1 + N_{Ri}^3)$$

$$MA_{E0}^1 = \sum_i (z_i^2 SA_{Ei}^3 - z_i^3 SA_{Ei}^2 + N_{Ei}^1)$$

$$MA_{E0}^2 = \sum_i (z_i^3 SA_{Ei}^1 - z_i^1 SA_{Ei}^3 + N_{Ei}^2)$$

$$MA_{CO}^3 = \sum_i (z_i^1 SA_{Ei}^2 - z_i^2 SA_{Ei}^1 + N_{Ei}^3)$$

$$MA_{RO}^1 = \sum_i (z_i^2 SA_{Ri}^3 - z_i^3 SA_{Ri}^2 + N_{Ri}^1)$$

$$MA_{RO}^2 = \sum_i (z_i^3 SA_{Ri}^1 - z_i^1 SA_{Ri}^3 + N_{Ri}^2)$$

$$MA_{RO}^3 = \sum_i (z_i^1 SA_{Ri}^2 - z_i^2 SA_{Ri}^1 + N_{Ri}^3)$$

$$MA_{EO}^1 = \sum_i (z_i^2 SA_{Ei}^3 - z_i^3 SA_{Ei}^2 + N_{Ei}^1)$$

$$MA_{EO}^2 = \sum_i (z_i^3 SA_{Ei}^1 - z_i^1 SA_{Ei}^3 + N_{Ei}^2)$$

$$MA_{EO}^3 = \sum_i (z_i^1 SA_{Ei}^2 - z_i^2 SA_{Ei}^1 + N_{Ei}^3)$$

12. For the computation of structural loads due to thrust forces, including only the engines called for in data submittal 8 (p. 113),

$$SJ_R^1 = \sum_i T_{xi}$$

$$SJ_R^2 = \sum_i T_{yi}$$

$$SJ_R^3 = \sum_i T_{zi}$$

$$MJ_{RO}^1 = \sum_i (z_i^2 T_{zi} - z_i^3 T_{yi})$$

$$MJ_{RO}^2 = \sum_i (z_i^3 T_{xi} - z_i^1 T_{zi})$$

$$MJ_{\text{rod}}^3 = \sum (z_i^1 T_{y_i} - z_i^2 T_{x_i})$$

13 For the computation of structural loads due to inertial forces, including only the tanks, sections, and particles called for in data submittal 8 (p. 113).

$$A^1 = \dot{V}_c^1 + \Omega^2 V^3 - \Omega^3 V^2 - \dot{\Omega}^2 z_c^3 + \dot{\Omega}^3 z_c^2$$

$$A^2 = \dot{V}_c^2 + \Omega^3 V^1 - \Omega^1 V^3 - \dot{\Omega}^3 z_c^1 + \dot{\Omega}^1 z_c^3$$

$$A^3 = \dot{V}_c^3 + \Omega^1 V^2 - \Omega^2 V^1 - \dot{\Omega}^1 z_c^2 + \dot{\Omega}^2 z_c^1$$

$$m_i^1 = \sum_k m_{ik}$$

m_i^1 : if all particles of a section are used
 $= M_{F_1}$ for tank i.

$$\tau_i^{1r} = \sum_k m_{ik} v_{ik}^{1r}$$

= 0 if all particles are used, or for a tank

$$\psi_{ki}^{1r} = \sum_k m_{ik} \sigma_{kik}^{1r}$$

= 0 if all particles are used, or for a tank

$$Y_i^r = m_i^1 z_i^r + \sum_{s=1}^3 e_{si}^r \tau_i^{1s} \quad \text{for a structural section}$$

$\gamma^r = m_i \gamma_F^r$ for a tank (that is, the fuel in the tank)

$$\gamma^r = \sum_i \gamma_i^r$$

$$m' = \sum_i m_i$$

$$\gamma_{kl}^r = \sum_i \left[m_i h_{ki}^r + \sum_{s=1}^3 e_{si}^r \psi_{ki}^{s3} + e_{ii}^r (\alpha_{ki}^{12} \tau_i^{13} - \alpha_{ki}^{13} \tau_i^{12}) \right. \\ \left. + e_{2i}^r (\alpha_{ki}^{13} \tau_i^{11} - \alpha_{ki}^{11} \tau_i^{13}) + e_{3i}^r (\alpha_{ki}^{12} \tau_i^{12} - \alpha_{ki}^{12} \tau_i^{11}) \right]$$

$$\gamma_{kl}^r = \sum_i \left[e_{ii}^r (\alpha_{ki}^{12} \psi_{li}^{13} - \alpha_{ki}^{13} \psi_{li}^{12} + \alpha_{li}^{12} \psi_{ki}^{13} - \alpha_{li}^{13} \psi_{ki}^{12}) \right. \\ \left. + e_{2i}^r (\alpha_{ki}^{13} \psi_{li}^{11} - \alpha_{ki}^{11} \psi_{li}^{13} + \alpha_{li}^{13} \psi_{ki}^{11} - \alpha_{li}^{11} \psi_{ki}^{13}) \right. \\ \left. + e_{3i}^r (\alpha_{ki}^{12} \psi_{li}^{12} - \alpha_{ki}^{12} \psi_{li}^{11} + \alpha_{li}^{12} \psi_{ki}^{12} - \alpha_{li}^{12} \psi_{ki}^{11}) \right. \\ \left. + \frac{1}{2} \sum_{s=1}^3 \sum_{t=1}^3 e_{st}^r (\alpha_{ki}^{1s} \alpha_{li}^{1t} + \alpha_{li}^{1s} \alpha_{ki}^{1t}) \tau_i^{1t} \right. \\ \left. - \left(\sum_{s=1}^3 e_{si}^r \tau_i^{1s} \right) \left(\sum_{t=1}^3 \alpha_{ki}^{1t} \alpha_{li}^{1t} \right) \right]$$

$$S I_R^1 = -A^1 m' - \Omega^1 \sum_{s=1}^3 \Omega^s \gamma^s + \gamma^1 \sum_{s=1}^3 \Omega^s \Omega^s - \Omega^2 \gamma^3 + \Omega^3 \gamma^2$$

$$S I_R^2 = -A^2 m' - \Omega^2 \sum_{s=1}^3 \Omega^s \gamma^s + \gamma^2 \sum_{s=1}^3 \Omega^s \Omega^s - \Omega^3 \gamma^1 + \Omega^1 \gamma^3$$

$$S I_R^3 = -A^3 m' - \Omega^3 \sum_{s=1}^3 \Omega^s \gamma^s + \gamma^3 \sum_{s=1}^3 \Omega^s \Omega^s - \Omega^1 \gamma^2 + \Omega^2 \gamma^1$$

$$S I_E^1 = -2 \sum_{k=1}^n (\Omega^2 \gamma_k^3 - \Omega^3 \gamma_k^2) \dot{q}^k - \sum_{k=1}^n \sum_{l=1}^n \gamma_{kl}^1 \dot{q}^k \dot{q}^l - \sum_{k=1}^n \gamma_k^1 \dot{q}^k$$

$$S I_E^2 = -2 \sum_{k=1}^n (\Omega^3 \gamma_k^1 - \Omega^1 \gamma_k^3) \dot{q}^k - \sum_{k=1}^n \sum_{l=1}^n \gamma_{kl}^2 \dot{q}^k \dot{q}^l - \sum_{k=1}^n \gamma_k^2 \dot{q}^k$$

$$S I_E^3 = -2 \sum_{k=1}^n (\Omega^1 Y_k^2 - \Omega^2 Y_k^1) \dot{q}^k - \sum_{k=1}^n \sum_{l=1}^n Y_{kl}^3 q^k \dot{q}^l - \sum_{k=1}^n Y_k^3 q^k$$

$$\Pi_{rs} = \sum_k m_{ik} v_{ik}^r v_{ik}^s$$

= Γ_{rsi} if all particles of i -section are used

= Γ_{rsi} for tank i .

$$Y_i^{rs} = z_i^r Y_i^{rs} + \sum_{t=1}^3 e_{ti}^r \Pi_{tsi}$$

$$\Phi_{rs} = \sum_i (Y_i^r z_i^s + \sum_{t=1}^3 Y_i^{rt} e_{ti}^s)$$

$$\Pi_{rs} = \delta_{rs} \sum_{t=1}^3 \Phi_{tt} - \Phi_{rs}, \text{ where } \delta_{rs} = 1 \text{ when } r = s$$

= 0 when $r \neq s$

$$M I_{R\theta}^1 = A^2 Y^3 - A^3 Y^2 - \sum_{s=1}^3 (\Omega^2 \Pi_{3s} - \Omega^3 \Pi_{2s}) \Omega^s - \sum_{s=1}^3 \Pi_{1s} \dot{\Omega}^s$$

$$M I_{R\theta}^2 = A^3 Y^1 - A^1 Y^3 - \sum_{s=1}^3 (\Omega^3 \Pi_{1s} - \Omega^1 \Pi_{3s}) \Omega^s - \sum_{s=1}^3 \Pi_{2s} \dot{\Omega}^s$$

$$M I_{R\theta}^3 = A^1 Y^2 - A^2 Y^1 - \sum_{s=1}^3 (\Omega^1 \Pi_{2s} - \Omega^2 \Pi_{1s}) \Omega^s - \sum_{s=1}^3 \Pi_{3s} \dot{\Omega}^s$$

$$A_{ki}^{rst} = \sum_k m_{ik} v_{ik}^r v_{ik}^s \sigma_{kik}^{rt}$$

= Λ_{ki}^{rst} if all particles are used, or $\hat{\Lambda}$ for a tank

$$q_{ki}^{rs} = \delta_{rs} \sum_{t=1}^3 \Lambda_{ki}^{rtt} - \Lambda_{ki}^{rs}$$

$$W_{kl}^r = \sum_i \left[\sum_{s=1}^3 \sum_{t=1}^3 \sum_{u=1}^3 e_{si}^r e_{ti}^s z_i^t (\alpha_{ki}^{rs} \psi_{li}^{tu} - \psi_{li}^{rs} \alpha_{ki}^{tu}) \right]$$

(continued on next page)

$$+ W_{KL}^r + \sum_{s=1}^3 \sum_{t=1}^3 \alpha_{ki}^{ts} \gamma_{it}^t e_{it}^r$$

$$- \frac{1}{2} \sum_{s=1}^3 (\alpha_{Li}^{12} \Pi_{s3i} - \alpha_{Li}^{13} \Pi_{s2i}) \alpha_{ki}^{1s}$$

$$- \frac{1}{2} e_{2i}^r \sum_{s=1}^3 (\alpha_{Li}^{13} \Pi_{s1i} - \alpha_{Li}^{11} \Pi_{s3i}) \alpha_{ki}^{1s}$$

$$- \frac{1}{2} e_{3i}^r \sum_{s=1}^3 (\alpha_{Li}^{11} \Pi_{s2i} - \alpha_{Li}^{12} \Pi_{s1i}) \alpha_{ki}^{1s} \Big],$$

$$\text{where } W_{KL}^1 = \frac{1}{2} \sum_{s=1}^3 \sum_{t=1}^3 (z_i^2 e_{si}^3 - z_i^3 e_{si}^2) (\alpha_{Li}^{12} T_i^{1t} - T_i^{1s} \alpha_{Li}^{1t}) \alpha_{ki}^{1t}$$

$$W_{KL}^2 = \frac{1}{2} \sum_{s=1}^3 \sum_{t=1}^3 (z_i^3 e_{si}^1 - z_i^1 e_{si}^3) (\alpha_{Li}^{13} T_i^{1t} - T_i^{1s} \alpha_{Li}^{1t}) \alpha_{ki}^{1t}$$

$$\text{and } W_{KL}^3 = \frac{1}{2} \sum_{s=1}^3 \sum_{t=1}^3 (z_i^1 e_{si}^2 - z_i^2 e_{si}^1) (\alpha_{Li}^{11} T_i^{1t} - T_i^{1s} \alpha_{Li}^{1t}) \alpha_{ki}^{1t}$$

$$Y_{ki}^{rs} = z_i^r \psi_{ki}^{1s} + \sum_{t=1}^3 e_{it}^r \Delta_{ki}^{1ts}$$

$$D_{rsk} = \sum_i \left[Y_i^r h_{ki}^s + \sum_{t=1}^3 Y_{ki}^{rt} e_{it}^s + (Y_i^{r2} e_{si}^s - Y_i^{r3} e_{2i}^s) \alpha_{ki}^{11} \right. \\ \left. + (Y_i^{r3} e_{1i}^s - Y_i^{r1} e_{3i}^s) \alpha_{ki}^{12} + (Y_i^{r1} e_{2i}^s - Y_i^{r2} e_{1i}^s) \alpha_{ki}^{13} \right]$$

$$P_{rsk} = \delta_{rs} \sum_{t=1}^3 \Theta_{tck} - D_{rsk}$$

$$H_{rsi} = \delta_{rs} \sum_{t=1}^3 \Pi_{tci} - \Pi_{rsi}$$

$$R_k^1 = \sum_i \left[Y_i^2 h_{ki}^3 - Y_i^3 h_{ki}^2 + \sum_{r=1}^3 (Y_{ki}^{2r} e_{ri}^3 - Y_{ki}^{1r} e_{ri}^2) \right]$$

$$R_k^2 = \sum_i \left[Y_i^3 h_{ki}^1 - Y_i^1 h_{ki}^3 + \sum_{r=1}^3 (Y_{ki}^{3r} e_{ri}^1 - Y_{ki}^{1r} e_{ri}^3) \right]$$

$$R_k^3 = \sum_i \left[Y_i^1 h_{ki}^2 - Y_i^2 h_{ki}^1 + \sum_{r=1}^3 (Y_{ki}^{1r} e_{ri}^2 - Y_{ki}^{2r} e_{ri}^1) \right]$$

$$R_k^r = R_k^r + \sum_i \left[\sum_{s=1}^3 \sum_{t=1}^3 e_{it}^r H_{sti} \alpha_{ki}^{1s} + \sum_{s=1}^3 \sum_{t=1}^3 \sum_{u=1}^3 e_{it}^r e_{iu}^s z_i^t (\alpha_{Li}^{1s} T_i^{1u} - T_i^{1s} \alpha_{Li}^{1u}) \right]$$

$$M_{\underline{I}}^r = -2 \left(\sum_{k=1}^n \sum_{s=1}^3 \Omega^s \Pi_{srk} q^k + \sum_{k=1}^n \sum_{s=1}^3 V_{srk} q^k \right) - \sum_{k=1}^n R_k^r \ddot{q}^k$$

For the computation of structural loads due to gravity,

$$S_{G^r} = g_a^r m' \quad (r = 1, 2, 3)$$

$$M_{G^1} = y^2 g_a^3 - y^3 g_a^2$$

$$M_{G^2} = y^3 g_a^1 - y^1 g_a^3$$

$$M_{G^3} = y^1 g_a^2 - y^2 g_a^1$$

14. For the computation of structural loads to be printed,

$$y_{eg}^r = z_e^r + \sum_{s=1}^3 e_{ese}^r v_{eg}^s$$

Case 1

$$S_{RI}^r = S_{A_R}^r + S_{I_R}^r + S_{G^r}$$

$$M_{RI}^1 = M_{A_{R0}}^1 + M_{I_{R0}}^1 + M_{G^1} - y_{eg}^2 S_{RI}^3 + y_{eg}^3 S_{RI}^2$$

$$M_{RI}^2 = M_{A_{R0}}^2 + M_{I_{R0}}^2 + M_{G^2} - y_{eg}^3 S_{RI}^1 + y_{eg}^1 S_{RI}^3$$

$$M_{RI}^3 = M_{A_{R0}}^3 + M_{I_{R0}}^3 + M_{G^3} - y_{eg}^1 S_{RI}^2 + y_{eg}^2 S_{RI}^1$$

Case 2

$$S_{R2}^1 = S_{RI}^r + S_{JR}^r$$

$$M_{RE}^1 = M_{RI}^1 + M_{JR}^1 - y_{eg}^2 S_{JR}^3 + y_{eg}^3 S_{JR}^2$$

$$M_{RE}^2 = M_{RI}^2 + M_{JR}^2 - y_{eg}^3 S_{JR}^1 + y_{eg}^1 S_{JR}^3$$

$$M_{RE}^3 = M_{RI}^3 + M_{JR}^3 - y_{eg}^1 S_{JR}^2 + y_{eg}^2 S_{JR}^1$$

Case 3

$$S_{RL}^r = S_{ARL}^r + S_{JR}^r + S_{IR}^r + S_{GR}^r$$

$$M_{RL}^1 = M_{ARL}^1 + M_{JR}^1 + M_{IR}^1 + M_{GR}^1 - y_{eg}^2 S_{RL}^3 + y_{eg}^3 S_{RL}^2$$

$$M_{RL}^2 = M_{ARL}^2 + M_{JR}^2 + M_{IR}^2 + M_{GR}^2 - y_{eg}^3 S_{RL}^1 + y_{eg}^1 S_{RL}^3$$

$$M_{RL}^3 = M_{ARL}^3 + M_{JR}^3 + M_{IR}^3 + M_{GR}^3 - y_{eg}^1 S_{RL}^2 + y_{eg}^2 S_{RL}^1$$

Case 4

$$S_E^r = S_{AE}^r + S_{IE}^r \quad (\text{do not print})$$

$$S^r = S_{RE}^r + S_E^r$$

$$M^1 = M_{RE}^1 + M_{AE}^1 + M_{IE}^1 - y_{eg}^2 S_E^3 + y_{eg}^3 S_E^2$$

$$M^2 = M_{RE}^2 + M_{AE}^2 + M_{IE}^2 - y_{eg}^3 S_E^1 + y_{eg}^1 S_E^3$$

$$M^3 = M_{RE}^3 + M_{AE}^3 + M_{IE}^3 - y_{eg}^1 S_E^2 + y_{eg}^2 S_E^1$$

Case 5

$$S_{EL}^r = SA_{EL}^r + SI_E^r \quad (\text{do not print})$$

$$C_B^r = S_{RL}^r + S_{EL}^r$$

$$M_B^1 = M_{RL}^1 + MA_{EL\theta}^1 + MI_{E\theta}^1 - y_{E\theta}^1 S_{EL}^1 + y_{E\theta}^2 S_{EL}^2$$

$$M_B^2 = M_{RL}^2 + MA_{EL\theta}^2 + MI_{E\theta}^2 - y_{E\theta}^2 S_{EL}^2 + y_{E\theta}^3 S_{EL}^3$$

$$M_B^3 = M_{RL}^3 + MA_{EL\theta}^3 + MI_{E\theta}^3 - y_{E\theta}^3 S_{EL}^3 + y_{E\theta}^4 S_{EL}^4$$

15. For the computation of accelerations and deflections, including only the points designated in data submittal 9 (p. 113),

$$y_{ih}^r = z_i^r + \sum_{s=1}^3 e_s^r \cdot v_{is}^r$$

Cases 1, 2, 3 (Use A^1, A^2, A^3 from 13, p. 147)

$$A_{R,i,h}^1 = A^1 - y_{ih}^1 (\Omega^2 \Omega^2 + \Omega^3 \Omega^3) + y_{ih}^2 (\Omega^1 \Omega^2 - \Omega^3) + y_{ih}^3 (\Omega^3 \Omega^1 - \Omega^2)$$

$$A_{R,i,h}^2 = A^2 - y_{ih}^2 (\Omega^3 \Omega^3 + \Omega^1 \Omega^1) + y_{ih}^3 (\Omega^2 \Omega^3 - \Omega^1) + y_{ih}^4 (\Omega^1 \Omega^2 + \Omega^3)$$

$$A_{R,i,h}^3 = A^3 - y_{ih}^3 (\Omega^1 \Omega^1 + \Omega^2 \Omega^2) + y_{ih}^4 (\Omega^3 \Omega^1 - \Omega^2) + y_{ih}^5 (\Omega^2 \Omega^3 - \Omega^1)$$

(Print the above 3 quantities for each point)

Cases 4&5

$$T_{kih}^r = h_{ki}^r + \sum_{s=1}^3 e_{si}^r \sigma_{kih}^{rs} \quad \begin{array}{l} e_{1i}^r \alpha_{ki}^{r1} \sigma_{kih}^{r1} \\ e_{2i}^r \alpha_{ki}^{r2} \sigma_{kih}^{r2} \\ e_{3i}^r \alpha_{ki}^{r3} \sigma_{kih}^{r3} \end{array}$$

$$D_{ih}^r = \sum_{k=1}^n T_{kih}^r q^k \quad (\text{print})$$

$$v_{Eih}^r = \sum_{k=1}^n T_{kih}^r \dot{q}^k$$

$$w_{Eih}^r = \sum_{k=1}^n T_{kih}^r \ddot{q}^k$$

$$\Phi_{kljh}^r = \frac{1}{2} \sum_{s=1}^3 \sum_{t=1}^3 e_{si}^r \alpha_{ki}^{rt} (\alpha_{li}^{st} v_{ih}^{it} - \alpha_{li}^{st} \dot{v}_{ih}^{it})$$

$$+ \begin{vmatrix} e_{1i}^r & \alpha_{ki}^{r1} & \sigma_{lih}^{r1} \\ e_{2i}^r & \alpha_{ki}^{r2} & \sigma_{lih}^{r2} \\ e_{3i}^r & \alpha_{ki}^{r3} & \sigma_{lih}^{r3} \end{vmatrix}$$

$$\chi_{Eih}^r = 2 \sum_{k=1}^n \sum_{l=1}^n \Phi_{kljh}^r \dot{q}^k \dot{q}^l$$

$$A_{Eih}^1 = 2 (\Omega^1 v_{Eih}^1 - \Omega^2 v_{Eih}^2) + w_{Eih}^1 \cdot 1^1$$

$$A_{Eih}^2 = 2 (\Omega^2 v_{Eih}^2 - \Omega^3 v_{Eih}^3) + w_{Eih}^2 \cdot 1^2$$

$$A_{Eih}^3 = 2 (\Omega^3 v_{Eih}^3 - \Omega^1 v_{Eih}^1) + w_{Eih}^3 \cdot 1^3$$

$$\alpha_{ih}^r = A_{Rih}^r - A_{Eih}^r \quad (\text{print})$$