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REPORT 372

ADVISORY GROUP FOR AERONAUTICAL RESEARCH AND DEVELOPMENT

64 RUE DE VARENNE, PARIS VII

REPORT 372

THEORY OF THE FLIGHT [redacted] AIRPLANES [redacted] ISOTROPIC TURBULENCE-
REVIEW AND EXTENSION

by

B. ETKIN

APRIL 1961

NORTH ATLANTIC TREATY ORGANISATION

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SUMMARY

Recent experimental information on low-level atmospheric turbulence is first reviewed. It is suggested that the assumptions of homogeneity and isotropy customarily adopted for high altitudes are still useful in this régime, and that the integral scale is roughly equal to 9/10 of the altitude up to about 1000 ft. Next, the previously published theory of the 'power-series approximation' as applied to the vertical component of the gust is extended to include all three velocity components simultaneously. Fourteen different one-dimensional input power spectra and cross spectra are found of which only five are important. Of these five, only one is a cross-spectrum involving two different velocity components (u and v) . Formulae for them are calculated and curves are presented. The 'gust derivatives' required for calculating airplane response are defined and discussed, and the most important ones are shown to be simply the negatives of classical stability derivatives. Methods of approach for calculating the remaining ones are suggested. Finally, it is shown that the dispersion, or probable error of position, is fundamentally different when the controlled variable is velocity or heading than when it is position.

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NOTATION

$[A_1]$, $[A_2]$	matrices of aerodynamic gust derivatives
A, B, C	airplane moments of inertia
$[B_1]$, $[B_2]$	matrices of equations of motion
E	airplane product of inertia
$\{F_1\}$, $\{F_2\}$	column matrices
$\{g_1\}$, $\{g_2\}$	column matrix of gust inputs
G_{ni}	overall transfer function relating n^{th} output to i^{th} gust input
$[G_1]$, $[G_2]$	matrices of overall transfer functions
h	altitude
H_e, H_a, H_r	hinge moment on elevator, aileron, rudder, respectively
I_e, I_a, I_r	effective inertia of elevator, aileron, rudder systems, respectively
k_i	dimensionless wave number $L\Omega_i$
k	$= (1 + k_1^2 + k_2^2 + k_3^2)^{1/2}$
k_2'	$= L\Omega_2'$
k'	$= (1 + k_1^2 + k_2'^2)^{1/2}$
L	integral scale of turbulence
l	characteristic dimension of airplane
l_t	tail length
l_1	overall length of wing
L, M, N	aerodynamic moments acting on airplane
m	mass of airplane
p, q, r	angular velocity components of airplane
s	Laplace transform variable
t	time
$d\vec{U}$	vector amplitude of elementary spectral component

(u_1, u_2, u_3)	velocity components of aircraft $[(\equiv (u, v, w)]$
(u'_1, u'_2, u'_3)	velocity components of atmosphere $[(\equiv (u', v', w')]$
u_0	reference (mean) speed of airplane
u'_x	$= \partial u'/\partial x$, and similarly for the remaining elements of the gust input matrices $\{g_1\}$ and $\{g_2\}$
$\vec{x}$	position vector
(x_1, x_2, x_3)	air-fixed co-ordinates
(x, y, z)	body-fixed co-ordinates
X, Y, Z	components of resultant aerodynamic force
$X_{u'}$	$= \bar{X}(s)/\bar{u}'(s)$, and similarly for remaining elements of the gust-derivative matrices $[A_1]$ and $[A_2]$
$\vec{\Omega}$	wave-number vector
Ω_i	component wave number, $2\pi/\lambda_i$
λ_i	component wave length
σ	root mean square gust velocity
ω	circular frequency $(= \Omega_1 u_0)$, rad/sec
θ	pitch angle
φ	bank angle
ξ, η, ζ	control surface angles
Λ	angle of sweep
Γ	dihedral angle
$\Phi_{ij}(k_1, k_2, k_3)$	three-dimensional spectrum function of $u_i u_j$
$\Phi_{\alpha\beta}(k_1, k_2, k_3)$	three-dimensional spectrum function of $\alpha\beta$
$\Theta_{\alpha\beta}(k_1)$	one-dimensional spectrum function of $\alpha\beta$
$(\bar{\ })$	denotes Laplace transform

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THEORY OF THE FLIGHT OF AIRPLANES IN ISOTROPIC TURBULENCE - REVIEW AND EXTENSION

B. Etkin*

1. INTRODUCTION

The flight of airplanes through turbulent air has been a subject of prime concern to aeronautical engineers since the beginning of flight itself. The attendant problems of structural integrity, flying qualities and performance receive continuing study. The application of statistical methods during the past decade, in particular the methods of power-spectral analysis and the theory of isotropic turbulence, have brought about a significant advance in our understanding of these problems.

The theoretical approaches to analysis fall into two categories, according to the manner of specifying the 'unit' element of the gust. The first† uses a 'gust impulse' as the basic element, as shown in Figure 1. References 1, 2 and 3 are representative of analyses based on this method. The second approach†† uses the elementary spectral (Fourier) component illustrated in Figure 2 as the basic element. This is the one which has been taken in References 4, 5 and 6. It should be emphasized that there is no fundamental opposition between the two formulations; both can in principle lead to the same results, the accuracy of which depends not on this choice, but rather on the details of the approximations subsequently made in the analysis. It is the opinion of the author that the second method has some advantages, viz:

(1) The mathematical formulation is simpler, and hence easier to understand and to use;

(2) It is easier to separate the elements of the theory that are essentially wing theory from those that are essentially the representation of the turbulence;

(3) By using the power-series approximation of References 5 and 6, extended herein, the accumulated knowledge of aerodynamics embodied in stability and flutter derivatives is easily incorporated;

(4) Approximations involving certain parts of the frequency spectrum are easily incorporated.

This report presents a brief review of the information on atmospheric turbulence in Section 2. It follows in Section 3 with a semi-qualitative description of the two basic methods of analysis mentioned above. Section 4 contains an extension and generalization of Reference 6 to cover the case of simultaneous inputs of all three gust components, and Section 5 presents some information on the flight path of a vehicle flying in isotropic turbulence.

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* *Professor of Aeronautical Engineering, University of Toronto, Canada*

† Due to H.W. Liepmann, Reference 1

†† Due to H.S. Ribner, Reference 4

2. THE STRUCTURE OF ATMOSPHERIC TURBULENCE

It is obvious that if we wish to study flight in turbulent air theoretically we must know enough about atmospheric turbulence to construct a reasonable mathematical model of it. For this purpose, the atmosphere close to the ground (in the boundary layer produced by the wind) needs to be considered separately from that higher up.

2.1 Outside the Boundary Layer

There is little to be added to the picture of turbulence at higher altitudes which has already been so competently given by Press and his co-workers at the N.A.S.A.[7,8] (An abbreviated account is given in Reference 5). In short, it is a reasonable approximation to regard high-level turbulence as homogeneous and isotropic in patches of limited extent. The velocity of airplanes is normally sufficiently high that the turbulent field within one of these patches may be regarded as constant during the time of passage; and the statistical properties of the input to the airplane are assumed to be independent of the response of the airplane itself, i.e. they are the same as would be obtained in rectilinear translation at constant speed. (This is not to say that the response of the airplane is neglected in calculating the aerodynamic forces. The forces associated with motion of the airplane are included as usual.) The probability distribution of the intensity σ of the turbulent patches is dependent on the route, season, altitude, etc. The one-dimensional spectrum function for the lateral component of the turbulence which is now widely accepted is

$$\Theta_{33}(k_1) = \Theta_{22}(k_1) = \frac{\sigma^2 L}{2\pi} \frac{1 + 3k_1^2}{(1 + k_1^2)^2} \tag{1}$$

According to the theory of isotropic turbulence[9] the above is derivable from the more basic energy spectrum-function. The latter is

$$E(k) = \frac{8}{\pi} \sigma^2 L \frac{k^4}{(k^2 + 1)^3} \tag{2}$$

In terms of $E(k)$, the one-dimensional spectrum is calculated from the relations

$$\Theta_{ij}(k_1) = \frac{1}{L^2} \iint_{-\infty}^{\infty} \Phi_{ij}(k_1, k_2, k_3) dk_2 dk_3 \tag{3}$$

and

$$\Phi_{ij}(k_1, k_2, k_3) = \frac{L^2}{4\pi} \frac{E(k)}{k^4} (k^2 \delta_{ij} - k_i k_j) \tag{4}$$

where δ_{ij} is the Kronecker delta. When Equation (2) is substituted into Equation (4) we get

$$\Phi_{ij}(k_1, k_2, k_3) = \frac{2}{\pi^2} \sigma^2 L^3 \frac{k^2 \delta_{ij} - k_i k_j}{(k^2 + 1)^3} \tag{5}$$

and Equation (3) becomes, for the particular energy spectrum adopted,

$$\Theta_{ij}(k_1) = \frac{2\sigma^2 L}{\pi^2} \iint_{-\infty}^{\infty} \frac{k^2\delta_{ij} - k_i k_j}{(k^2 + 1)^3} \, dk_2 dk_3 \tag{6}$$

From Equation (6) we may also obtain the companion to Equation (1), i.e. the longitudinal one-dimensional spectrum

$$\Theta_{11}(k_1) = \frac{\sigma^2 L}{\pi} \frac{1}{1 + k_1^2} \tag{7}$$

The cross-spectra Θ_{12}, Θ_{23}, Θ_{31} are all zero, since for these cases the integrand of Equation (6) is antisymmetrical with respect to one or both of k_2, k_3.

Unfortunately, there is insufficient information available on the scale L of the turbulence in the atmosphere. The value L = 1000 ft has been assumed by Press and others to be reasonably representative but much more experimental information is needed. It should be pointed out that this is a very important parameter, since it may exert a dominant influence on the energy available at the resonant frequencies of the airplane. This effect is shown in Figure 3, taken from a Douglas Company report[10]. Furthermore, the accuracy of the power-series method (Sec.5) is dependent on the ratio of airplane size to turbulence scale.

2.2 Near the Ground

At low levels, the turbulence resembles that which occurs in boundary layers adjacent to rough surfaces and is strongly affected by the terrain. The scale and intensity both vary rapidly with height above the ground, and in general the field is neither homogeneous nor isotropic. A number of measurements have recently been reported of statistical properties of low-level turbulence[10-13] from which two useful general conclusions can be drawn. The first is that Equations (1) and (7) are fair approximations to the lateral and longitudinal one-dimensional spectrum functions. The second is that the scale factor L in these equations, up to 1000 ft altitude, may be approximated roughly by

$$L = 0.9 h \tag{8}$$

where h is the altitude. The evidence for these conclusions is given in Figures 4 to 6. Figures 4 and 5 show comparisons made in the USAF-supported Douglas study between measured spectra, and those given by the equations. The agreement as to shape is encouraging. Figure 6, which contains more detail at the low wave numbers, is another comparison, using $k_1\Theta(k_1)$ as the ordinate, and the ratio altitude/wavelength as abscissa. The experimental data are those of Panofsky[11] and the heavy line is Equation (1) with $L = 0.93$ h. This value of L corresponds to a maximum of the curve at $h = 0.25$. This seems to give the shape of the experimental curves well enough at heights as diverse as 1 metre and 300 metres. No importance should be attached to the actual ordinates of the curves in these Figures, since none of them has been normalized, and there are wide variations in σ (which is the area under the curve when plotted to linear co-ordinate scales); only the shapes are significant. Panofsky also gives a semi-empirical formula for the variation of intensity with height and ground roughness under unstable meteorological conditions. This is

$$\sigma = 0.226 \frac{\bar{V}}{\log h/h_0}$$

where $\bar{V}$ = mean wind at height h

h_0 = characteristic roughness length.

The questions of homogeneity and isotropy are more troublesome. The evidence shows quite clearly that low-level turbulence reflects the nature of the terrain. If the latter is homogeneous and isotropic, then the turbulence will be closely axisymmetric, i.e. independent of rotation about a vertical axis, and homogeneous with respect to translations in the horizontal plane. However, the scale and intensity in general vary with height, and hence the turbulence is not truly isotropic and the theory leading to the one-dimensional spectrum given in Equation (6) is not valid. In spite of this, there would seem to be no recourse, in the present state of the subject, but to use the isotropic model for the low-level case as well as for high altitudes. The complexity of the problem is even then quite sufficient!

Equation (8) indicates that we must be concerned with turbulence having scales as small as 200-300 ft. At such small scales, the variation in gust velocity over the airplane becomes important, and analytical methods of some refinement and complexity are indicated.

3. THE TWO BASIC METHODS OF ANALYSIS

3.1 The 'Impulse' Method

Let $ox_1 x_2 x_3$ be a co-ordinate system so chosen that the mean wind in it is zero, and such that ox_1 is the mean flight path. Let the airplane be regarded as planar, so that only the distribution of atmospheric motion (u_1', u_2', u_3') over the horizontal plane $ox_1 x_2$ is of interest. The impulsive gust element at point (x_1, x_2) then has components

$$u_1' dx_1 dx_2, \quad u_2' dx_1 dx_2, \quad u_3' dx_1 dx_2$$

of which we consider one at a time (as for example in Figure 1). Now let the airplane come under the influence of the gust element when the c.g. is at position $(\xi, 0, 0)$. Then a typical aerodynamic force or moment associated with it, e.g. the Z component (the negative of the lift) is

$$u_3' dx_1 dx_2 h(x_{1_{c.g.}} - \xi, x_2)$$

where $h(\Delta x_1, x_2)$ is the response function for a unit-impulse gust, and is zero for $\Delta x_1 < 0$. The total force $Z(x_{1_{c.g.}})$ acting on the airplane is then obtained by integrating with respect to x_2 across the span and with respect to ξ from $-\infty$ to ∞. The autocorrelation of $Z(x_{1c.g.})$ is next obtained, viz.

$$R_{ZZ}(\Delta x_1) = \frac{1}{\overline{Z^2}} \overline{Z(x_1)Z(x_1 + \Delta x_1)}$$

and finally the spectral density (which is the quantity sought) is obtained by taking the Fourier Transform of the autocorrelation. This procedure entails some quite complicated mathematics. It is worth noting that the basic aerodynamic information is all bound up in the function $h(\Delta x_1, x_2)$.

Thus the method does not lend itself readily to incorporating aerodynamic information (experimental or theoretical) which is in the form of stability or flutter derivatives. There is a large body of such information, and to be able to draw on it easily is an advantage. Furthermore, when we wish to extend the impulse method to include the three velocity components simultaneously, the complexity is further increased by the presence of non-vanishing two-point cross-correlations between the u_1', u_2' and u_3' components.

3.2 The 'Fourier Component' Method

In this method the basic element of the turbulent velocity field is a wave of shearing motion, described by the expression

$$e^{i\vec{\Omega}\vec{x}} d\vec{U}(\vec{\Omega}) \tag{9}$$

The corresponding distribution of downwash over the ox_1x_2 plane, for example, is shown in Figure 7. Once the lift and other relevant aerodynamic forces or moments have been determined for such basic velocity fields, the formalism for writing down the spectra of the inputs to the airplane system is quite straightforward. However, in itself this step does not make the *determination* of the basic lift element any easier. It replaces the problem of finding $h(\Delta x_1, x_2)$ with that of finding the periodic lift (or other force) associated with a running-wave boundary condition. In fact, the latter solutions may be constructed by a suitable integration of the former. Examples of solutions of this kind of wing-lift problem are found in References 14 and 15.

3.2.1 The Power-Series Approximation

A simplifying approximation introduced in Reference 5 and extended in Reference 6 is based on representing the gust-velocity field over the airplane by a modified Taylor series. It was shown in Reference 6 that by keeping terms in the series up to the second order, the velocity distribution can be represented adequately for spectral components whose wavelengths on the two axes (λ_1 and λ_2, Fig.7) exceed twice the corresponding airplane dimension (length or span)*. It was further shown that the cut-off frequency obtained by excising the higher wave numbers is high enough to allow inclusion of important elastic modes, and that the error due to using a

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* It is shown later (Sec.4.1.1) that the wave-length limitation is actually less restrictive than this.

truncated spectrum is not serious provided that the ratio L/l is not less than about 3. The value of l for a large sweptwing airplane is about 100 ft, so the turbulence scale L may be as small as 300 ft for such aircraft. For smaller machines, L may be correspondingly less.

It should be noted that it may frequently not be necessary to retain the second order terms in the power-series development. From the examples shown in Reference 5 it can be seen that cutting off the spectrum at component wave-lengths less than eight times the wing chord and wing span respectively may still provide sufficiently good results for motion in the rigid-body modes. This requires only that the zero order and linear terms of the series be retained. Furthermore, it will be seen in the following that the input spectra associated with the second-order terms are very small.

4. EXTENSION OF THE POWER-SERIES APPROXIMATION

In References 5 and 6 only the vertical component of the turbulence ($u_3' = w'$) was considered to be present. However, the simultaneous occurrence of all three velocity components must be considered for a complete theory. Thus we take as the description of a single spectral component of the gust field the Taylor series

$$u_i' = (u_i')_0 + \left(\frac{\partial u_i'}{\partial x_j}\right)_0 x_j + \frac{1}{2}\left(\frac{\partial^2 u_i'}{\partial x_j \partial x_k}\right)_0 x_j x_k \qquad \begin{matrix} i = 1,2,3 \\ j,k = 1,2 \end{matrix} \tag{10}$$

where the summation convention for repeated suffices is implied. The subscript o denotes the airplane C.G., i.e. the point $(u_0 t, o, o)$. Thus u_i' and its derivatives are periodic, with circular frequency $\Omega_1 u_0 = \omega$.

As in Section 3.1, we consider the airplane to be a planar body, so that only the variations in the $x_1 x_2$ plane are of interest - hence the restriction of j, k to 1, 2 in Equation (10). In Reference 6 a refinement was included which improved the fit obtained with this approximation to the actual sinusoidal velocity distribution. The refinement was to multiply the linear terms by suitably chosen frequency-dependent factors. This had the same effect on the input spectrum functions as would adding certain third order terms to Equation (10). Although there is certainly some gain in accuracy achieved thereby, this refinement adds undesirable complexity, and is not included herein.

The point of view taken is that each term of Equation (10) when it is applied to a single spectral component) represents a periodic relative velocity field of simple form, which results in periodic aerodynamic forces and moments. These are expressed quite generally by a set of 'gust derivatives' or 'gust transfer functions' which are analogous to (some are identical to) the familiar stability and flutter derivatives which have been in use for so long.

Consideration of the symmetry of the velocity distributions represented by the individual terms of Equation (10) permits separation of the associated aerodynamic forces and moments into the usual longitudinal and lateral groups. The following matrix equations serve to define the 'gust derivatives' (note that the gust velocities are now denoted by u', v', w') :

$$\{F_1\} = [A_1]\{g_1\} \tag{11}$$

$$\{F_2\} = [A_2]\{g_2\} \tag{12}$$

where

$$\{F_1\} = \begin{bmatrix} \bar{X} \\ \bar{Z} \\ \bar{M} \\ \bar{H}_e \end{bmatrix} \tag{13}$$

$$\{F_2\} = \begin{bmatrix} \bar{Y} \\ \bar{L} \\ \bar{N} \\ \bar{H}_a \\ \bar{H}_r \end{bmatrix} \tag{14}$$

$$\{g_1\} = \begin{bmatrix} \bar{u}' \\ \bar{w}' \\ \bar{u}'_x \\ \bar{w}'_x \\ \bar{v}'_y \\ \hdashline \bar{u}'_{xx} \\ \bar{w}'_{xx} \\ \bar{v}'_{xy} \\ \bar{u}'_{yy} \\ \bar{w}'_{yy} \end{bmatrix} \tag{15}$$

$$\{g_2\} = \begin{bmatrix} \bar{v}' \\ \bar{v}'_x \\ \bar{u}'_y \\ \bar{w}'_y \\ \hdashline \bar{v}'_{xx} \\ \bar{u}'_{xy} \\ \boxed{\bar{w}'_{xy}} \\ v'_{yy} \end{bmatrix} \tag{16}$$

where $w'_x = \partial w'/\partial x$, etc. It will be seen subsequently that the input spectra associated with $\bar{w}'_{xy}$ are negligible, and hence that term may be dropped.

$$[A_1] = \left[\begin{array}{ccccc|ccccc} X_{u'} & X_{w'} & X_{u'_x} & X_{w'_x} & X_{v'_y} & X_{u'_{xx}} & X_{w'_{xx}} & X_{v'_{xy}} & X_{u'_{yy}} & X_{w'_{yy}} \\ Z_{u'} & Z_{w'} & Z_{u'_x} & Z_{w'_x} & Z_{v'_y} & Z_{u'_{xx}} & Z_{w'_{xx}} & Z_{v'_{xy}} & Z_{u'_{yy}} & Z_{w'_{yy}} \\ M_{u'} & M_{w'} & M_{u'_x} & M_{w'_x} & M_{v'_y} & M_{u'_{xx}} & M_{w'_{xx}} & M_{v'_{xy}} & M_{u'_{yy}} & M_{w'_{yy}} \\ H_{e_{u'}} & H_{e_{w'}} & H_{e_{u'_x}} & H_{e_{w'_x}} & H_{e_{v'_y}} & H_{e_{u'_{xx}}} & H_{e_{w'_{xx}}} & H_{e_{v'_{xy}}} & H_{e_{u'_{yy}}} & H_{e_{w'_{yy}}} \end{array}\right] \qquad (17)$$

$$[A_2] = \left[\begin{array}{cccc|cc|c|c} Y_{v'} & Y_{v'_x} & Y_{u'_y} & Y_{w'_y} & Y_{v'_{xx}} & Y_{u'_{xy}} & Y_{w'_{xy}} & Y_{v'_{yy}} \\ L_{v'} & L_{v'_x} & L_{u'_y} & L_{w'_y} & L_{v'_{xx}} & L_{u'_{xy}} & L_{w'_{xy}} & L_{v'_{yy}} \\ N_{v'} & N_{v'_x} & N_{u'_y} & N_{w'_y} & N_{v'_{xx}} & N_{u'_{xy}} & N_{w'_{xy}} & N_{v'_{yy}} \\ H_{a_{v'}} & H_{a_{v'_x}} & H_{a_{u'_y}} & H_{a_{w'_y}} & H_{a_{v'_{xx}}} & H_{a_{u'_{xy}}} & H_{a_{w'_{xy}}} & H_{a_{v'_{yy}}} \\ H_{r_{v'}} & H_{r_{v'_x}} & H_{r_{u'_y}} & H_{r_{w'_y}} & H_{r_{v'_{xx}}} & H_{r_{u'_{xy}}} & H_{r_{w'_{xy}}} & H_{r_{v'_{yy}}} \end{array}\right] \qquad (18)$$

In the above expressions $\{F_1\}$ and $\{F_2\}$ are the column matrices of the Laplace transforms of the longitudinal and lateral aerodynamic forces respectively, $\{g_1\}$ and $\{g_2\}$ are the matrices of the Laplace transforms of the gust-velocity inputs for the longitudinal and lateral equations, and $[A_1]$ and $[A_2]$ are the matrices of 'gust transfer functions' defined by Equations (11) and (12). These transfer functions might frequently be approximated by simple derivatives, e.g. $Y_{v'} = \partial Y/\partial v'$ (see Ref.5, Sec.4.16). The matrices $[A_1]$ and $[A_2]$ are written out above with maximum generality, in which case there are a total of 80 transfer functions! The dashed lines in Equations (15) to (18) indicate those portions of the matrices (to the right of the line) which would be neglected in a first-order theory. The number of transfer functions is then reduced to 40. If only control-fixed conditions are of interest, a further reduction to 27 is effected by dropping all the H terms. Additional simplifications of the sort common in stability and control work might frequently be in order: for example, neglecting the X-force equation altogether in the longitudinal equations of motion, and dropping certain transfer functions which experience or analysis indicate are small.

4.1 The One-Dimensional Input Spectra

Since the 'inputs' $\{g_1\}$ and $\{g_2\}$ contain more than one element, the airplane system is subjected to a set of simultaneous random inputs. Figure 8 illustrates the general case, with inputs $x_i(t)$, $i = 1$ to n, output $y_n(t)$, and transfer functions $G_{ni}(s)$. The output is given by

$$\bar{y}_n(s) = \sum_i G_{ni}(s)\bar{x}_i(s)$$

and, as shown in Reference 16 the (one-dimensional) power spectral density of y_n is given by

$$\Theta_{y_n y_n}(\omega) = \sum_i \sum_j G^*_{ni}(i\omega)G_{nj}(i\omega)\Theta_{ij}(\omega) \tag{19}$$

The star denotes the conjugate complex number, so that, for example, the term $G^*_{ni}G_{ni}\Theta_{ii} = |G_{ni}|^2\Theta_{ii}$, which is the familiar result for a single input. Θ_{ij} is the cross-spectrum of x_i and x_j, i.e. the spectral distribution of $x_i x_j$, or the Fourier transform of the cross-correlation of x_i and x_j. In using Equation (19) it is important to note that

$$\Theta_{ij}(\omega) = \Theta^*_{ji}(\omega) \tag{20}$$

If y_n is one of the airplane response quantities such as roll angle, load factor, wing stress, etc., then $G_{ni}(\omega)$ is the overall transfer function relating this particular response to the input x_i (e.g. w'_y). The evaluation of these transfer functions is performed by applying the forces $\{F_1\}$ or $\{F_2\}$ to the appropriate equations of motion, e.g. for the horizontal flight of a rigid airplane with six rigid-body and three control system degrees of freedom:

$$[B_1]\begin{bmatrix}\bar{u}\\ \bar{w}\\ \bar{\theta}\\ \bar{\eta}\end{bmatrix} = \{F_1\} \tag{21}$$

$$[B_2]\begin{bmatrix}\bar{v}\\ \bar{\varphi}\\ \bar{r}\\ \bar{\xi}\\ \bar{\zeta}\end{bmatrix} = \{F_2\} \tag{22}$$

where

$$[B_1] = \begin{bmatrix}(ms - X_u) & X_w & mg & -X_\eta\\ -Z_u & (ms - Z_w) & -(mu_0 + Z_q s) & -Z_\eta\\ -M_u & -M_w & (Bs^2 - M_q s) & -M_\eta\\ -H_{e_u} & -H_{e_w} & H_{e_q}s & (I_e s^2 + H_{e_\eta})\end{bmatrix} \tag{23}$$

$$[B_2] = \begin{bmatrix} (ms - Y_v) & -(mg + Y_p s) & (mu_0 - Y_r) & 0 & -Y_\zeta \\ -L_v & (As^2 - L_p s) & -(Es + L_r) & -L_\xi & -L_\zeta \\ -N_v & -(Es + N_p) & (Cs - N_r) & -N_\xi & -N_\zeta \\ 0 & -2H_{a_p} & -2H_{a_r} & (I_a s^2 - 2H_{a_\xi}) & 0 \\ -H_{r_v} & -H_{r_p} & -H_{r_r} & 0 & (I_r s^2 - H_{r_\zeta}) \end{bmatrix} \tag{24}$$

It follows from Equations (11), (12), (21) and (24) that $[G_1] = [B_1^{-1}][A_1]$ and $[G_2] = [B_2^{-1}][A_2]$, where $[G_1]$ and $[G_2]$ are the matrices of the overall transfer functions. G_{hi} for the two sets of equations. In the above equations, the quantities Z_w , etc., are to be interpreted as transfer functions, i.e.

$$Z_w = \frac{\partial Z}{\partial \dot{w}} s + \frac{\partial Z}{\partial w} \text{, etc.} \tag{25}$$

The equations do not include any automatic control elements, but the addition of these in particular cases is usually fairly straightforward.

We must now consider the input spectrum functions which occur in Equation (19). These are the cross-spectra of all the inputs that occur in $\{g_1\}$ or $\{g_2\}$, that is, among the velocity components and their first and second derivatives. Many of these cross-spectra are zero by virtue of the fact that the two quantities involved are uncorrelated (see after Eqn.(7)). However, a number of them remain, and these must be calculated. Let the spectrum function corresponding to any pair of entries in $\{g_1\}$ or $\{g_2\}$ be identified by a corresponding pair of subscripts. For example, $\Theta_{u_x v_{xy}}$ is the cross-spectrum of u'_x and v'_{xy} which occur in $\{g_1\}$. The expression for the three-dimensional cross-spectrum of two scalar components of a vector of the form given by Equation (9) is given by Batchelor (Ref.9, Eqn.2.5, 5) as

$$\Phi_{ij}(\vec{\Omega}) = \lim_{d\vec{\Omega}\to 0} \frac{\overline{dU_i^*(\vec{\Omega})\,dU_j(\vec{\Omega})}}{d\Omega_1 d\Omega_2 d\Omega_3} \tag{26}$$

The cross-spectra of elements containing derivatives can be written down directly from Equation (26). For example, the spectral component of u_x , from Equation (9) is given by the x_1 derivative of the u_1 component, viz.

$$e^{i\vec{\Omega}\vec{x}}(i\Omega_1 dU_1) \tag{27}$$

whence for example

$$\Phi_{u_x v_{xy}}(\vec{\Omega}) = i\Omega_1^2\Omega_2\Phi_{12}(\vec{\Omega}) \tag{28}$$

The general rule is seen to be that for each derivative with respect to x_k the spectrum function Φ_{ij} is multiplied by $\pm i\Omega_k$. The plus sign is for derivatives of the second subscript velocity component (v_{xy} in Eqn. (28)), and the minus sign is for derivatives of the first (u_x). The difference in sign occurs because the *conjugate* of the first amplitude element is used in Equation (26). The corresponding one-dimensional spectrum function, continuing with the same example, is then (cf. Eqn. (3))

$$\Theta_{u_x v_{xy}}(\Omega_1) = i\Omega_1^2 \iint_{-\infty}^{\infty} \Omega_2 \Phi_{12}(\vec{\Omega}) d\Omega_2 d\Omega_3 \qquad \text{(a)} \qquad (29)$$

or

$$\Theta_{u_x v_{xy}}(k_1) = i \frac{k_1^2}{L^5} \iint_{-\infty}^{\infty} k_2 \Phi_{12}(k_1, k_2, k_3) dk_2 dk_3 \qquad \text{(b)}$$

In the theory presented herein, we exclude that portion of the spectrum for which $\Omega_2 > \Omega_2'$ and $\Omega_1 > \Omega_1'$ where Ω_2' and Ω_1' correspond to the wave-length limits for which the power-series approximation is valid. It must be noted that some of the integrals of the type contained in Equation (29) are divergent when the limits are infinite and the truncation is therefore essential, and not a matter of choice. The expression for the truncated spectrum is

$$\Theta_{u_x v_{xy}}(k_1) = i \frac{k_1^2}{L^5} \int_{-k_2'}^{k_2'} k_2 dk_2 \int_{-\infty}^{\infty} \Phi_{12}(k_1, k_2, k_3) dk_3 \qquad (30)$$

With the value of Φ_{ij} given in Equation (5) this integral, and the others like it which occur in the equations, can all be evaluated quite simply. The integrand in the majority of cases is an odd function of one or both of k_2 and k_3 and for these the integral is zero. Of those which remain, some can be discarded on the basis of the following order-of-magnitude analysis.

The general form for $\Theta(k_1)$ (apart from sign) is

$$\Theta(k_1) = \frac{1}{L^{n+2}} \int_{-k_2'}^{k_2'} dk_2 \int_{-\infty}^{\infty} (ik_1)^{\alpha} (ik_2)^{\beta} \Phi_{ij} dk_3 \qquad (31)$$

where $n = \alpha + \beta$, and α and β are the orders of the two velocity derivatives involved. When the expression for Φ_{ij} given by Equation (5) is inserted, we get

$$\Theta(k_1) = \frac{2\sigma^2}{\pi^2} L^{1-n} (ik_1)^{\alpha} \int_{-k_2'}^{k_2'} (ik_2)^{\beta} dk_2 \int_{-\infty}^{\infty} \frac{k^2 \delta_{ij} - k_i k_j}{(k^2+1)^3} dk_3 \qquad (32)$$

Depending on the values of i and j, the integration with respect to k_3 leads to zero, or one or both of the following terms

$$\frac{3\pi}{8}(k_1^2 + k_2^2 + 1)^{-5/2}\;; \qquad \frac{\pi}{8}(k_1^2 + k_2^2 + 1)^{-3/2} \tag{33}$$

Since we are interested in values of k_1 and k_2 up to about 100, we see that the magnitude of Θ is characterized by the numbers

$$L^{1-n}(100)^{n-4} \quad \text{or} \quad L^{1-n}(100)^{n-2}$$

of which the larger one is the second.

Thus the relative values of Θ with ascending n are characterized as shown in the following table:

Table I

n	0	1	2	3	
Relative Θ	1	0.1	0.01	0.001	for L = 1000
	1	0.5	0.25	0.125	for L = 200

On the strength of these values, and noting that $L = 200$ is a rather small scale, we may neglect all cross-spectra for which $n > 2$. The remaining non-zero spectra (25 in all) have been calculated and are given below.

For the Longitudinal Equations

<u>n = 0</u>

$$\Theta_{uu} = \frac{\sigma^2 L}{2\pi}\frac{1}{1+k_1^2}\frac{k_2'}{k'}\left[1+\left(\frac{k_2'}{k'}\right)^2\right] \tag{34}$$

$$\Theta_{ww} = \frac{3\sigma^2 L}{2\pi}\frac{1}{1+k_1^2}\frac{k_2'}{k'}\left\{\frac{1}{3}\left(\frac{k_2'}{k'}\right)^2 + \frac{k_1^2}{1+k_1^2}\left[1-\frac{1}{3}\left(\frac{k_2'}{k'}\right)^2\right]\right\} \tag{35}$$

<u>n = 1</u>

$$\Theta_{uu_x} = i\frac{\sigma^2}{2\pi}\frac{k_1 k_2'}{k'(1+k_1^2)}\left[1+\left(\frac{k_2'}{k'}\right)^2\right] \tag{36}$$

$$\Theta_{ww_x} = i\frac{3}{2}\frac{\sigma^2}{\pi}\frac{k_1 k_2'}{k'(1+k_1^2)}\left[\frac{k_1^2}{1+k_1^2} - \frac{1}{3}\frac{k_1^2}{1+k_1^2}\left(\frac{k_2'}{k'}\right)^2 + \frac{1}{3}\left(\frac{k_2'}{k'}\right)^2\right] \tag{37}$$

$$\Theta_{uv_y} = -i\frac{\sigma^2}{2\pi}\frac{k_1}{1+k_1^2}\left(\frac{k_2'}{k'}\right)^3 \tag{38}$$

n = 2

$$\Theta_{uu_{xx}} = i\frac{k_1}{L}\Theta_{uu_x} \tag{39}$$

$$\Theta_{uu_{yy}} = \frac{\sigma^2}{2\pi L}\left[4\frac{k_2'}{k'} + \left(\frac{k_2'}{k'}\right)^3 - 2\ln\frac{k'+k_2'}{k'-k_2'}\right] \tag{40}$$

$$\Theta_{uv_{xy}} = \frac{\sigma^2}{2\pi L}\frac{k_1^2}{1+k_1^2}\left(\frac{k_2'}{k'}\right)^3 \tag{41}$$

$$\Theta_{u_x v_y} = -\Theta_{uv_{xy}} \tag{42}$$

$$\Theta_{u_x u_x} = -\Theta_{uu_{xx}} \tag{43}$$

$$\Theta_{v_y v_y} = \frac{\sigma^2}{2\pi L}\left[\frac{k_1^2}{1+k_1^2}\left(\frac{k_2'}{k'}\right)^3 - \left(\frac{k_2'}{k'}\right) + \frac{1}{2}\ln\frac{k'+k_2'}{k'-k_2'}\right] \tag{44}$$

$$\Theta_{w_x w_x} = -i\frac{k_1}{L}\Theta_{ww_x} \tag{45}$$

$$\Theta_{ww_{xx}} = i\frac{k_1}{L}\Theta_{ww_x} \tag{46}$$

$$\Theta_{ww_{yy}} = \frac{3\sigma^2}{4\pi L}\left[\frac{2}{3}\frac{k_1^2}{1+k_1^2}\left(\frac{k_2'}{k'}\right)^3 - 2\left(\frac{k_2'}{k'}\right) - \frac{2}{3}\left(\frac{k_2'}{k'}\right)^3 + \ln\frac{k'+k_2'}{k'-k_2'}\right] \tag{47}$$

For Lateral Equations

n = 0

$$\Theta_{vv} = \frac{3\sigma^2 L}{2\pi}\frac{1}{1+k_1^2}\frac{k_2'}{k'}\left\{\frac{k_1^2}{1+k_1^2}\left[1 - \frac{1}{3}\left(\frac{k_2'}{k'}\right)^2\right] + \frac{1}{3}\right\} \tag{48}$$

n = 1

$$\theta_{vv_x} = i\frac{k_1}{L}\theta_{vv} \tag{49}$$

$$\theta_{vu_y} = \theta_{uv_y} \tag{50}$$

n = 2

$$\theta_{u_y v_x} = -\theta_{uv_{xy}} \tag{51}$$

$$\theta_{vv_{xx}} = i\frac{k_1}{L}\theta_{vv_x} \tag{52}$$

$$\theta_{vu_{xy}} = \theta_{uv_{xy}} \tag{53}$$

$$\theta_{vv_{yy}} = -\theta_{v_y v_y} \tag{54}$$

$$\theta_{v_x u_y} = -\theta_{uv_{xy}} \tag{55}$$

$$\theta_{v_x v_x} = -\theta_{vv_{xx}} \tag{56}$$

$$\theta_{u_y u_y} = -\theta_{uu_{yy}} \tag{57}$$

$$\theta_{w_y w_y} = -\theta_{ww_{yy}} \tag{58}$$

The spectra given above are plotted against k_1 for several values of k_2' in Figures 9 to 22. It may be noted that none of them are complex - they are either real or pure imaginaries. There are 25 non-zero power spectra and cross-spectra listed above. Many of these are equal or merely opposite in sign to others, so that there are only fourteen essentially different ones. Of these fourteen, three are zero-order (θ_{uu} , θ_{vv} , θ_{ww}) four are first-order (θ_{uu_x} , θ_{uv_y} , θ_{vv_x} , θ_{ww_x}) and the remaining seven are all second-order. Of the first-order spectra, only one is a cross-spectrum involving two different velocity components, i.e. θ_{uv_y} . Hence in a first-order theory, this remains as the *only* cross-term between velocity components, and if it is neglected, complete statistical separation of the response to the three components of the turbulence results.

4.1.1 The Wavelength Limitation

Examination of Figures 9 to 22 and Table 1 shows that the order of magnitude of the spectrum peak is given by L^{1-n} . Now if the basic series giving the velocity (Eqn.10) had been extended to include higher-order terms, the effect would simply have been to add additional higher-order spectra $(n \geq 3)$ to the list already calculated. It is evident that these higher-order spectra would be negligibly small for the frequency range $k < 1$ and for the scale $L > 100$. In the range $1 \leq k \leq 100$ they would ultimately become large as n increased indefinitely. Thus it appears that the spectra presented are actually valid for a series representation of the velocity containing terms of at least the third, and probably higher order. The wavelength limitations may therefore reasonably be taken as

$$\lambda_1' = l_1$$

$$\lambda_2' = b$$

where l_1 is the overall length of the wing. Hence

$$k_1' = 2\pi\frac{L}{\lambda_1'} = 2\pi\frac{L}{l_1}$$

$$k_2' = 2\pi\frac{L}{\lambda_2'} = 2\pi\frac{L}{b}$$

For example, if $L = 1000$ and $b = 100$, then $k_2' = 20\pi = 62.8$.

Finally, it may be remarked that for large L, (i.e. 1000 ft) the second-order spectra are less at medium wave numbers $(k \simeq 1)$ by a factor of order 10^9 than the zero order spectra. Thus, unless relatively high frequency responses are of interest (e.g. elastic modes) the second-order spectra are not at all important.

4.2 The 'Gust Derivatives'

Equations (17) and (18) indicate that the general second-order theory, when applied to the rigid-body motion of an airplane with three additional control degrees of freedom, involved 80 aerodynamic transfer functions (which we have termed 'gust derivatives'). Should additional elastic degrees of freedom of the airplane be included (as in Ref.6), then still additional gust derivatives would be required. As has already been mentioned, however, substantial simplifications can be made in many practical analyses, such as dropping the X-force equation, keeping only the first-order derivatives, etc. These simplifications must always be determined by the particular circumstances, and it is not within the scope of this paper to anticipate all the possibilities. Neither is it within its scope to present a collection of data on the derivatives, although it is hoped that some research at the Institute of Aerophysics will be directed to that end. Nevertheless, a discussion of the derivatives is given in Sections 4.2.1 to 4.2.3.

4.2.1 The Zero-Order Derivatives

The zero-order derivatives, which are the most important ones, are those with respect to the gust-velocity components themselves, e.g. $M_{v'}$, $L_{v'}$, etc. They are the elements of the first two columns of $[A_1]$ and the first column of $[A_2]$.

These are simply the aerodynamic transfer functions (stability derivatives) of classical aerodynamic theory, with opposite sign, i.e.

$$Z_{w'} = -Z_w \text{ , etc.} \tag{59}$$

where Z_w is given by Equation (25). The reversal of sign is because w is the velocity of the airplane in the z-direction, and w' is the velocity of the air in the same direction; hence the relative motion is given by $(w - w')$. This group of derivatives embodies the major aerodynamic effects of gusty air, and a simplified calculation in which all others are neglected would still be of considerable value, especially for small airplanes in large-scale turbulence.

4.2.2 *The First-Order Derivatives*

Columns 3 to 5 of $[A_1]$ and columns 2 to 4 of $[A_2]$ contain the elements in which there appears a first-order derivative of the velocity components. These describe the influence of the 'gust gradient' on the airplane, and are no doubt important for large airplanes, especially near the ground in small-scale turbulence. It has already been shown[6] that the derivatives with respect to w_x' and w_y' are identical with the classical pitch and roll stability derivatives, viz.:

$$M_{w_x'} = M_q \text{ , etc.}$$

and (60)

$$L_{w_y'} = -L_p \text{ , etc.}$$

No correspondingly simple interpretation is in general possible for the velocity fields associated with u_x' and u_y' . For unswept wings of high aspect ratio, the derivative u_x' would presumably be significant only in introducing a relative wind at the tail different from that at the c.g., i.e.

$$\Delta u_{rel} = -u_x' l_t$$

This would modify the tail lift, and hence the lift and pitching moment of the airplane, as expressed in the derivatives $Z_{u_x'}$ and $M_{u_x'}$. For sweptback high-aspect-ratio wings it introduces a variable (linear) relative wind along the span, which could be treated by a suitably modified lifting-line theory. The same theoretical wing problem is presented by the velocity field associated with u_y' , with the difference that the spanwise velocity variation associated with the latter exists for all wings, whether swept or not. For the particular case of a straight lifting line, the forces corresponding to u_y' are just those given by the classical yaw-rate derivatives, viz.:

$$L_{u_y'} = -L_r \text{ , etc.}$$

The effects of the linear velocity fields associated with v_x' and v_y' on the contributions of the vertical tail to the aerodynamic forces can readily be estimated, since they merely change the average relative sidewind wind at the tail and hence the angle of attack of the vertical tail. Their effects on the wings are rather more involved. v_x' would not be expected to be of much importance for unswept wings, but for swept wings v_x' and v_y' both have the effect of modifying the wing angle of attack distribution, when it has dihedral, in the manner illustrated in Figure 23. Again, for high-aspect-ratio wings, lifting-line theory could be used to calculate these effects in a rather straightforward manner. For more general cases, lifting-surface theory would have to be employed.

When the wing is swept there is, in addition to the α changes described above, the important variation of the magnitude of the component of the relative wind normal to the line of aerodynamic centres. This is given by

$$\Delta V_n = \mp v' \sin\Lambda , \quad y \gtrless 0$$

and the distributions of ΔV_n associated with v_x' and v_y' have exactly the same form as those shown for $\Delta\alpha$ in Figure 23. Thus the two effects will be additive in producing rolling moment, side force and yawing moment.

4.2.3 *The Second-Order Derivatives*

Columns 5 to 10 of $[A_1]$ and 4 to 8 of $[A_2]$ contain the second-order elements. By virtue of the assumption made in Section 4.1, i.e. neglecting all input spectra having $n > 2$, one column of these derivatives is not required. That is the seventh column in $[A_2]$, containing derivatives with respect to w_{xy}'. The reason for this is that the lowest-order spectrum function which contains the input w_{xy}' is $\Theta_{vw_{xy}'}$ $(n = 2)$ and it is identically zero. Hence this particular input is of negligible importance and the associated derivatives are not of interest.

Of the remaining derivatives, those involving w_{xx}' and w_{yy}' have already been discussed in Reference 6 (using a different nomenclature). They are shown to give the aerodynamic forces resulting from a periodic cambering or chordwise bending of the wing (w_{xx}') and a flapping or spanwise bending (w_{yy}'). Values of the lift and pitching moment on a two-dimensional wing in incompressible flow are given there for the w_{xx}' case. The calculation of forces due to the w_{yy}' field could be accomplished by a relatively straightforward application of the appropriate method of wing theory.

The elevator and rudder hinge-moment derivatives contained in $[A_1]$ and $[A_2]$ could all be calculated relatively easily on the assumption that the surface in question experiences an angle of attack or velocity change equal to that at the mean aerodynamic centre of the surface. The calculation of aileron hinge-moment derivatives (the 4th row of $[A_2]$) would take more effort except when simple strip theory is acceptable.

Generally speaking, since the input spectra corresponding to $n = 2$ are relatively so weak, it appears that rough estimates of the second-order derivatives will serve well enough for analysis. A note of caution must be sounded in this connection, however, when elastic modes of the aircraft are involved, for then the second-order terms may be more important.

5. DISPERSION OF THE FLIGHT PATH

When the aircraft is flown by a human or automatic pilot so as to traverse a specified track (e.g., as given by a radio beam), at a specified altitude (e.g., as given by an altimeter), then the controlled variables may be considered as x_3 (altitude) and x_2 (lateral displacement). These will be random variables, having mean-square values which, when used in the normal (Gaussian) probability distribution, give the probability of dispersion of the aircraft from the desired (rectilinear) flight path. In a homogeneous isotropic atmosphere this probability function applies equally well to all portions of the path. However, when the controlled variable is a velocity, rather than a displacement, the situation is fundamentally different. This would normally be the case for the x_1 degree of freedom; that is, forward speed, not distance flown, is the controlled variable. Likewise, if a heading reference only is used for navigation (e.g. magnetic compass), then u_2, not x_2, becomes the random output. In such a case the displacement in a given direction is the integral of the corresponding random velocity component, i.e.

$$x_1 = \int_0^t u_1 \, dt \qquad (61)$$

If we consider a very large number of flight paths through the turbulent field, and take an ensemble average, denoted by $\langle \, \rangle$, then

$$\langle x_1 \rangle = \int_0^t \langle u_1 \rangle dt = 0 \qquad (62)$$

since the ensemble average is equal to the space average. The mean square co-ordinate, however, does not vanish:

$$x_1^2(t) = \iint_0^t u_1(\alpha) u_1(\beta) d\alpha d\beta \qquad (63)$$

The ensemble average (average over many flights at given time t) is

$$\langle x_1^2(t) \rangle = \iint_0^t \langle u_1(\alpha) u_1(\beta) \rangle d\alpha d\beta \qquad (64)$$

But the mean product $\langle u_1(\alpha) u_1(\beta) \rangle$ is known from the autocorrelation:

$$R(\alpha - \beta) = \frac{\langle u_1(\alpha) u_1(\beta) \rangle}{\overline{u_1^2}} \qquad (65)$$

Therefore

$$\langle x_1^2(t)\rangle = \overline{u_1^2} \iint_0^t R(\alpha - \beta)\, d\alpha\, d\beta \qquad (66)$$

The integral can be shown to have the value

$$2t \int_0^t R(\tau)\, d\tau - 2 \int_0^t \tau R(\tau)\, d\tau \qquad (67)$$

At large values of t the second term becomes negligible, and we have the final result

$$\langle x_1^2\rangle = 2\overline{u_1^2} A t \qquad (68)$$

where $A = \int_0^\infty R(\tau)\, d\tau$ is the area under the autocorrelation curve of $u_1(t)$. The latter is directly related to the output power spectrum of u_1, and can be calculated from it, i.e.

$$R(\tau) = \frac{1}{\overline{u_1^2}} \int_{-\infty}^{\infty} \Theta_{u_1 u_1}(\omega) e^{i\omega\tau}\, d\omega \qquad (69)$$

where $\Theta_{u_1 u_1}(\omega)$ is the power spectral density of u_1. The significance of the result given in Equation 68 is that the r.m.s. dispersion $(\langle x^2\rangle)^{1/2}$ varies as $\sqrt{t}$. This is the same result as in the classical problem of the 'random walk'. Thus the probable error in the lateral position of a compass-controlled flight path increases with the square root of the time, or distance flown. The same would be true of the distance flown itself in a speed-controlled flight. However, since an altitude reference is almost invariably used in the flight of airplanes, the probable error in the height remains constant with time. The dispersion of entirely unguided bodies, e.g. ballistic missiles, would vary as $\sqrt{t}$ in all three co-ordinates.

ACKNOWLEDGMENT

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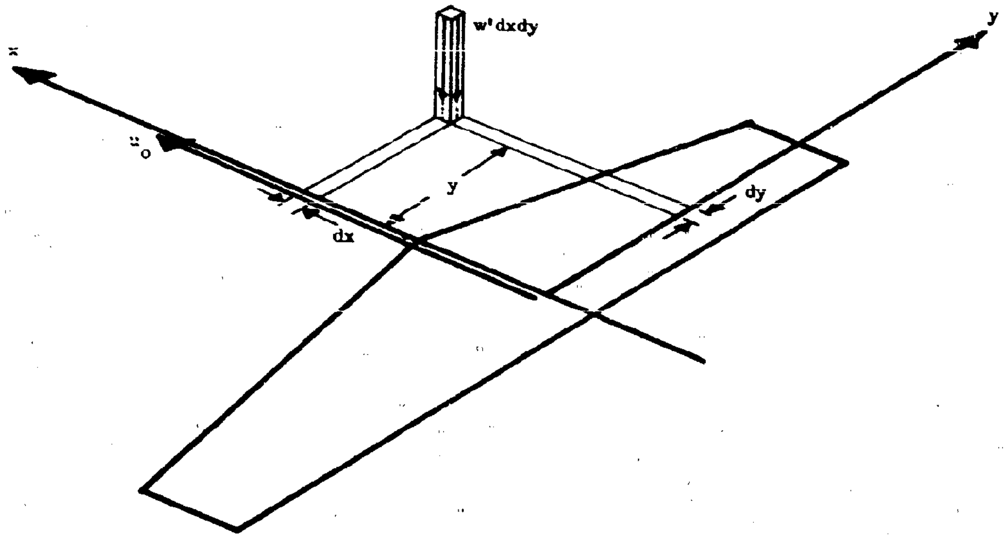


Fig. 1 Vertical-gust impulse (reproduced from Diederich, Ref. 2)

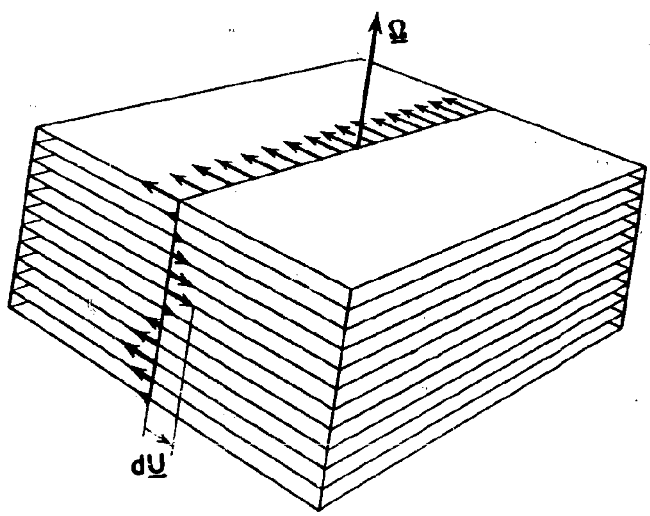


Fig. 2 A single spectral component (reproduced from NACA TN. 3255, 1954, by H.S.Ribner)

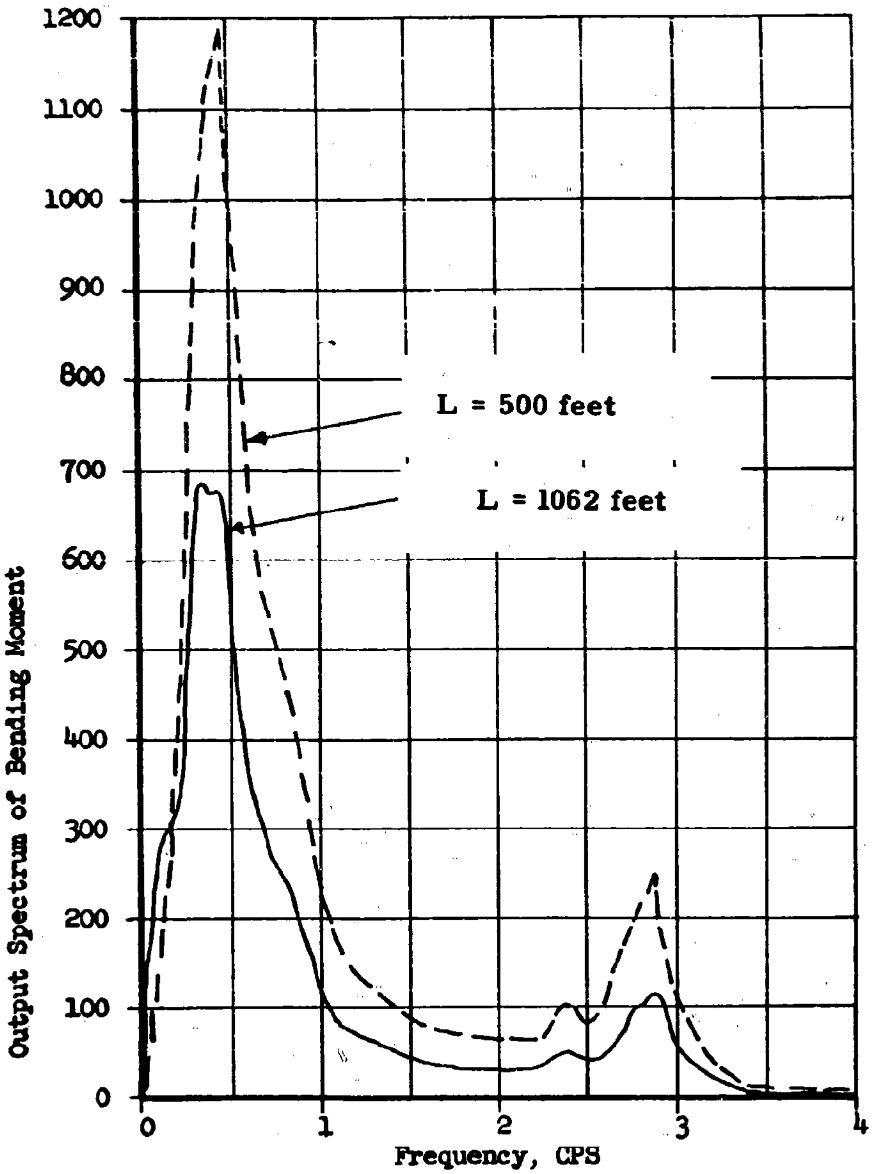


Fig. 3 Effect of scale of turbulence on output spectra. Mean-square turbulence = $12.4 (fps)^2$. (Reproduced from Ref. 10, Douglas Aircraft Co.)

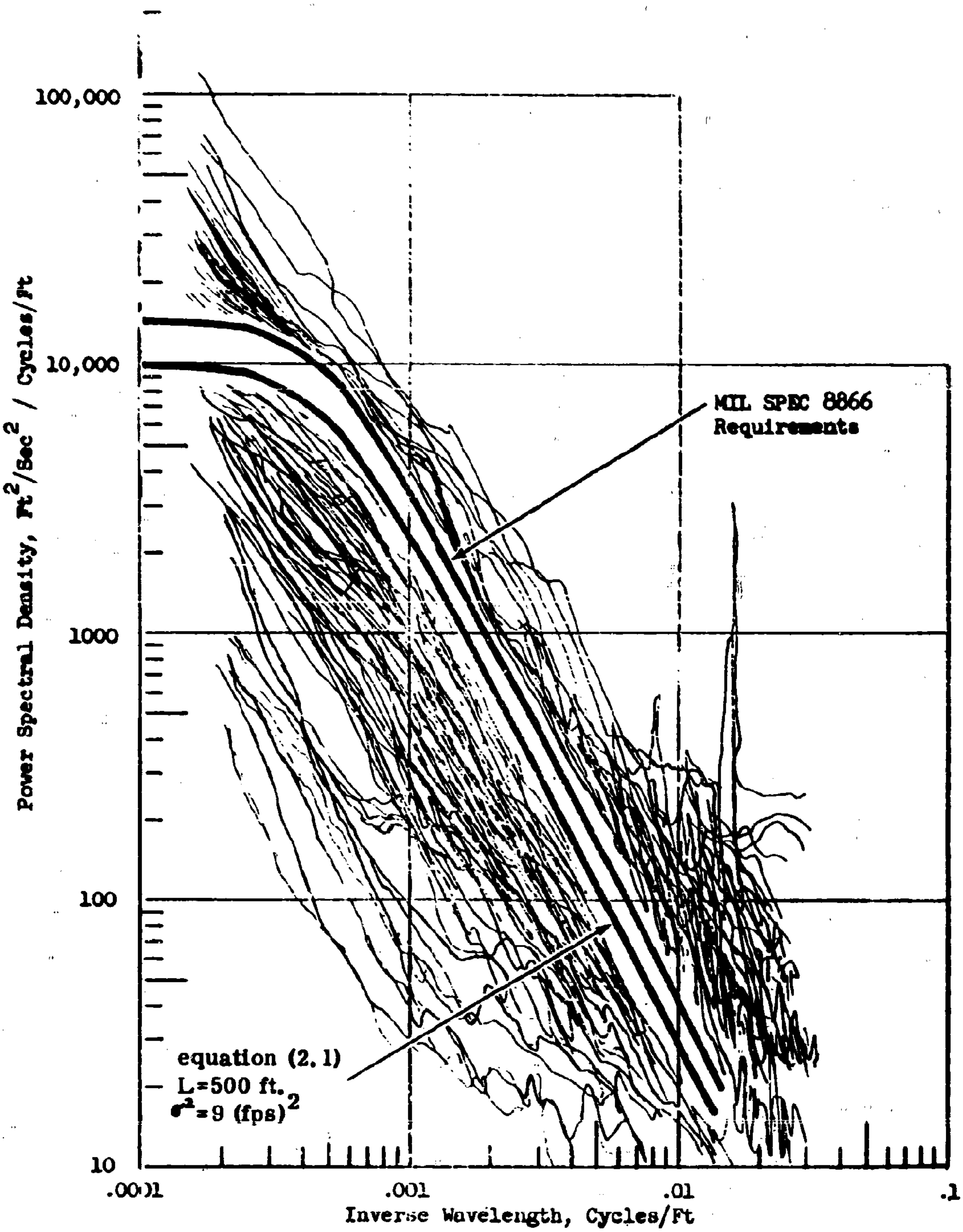


Fig. 4 Composite and expected spectrum of vertical gust velocities from test data and Mil Spec 8866 Requirements. (Reproduced from Ref. 10, Douglas Aircraft Co.)

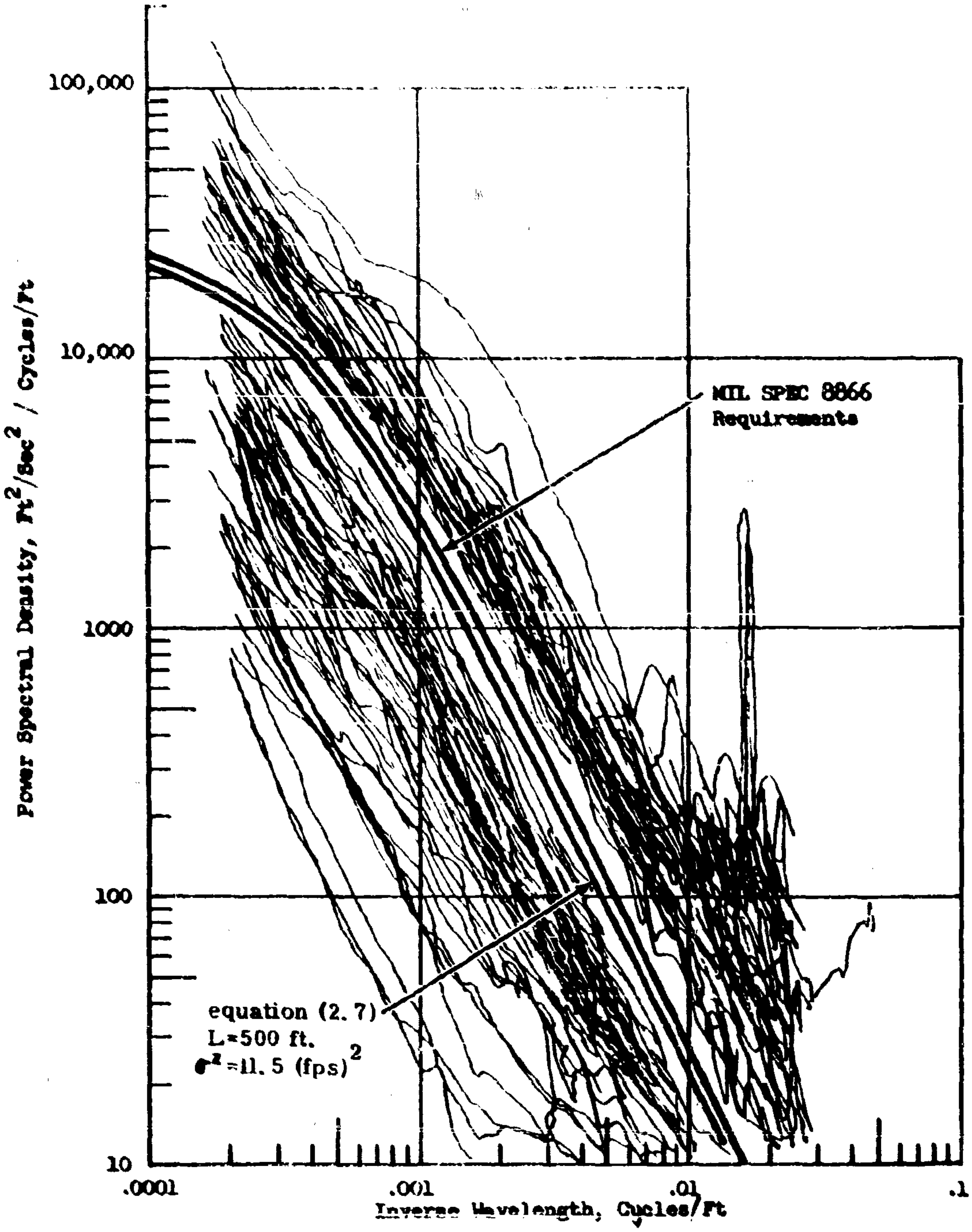


Fig. 5 Composite and expected spectrum of forward gust velocities from test data and Mil Spec 8866 requirements. (Reproduced from Ref. 10, Douglas Aircraft Co.)

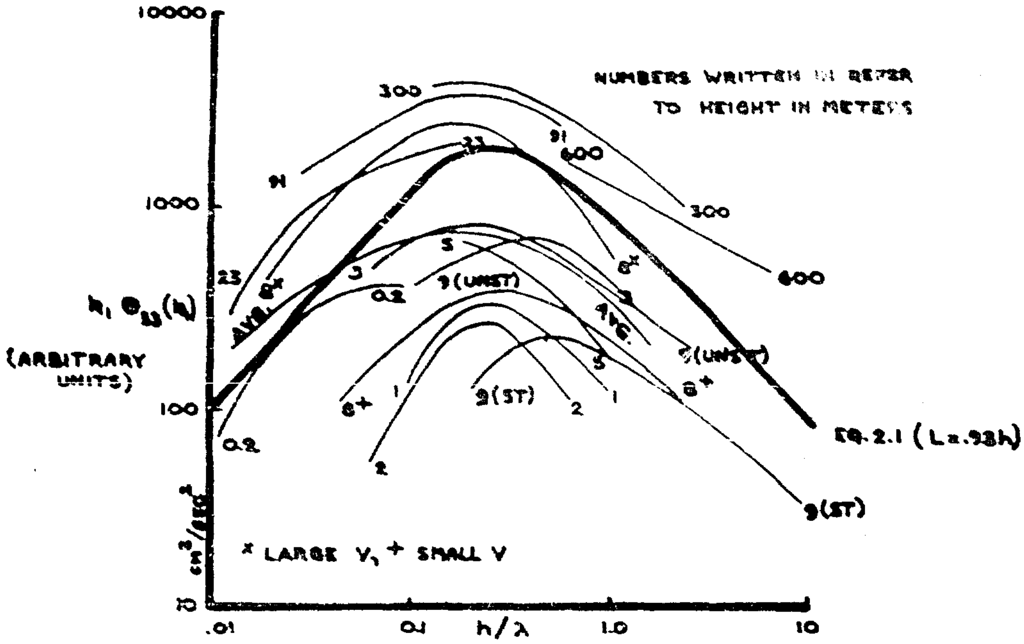


Fig.6 Low-level spectra-comparison of data of Panofsky & McCormick, Ref.11, with Equation (1)

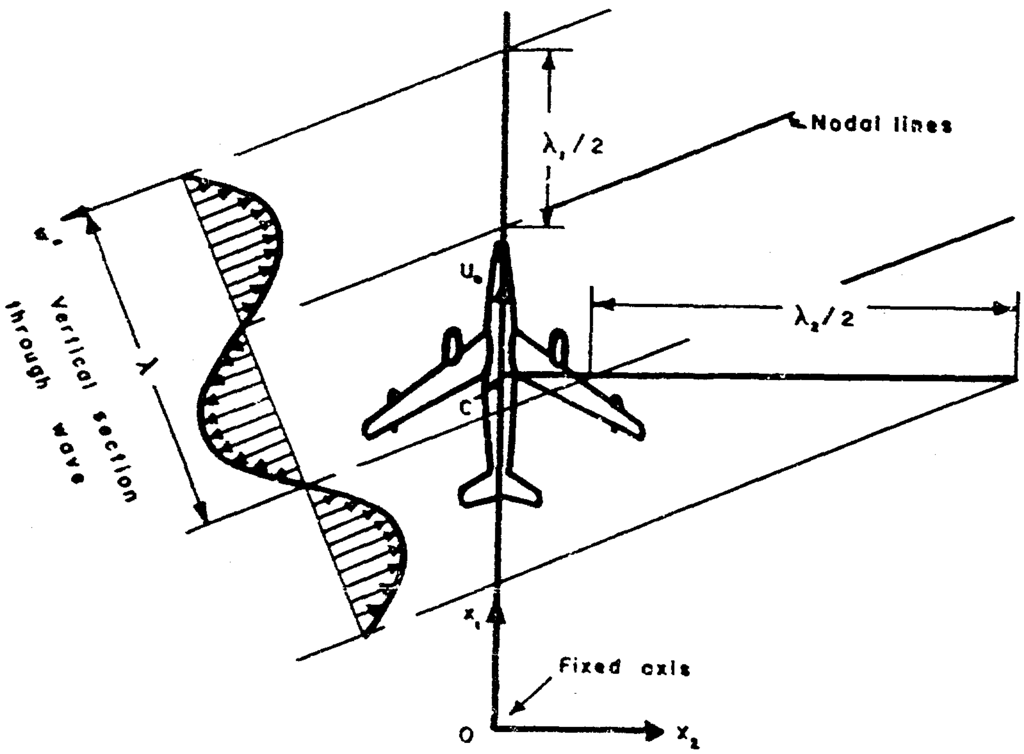


Fig.7 Variation of downwash in x_1x_2 plane for a single spectral component (after H.S. Ribner, Ref.4)

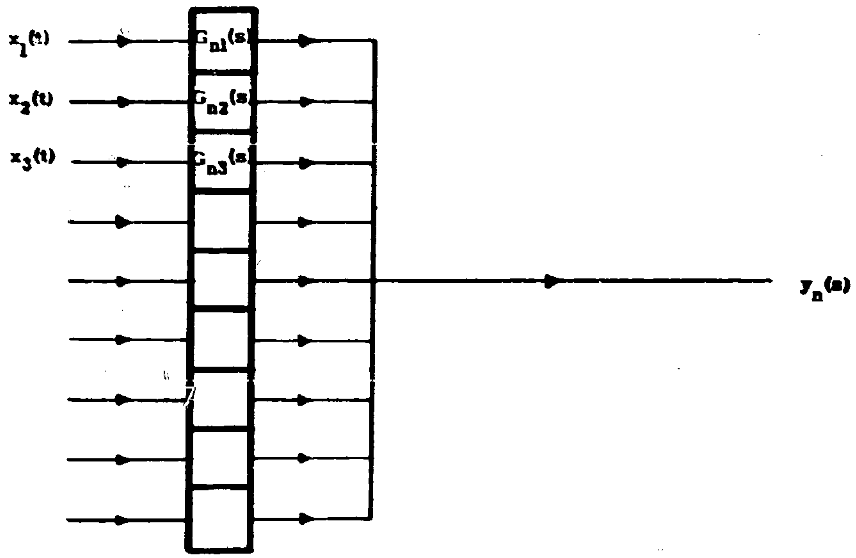


Fig. 8 Linear system with many inputs

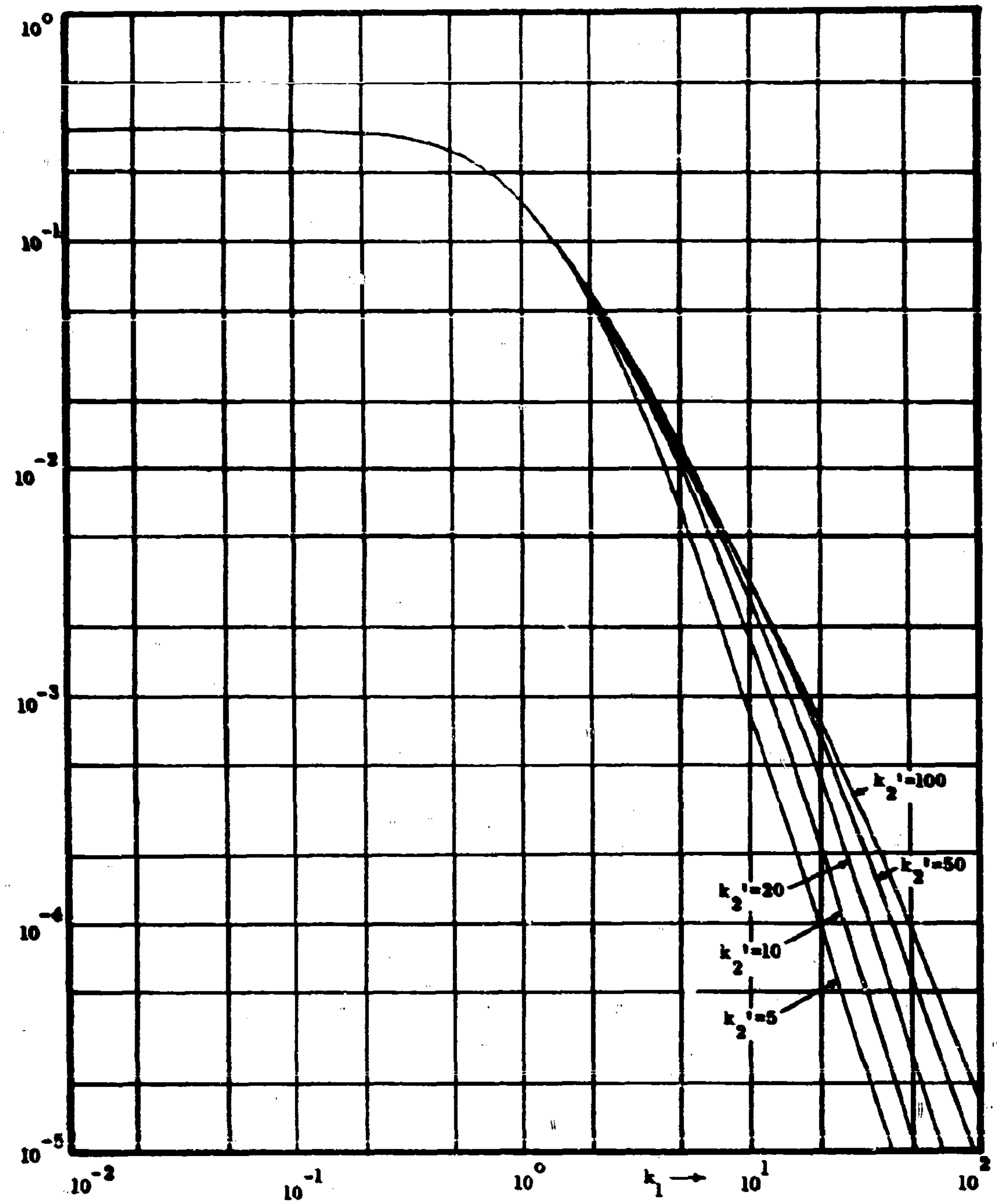


Fig. 9 Spectrum function $\Theta_{uu}/\sigma^2 L$

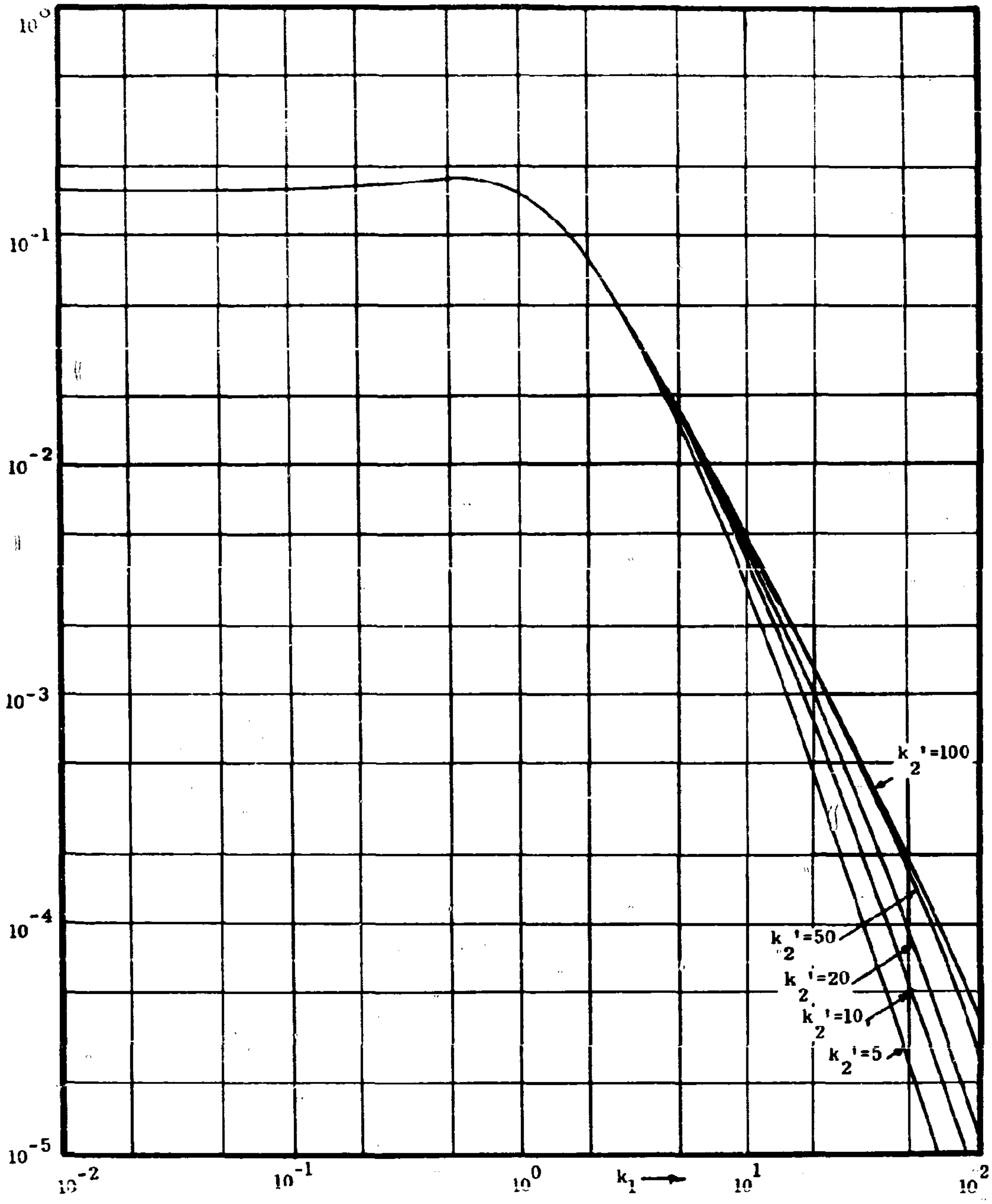


Fig. 10 Spectrum function Θ_{vv}/σ^2L

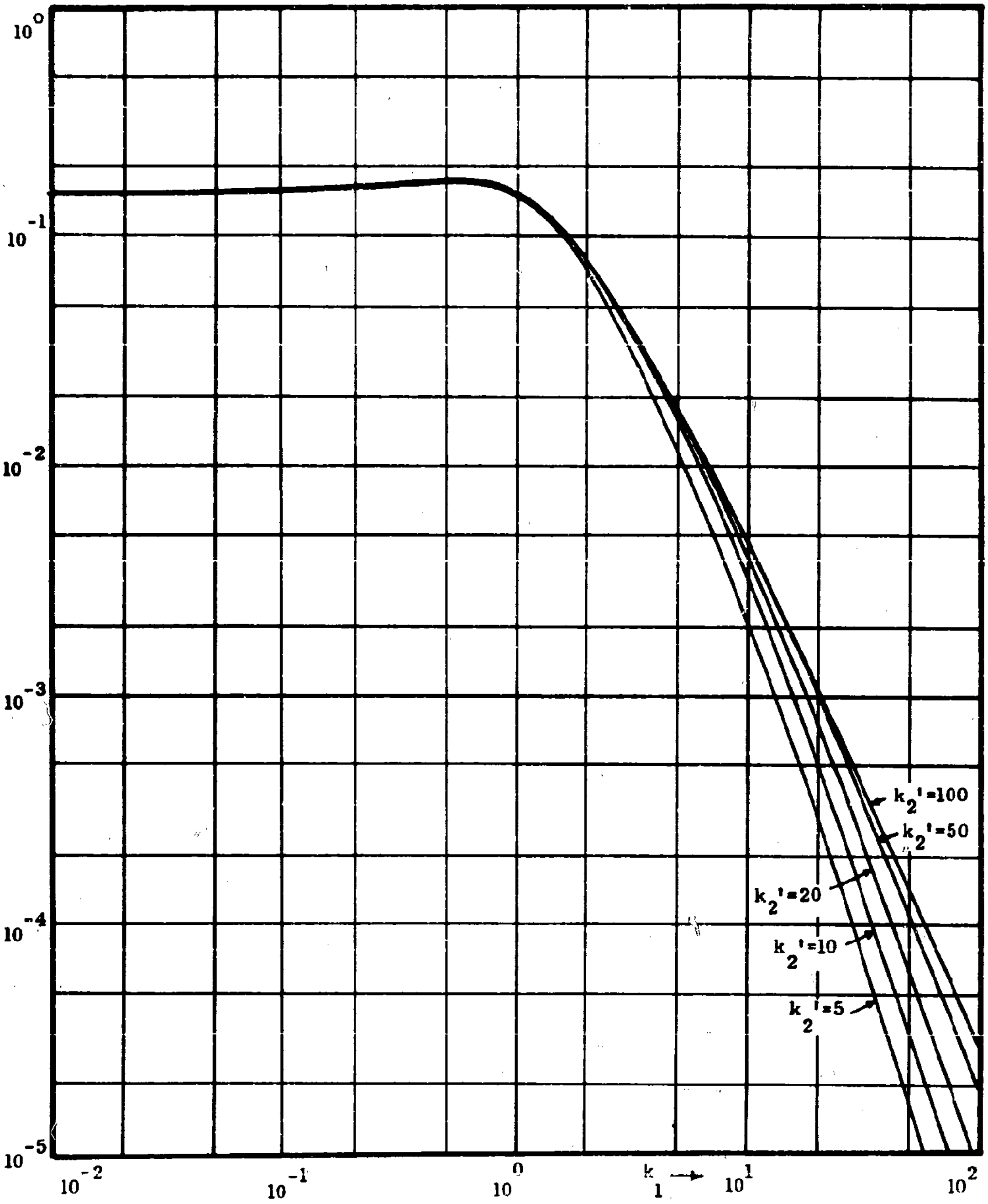


Fig. 11 Spectrum function $\Theta_{ww}/\sigma^2 L$

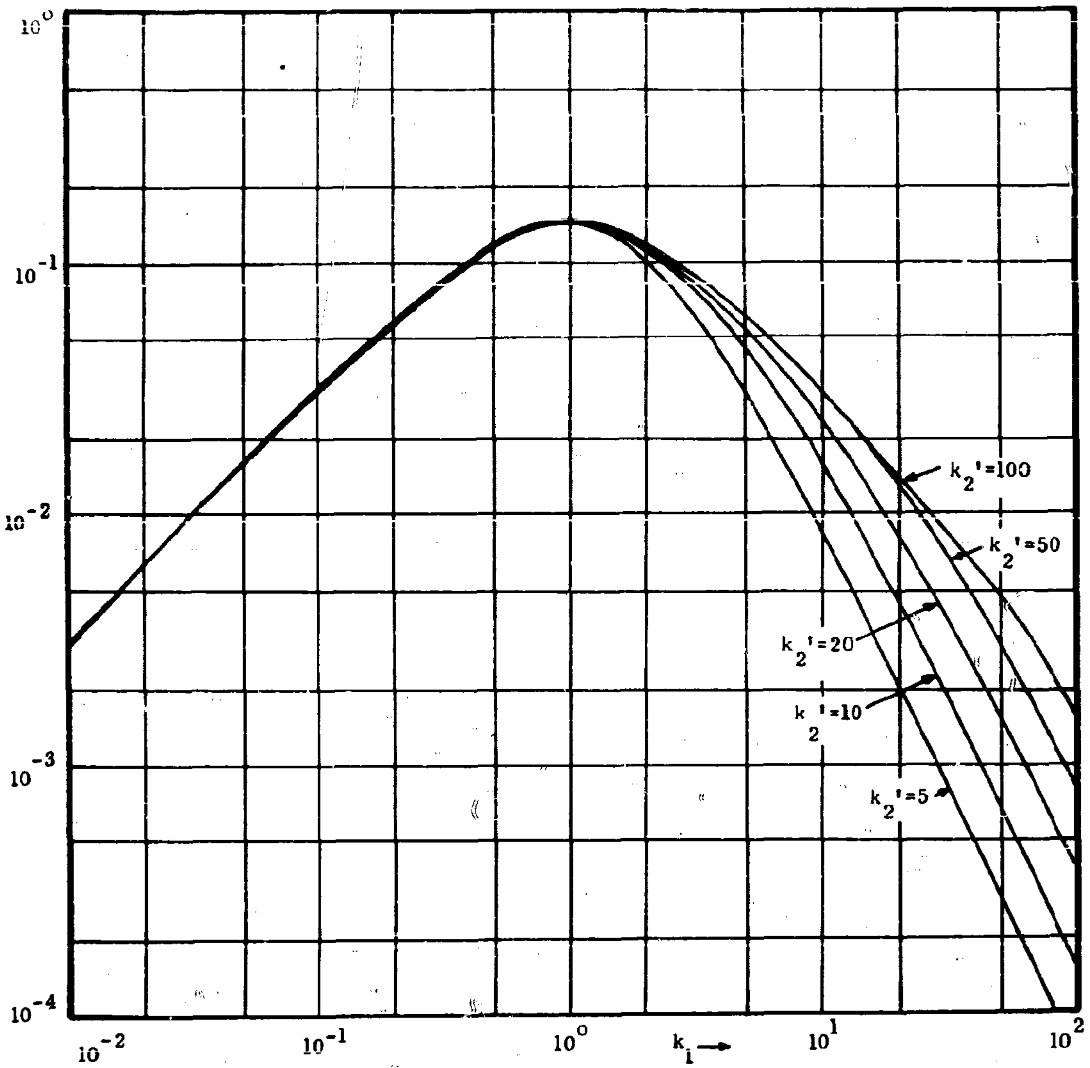


Fig. 12 Spectrum function $\Theta_{uu_x}/i\sigma^2$

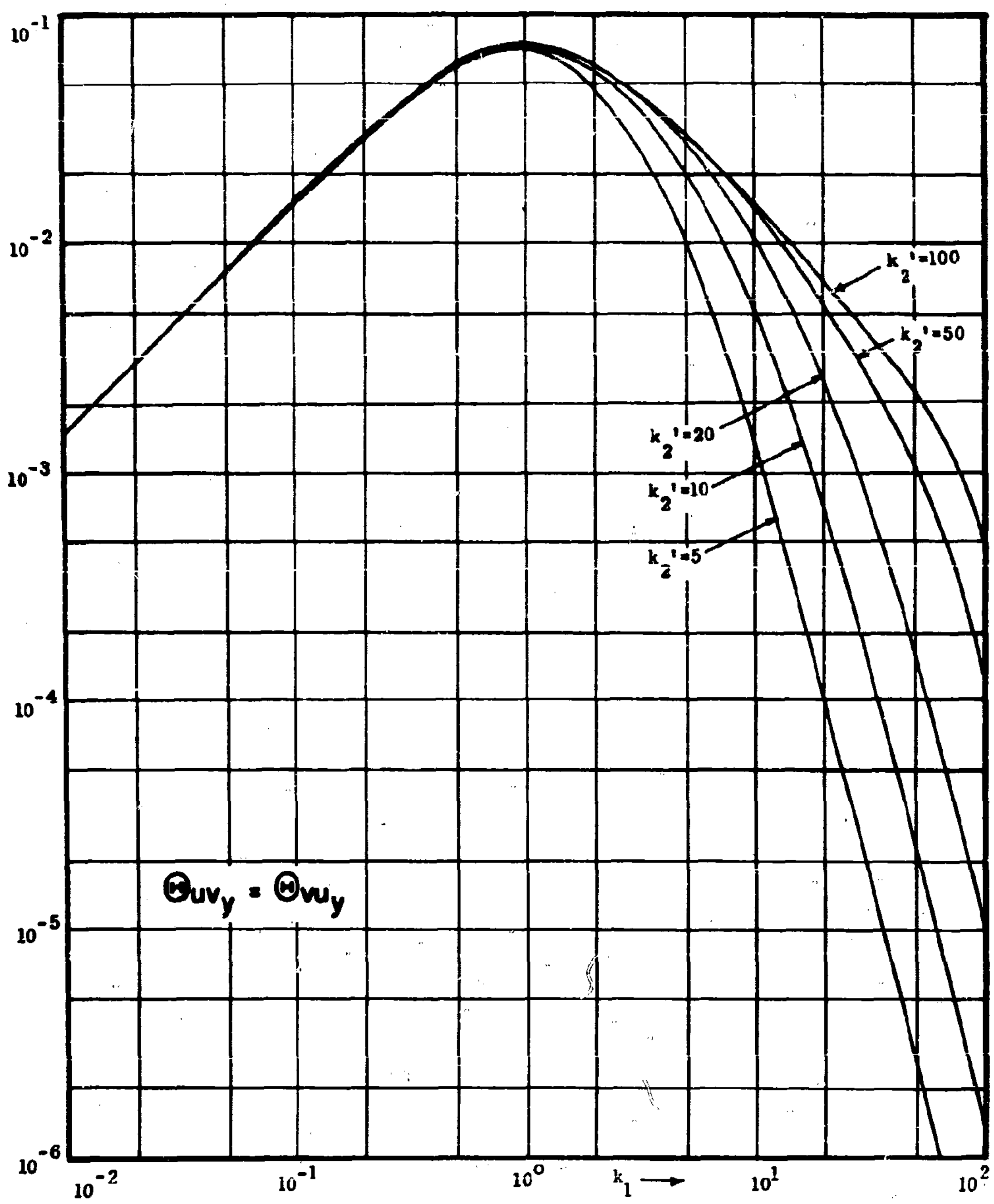


Fig. 13 Spectrum function $-\Theta_{uv_y}/i\sigma^2$

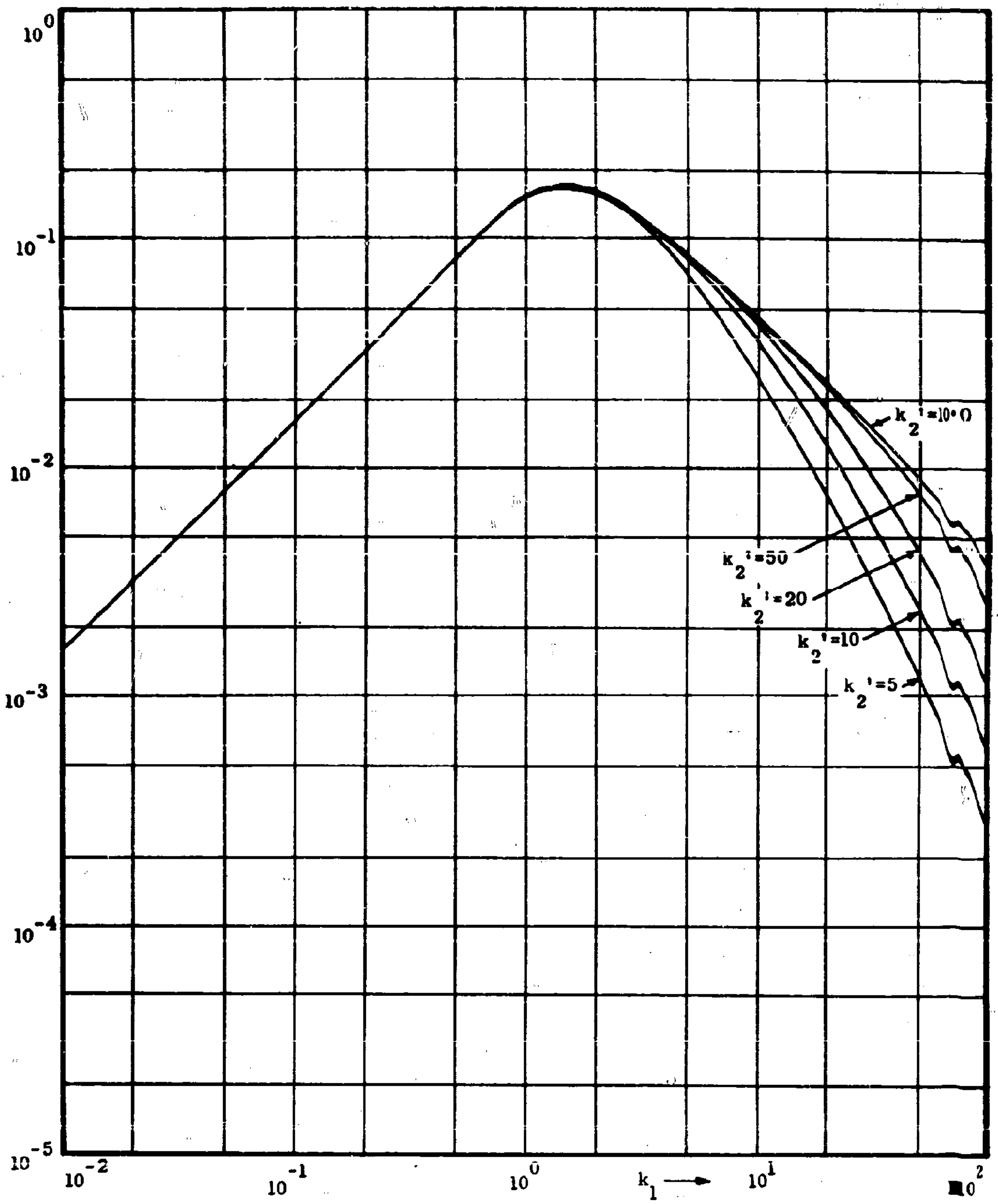


Fig. 14 Spectrum function $\Theta_{vv_x}/i\sigma^2$

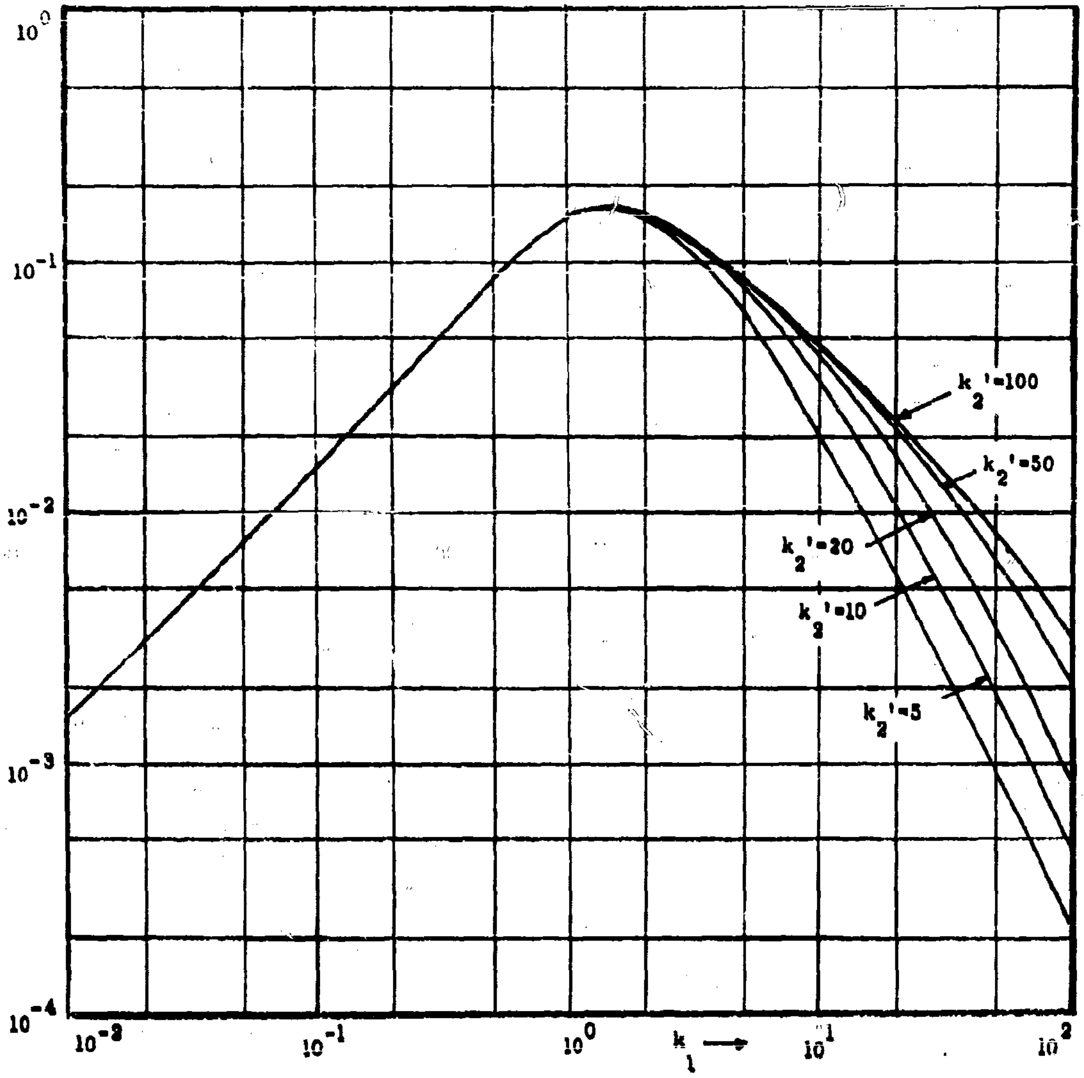


Fig. 15 Spectrum function $\Theta_{ww_x}/i\sigma^2$

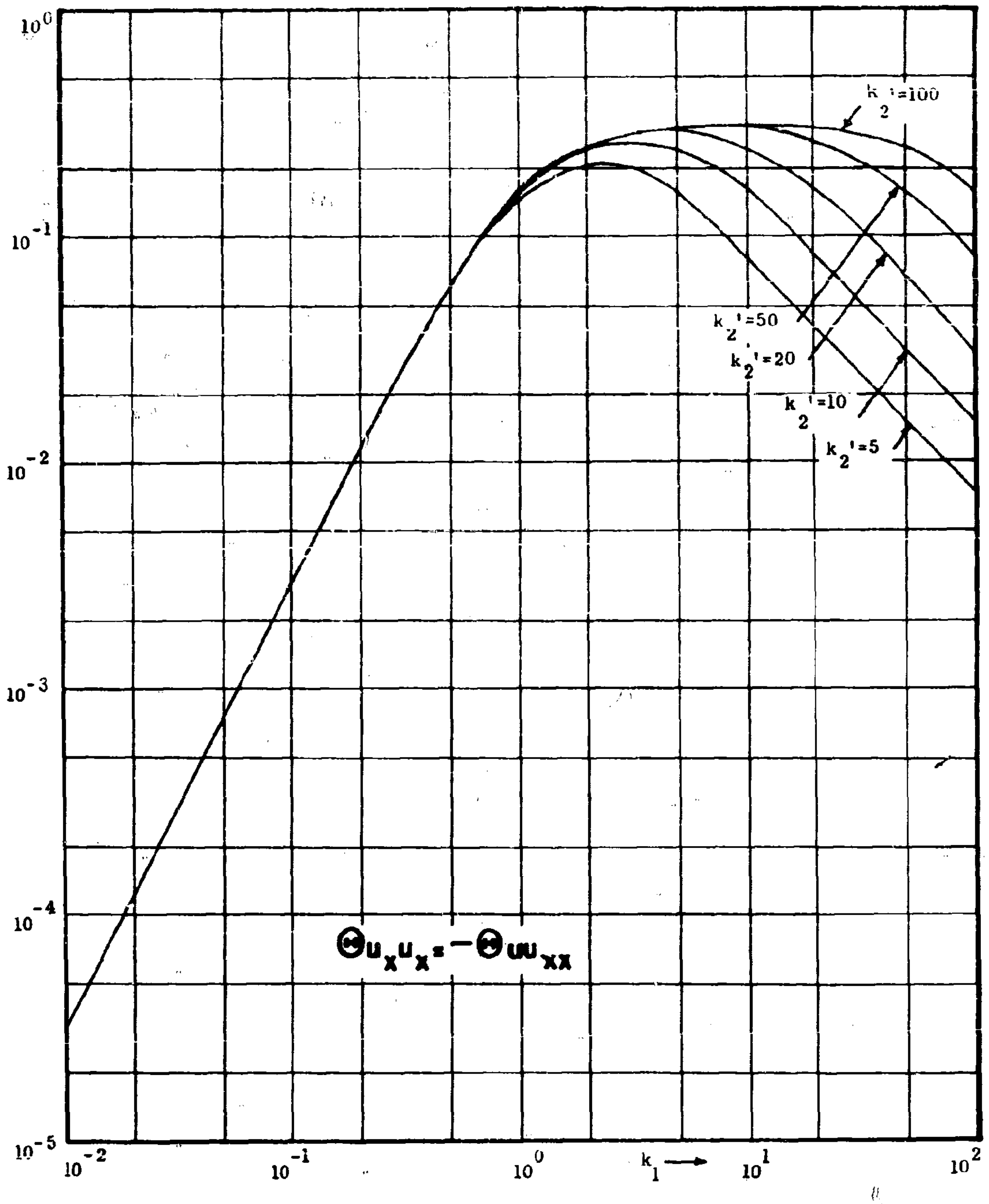


Fig. 16 Spectrum function $\Theta_{u_x u_x} L/\sigma^2$

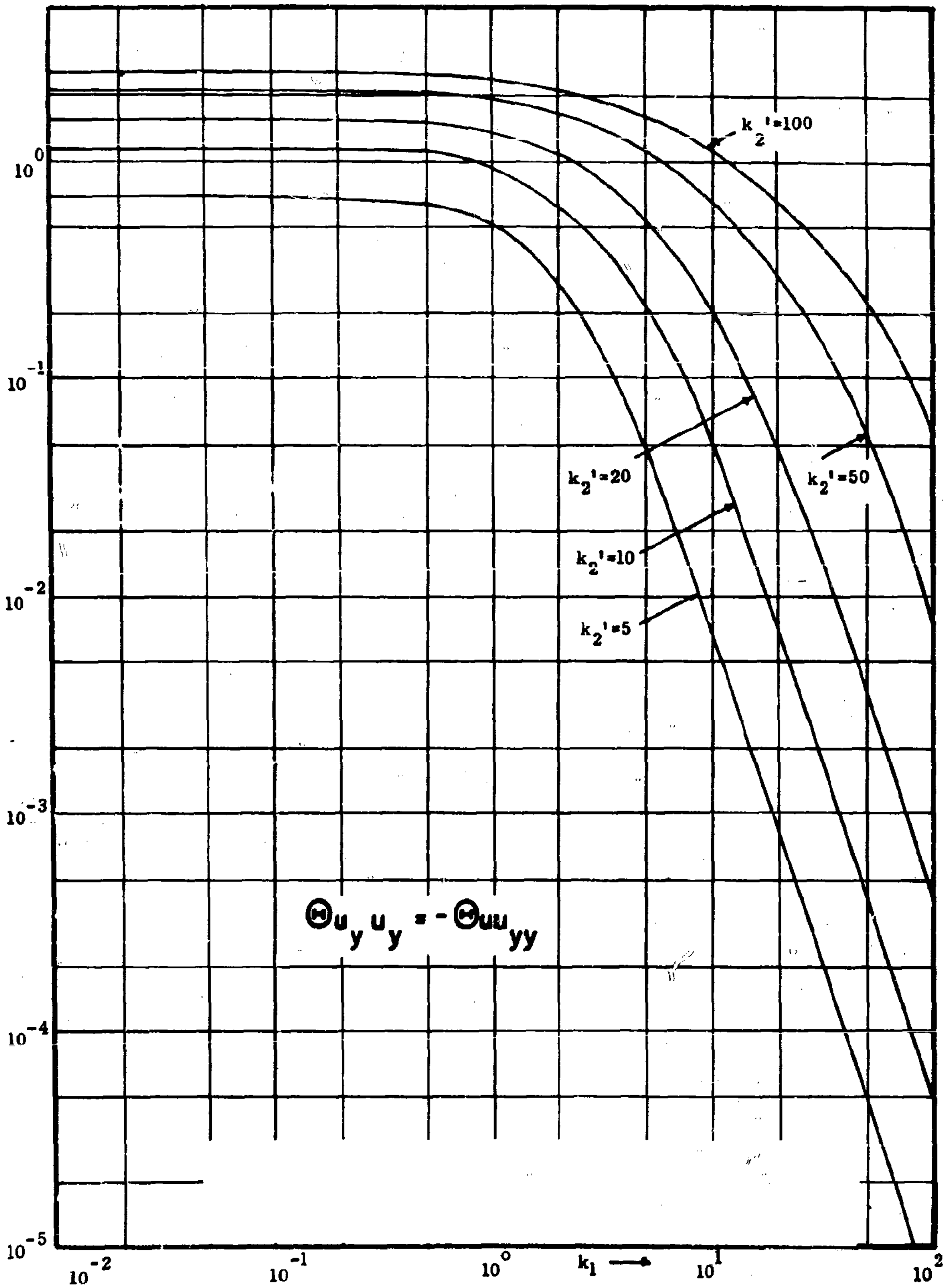


Fig. 17 Spectrum function $\Theta_{u_y u_y} L/\sigma^2$

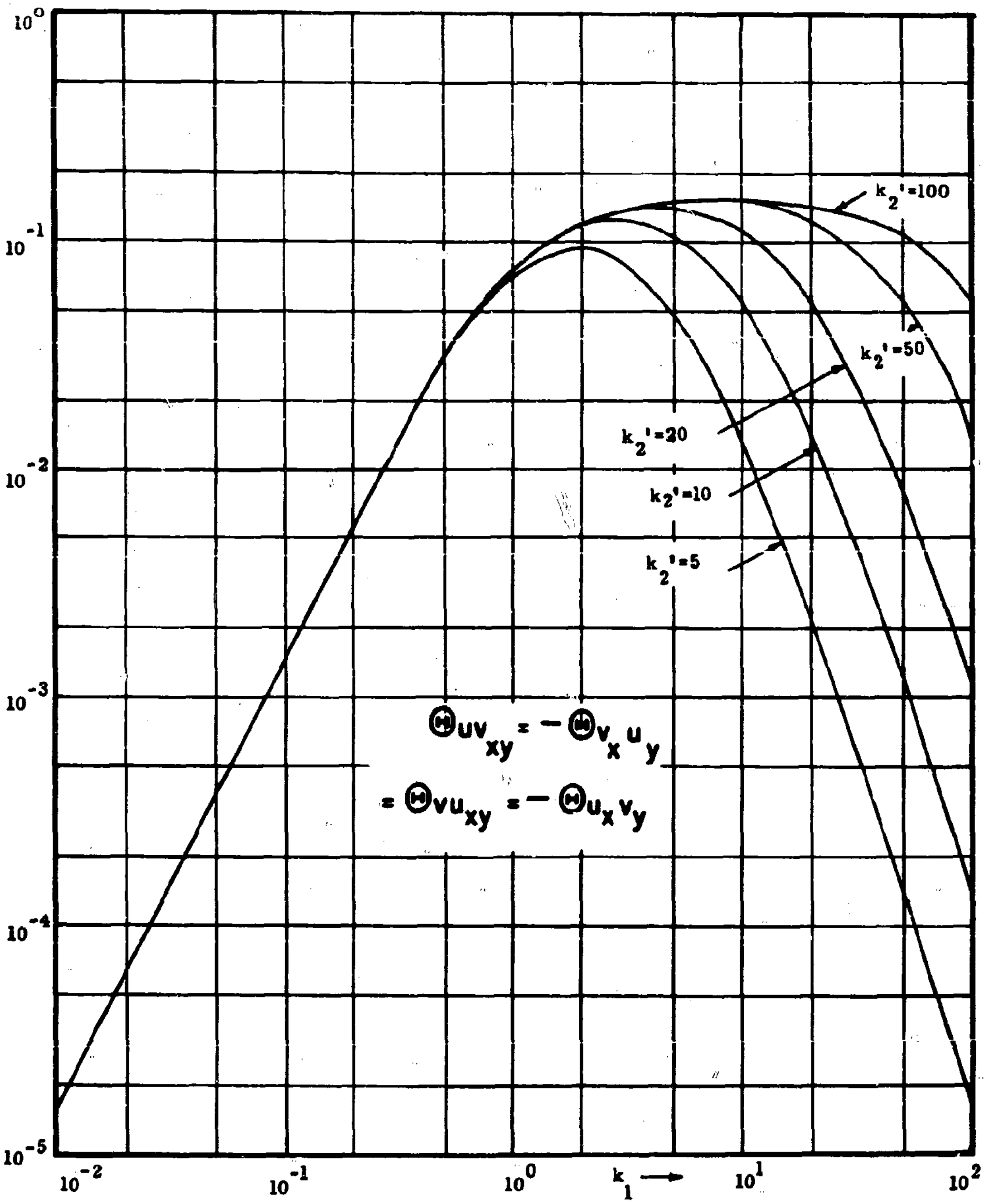


Fig. 18 Spectrum function $\Theta_{uv_{xy}} L/\sigma^2$

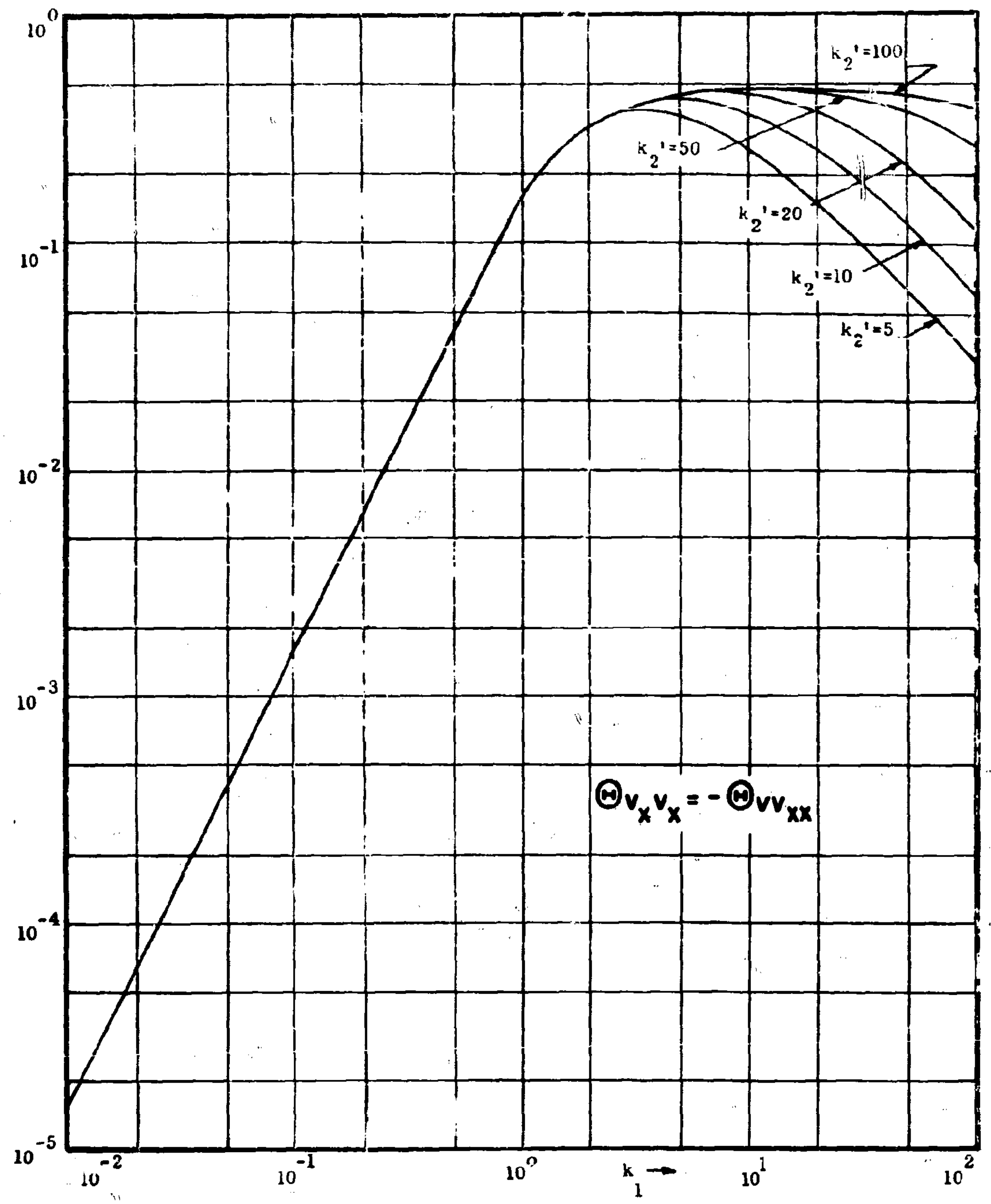


Fig. 19 Spectrum function $\Theta_{v_x v_x} L/\sigma^2$

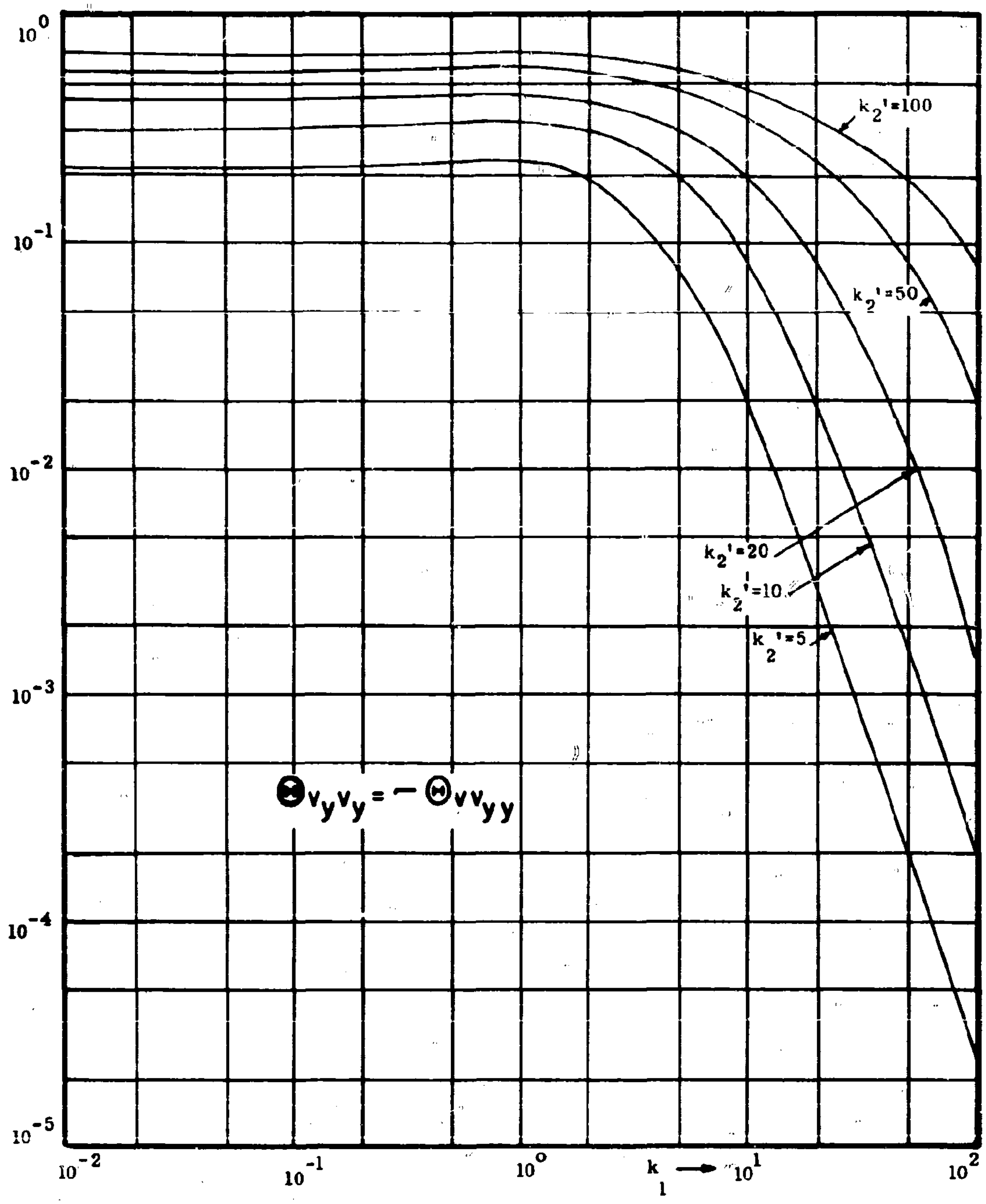


Fig. 20 Spectrum function $\Theta_{v_y v_y} L/\sigma^2$

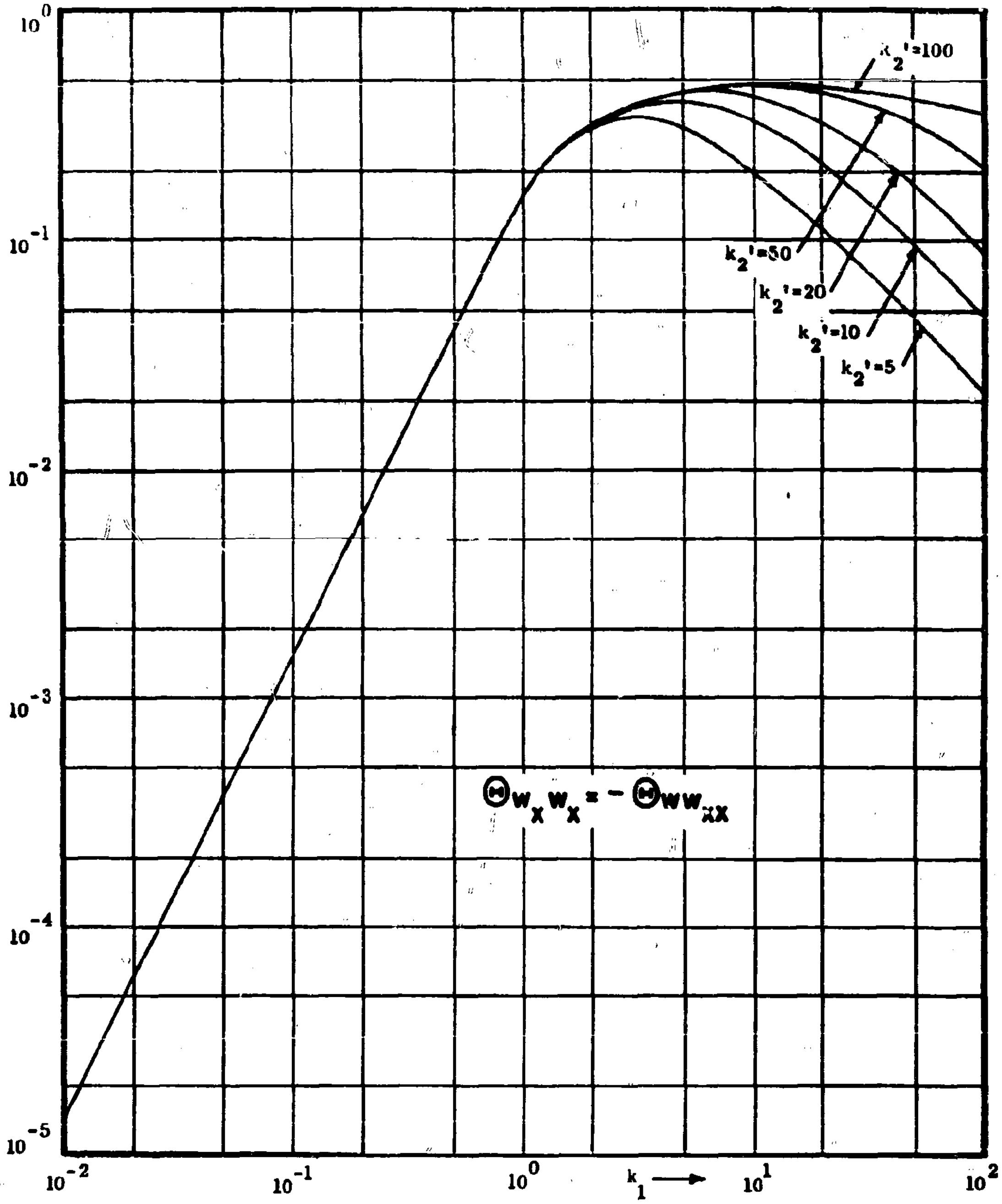


Fig. 21 Spectrum function $\Theta_{w_x w_x} L/\sigma^2$

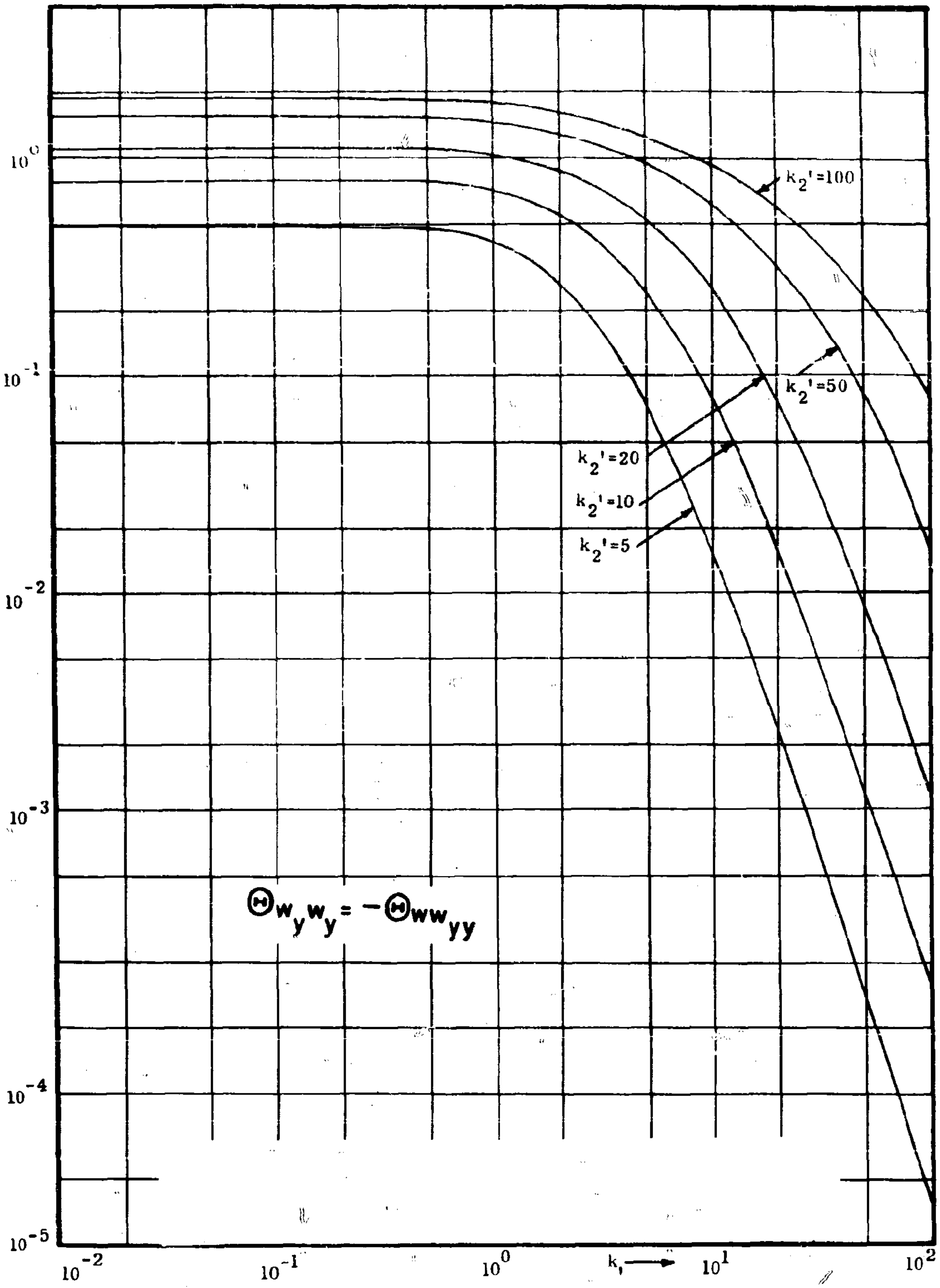
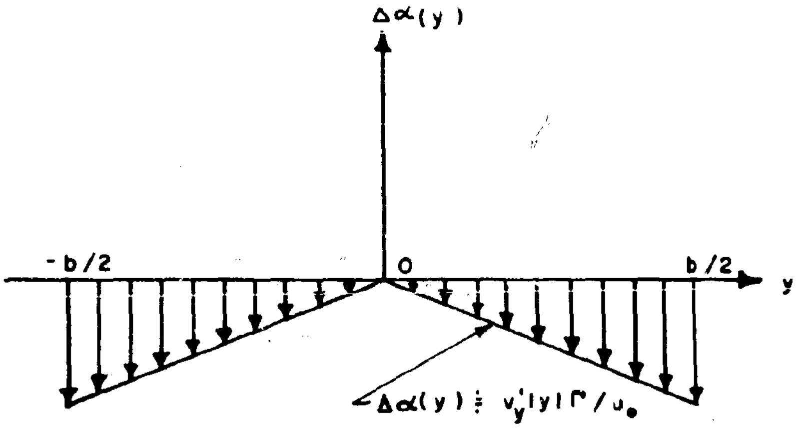
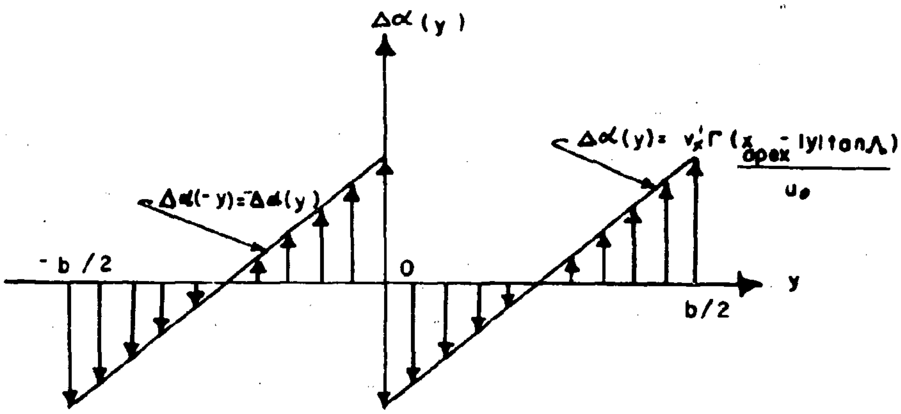


Fig. 22 Spectrum function $\Theta_{w_y w_y} L/\sigma^2$



(a) $\Delta\alpha$ distribution associated with v_y'



(b) $\Delta\alpha$ distribution associated with v_x' for a sweptback wing

Fig. 23 Variations of angle of attack along span of a high-aspect-ratio wing associated with the v-derivatives

DISCUSSION

O.E. Michaelsen (Canada): I believe Professor Etkin mentioned that the difference between the simple one-dimensional gust and the random turbulence may be important, particularly with regard to V/STOL aircraft. I would believe that this indeed is the case since the changes in inflow angles to the aircraft due to gusts are of the same order as the basic flow angle in the steady-state condition. In addition, an amplification of the turbulence effects can occur as a result of the induced changes in the slipstream flow angle to a tilt, or deflected slipstream wing, I would appreciate it if Professor Etkin would comment upon this.

Reply by Author: I agree that the behaviour of vehicles in hovering and low-speed flight subjected to low-level turbulence is important to understand in connection with V/STOL and G/TOL aircraft. As Mr. Michaelsen points out, the inflow angles may be so large that non-linear aerodynamics is involved. This will certainly present a serious difficulty. It may also be expected that all three turbulence components will be of comparable importance, and that the correlations associated with turbulent shear flow (R_{uw} , R_{vw}) will have to be taken into consideration. I think we are a long way yet from fully understanding the behaviour of all kinds of airborne vehicles in atmospheric turbulence.

ADDENDUM

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on

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Complete List of Papers Presented

Following is a list of the titles and authors of the 41 papers presented at the Stability and Control Meeting held in Brussels in April, 1960, together with the AGARD Report number covering the publication of each paper.

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The Aeroplane Designer's Approach to Stability and Control, by G.H.Lee (United Kingdom) .. .. Report 334

The Missile Designer's Approach to Stability and Control Problems, by M.W.Hunter and J.W.Hindes (United States) .. .. Report 335

DESIGN REQUIREMENTS

Flying Qualities Requirements for United States Navy and Air Force Aircraft, by W.Koven and R.Wasicko (United States) .. .. .. Report 336

Design Aims for Stability and Control of Piloted Aircraft, by H.J.Allwright (United Kingdom) .. .. Report 337

Design Criteria for Missiles, by L.G.Evans (United Kingdom) .. Report 338

AERODYNAMIC DERIVATIVES

State of the Art of Estimation of Derivatives, by H.H.B.M.Thomas (United Kingdom) .. .. Report 339

The Estimation of Oscillatory Wing and Control Derivatives, by W.E.A.Acum and H.C.Garner (United Kingdom) .. .. Report 340

Current Progress in the Estimation of Stability Derivatives, by L.V.Malthan and D E.Hoak (United States) .. .. Report 341

Calculation of Non-Linear Aerodynamic Stability Derivatives of Aeroplanes, by K.Gersten (Germany) .. .. Report 342

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