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ON THE COMPUTATIONAL SOLUTION OF TWO-POINT BOUNDARY-VALUE PROBLEMS

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PREFACE

Part of the Project RAND research program consists of basic supporting studies in mathematics. In this Memorandum the authors discuss a method for solving large systems of differential equations where the solution is subject to certain boundary conditions.

SUMMARY

Two-point boundary-value problems for second-order systems of linear differential equations are usually solved by a process involving the inversion of a certain matrix. If the system is too large, it may be difficult to compute this inverse to a high degree of accuracy. The purpose of this paper is to demonstrate that this difficulty can in some cases be circumvented by applying a method like that of Bodewig and Hotelling.

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ON THE COMPUTATIONAL SOLUTION OF TWO-POINT BOUNDARY-VALUE PROBLEMS

1. INTRODUCTION

Consider (as in [1]) the n-dimensional vector differential equation

$$x'' + A(t)x = 0 , \tag{1.1}$$

where the solution is subject to the boundary conditions

$$x(0) = c, \quad x(1) = d. \tag{1.2}$$

The problem is generally solved as follows. Let X_1 and X_2 denote the matrix solution of

$$X'' + A(t)X = 0 \tag{1.3}$$

satisfying the initial conditions

$$\begin{aligned} X_1(0) &= I, \quad X_1'(0) = 0, \\ X_2(0) &= 0, \quad X_2'(0) = I. \end{aligned} \tag{1.4}$$

If g represents the (unknown) value of $x'(0)$, where $x(t)$ is the solution to the problem, then

$$g = X_2(1)^{-1}[d - X_1(1)c]. \tag{1.5}$$

If $X_2(1)$ is singular, then there may be many solutions, or none, and (1.5), of course, makes no sense.

If n is large, it may be difficult to compute $X_2^{-1}(1)$ to a high degree of accuracy. The purpose of this paper is to discuss a method of overcoming this difficulty.

2. AN ITERATIVE TECHNIQUE

Let $X_2^*(1)$ be some approximation to $X_2^{-1}(1)$. Define

$$(2.1) \qquad g_1 = X_2^*(1)[d - X_1(1)c],$$

$$g_n = X_2^*(1)[d - X_1(1)c - X_2(1)g_{n-1}] + g_{n-1}.$$

Then we have the following theorem:

Theorem. If the spectral radius of $I - X_2^*(1)X_2(1)$ *is less than one, then the sequence* $\{g_n\}$ *defined by* (2.1) *converges to* g, *the unique solution of* (1.5).

Proof. First note that if $I - X_2^*(1)X_2(1)$ has spectral radius less than one, then $X_2^*(1)X_2(1)$ must be nonsingular. Thus $X_2^*(1)$ and $X_2(1)$ are nonsingular, which means that (1.5) has a unique solution. If g is the unique solution of (1.5), then

$$(2.2) \qquad \begin{aligned} g_n - g &= X_2^*(1)[d - X_1(1)c - X_2(1)g_{n-1}] + g_{n-1} - g \\ &= X_2^*(1)[d - X_1(1)c - X_2(1)g_{n-1}] \\ &\quad - X_2^*(1)[d - X_1(1)c - X_2(1)g] + g_{n-1} - g \\ &= (I - X_2^*(1)X_2(1))(g_{n-1} - g). \end{aligned}$$

If the spectral radius of $I - X_2^*(1)X_2(1)$ is less than one, this shows that $\{g_n - g\}$ goes to zero as n goes to infinity, and this concludes the proof. This theorem may be viewed as an application of a method of matrix inversion like that of Bodewig and Hotelling (see [3], [4] for additional references).

Corollary. *If* $A(t) = B^2$, *a constant positive-definite matrix, then taking* $X_2^*(1) = X_2(1)$ *makes* $\{g_n\}$ *converge to the solution.*

Proof. Since $X_2(1) = B^{-1} \sin B$, it follows that the eigenvalues of $X_2(1)$ all have absolute value less than one, and thus all the eigenvalues of $X_2^2(1)$ are between 0 and one.

Corollary. *If each element of* $I - X_2^*(1)X_2(1)$ *is less in absolute value than* $1/n$, *then* $\{g_n\}$ *converges to the solution.*

Corollary. *If* $A(t) = -B^2$, *where* B *is a matrix with only real eigenvalues each of which is greater than zero, then taking* $X_2^*(1) = 2Be^{-B}$ *makes* $\{g_n\}$ *converge to the solution.*

Proof. $X_2(t) = B^{-1}(\frac{e^{Bt} - e^{-Bt}}{2})$, whence $X_2^*(1)X_2(1)$ equals $I - e^{-2B}$.

Corollary. *If* $Y_1(t)$, $Y_2(t)$ *are solutions to* $Y'' + A(1-t)Y = 0$ *satisfying initial conditions like* (1.4), *then taking* $X_2^*(1) = Y_1'(1)$ *will make* $\{g_n\}$ *converge to the solution if* $Y_2'(1)X_2'(1)$ *has spectral radius less than one.*

Proof. $Y_2'(1)X_2'(1) = I - Y_1'(1)X_2(1)$.

Corollary. If $X_2^*(1) = dA$, where A is the transpose of $X_2(1)$ and d is a positive constant chosen to be less than twice the reciprocal of the sum of the absolute values of each row of $AX_2(1)$, then $\{g_n\}$ converges to the solution.

Note that this last corollary is not apt to be computationally useful, however, since if $X_2(1)$ has some very small eigenvalues (and thus is hard to invert), under the above procedure $I - X_2^*(1)X_2(1)$ will have spectral radius very close to one, so that convergence will be slow.

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