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MEMORANDUM  
RM-3857-PR  
APRIL 1964

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# ON THE COMPUTATIONAL SOLUTION OF TWO-POINT BOUNDARY-VALUE PROBLEMS

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PREPARED FOR:  
UNITED STATES AIR FORCE PROJECT RAND

The **RAND** Corporation  
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This research is sponsored by the United States Air Force under Project RAND—contract No. AF 49(638)-700 monitored by the Directorate of Development Planning, Deputy Chief of Staff, Research and Development, Hq USAF. Views or conclusions contained in this Memorandum should not be interpreted as representing the official opinion or policy of the United States Air Force.

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PREFACE

Part of the Project RAND research program consists of basic supporting studies in mathematics. In this Memorandum the authors discuss a method for solving large systems of differential equations where the solution is subject to certain boundary conditions.

SUMMARY

Two-point boundary-value problems for second-order systems of linear differential equations are usually solved by a process involving the inversion of a certain matrix. If the system is too large, it may be difficult to compute this inverse to a high degree of accuracy. The purpose of this paper is to demonstrate that this difficulty can in some cases be circumvented by applying a method like that of Bodewig and Hotelling.

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ON THE COMPUTATIONAL SOLUTION OF  
TWO-POINT BOUNDARY-VALUE PROBLEMS

1. INTRODUCTION

Consider (as in [1]) the  $n$ -dimensional vector differential equation

$$(1.1) \quad x'' + A(t)x = 0,$$

where the solution is subject to the boundary conditions

$$(1.2) \quad x(0) = c, \quad x(1) = d.$$

The problem is generally solved as follows. Let  $X_1$  and  $X_2$  denote the matrix solution of

$$(1.3) \quad X'' + A(t)X = 0$$

satisfying the initial conditions

$$(1.4) \quad X_1(0) = I, \quad X_1'(0) = 0,$$

$$X_2(0) = 0, \quad X_2'(0) = I.$$

If  $g$  represents the (unknown) value of  $x'(0)$ , where  $x(t)$  is the solution to the problem, then

$$(1.5) \quad g = X_2(1)^{-1}[d - X_1(1)c].$$

If  $X_2(1)$  is singular, then there may be many solutions, or none, and (1.5), of course, makes no sense.

If  $n$  is large, it may be difficult to compute  $X_2^{-1}(1)$  to a high degree of accuracy. The purpose of this paper is to discuss a method of overcoming this difficulty.



2. AN ITERATIVE TECHNIQUE

Let  $X_2^*(1)$  be some approximation to  $X_2^{-1}(1)$ .

Define

$$(2.1) \quad \begin{aligned} g_1 &= X_2^*(1)[d - X_1(1)c], \\ g_n &= X_2^*(1)[d - X_1(1)c - X_2(1)g_{n-1}] + g_{n-1}. \end{aligned}$$

Then we have the following theorem:

Theorem. If the spectral radius of  $I - X_2^*(1)X_2(1)$  is less than one, then the sequence  $\{g_n\}$  defined by (2.1) converges to  $g$ , the unique solution of (1.5).

Proof. First note that if  $I - X_2^*(1)X_2(1)$  has spectral radius less than one, then  $X_2^*(1)X_2(1)$  must be nonsingular. Thus  $X_2^*(1)$  and  $X_2(1)$  are nonsingular, which means that (1.5) has a unique solution. If  $g$  is the unique solution of (1.5), then

$$(2.2) \quad \begin{aligned} g_n - g &= X_2^*(1)[d - X_1(1)c - X_2(1)g_{n-1}] + g_{n-1} - g \\ &= X_2^*(1)[d - X_1(1)c - X_2(1)g_{n-1}] \\ &\quad - X_2^*(1)[d - X_1(1)c - X_2(1)g] + g_{n-1} - g \\ &= (I - X_2^*(1)X_2(1))(g_{n-1} - g). \end{aligned}$$

If the spectral radius of  $I - X_2^*(1)X_2(1)$  is less than one, this shows that  $\{g_n - g\}$  goes to zero as  $n$  goes to infinity, and this concludes the proof. This

theorem may be viewed as an application of a method of matrix inversion like that of Bodewig and Hotelling (see [3], [4] for additional references).

Corollary. If  $A(t) = B^2$ , a constant positive-definite matrix, then taking  $X_2^*(1) = X_2(1)$  makes  $\{g_n\}$  converge to the solution.

Proof. Since  $X_2(1) = B^{-1} \sin B$ , it follows that the eigenvalues of  $X_2(1)$  all have absolute value less than one, and thus all the eigenvalues of  $X_2^2(1)$  are between 0 and one.

Corollary. If each element of  $I - X_2^*(1)X_2(1)$  is less in absolute value than  $1/n$ , then  $\{g_n\}$  converges to the solution.

Corollary. If  $A(t) = -B^2$ , where  $B$  is a matrix with only real eigenvalues each of which is greater than zero, then taking  $X_2^*(1) = 2Be^{-B}$  makes  $\{g_n\}$  converge to the solution.

Proof.  $X_2(t) = B^{-1} \left( \frac{e^{Bt} - e^{-Bt}}{2} \right)$ , whence  $X_2^*(1)X_2(1)$  equals  $I - e^{-2B}$ .

Corollary. If  $Y_1(t), Y_2(t)$  are solutions to  $Y'' + A(1-t)Y = 0$  satisfying initial conditions like (1.4), then taking  $X_2^*(1) = Y_1'(1)$  will make  $\{g_n\}$  converge to the solution if  $Y_2'(1)X_2'(1)$  has spectral radius less than one.

Proof.  $Y_2'(1)X_2'(1) = I - Y_1'(1)X_2(1)$ .

Corollary. If  $X_2^*(1) = dA$ , where  $A$  is the transpose of  $X_2(1)$  and  $d$  is a positive constant chosen to be less than twice the reciprocal of the sum of the absolute values of each row of  $AX_2(1)$ , then  $\{g_n\}$  converges to the solution.

Note that this last corollary is not apt to be computationally useful, however, since if  $X_2(1)$  has some very small eigenvalues (and thus is hard to invert), under the above procedure  $I - X_2^*(1)X_2(1)$  will have spectral radius very close to one, so that convergence will be slow.

REFERENCES

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