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ROYAL AIRCRAFT ESTABLISHMENT

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THE LONG-PERIOD MOTION OF THE PLANE OF A DISTANT CIRCULAR ORBIT

by

R. R. Allan and G. E. Cook

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SUMMARY

We consider Earth satellites in the region where the perturbing effects due to Earth's oblateness and luni-solar gravitational forces are comparable. A general solution is obtained for simultaneous precession about any number of fixed axes; this is an extension of Laplace's treatment for the motion of Iapetus about Saturn. Results are given for general orbits on the assumption that the lunar orbit lies in the ecliptic. Synchronous orbits are considered in greater detail.

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1 INTRODUCTION

The main perturbations for a relatively distant Earth satellite are due to the Earth's oblateness and the gravitational attractions of the Sun and Moon. We consider satellites at distances of a few Earth radii, where the effect of the Earth's oblateness is of the same order as the effects of the distant bodies. This range covers a major region of interest for communication satellites. Two aspects of this problem have already been considered by the authors separately[1,2], and the present work gives a more complete solution of the general problem.

The classical problem of this type concerns the motion of Iapetus[3] which, apart from the very distant and retrograde satellite Phoebe, is the outermost satellite of Saturn. The perturbing force due to the oblateness of Saturn (suitably modified to include the attraction of the rings and the inner satellites) is comparable with the perturbing force due to the gravitational attraction of the Sun. By itself the former would cause the satellite's orbital plane to precess around the axis of rotation of Saturn, while the latter by itself would cause the orbital plane to precess around the pole of the orbit of Saturn. The motion of Iapetus, which is a combination of simultaneous precessions about two different axes, was first considered by Laplace[4] and later discussed in considerable detail by Tisserand[5].

We take the disturbing function to include the second zonal harmonic of the Earth's gravitational potential and the lowest order term, i.e. the second harmonic, in the expansion of each of the solar and lunar disturbing functions. We consider only initially circular satellite orbits and, since the second harmonic has no secular effect on the eccentricity of a circular orbit, the orbits will remain circular throughout. In fact the higher harmonics in the lunar disturbing function become more important as the orbital radius increases, and their neglect sets an upper limit to the radius of the satellite orbit for which the theory is applicable, especially since the parallactic term (third harmonic) will change the eccentricity of a circular orbit. A small initial eccentricity, such as may be built up by the parallactic term, may increase rapidly if the orbit is sufficiently large[6].

The disturbing functions are averaged analytically over the mean anomalies of both the satellite and the disturbing bodies. For this procedure to give a good representation of the satellite's motion, the period of the predicted long-period motion of the orbital plane should be reasonably long compared with the periods of the disturbing bodies.

As a result of the limitations discussed in the last two paragraphs our treatment is most useful for satellites at distances between about 3 and 10 Earth radii.

2 DISTURBING FUNCTION AND EQUATIONS OF MOTION

The perturbing effect on the motion of a satellite about the Earth due to the gravitational attraction of a third body of mass m_j ($j = 1$ for Sun, $j = 2$ for Moon) is given by the disturbing function

$$\mu_j \left\{ |\underline{r} - \underline{r}_j|^{-1} - r_j^{-3}(\underline{r} \cdot \underline{r}_j) \right\} , \qquad (1)$$

where $\underline{r}$, $\underline{r}_j$ are the positions of the satellite and the disturbing body relative to the Earth, and $\mu_j = Gm_j$. We assume $r \ll r_j$ and retain only the lowest-order term in the expansion of this disturbing function in powers of r/r_j, viz.

$$\frac{\mu_j r^2}{r_j^3} P_2(\cos \theta_j) = \frac{\mu_j}{r_j^5} \left\{ \frac{3}{2} (\underline{r} \cdot \underline{r}_j)^2 - \frac{1}{2} r^2 r_j^2 \right\}, \tag{2}$$

where θ_j is the angle between $\underline{r}$ and $\underline{r}_j$. Including the second harmonic in the Earth's gravitational potential, the complete disturbing function may be written

$$U = -\frac{\mu J_2 R_E^2}{r^3} P_2(\cos \theta_o) + \sum_{j=1,2} \frac{\mu_j r^2}{r_j^3} P_2(\cos \theta_j) . \tag{3}$$

Here $\mu = Gm$, where m is the mass of the Earth, J_2 is the coefficient of the second zonal harmonic, R_E is the mean equatorial radius of the Earth, and θ_o is the angle between $\underline{r}$ and the polar axis of the Earth.

For the long-period motion it is permissible to average the disturbing function over the mean anomalies of the satellite and of the disturbing bodies. Equation (3) is first written in the form

$$U = -\frac{\mu J_2 R_E^2}{2r^5} \left\{ 3(\underline{r} \cdot \underline{R}_o)^2 - r^2 \right\} + \sum_{j=1,2} \frac{\mu_j}{2r_j^5} \left\{ 3(\underline{r} \cdot \underline{r}_j)^2 - r^2 r_j^2 \right\}, \tag{4}$$

where $\underline{R}_o$ is a unit vector along the Earth's axis. To average over the mean anomaly ℓ of the satellite the following integrals will be used (cf. Musen[7]).

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{1}{r^3} d\ell = \frac{1}{a^3(1 - e^2)^{3/2}}, \tag{5a}$$

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{1}{r^5} \underline{r}\,\underline{r}\, d\ell = \frac{1}{2a^3(1 - e^2)^{3/2}} (\underline{\underline{1}} - \underline{R}\,\underline{R}), \tag{5b}$$

$$\frac{1}{2\pi}\int_0^{2\pi} r^2 \, d\ell = a^2\left(1 + \frac{3}{2}e^2\right), \tag{6a}$$

$$\frac{1}{2\pi}\int_0^{2\pi} \underline{r}\,\underline{r} \, d\ell = \tfrac{1}{2}a^2\left\{(1 + 4e^2)\,\underline{P}\,\underline{P} + (1 - e^2)\,\underline{Q}\,\underline{Q}\right\}, \tag{6b}$$

where a, e are the semi-major axis and eccentricity of the satellite's orbit, and $\underline{P}$, $\underline{Q}$, $\underline{R}$ are the natural unit vectors associated with the satellite orbit, with $\underline{R}$ along the positive normal to the orbital plane, $\underline{P}$ in the orbital plane in the direction of perigee, and $\underline{Q} = \underline{R} \wedge \underline{P}$. $\underline{\underline{1}}$ is the idemtensor. Equations (5a,b) may easily be verified taking the true anomaly as the variable of integration, and equations (6a,b) by taking the eccentric anomaly. It is immaterial in what order the double averaging is performed, and it proves rather simpler to average first over the mean anomalies of the disturbing bodies using the analogues of (5a) and (5b) so that the singly-averaged disturbing function may be written as

$$U = -\frac{\mu J_2 R_E^2}{2r^5}\left\{3(\underline{r}\cdot\underline{R}_o)^2 - r^2\right\} - \sum_{j=1,2}\frac{\mu_j}{4a_j^3(1 - e_j^2)^{3/2}}\left\{3(\underline{r}\cdot\underline{R}_j^2) - r^2\right\}, \tag{7}$$

where a_j, e_j are the semi-major axis and eccentricity of the orbit of the j^{th} disturbing body, and $\underline{R}_j$ is the unit vector normal to the orbital plane, i.e. $\underline{R}_1$ lies along the pole of the ecliptic and $\underline{R}_2$ is normal to the lunar orbital plane.

After taking the average over the mean anomaly of the satellite using (5a) to (6b), the doubly-averaged disturbing function takes the form

$$U = n\,a^2\,U^*, \tag{8}$$

where

$$U^* = \omega_o(1 - e^2)^{-3/2}\left\{\tfrac{1}{2}(\underline{R}\cdot\underline{R}_o)^2 - \tfrac{1}{6}\right\} + \sum_{j=1,2}\omega_j\left[\tfrac{1}{2}(1 - e^2)(\underline{R}\cdot\underline{R}_j)^2 + e^2\left\{1 - \tfrac{5}{2}(\underline{P}\cdot\underline{R}_j)^2\right\}\right], \tag{9}$$

where n is the mean motion of the satellite, and

$$\omega_0 = \frac{3n\, J_2\, R_E^2}{2a^2}\,, \quad \omega_j = \frac{3\mu_j}{4na_j^3(1-e_j^2)^{3/2}}\;. \tag{10}$$

The semi-major axis a and the mean motion n are constant in the long-period motion, and they have been freely incorporated in (10). In writing (9) we have used the tensor identity

$$\underline{P}\,\underline{P} + \underline{Q}\,\underline{Q} + \underline{R}\,\underline{R} = \underline{\underline{1}}\,,$$

and dropped an irrelevant constant term from the luni-solar part of the disturbing function.

If we choose a particular reference plane, we can express the disturbing function in terms of the corresponding elements e, i, Ω, ω and give the Lagrange equations for their variation. However this procedure is necessarily very complicated if the two disturbing orbit planes are taken as distinct, and it is much simpler to use vectorial elements which are not tied to any particular frame. The vectorial elements are taken as $\underline{e}$ and $\underline{h}$ defined by

$$\underline{e} = e\,\underline{P}\,, \quad \text{and} \quad \underline{h} = (1-e^2)^{\frac{1}{2}}\,\underline{R}\,, \tag{11}$$

so that

$$\underline{e}\cdot\underline{h} = 0, \quad \text{and} \quad \underline{e}^2 + \underline{h}^2 = 1\,. \tag{12}$$

In terms of these elements the Lagrange equations take the compact and symmetrical form[8]

$$\begin{aligned} \dot{\underline{h}} &= \underline{h}\wedge\frac{\partial U^*}{\partial \underline{h}} + \underline{e}\wedge\frac{\partial U^*}{\partial \underline{e}}\,, \\ \dot{\underline{e}} &= \underline{e}\wedge\frac{\partial U^*}{\partial \underline{h}} + \underline{h}\wedge\frac{\partial U^*}{\partial \underline{e}} \end{aligned} \tag{13}$$

where $U^* = U/na^2$. From (9) the disturbing function U^* may be written in terms of these elements as

$$U^* = \omega_0\left\{\tfrac{1}{2}(1-e^2)^{-5/2}\,(\underline{h}\cdot\underline{R}_0)^2 - \tfrac{1}{6}\,(1-e^2)^{-3/2}\right\}$$
$$+ \sum_{j=1,2}\omega_j\left\{\tfrac{1}{2}(\underline{h}\cdot\underline{R}_j)^2 + e^2 - \tfrac{5}{2}\,(\underline{e}\cdot\underline{R}_j)^2\right\}. \tag{14}$$

From (13) and (14) the equation for $\underline{h}$ is

$$\dot{\underline{h}} = -\omega_0(1-e^2)^{-5/2}(\underline{h}\cdot\underline{R}_0)(\underline{R}_0\wedge\underline{h})$$

$$-\sum_{j=1,2}\omega_j(\underline{h}\cdot\underline{R}_j)(\underline{R}_j\wedge\underline{h}) + 5\sum_{j=1,2}\omega_j(\underline{e}\cdot\underline{R}_j)(\underline{R}_j\wedge\underline{e}) \quad (15)$$

It is obvious from the form of U* that $\partial U^*/\partial\underline{e}$ vanishes for $e = 0$, so that $\dot{\underline{e}}$ must vanish for a circular orbit (with our assumptions regarding the disturbing function).

We now consider only circular satellite orbits, setting $e = 0$ in (15) and replacing $\underline{h}$ by $\underline{R}$, so that we can write

$$\dot{\underline{R}} = -\sum_{j=0}^{2}\omega_j(\underline{R}\cdot\underline{R}_j)(\underline{R}_j\wedge\underline{R}) \quad (16)$$

If only one of the terms on the right hand side of (16) were present, the normal to the orbit $\underline{R}$ would simply regress around the appropriate $\underline{R}_j$ at a constant rate $\omega_j \cos\gamma_j$, where γ_j is the angle between $\underline{R}$ and $\underline{R}_j$. The rate of precession is zero when $\gamma_j = 90^\circ$.

For a distant Earth satellite the motion of the orbital plane is a combination of rates of precession about three different axes. Moreover the lunar orbital plane is inclined to the ecliptic at $5^\circ 9'$ and is itself regressing about the pole of the ecliptic with a period of 18.6 years. It is usually adequate to approximate by assuming that the lunar orbit lies in the ecliptic, or alternatively by taking a mean position for the pole of the lunar orbit if the time interval of interest is short.

3 GENERAL SOLUTION FOR ANY NUMBER OF FIXED AXES

We shall show that equation (16) can be solved exactly for simultaneous precession about any number of fixed axes. However we first consider the simple approximation of assuming that the angles γ_j between $\underline{R}$ and the various axes $\underline{R}_j$ are all small. This requires that the axes $\underline{R}_j$ themselves must be close together, which is approximately true in practice since $\underline{R}_0$ and $\underline{R}_1$ are inclined at 23°, $\underline{R}_1$ and $\underline{R}_2$ at 5°, and $\underline{R}_0$ and $\underline{R}_2$ at between 18° and 28°. If the various cosine factors $(\underline{R}\cdot\underline{R}_j) = \cos\gamma_j$ are simply replaced by unity, equation (16) becomes

$$\dot{\underline{R}} \simeq -\ddot{\omega}\,\overline{\underline{R}}\wedge\underline{R}, \quad (17)$$

where

$$\bar{\omega}\,\underline{\bar{R}} = \sum_j \omega_j\,\underline{R}_j\,, \tag{18}$$

and $\underline{\bar{R}}$ is a unit vector. Then $\underline{R}$ regresses at a constant inclination around the mean pole $\underline{\bar{R}}$ with the precession velocity $\bar{\omega}$.

To obtain an exact solution of (16), we note that the equation simplifies considerably if it is referred to the principal axes of the symmetric tensor $\Sigma\,\omega_j\,\underline{R}_j\,\underline{R}_j$. Alternatively one can ask whether there are any positions where $\underline{R}$ will remain at rest. In fact $\dot{\underline{R}}$ will vanish if the scalar product of $\underline{R}$ with the tensor either vanishes or is a multiple of $\underline{R}$. In either case $\underline{R}$ is an eigenvector of the tensor, i.e.

$$(\Sigma\,\omega_j\,\underline{R}_j\,\underline{R}_j - \lambda\,\underline{\underline{1}})\cdot\underline{R} = 0\,, \tag{19}$$

where $\underline{\underline{1}}$ is the idemtensor, and λ is the eigenvalue.

To obtain an idea of the magnitudes involved, the eigenvalues have been evaluated using the approximation that the lunar orbit lies in the ecliptic, i.e. that $\underline{R}_2$ coincides with $\underline{R}_1$. The $\underline{R}_j$ are first referred to a right-handed system of equatorial axes with one axis towards the vernal equinox. The components of $\underline{R}_o$ are $(0,0,1)$, while those of $\underline{R}_1$ are $(0, -\sin\varepsilon, \cos\varepsilon)$, ε being the obliquity of the ecliptic. Then, if we set $\omega_1 + \omega_2 = \omega^*$, the eigenvalues are the roots of the equation

$$\begin{vmatrix} -\lambda & 0 & 0 \\ 0 & \omega^*\sin^2\varepsilon - \lambda & -\omega^*\sin\varepsilon\cos\varepsilon \\ 0 & -\omega^*\sin\varepsilon\cos\varepsilon & \omega_o + \omega^*\cos^2\varepsilon - \lambda \end{vmatrix} = 0\,.$$

We label the eigenvalues λ_1, λ_2 and λ_3 and arrange them in the order $\lambda_3 > \lambda_2 > \lambda_1$. The suffixes 1, 2 and 3 used here for the eigenvalues, and later for the principal axes, have no connexion with the suffixes 1 and 2 used for the Sun and Moon. We find

$$\lambda_1 = 0,$$

$$\left.\begin{matrix}\lambda_2\\ \lambda_3\end{matrix}\right\} = \tfrac{1}{2}(\omega_o + \omega^*) \mp \tfrac{1}{2}\left\{(\omega_o + \omega^*)^2 - 4\omega_o\,\omega^*\sin^2\varepsilon\right\}^{\frac{1}{2}}.$$

The values of ω_o, ω_1 and ω_2 and the eigenvalues λ_2 and λ_3 are plotted as functions of a/R_E in Fig.1.

The directions of the corresponding principal axes are readily discovered from (16). That corresponding to the eigenvalue zero is clearly perpendicular to $\underline{R}_o$ and $\underline{R}_1$, i.e. it is directed towards (or away from) the vernal equinox. The remaining two principal axes lie in the plane of $\underline{R}_o$ and $\underline{R}_1$. If α is the angle between $\underline{R}_o$ and the principal axis corresponding to λ_3 (or λ_2), we have

$$\tan 2\alpha = \omega^* \sin 2\varepsilon/(\omega_o + \omega^* \cos 2\varepsilon) .$$

The orientation of the principal axes 0123 is shown incidentally in Fig.3, and the value of the angle α is given as a function of a/R_E in Fig.2.

Assuming that, in the general case, we have found the principal axes 01, 02, 03 and the eigenvalues λ_1, λ_2, λ_3, equation (16) becomes in scalar form

$$\left.\begin{aligned} \dot{x}_1 &= (\lambda_3 - \lambda_2)\, x_2\, x_3 \\ \dot{x}_2 &= (\lambda_1 - \lambda_3)\, x_3\, x_1 \\ \dot{x}_3 &= (\lambda_2 - \lambda_1)\, x_1\, x_2 \end{aligned}\right\} \quad , \tag{20}$$

where x_1, x_2, x_3 are the components of $\underline{R}$ along the principal axes. Multiplying each equation by the corresponding x_j and adding, we have

$$x_1\,\dot{x}_1 + x_2\,\dot{x}_2 + x_3\,\dot{x}_3 = 0 ,$$

which merely leads to the known condition that $\underline{R}$ is a unit vector:

$$x_1^2 + x_2^2 + x_3^2 = 1 . \tag{21}$$

On the other hand multiplying each equation by $\lambda_j\, x_j$ and adding gives

$$\lambda_1\, x_1\, \dot{x}_1 + \lambda_2\, x_2\, \dot{x}_2 + \lambda_3\, x_3\, \dot{x}_3 = 0 ,$$

which yields the non-trivial integral

$$\lambda_1\, x_1^2 + \lambda_2\, x_2^2 + \lambda_3\, x_3^2 = \lambda_o , \tag{22}$$

where λ_o is a constant lying between λ_1 and λ_3, from (21).

The integral (22) can be found without diagonalising the tensor, in two other ways. First, multiplying (16) scalarly by $\sum_k \omega_k(\underline{R} \cdot \underline{R}_k)\, \underline{R}_k$ gives

$$\sum_k \omega_k(\underline{R} \cdot \underline{R}_k)(\dot{\underline{R}} \cdot \underline{R}_k) = \sum_{j,k} \omega_j\, \omega_k(\underline{R} \cdot \underline{R}_j)(\underline{R} \cdot \underline{R}_k) \left\{(\underline{R} \wedge \underline{R}_j) \cdot \underline{R}_k\right\} .$$

The right-hand side of this equation must vanish since it is anti-symmetric in j and k so that we obtain

$$\sum \omega_j(\underline{R} \cdot \underline{R}_j)^2 = \sum \omega_j \cos^2\gamma_j = \lambda_o \quad , \tag{23}$$

which is the form in which the integral is given for simultaneous precession about two different axes in the classical problem of the motion of Iapetus[3].

Secondly, we note that the doubly-averaged Hamiltonian of the system is $H = -(\mu/2a) - U$. If the axes $\underline{R}_j$ are fixed, then H is independent of explicit time and must be a constant of the motion. Since a is also a constant of the long-period motion, U itself must be constant. Using the knowledge that an initially circular orbit remains circular, we recover the integral (22) on setting $e = 0$ in (9).

4 NATURE OF THE MOTION

The equations (20) are similar to the Euler equations for the motion of a rigid body under no forces, and corresponding results can be derived for the nature of the motion and the period. We recall that the eigenvalues are arranged in the order $\lambda_3 > \lambda_2 > \lambda_1$ and that λ_o lies between λ_1 and λ_3. In practice λ_3 is much greater than the other two eigenvalues (see Fig.1). Equation (22) gives a one-parameter family of curves on the unit sphere which are the possible trajectories of the pole $\underline{R}$; the initial position of the pole determines the value of λ_o. The motion is periodic, and the trajectories divide into two types depending on whether λ_o is greater or less than λ_2.

<u>Case (i)</u>: If $\lambda_3 > \lambda_o > \lambda_2$, (21) and (22) cannot be satisfied simultaneously if $x_3 = 0$, so that the pole $\underline{R}$ can never attain or cross the plane $x_3 = 0$.

<u>Case (ii)</u>: Similarly if $\lambda_2 > \lambda_o > \lambda_1$, the pole can never attain or cross the plane $x_1 = 0$.

Eliminating x_2 between (21) and (22), we find

$$(\lambda_3 - \lambda_2)x_3^2 - (\lambda_2 - \lambda_1)x_1^2 = (\lambda_o - \lambda_2) , \qquad (24)$$

so that, on setting $\lambda_o = \lambda_2$, the bounding curve between the two types of trajectory is the intersection of the pair of planes

$$x_3/x_1 = \pm \left\{(\lambda_2 - \lambda_1)/(\lambda_3 - \lambda_2)\right\}^{\frac{1}{2}} \qquad (25)$$

with the unit sphere. The half-angle between the planes is plotted (under the legend $\pi/2 - \phi_2$) in Fig.2 as a function of a/R_E.

An idea of the shapes of the trajectories is given by their projections on the coordinate planes; eliminating x_3 and x_1 in turn from (21) and (22) we find

$$(\lambda_3 - \lambda_1)x_1^2 + (\lambda_3 - \lambda_2)x_2^2 = (\lambda_3 - \lambda_o) \qquad (26a)$$

$$(\lambda_2 - \lambda_1)x_2^2 + (\lambda_3 - \lambda_1)x_3^2 = (\lambda_o - \lambda_1) . \qquad (26b)$$

For $\lambda_3 > \lambda_o > \lambda_2$, the trajectory encircles the axis 03, and its projection on 012 is an ellipse of small eccentricity (assuming $\lambda_3 >> \lambda_2, \lambda_1$). Likewise for $\lambda_2 > \lambda_o > \lambda_1$, the trajectory encircles the axis 01, and its projection on 023 is an ellipse of large eccentricity. Trajectories of the pole R are shown in Fig.3.

Laplace considered the case where the pole of the satellite orbit is very near the principal axis 03, when the trajectory on the unit sphere is very nearly a circle around the pole 03. This is equivalent to Laplace's result, which is that the satellite orbit regresses at nearly constant rate and inclination on the proper or Laplacian plane which is here simply the coordinate plane 012.

5 PERIOD OF OSCILLATION

Using (26a) and (26b) to eliminate x_1 and x_3 from the second of (20), we find

$$\dot{x}_2 = \pm \left\{(\lambda_3 - \lambda_o) - (\lambda_3 - \lambda_2)x_2^2\right\}^{\frac{1}{2}} \left\{\lambda_o - \lambda_1) - (\lambda_2 - \lambda_1)x_2^2\right\}^{\frac{1}{2}} .$$

The quarter-period of the oscillation is given by inverting and integrating from $x_2 = 0$ to $\dot{x}_2 = 0$; the upper limit as a function of x_2 depends on whether $\lambda_o > \lambda_2$ or not. For instance if $\lambda_o > \lambda_2$, the upper limit of integration is determined by the first square root, and the period of a complete oscillation is

$$T = 4\left\{(\lambda_3 - \lambda_o)(\lambda_o - \lambda_1)\right\}^{-\frac{1}{2}} \int_0^{v_1} (1 - x_2^2/v_1^2)^{-\frac{1}{2}} (1 - x_2^2/v_2^2)^{-\frac{1}{2}} \, dx_2 ,$$

where
$$v_1^2 = (\lambda_3 - \lambda_o)/(\lambda_3 - \lambda_2) ,$$

and
$$v_2^2 = (\lambda_o - \lambda_1)/(\lambda_2 - \lambda_1) .$$

The transformation $x_2 = v_1 \sin \psi$ reduces the above to the standard form

$$T = 4(\lambda_3 - \lambda_2)^{-\frac{1}{2}} (\lambda_o - \lambda_1)^{-\frac{1}{2}} \int_0^{\pi/2} (1 - k^2 \sin^2\psi)^{-\frac{1}{2}} \, d\psi ,$$

where

$$k^2 = v_1^2/v_2^2 = \left\{(\lambda_3 - \lambda_o)(\lambda_2 - \lambda_1)\right\} \Big/ \left\{(\lambda_3 - \lambda_2)(\lambda_o - \lambda_1)\right\} ,$$

so that

$$T = 4(\lambda_3 - \lambda_2)^{-\frac{1}{2}} (\lambda_o - \lambda_1)^{-\frac{1}{2}} K(k), \qquad \lambda_o > \lambda_2 , \tag{27a}$$

where $K(k)$ is the complete elliptic integral of the first kind. As λ_o approaches its upper limit λ_3, the trajectories shrink towards the stationary point on the axis $O3$; k tends to zero, and the period tends to a limiting value which we will call T_3, given by

$$T_3 = 2\pi(\lambda_3 - \lambda_1)^{-\frac{1}{2}} (\lambda_3 - \lambda_2)^{-\frac{1}{2}} , \tag{28a}$$

since $K(0) = \pi/2$. On the other hand as λ_o approaches λ_2, k tends to 1, and the period tends to infinity.

The period can be evaluated similarly for $\lambda_o < \lambda_2$ giving

$$T = 4(\lambda_3 - \lambda_o)^{-\frac{1}{2}} (\lambda_2 - \lambda_1)^{-\frac{1}{2}} K(k'), \qquad \lambda_o < \lambda_2 , \tag{27b}$$

where

$$k'^2 = \left\{(\lambda_3 - \lambda_2)(\lambda_o - \lambda_1)\right\} \Big/ \left\{(\lambda_3 - \lambda_o)(\lambda_2 - \lambda_1)\right\} .$$

As λ_o approaches its lower limit λ_1, the trajectories shrink towards the stable point on the axis 01; k' tends to zero and the period tends to the limiting value T_1 where

$$T_1 = 2\pi(\lambda_3 - \lambda_1)^{-\frac{1}{2}} (\lambda_2 - \lambda_1)^{-\frac{1}{2}} . \tag{28b}$$

Fig. 4 gives the limiting periods T_3 and T_1 as functions of a/R_E, again on the assumption that the lunar orbit lies in the ecliptic ($\lambda_1 = 0$).

6 SYNCHRONOUS ORBITS

For orbits with a period of one sidereal day (at a height $a/R_E = 6.6108$) we have the following values

$$\omega_o = 4.900°/\text{yr.}, \quad \omega_1 = 0.738°/\text{yr.}, \quad \omega_2 = 1.611°/\text{yr.}$$

We consider first the approximation that the lunar orbit lies in the ecliptic, so that the satellite orbit is subjected to simultaneous precession forces about only two axes, the Earth's axis and the pole of the ecliptic. If we take the simplest approximation of equation (17), then the mean pole $\bar{\underline{R}}$ lies in the plane of $\underline{R}_o$ and $\underline{R}_1$ at an angle of 7°33' from the Earth's axis, and $\bar{\omega} = 7.116°/\text{yr}$. The shortest period, when $\underline{R}$ is very close to $\bar{\underline{R}}$, is 50.6 yr. If the satellite orbit is initially equatorial, the rate of precession is reduced by the cosine of the inclination to $\bar{\underline{R}}$, and the period is 51.0 yr.

In the general treatment of Section 3, but still assuming the lunar orbit lies in the ecliptic, the eigenvalues are

$$\lambda_1 = 0, \quad \lambda_2 = 0.261°/\text{yr.}, \quad \lambda_3 = 6.988°/\text{yr.},$$

and the principal axis 03 lies in the plane of $\underline{R}_0$ and $\underline{R}_1$ at an angle $\alpha = 7^{\circ}23'$ to the Earth's axis. From (28a) the limiting value of the period when $\underline{R}$ is very close to 03 is 52.5 yr. On the other hand if the satellite orbit is initially equatorial,

$$\lambda_0 = 6.877^{\circ}/\mathrm{yr.}, \quad k^2 = 6.26 \times 10^{-3} ,$$

and the period is 52.9 yr. Thus the simplest approximation of a mean pole is remarkably accurate for this case.

The period of the oscillation can be calculated from (27a) and (27b) for all possible trajectories at synchronous height. Since all trajectories intersect the plane 013, a convenient parameter here is the angle ϕ between 03 and the intersection of the trajectory with 013. Setting

$$x_1 = \sin\phi, \quad x_2 = 0, \quad x_3 = \cos\phi ,$$

in (22) we have

$$\lambda_0 = \lambda_1 \sin^2\phi + \lambda_3 \cos^2\phi .$$

The value of ϕ corresponding to $\lambda_0 = \lambda_2$ is denoted by ϕ_2 so that the half-angle between the bounding planes between the two regions of motion in Fig.3 is $\pi/2 - \phi_2$ (which has already been plotted in Fig.2). Fig.5 shows the period of oscillation as a function of ϕ for synchronous orbits.

The lunar orbital plane is not fixed in the ecliptic but is itself regressing around the pole of the ecliptic with a period of 18.6 years at an inclination of $5^{\circ}9'$ to the ecliptic. For each position of the lunar plane it would be possible to find the eigenvalues λ_1, λ_2, λ_3 ($\lambda_1 \neq 0$) and the directions of the principal axes, but these are varying all the time. However it is simple to integrate (16) numerically, and we have performed this integration for an initially equatorial circular satellite orbit for four different initial positions of the lunar plane. Since the polar coordinates of $\underline{R}$ with respect to an equatorial system are i and $(\Omega-\pi/2)$, where i and Ω are the inclination and the right ascension of the ascending node on the equatorial plane, these provide a useful way of illustrating the results. We have given in Fig.6 a polar plot of the coordinates i and $(\Omega-\pi/2)$ of the pole $\underline{R}$. The four different curves refer to the different initial conditions given by $\Omega' = 90^{\circ}$ (July 1964), $\Omega' = 0^{\circ}$ (March 1969), $\Omega' = 270^{\circ}$ (Nov. 1973), $\Omega' = 180^{\circ}$ (July 1978), where Ω' is the longitude of the ascending node of the lunar orbit on the plane of the ecliptic measured from vernal equinox. The symbols on the curves mark the subsequent positions of the pole $\underline{R}$ at intervals of 5 years from the launch date. In general the trajectory does not quite pass through the Earth's axis after one circuit, and the time of closest passage varies between about 52.5 and 53.5 years.

The most significant feature revealed by Fig.6 is that the different trajectories intersect after about 28 years, and again after about 56 years. This arises from the near-commensurability of the period of lunar precession (18.6 years) and the period of the motion of the satellite orbital plane (~52.9 years). It is possible to give a perturbation treatment for the displacement of the pole R from the position it would have if the lunar orbit lay in the ecliptic. If the pole R is not too far away from O3, there are two large terms in the displacement with frequencies $\Omega' \pm \omega$, where Ω' is the rate of regression of the lunar node on the ecliptic and ω is a mean rate of precession of R around O3. Numerically

$$\Omega' \pm \omega = (19.34 \pm 6.80)^{\circ}/\text{yr.} = 26.14^{\circ}/\text{yr. or } 12.54^{\circ}/\text{yr.},$$

which gives periods of 13.8 years and 28.7 years. Consequently the two large terms in the displacement both vanish first after about 28 years.

7 CONCLUSIONS

The long-term evolution of a circular satellite orbit at a distance of a few Earth radii has been investigated. Only the leading terms are retained in the disturbing function, which is averaged over the mean anomalies of the satellite, the Sun and the Moon. Neglect of the third harmonic in the lunar disturbing function sets an upper limit to the radius of the orbit for which the theory is valid. On the other hand the period of the predicted motion must be appreciably longer than a year for the double averaging procedure to be justifiable; this sets a lower limit to the orbital radius. For these reasons the analysis is most useful for orbits at distances between 3 and 10 Earth radii.

The motion consists of simultaneous precession about the Earth's axis, the pole of the ecliptic and the pole of the lunar orbit, and can be described by a general solution if these axes are assumed fixed. The motion of the pole of the orbit is analogous to the motion of the instantaneous axis of rotation of a rigid body moving under no forces. For a circular orbit of given size, there are three mutually perpendicular directions in which the pole of the orbit will remain at rest, two of which are stable positions. In general the projection of the pole of the orbit on the unit sphere follows a fixed trajectory which encircles one or other of the two stable positions; in fact the unit sphere is divided into two different regions bounded by a pair of planes through the third and unstable axis and the trajectory must lie entirely in one or the other of these two regions. In practice the larger region is the more important and the corresponding stable position of the plane is the well-known Laplacian or proper plane. The significance of the proper plane is that the pole of an orbit inclined at a small angle to it will regress around the pole of the proper plane at nearly constant rate and inclination.

The analysis has been applied to Earth satellites assuming the lunar orbit lies in the ecliptic. The inclination of the proper plane to the equator increases from 0.19° at 3 Earth radii to 18.8° at 10 Earth radii, and the semi-angle between the bounding planes reaches a maximum value of nearly 12° at about 7.7 Earth radii, as shown in Fig.2. As the trajectory of the pole of the orbit shrinks towards either of the stable positions the period decreases to one or other of two limiting values. For a given size of orbit, the limiting value at the pole of the proper plane is the shortest possible period; this limiting period attains a maximum value of about 70 years near 8.9 Earth radii as shown in Fig.4. The limiting period around the other stable stationary position increases nearly linearly with radial distance from 121 to 403 years between 3 and 10 Earth radii.

Synchronous orbits have been considered in more detail. The period of the long-term motion is given for all initial positions in Fig.5, and the limiting values of the period are 52.5 and 267 years. The effect of the motion of the lunar orbit on an initially equatorial orbit at synchronous height has been found by numerical integration for four different initial positions of the lunar orbit, i.e. four different launch dates, and is illustrated in Fig.6.

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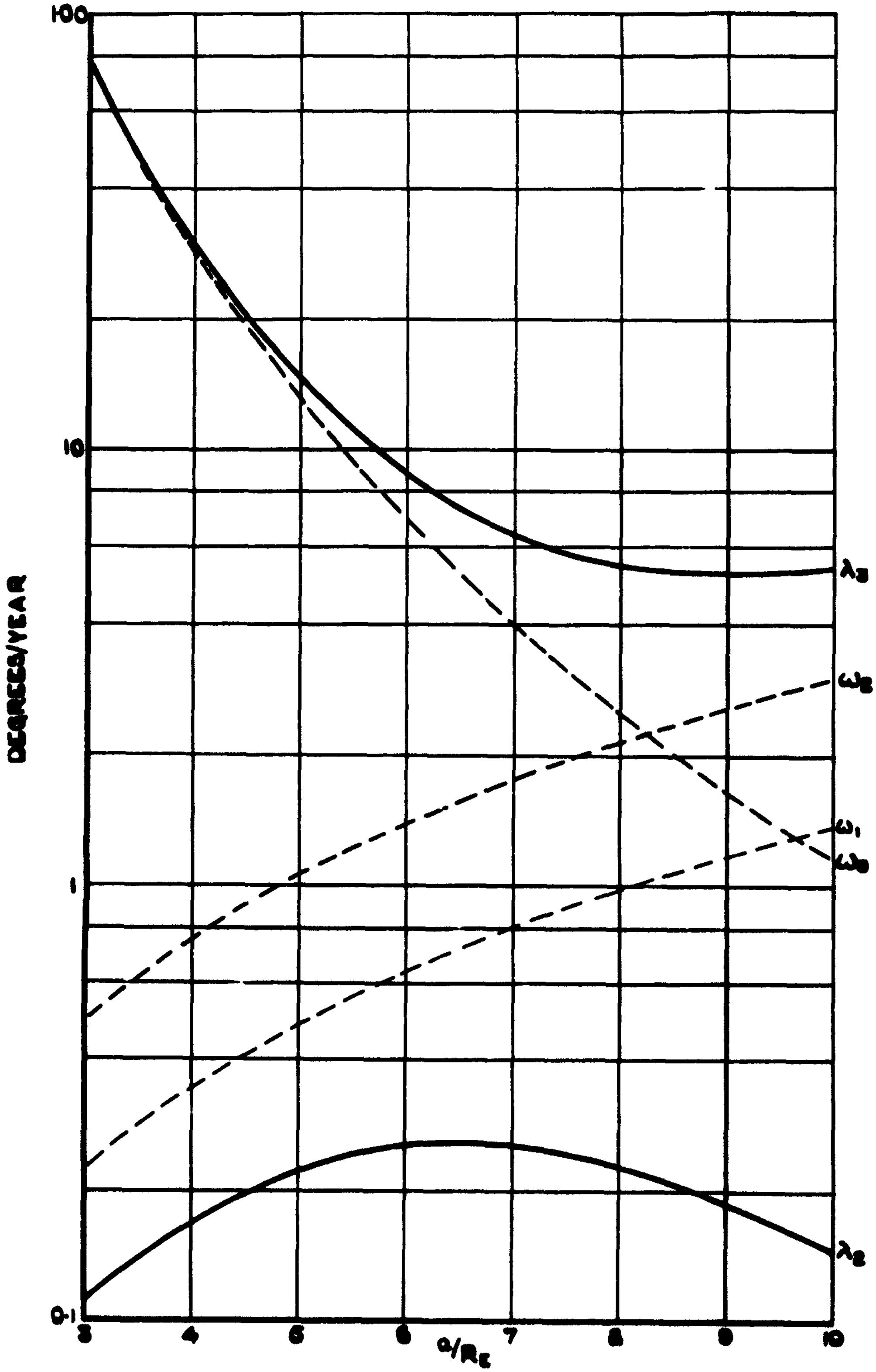

FIG.1. VALUES OF THE ω_j AND THE EIGENVALUES λ_2, λ_3

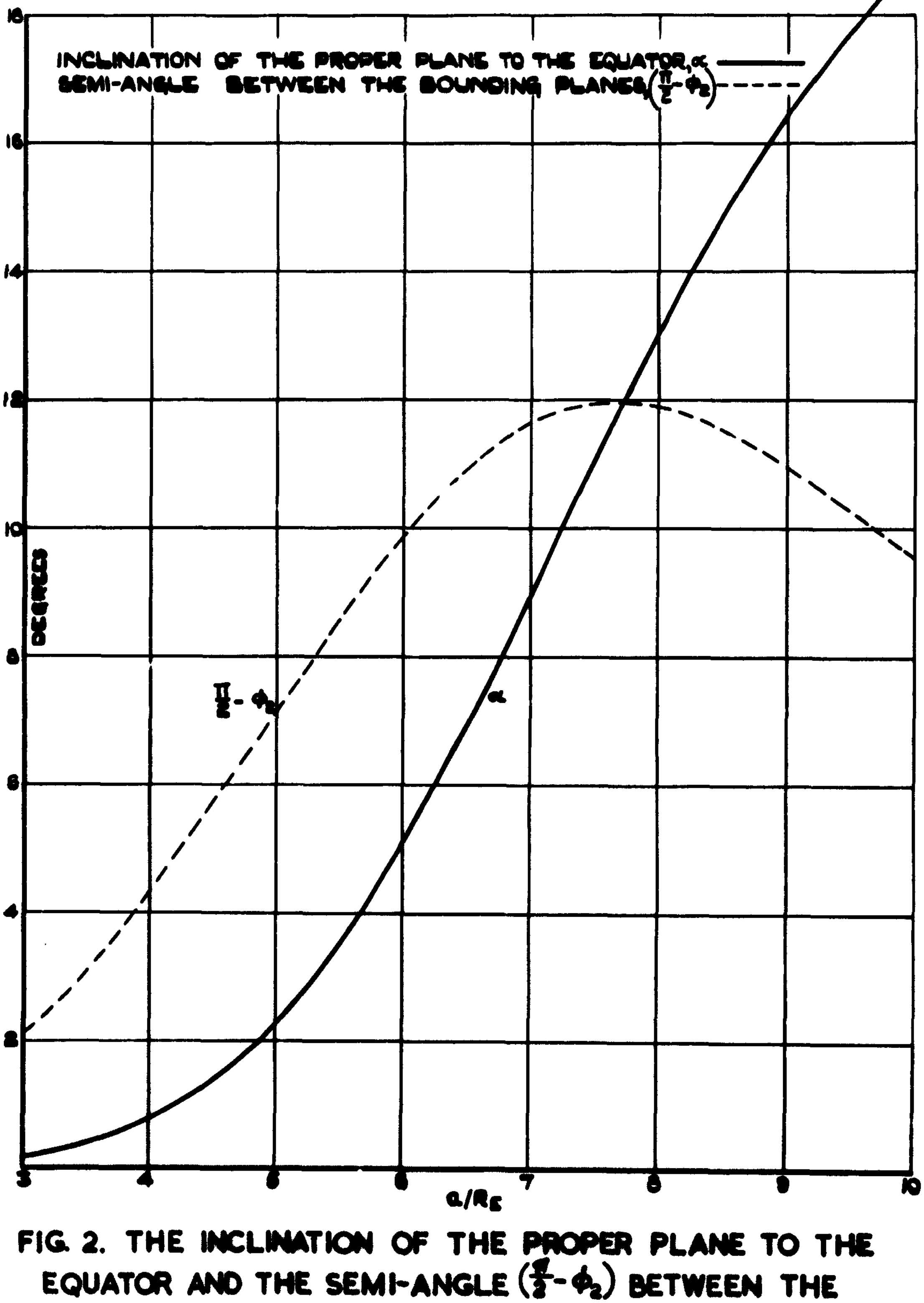

FIG. 2. THE INCLINATION OF THE PROPER PLANE TO THE EQUATOR AND THE SEMI-ANGLE $(\frac{\pi}{2}-\phi_2)$ BETWEEN THE BOUNDING PLANES AS A FUNCTION OF a/R_E

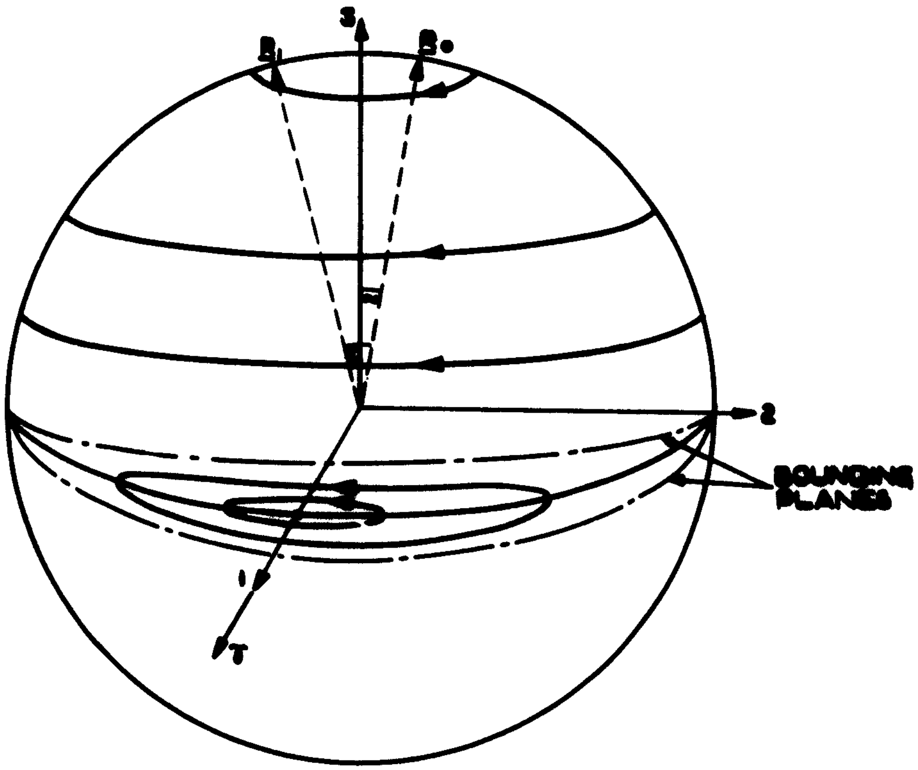

FIG. 3. TRAJECTORIES OF THE POLE B
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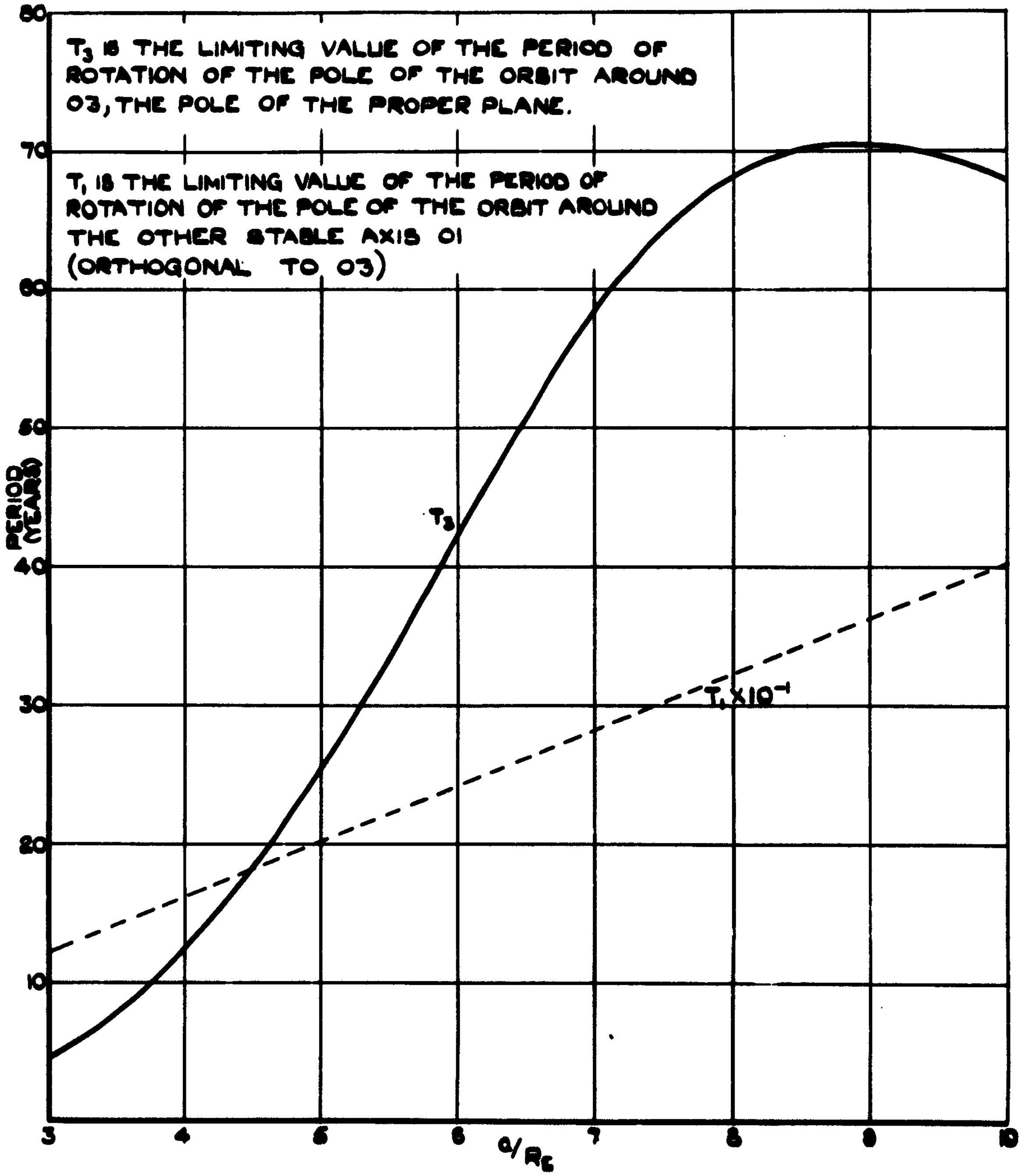


FIG.4. LIMITING VALUES OF THE PERIODS AS FUNCTIONS OF a/R_E

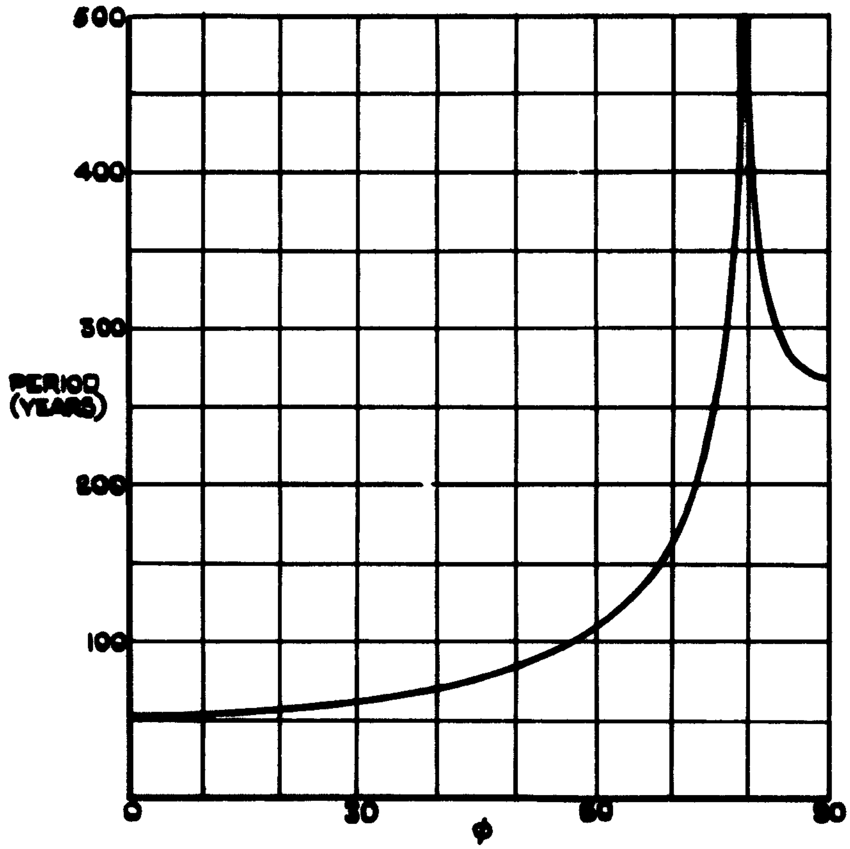
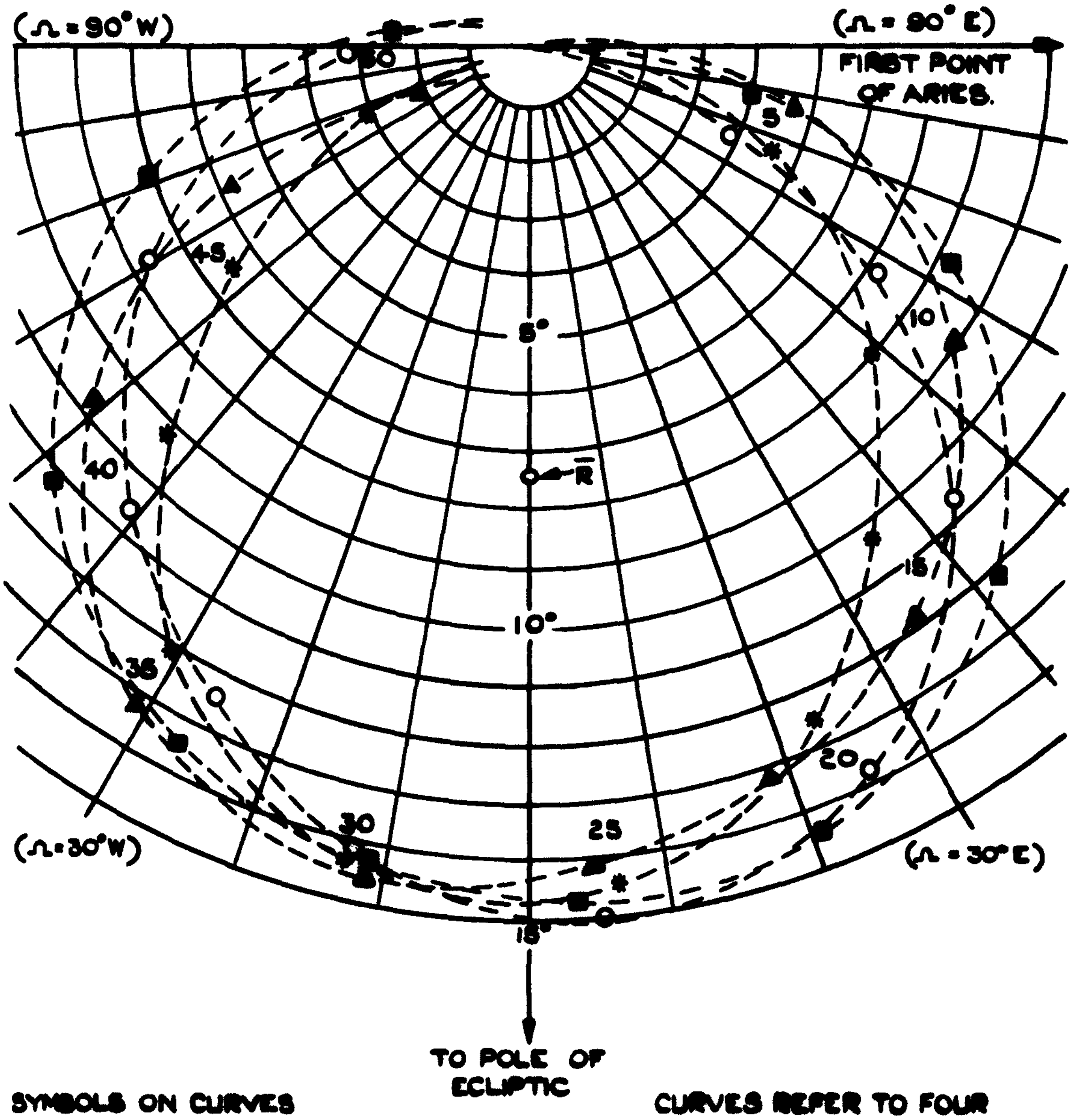


FIG.5. THE PERIOD OF THE LONG-TERM MOTION OF SYNCHRONOUS ORBITS FOR DIFFERENT VALUES OF ϕ



SYMBOLS ON CURVES MARK POSITION OF POLE OF ORBITAL PLANE AT INTERVALS OF FIVE YEARS FROM LAUNCH.

CURVES REFER TO FOUR DIFFERENT LAUNCH DATES AS FOLLOWS:

▲ JULY 1964 (☊' = 90°)
\# MARCH 1969 (☊' = 0°)
○ NOVEMBER 1973 (☊' = 270°)
■ JULY 1978 (☊' = 180°)

FIG. 6. LONG-PERIOD MOTION OF ORBITAL PLANE.

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