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The Influence of
Viscoelastic Material
Properties on Design Problems

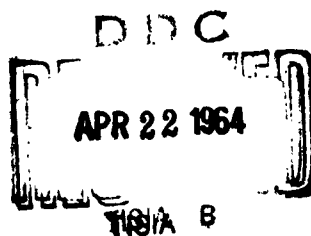


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THE INFLUENCE OF VISCOELASTIC MATERIAL
PROPERTIES ON DESIGN PROBLEMS*

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Abstract

Viscoelasticity comprises an irreversible relation between stress and strain which is governed by time effects. Stress distributions for viscoelastic bodies subjected to constant loads thus commonly vary with time. It is possible to utilize this flexibility of solution in some cases to meet stress design criteria. For example, the irreversibility embodied in viscoelasticity can lead to the production of a beneficial residual stress distribution following loading. Such an example is discussed. Temperature has a marked effect on viscoelastic characteristics, and thus supplies an additional controlling variable in selecting a design solution. Cooling slows down rate processes so that it may be possible to eliminate deleterious aspects of viscoelasticity in a particular design problem by selecting an appropriate temperature history. The example

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of the production of toughened glass by the interaction of viscoelasticity and temperature effects is cited. Although viscoelastic stress analysis is usually applied to polymers and glasses, its influence also arises in design problems for metals in the creep range.

1. Introduction

In recent years methods of solution have been developed which considerably broaden the range of stress distribution problems for viscoelastic materials which are amenable to analysis. The technological motivation has been mainly the increasing use of polymers and plastics for components which must transmit load, and in particular the need to analyse the stresses and strains in the grains of solid propellant rockets during both storage and firing. Temperature influences can have a dominant effect in such solutions because of the usually major influence of temperature on viscoelastic characteristics.

Because linear viscoelastic theory supplies a useful approximation to the response of many polymers at moderate strains, and because this theory permits the utilization of the powerful tools of linear mathematical analysis, the development of methods of solution of particular problems has been mainly restricted to linear material behavior. Creep of metals at high temperature, and the associated viscoelastic characteristics of recovery on unloading and stress relaxation at constant strain, exhibit essentially non-linear response, so that linear theory is not quantitatively applicable. However, the types of influence of viscoelastic material behavior on design, described in this paper, will also arise for creep in metals. A basis for the analysis for non-linear materials has been developed

by Green and Rivlin [1]* but to date only applications of this theory limited to simple configurations have been published.

The salient characteristics of viscoelastic material behavior are exhibited by the creep and unloading response of a material to a pulse of constant stress applied for a period and then removed as illustrated in Fig. 1. OA represents the instantaneous elastic response which is instantaneously recovered at CD on unloading. Delayed elasticity develops along AB and this is recovered gradually along DE. Viscous flow occurs along BC and this leads to permanent residual strain. These features comprise time or rate effects in the material response which influence the stress distribution developed when a body is loaded by surface forces. We will be concerned with quasi-static situations in which inertia forces are negligible compared with applied loads. Stress distributions in viscoelastic materials then usually differ essentially from those for elastic materials in that under constant surface tractions the time dependent material characteristics generate a varying stress field in contrast to the constant field for an elastic body. This situation is commonly considered as an unfortunate complicating feature of viscoelastic stress-analysis, and certainly the theory, reviewed in the next section, is more involved than the corresponding elastic analysis, due to the addition of the variable t and the differential and integral operators in this variable which occur. However, the variation with time of the stress field does introduce a flexibility into solutions which may be utilized to satisfy design requirements by means not available for elastic bodies. For example, the irreversibility

* Numbers in square brackets refer to the bibliography at the end of the paper.

inherent in viscoelastic response can lead to the development of a beneficial residual stress pattern after load application, which could not appear in an elastic body since reversibility dictates zero stress magnitudes after the load is removed. Such a situation is discussed in Section 3 of this paper. Greater control over the influence of the history of loading can be achieved by utilizing the marked effects of temperature variation on viscoelastic response, and some applications of this concept are presented in the last section of the paper.

2. Linear Viscoelastic Stress-Analysis Theory

Stress distribution problems for viscoelastic bodies are set in a similar form to those for elastic bodies, but with the addition of time t as an independent variable; prescribed tractions and displacement being given as functions of time as well as of position. Fig. 2 illustrates such a problem. With Cartesian axes x_i , $i = 1, 2, 3$, the stress distribution function $\sigma_{ij}(x, t)$ for the viscoelastic body V is to be found, where for conciseness x denotes the triad of space coordinates (x_1, x_2, x_3) . The history of prescribed traction $T_1(x, t)$ is given over the part of the surface S_1 , and of the prescribed displacement $u_1(x, t)$ over the remainder S_2 . Other combinations of stress and displacement components also can be used to prescribe particular problems as in elasticity. Initial conditions, usually that the body is undisturbed at zero time, are also needed.

The stress field must satisfy the equilibrium equations for quasi-static motion:

$$\frac{\partial \sigma_{1j}}{\partial x_j} + f_1(x,t) = 0 \quad (1)$$

where, according to the usual summation convention, the repeated j indicates summation over the indices 1, 2, and 3. Index i taking on the values 1, 2, and 3 gives the three equilibrium equations for components in the directions x_1 , x_2 and x_3 respectively.

We shall limit our analysis to small strains, and use infinitesimal strain theory to relate the strain and displacement components:

$$\epsilon_{1j} = \left(\frac{\partial u_1}{\partial x_j} + \frac{\partial u_j}{\partial x_1} \right) / 2 \quad (2)$$

The stress-strain relations for combined stresses are now needed to complete the formulation of the problem. Linear viscoelasticity implies that creep curves of the form OABC in Fig. 1, for different values of the applied stress magnitude σ_0 , have ordinates proportional to σ_0 at each time t . Such response indicates linear mathematical relations between stress and strain, and the applicability of superposition of effects for the combined influence of more than one loading function. For isotropic material behavior it is convenient to decompose the stress and strain into shear type and dilatational components. The former relate the deviator components of strain and stress:

$$e_{1j} = \epsilon_{1j} - \frac{1}{3} \epsilon_{kk} \delta_{1j} \quad , \quad s_{1j} = \sigma_{1j} - \frac{1}{3} \sigma_{kk} s_{1j} \quad (3)$$

where δ_{ij} is the Kronecker delta, and the latter the dilatation ϵ_{11} and the average of the normal stress components $\sigma_{11}/3$. Linear viscoelasticity then implies laws of the form

$$P(s_{1j}) = Q(e_{1j}) \quad (4)$$

$$P'(\sigma_{11}) = Q'(\epsilon_{11}) \quad (5)$$

where P, Q and P', Q' are independent pairs of linear viscoelastic operators which can take the form of differential operators related to spring-dashpot models, integral operators involving the creep compliance function $J(t)$ or the relaxation modulus function $G(t)$, or the complex algebraic relation for oscillatory stress and strain variation in terms of the complex modulus:

$$G^*(\omega) = G_1(\omega) + i G_2(\omega) \quad (6)$$

The operator pairs P, Q or P', Q' acting on a stress component σ and a strain component ϵ thus can take the forms:

$$\sum_0^{n_p} p_r D^r \sigma = \sum_0^{n_q} q_r D^r \epsilon \quad (7a)$$

where $D \equiv \frac{\partial}{\partial t}$

$$\epsilon(t) = \int_0^t J(t-\tau) \frac{\partial \sigma}{\partial \tau} d\tau \quad (7b)$$

$$\sigma(t) = \int_0^t G(t-\tau) \frac{\partial \epsilon}{\partial \tau} d\tau \quad (7c)$$

$$\sigma = G^*(\omega) \epsilon \quad (7d)$$

where in (7d) σ and ϵ are oscillatory with angular frequency ω and so can be represented in the forms $\sigma_0 e^{i\omega t}$ and $\epsilon_0 e^{i\omega t}$ respectively, σ_0 and ϵ_0 being in general complex constants. Depending on whether the operator pairs in (7) apply to (4) or (5), the characteristics used are associated with shear and dilatation respectively, and are measured independently or deduced from combined stress test results.

In addition to the field equations (1)-(5), boundary conditions are needed to determine the stress distribution. Over the part of the surface S_1 compatibility between the stress and the traction requires:

$$T_1 = \sigma_{1j} n_j \quad (8)$$

where n_j are the components of the unit external normal; and over S_2 , $u(x,t)$ is prescribed.

The most common form of solution of these equations applies to the special case when the surface regions S_1 and S_2 do not change with time, and then operation of the Laplace transform on all the equations transforms the problem to an associated elastic problem for the transformed variables [2]. Inversion of the solution then gives the required viscoelastic stress distribution. This procedure can be most easily carried out for low order differential operators (7a), but these often do not provide a satisfactory representation of material behavior [3]. The integral operators (7b) and (7c) represent accurately, arbitrary linear viscoelastic response within the time range over which the kernel functions $J(t)$ or $G(t)$ are measured. However, Muki and Sternberg [4] found use of the Laplace transform approach with measured values of the relaxation modulus $G(t)$ to be cumbersome.

Since equations (1) and (2) contain only space derivatives and (4) and (5) only time operators, it is sometimes possible to integrate independently with respect to the space and time variables either analytically [5] or by making use of numerical methods and directly introducing measured material properties [6]. The latter approach has been used to analyse stresses in an encased hollow cylinder subjected to internal pressure, which forms the basis for discussion of the design problem in the next section. The method applies for varying S_1 in the form of an ablating cavity, and for a more general type of boundary condition on the outer surface corresponding to an elastic casing:

$$\sigma_r(b,t) = - B \epsilon_\theta(b,t) \quad (9)$$

where the usual notation for an axially symmetrical problem in polar coordinates is used, and b is the casing radius and B a constant prescribing casing stiffness. The development of the theory is discussed in [7], and the solution for a body in terms of the relaxation moduli in shear and dilatation is given in [8]. Thus the problem discussed in the next section can be solved for arbitrary linear viscoelastic material characteristics and simple models are used for illustration merely for convenience.

3. Generating Residual Stresses in an Encased Cylinder

Consider a viscoelastic hollow cylinder with a supporting elastic casing as shown in Fig. 3. It is to be designed to withstand internal pressure p_0 in plane strain, and it could, for example, be a rocket motor which must not fracture under the internal gas pressure on firing [9]. The varying stress distribution for suddenly applied and maintained pressure is given in [10] for a cylinder exhibiting Maxwell type viscoelastic response in shear, with elasticity in dilatation. Maxwell response corresponds to the model and the creep and relaxation curves shown in Fig. 4. The resulting variation of radial and circumferential stresses are shown in Fig. 5.

It is seen that circumferential tensile stresses occur adjacent to the cavity surface immediately on loading, but that they, in common with stress components throughout, decrease algebraically and tend towards uniform hydrostatic compression of magnitude of the applied pressure. This change occurs since the cylinder is constrained against continued flow by the elastic casing and the plane strain condition, so that the shear stress relaxes to zero as in the relaxation test, Fig. 4c. The viscoelastic material is then stressed as a liquid under hydrostatic pressure, and

the full cavity pressure is transmitted to the casing. Since all normal stress components in the viscoelastic body are then negative, a strong inhibiting factor against fracture will exist. In fact, throughout the loading process, the only state exhibiting a tendency to fracture is the region of tensile circumferential stress adjacent to the cavity for the period shortly after load application. The negative growth of radial stress at the external radius will, however, generate an increasing tensile stress in the casing, which must be designed to carry this.

The stress field illustrated in Fig. 5 for a suddenly applied and maintained pressure can be used in conjunction with superposition to deduce the stress field for gradually applied pressure $p(t)$. As in the development of the Duhamel integral, the pressure growth $p(t)$ can be considered as the limit of pressure steps $\frac{dp(t)}{dt} dt$. Since all stresses in Fig. 5 are proportional to p_0 , we write:

$$\sigma_{\theta}(r,t) = p_0 \bar{\sigma}_{\theta}(r,t) \quad (10)$$

where $\bar{\sigma}_{\theta}$ is then the circumferential stress field for unit pressure. Deducing from (10) and linearity the response to each infinitesimal step of the pressure growth $p(t)$, gives the corresponding circumferential stress field:

$$\sigma_{\theta}(r,t) = \int_{-\infty}^t \bar{\sigma}_{\theta}(r,t-\tau) \frac{dp(\tau)}{d\tau} d\tau \quad (11)$$

where the lower limit can be taken as zero for a cylinder undisturbed prior to zero time.

It is clear from Fig. 5 that the internal cavity surface is subjected to a stress variation most likely to lead to fracture initiation since the maximum tensile stresses occur there. The variation of $\bar{\sigma}_\theta(a,t)$ is illustrated in Fig. 6 to facilitate the assessment of fracture tendencies in the case of gradually applied pressure. For $r = a$, the contribution to the integral in (11) for $\sigma_\theta(r,t)$ by the pressure steps immediately prior to the time t will be positive since $(t-\tau)$ will be small, and hence the integrand positive according to Fig. 6 for monotonically increasing pressure. For the more remote pressure increments, the contribution to the integral will be negative. Thus by sufficiently gradual application of the pressure, the positive contribution can be reduced arbitrarily by restricting the pressure rise for $t - \tau < t_0$, where t_0 is the duration of tensile stress shown in Fig. 6. If the pressure is build up gradually to p_0 and maintained constant, the stress will finally settle down to hydrostatic pressure p_0 as depicted in Fig. 5 for large t , since all stress increments in (11) will be effectively remote and the $\bar{\sigma}_\theta(r,t-\tau)$ will approach -1. Thus the hydrostatic pressure field in the viscoelastic cylinder, which inhibits fracture initiation, can be achieved without passing through a stage of appreciable cavity surface tension merely by gradual rather than sudden pressure application. The possibility of such an effect is due to the influence of loading history on the response of a viscoelastic body, and such an effect could not arise for an elastic body in which the stress field is determined by the current applied load and is otherwise independent of the load history.

If when the steady hydrostatic stress field has developed, the internal pressure is removed, the resulting stress distribution variation is given by superposing the solution for suddenly applied negative pressure of magnitude p_0 , that is superposing the field shown in Fig. 5 with sign reversed. This leads to a residual stress field with no applied surface loading. Compressive normal stresses occur throughout the cylinder, the circumferential stress at the cavity surface being compressive and of magnitude greater than p_0 . Thus the viscoelastic characteristics of the cylinder have generated a residual stress field beneficial from the standpoint of inhibiting fracture, and ensuring the compressive normal stresses throughout the cylinder when an internal pressure of magnitude p_0 is applied. Because of time effects illustrated by the varying stress field shown in Fig. 5, the residual stress field will gradually relax away, and maintenance of internal pressure would be needed to retain its full influence.

A similar situation would arise for a more general viscoelastic law, although a material which relaxes to a limiting non-zero shear stress would not settle to uniform hydrostatic pressure, but would retain some distribution of shear stress. In practice this is necessary to prevent continued flow from, for example, gravity forces, and is needed to maintain the cylindrical form. Behavior qualitatively of the type described will arise in general, and as referenced in the previous section, methods are available to analyse the situation for arbitrary linear viscoelastic behavior. Some comments on the application of this concept to rocket grain design are given in [9].

The generation of a residual stress field in a hollow cylinder which reduces deleterious stresses on application

of internal pressure is analogous to the process of autofrettage used to strengthen steel pressure vessels and gun tubes. In this case the irreversibility of plastic flow is utilized to generate a beneficial residual stress field. Compressive residual circumferential stresses are generated adjacent to the cavity surface, and the outer regions are stressed in tension. For the viscoelastic problem discussed above, the casing takes the place of such outer layers.

The influence of viscoelastic response in modifying stress distributions and generating residual stress fields is restricted by the limited time dependent viscoelastic characteristics available which, for example, permit a beneficial residual stress field to attenuate. Temperature change has a marked effect on the viscoelastic properties, and utilization of this additional variable provides greater control and permanency to such influences. The theory for non-isothermal conditions is given in the next section, with some design applications of these concepts in the last one.

4. Thermo-Viscoelastic Stress Distributions

In recent years the theory of stress analysis has been developed for viscoelastic bodies with temperature variations, for which the influence of temperature on the viscoelastic characteristics is taken into account in addition to the effect of thermal expansion [4,11,12]. So-called thermo-rheologically simple material behavior has been assumed, for which temperature rise causes a contraction in the time scale of all relaxation processes by a function of the temperature. This corresponds to temperature change causing a shift, without change of shape, of the relaxation modulus function when plotted on a log

(time) base, and is known as the Williams-Landel-Ferry law in the chemical literature. This law, combined with linear viscoelasticity, provides a satisfactory material representation for many polymers and glasses for certain ranges of stress and temperature.

This influence of temperature can be conveniently characterized by defining a reduced time ξ which incorporates the temperature dependent time scale factor, so that, in terms of ξ , the isothermal viscoelastic law applies corresponding to some chosen base temperature T_B . If $1/\phi(T)$ represents the time scale factor, so that $\phi(T_B) = 1$, and $\log[\phi(T)]$ is the shift of the relaxation modulus function $G(\log t)$ in the negative direction along the $\log(t)$ axis, then the reduced time for varying temperature as defined in [11] is:

$$\xi(t) = \int_0^t \phi[T(t')] dt' \quad (12)$$

and the stress-strain relation is given by (7c) with the relaxation modulus measured at the base temperature T_B and reduced time replacing real time:

$$\sigma(\xi) = \int_0^{\xi} G(\xi - \xi') \frac{\partial \epsilon}{\partial \xi'} d\xi' \quad (13)$$

At high temperature ϕ is large, so that large ξ , and hence appreciable relaxation of stress according to (13), occurs for small values of t . As the temperature is

reduced, $\phi(T)$ decreases, and much larger durations of real time are needed to permit appreciable relaxation.

For thermo-viscoelastic stress analysis theory equations (1) and (2) remain unchanged, (4) is replaced by a relation of the type (13) for deviator components, and assuming elastic response in dilatation, (5) is modified to allow for thermal expansion:

$$\sigma_{11} = 3K (\epsilon_{11} - 3\alpha T) \quad (14)$$

where K is the bulk modulus and α the coefficient of linear thermal expansion.

Problem types amenable to solution are much more restricted than in the isothermal case since, as discussed in [13], equations (1) and (2) contain partial space derivatives for constant real time t , so that if the reduced time variable ξ is used to introduce the simple relation (13) and include temperature variation effects, the expressions for partial spacial derivatives at constant real time are complicated. If real time is retained for the time variable, (13) no longer has the form of a convolution integral. For certain problems [4] the Laplace transform with respect to ξ can be utilized, and in others numerical integration is feasible. Solutions for slabs with stress and temperature a function of the depth only, and spheres with spherically symmetrical distributions have appeared in the literature [4,12,13] and form the basis for the discussion of design problems in the next section.

5. The Influence of Temperature Effects on Design

The acceleration of relaxation processes with increasing temperature has a marked effect in attenuating thermal stresses. For example [12] presents the thermo-viscoelastic stress distributions for a sphere with a concentric spherical cavity, initially at uniform temperature, heated internally. The cavity surface is held at constant higher temperature and is considered to ablate with an approximately constant velocity. The configuration represents an idealized model of a burning rocket grain, and the solution was calculated for polymethyl-methacrylate since for this material the relaxation modulus in shear and the temperature scale factor $\phi(T)$ are available in the literature. The corresponding elastic solution was also computed, and an example of the type of stress fields generated is illustrated in Fig. 7, which is taken from [12]. It is seen that due to the internal heating, circumferential compressive stress arises adjacent to the cavity surface due to the constraint of the unheated outer layer inhibiting thermal expansion. This leads to a maximum thermal stress at the surface in the elastic case, but for the viscoelastic body the higher cavity temperature has generated appreciable relaxation of stress. This study showed that even for a rapidly moving ablating front, appreciable thermal stress could be generated, but may only exist over a narrow region adjacent to the front.

$\phi(T)$ is (12) varies extremely rapidly with temperature for many materials. For polymethyl-methacrylate the measurements reproduced in [4] give a ratio of over 10^6 for a temperature change of 60°C , and commercial soda-lime glass gives a ratio of about 10^4 for a temperature change of 100°C [14]. Thus, on cooling, relaxation times

can grow exceedingly rapidly, and when long compared with the time period studied, the material behaves virtually elastically. This concept is now developed analytically.

When a body is cooled from a temperature at which marked viscoelastic characteristics are exhibited, $\xi(t)$ given by (12) at first grows rapidly due to $\phi(T)$ being large, but since ϕ can decrease by many orders of magnitude, the growth of ξ virtually ceases. When this barrier or "freezing time" is reached, subsequent changes in stress can be determined on the basis of thermo-elastic theory [15]. For if t_1 and t_2 are times after the barrier has been reach

$$\xi(t_1) \simeq \xi(t_2) \quad (15)$$

Since with appreciable temperature changes still taking place, stresses and strains can change at approximately constant ξ , (13) must be written in terms of real time t to preserve a bounded strain derivative in the integrand, leading to:

$$\sigma(t) = \int_0^{t_1} G[\xi(t) - \xi(t')] \frac{\partial \epsilon}{\partial t'} dt' \quad (16)$$

Thus

$$\begin{aligned} \sigma(t_1) - \sigma(t_2) &= \int_0^{t_1} G[\xi(t_1) - \xi(t')] \frac{\partial \epsilon}{\partial t'} dt' \\ &\quad - \int_0^{t_2} G[\xi(t_2) - \xi(t')] \frac{\partial \epsilon}{\partial t'} dt' \end{aligned}$$

But by (15) the integrands can be considered identical and

$$\sigma(t_1) - \sigma(t_2) = \int_{t_2}^{t_1} G[\xi(t_1) - \xi(t')] \frac{\partial \epsilon}{\partial t'} dt' \quad (17)$$

Since t' now lies only between t_2 and t_1 , $\xi(t')$ remains at the barrier (15), and the kernel in (17) can be replaced by $G(0)$, whence:

$$\sigma(t_1) - \sigma(t_2) = G(0)[\epsilon(t_1) - \epsilon(t_2)] \quad (18)$$

Thus the shear law for stress changes becomes the elastic law with the modulus associated with initial response. (14) gives a similar relation for dilatational components, and thermo-elastic theory thus determines changes of stress and strain.

This change to virtually thermo-elastic behavior on cooling provides a means of permanent retention of residual stress fields generated by viscoelasticity, and thus avoids the attenuation which occurs in the isothermal case (Section 3 above).

Examples of the generation of permanent residual stress fields through thermo-viscoelasticity by cooling have been given for a solid sphere [15] and a plate [16]. In both cases cooling from a surface generates compressive subsurface stress components parallel to the surface, and these would tend to inhibit fracture in subsequent stressing of the body. This concept is the basis for the

manufacture of toughened glass, the theory for which is presented in [16]. Hot glass in a viscoelastic state is cooled from the surfaces which then tend towards elastic behavior and to contract thermally. Since the latter is inhibited by the interior, still hot, layers, tension stresses develop adjacent to the surfaces, which produce compressive stresses in the interior since no external forces are applied, and compressive viscoelastic flow of the internal material occurs, thus relieving the elastic tensile sub-surface stress. As the cooling spreads through the plate, general thermal contraction occurs in addition to the afore-mentioned viscoelastic contraction, and residual compressive stresses are produced in the layers adjacent to the surface, balanced by internal tensile stresses. As the glass cools down below the temperature at which a barrier in ξ is reached, the stresses become virtually permanent. For glass cooled from 600°C , the ratio of relaxation times according to the thermorheologically simple law extrapolated from measurements in the neighborhood of 500°C is about 10^{24} , so that relaxation can be considered completely eliminated at room temperature. The residual stress distribution with sub-surface compressive stresses strongly inhibits fracture. The process is seen to depend on interaction of cooling temperature gradients, thermo-viscoelasticity including the transformation to thermo-elasticity at lower temperatures, and thermal contraction. These factors are included in the theory presented in the previous section with the addition of the heat conduction equation to determine the temperature field.

In assessing the application of such a process to meet safe stress criteria in use, it is important to study the entire history of stress during the process, since this

may prescribe limitations. For the example considered above, the tensile stress generated on first cooling can lead to surface fracture if the cooling is too rapid, and this limits the magnitudes of residual stress which can be achieved. These factors depend in an involved manner on the thermal and viscoelastic material characteristics, the plate thickness and cooling process, and it is essential to evaluate the problem before rational design decisions can be made.

Analogous situations will arise in connection with creep of metals of elevated temperatures, and in order to provide adequate analyses it is necessary to develop creep laws to include general loading history, as does the law used above for thermo-viscoelastically simple material response.

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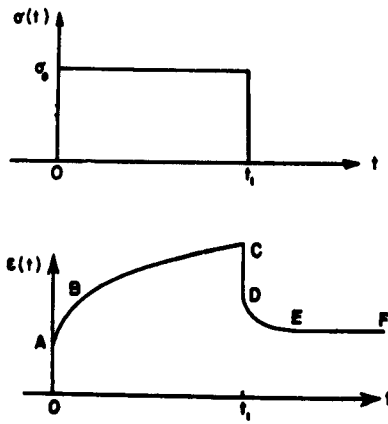


FIGURE 1

Strain response to a pulse of constant stress for a viscoelastic body.

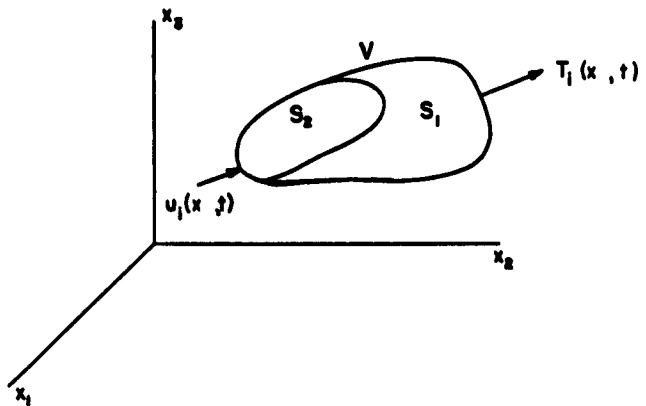


FIGURE 2

A boundary value problem for stress analysis.

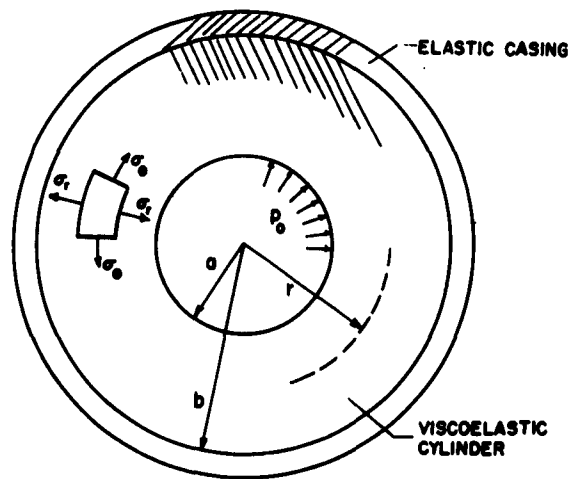


FIGURE 3

Pressurized hollow cylinder supported by an elastic casing.

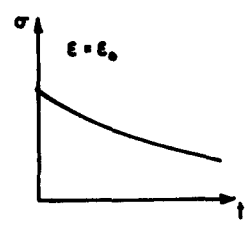
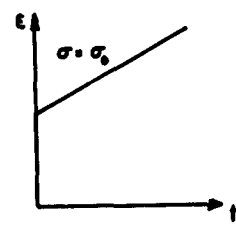
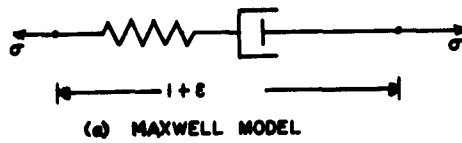


FIGURE 4

Shear behavior of the viscoelastic cylinder.

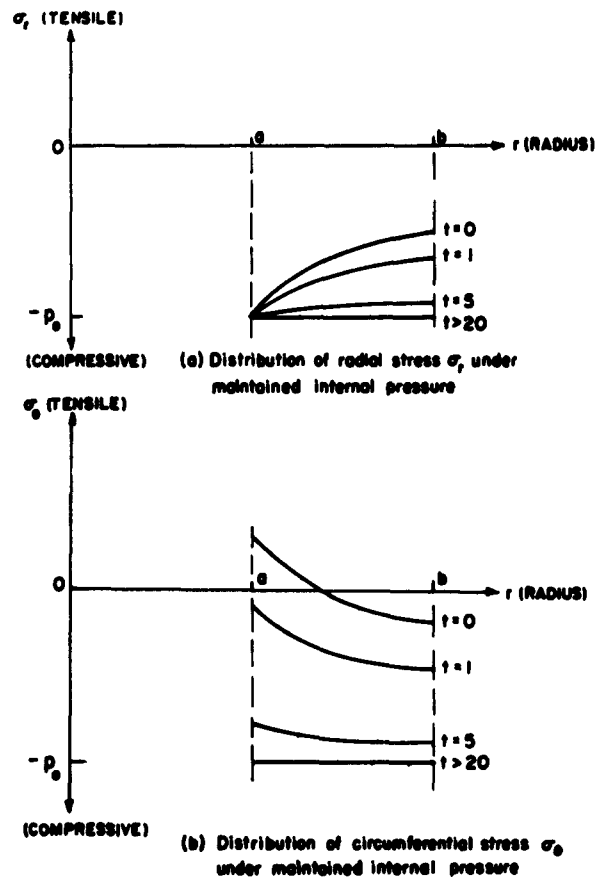


FIGURE 5

The varying stress distribution after sudden loading.

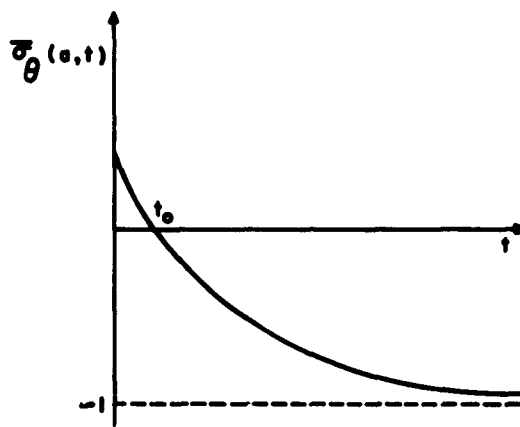


FIGURE 6

Variation of circumferential stress at the cavity surface for unit internal pressure.

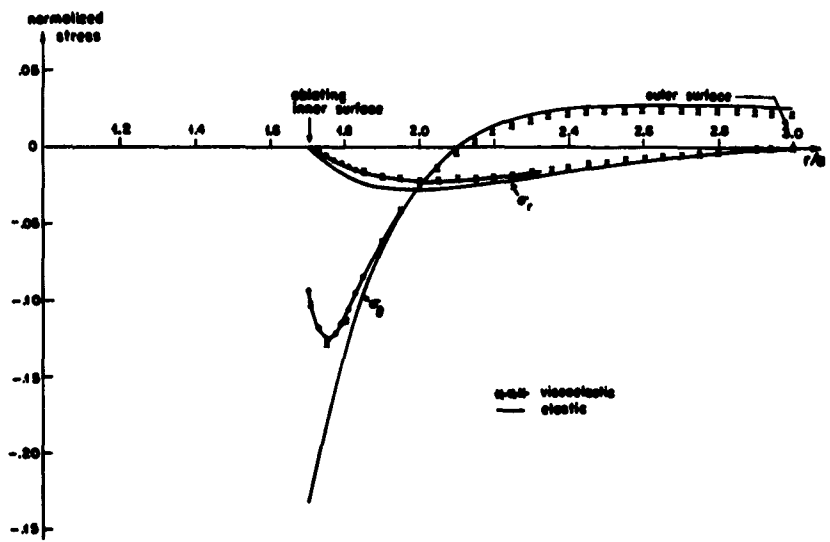


FIGURE 7

Stress distributions for an internally ablating sphere.