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COMPRESSIBLE PLASMA FLOW OVER A BIASED BODY

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COMPRESSIBLE PLASMA FLOW OVER A BIASED BODY

I. Introduction.

Recently there has been a revival of interest in using Langmuir probes as a measuring device in plasmas. However, the classical low density theory cannot be applied to high density plasma flows, which are encountered in devices such as shock tubes and plasma arcs. Additional interest in the probe problem arises from its relation to the hypersonic aerodynamic problem in which a plasma is generated behind the bow shock formed in front of a blunt body flying at hypersonic speeds. In the probe problem one is interested in the current-voltage characteristics from which one hopes to obtain some information regarding the properties of the plasma. In the blunt body aerodynamic problem the main interest is in the distribution of charged particles around the body and any change in the heat transfer characteristics which may occur due to the flux of charged particles to the body surface which, in most practical situations, is at the floating potential.

A continuum theory of electrostatic probes in a static isothermal plasma was given by Su and Lam[1] for negative probe potentials above the floating potential, and by Cohen[2] for moderate probe potentials (between the plasma and floating potential). In such analyses, the sheath was not assumed, a priori; rather it turned out to be a consequence of a careful asymptotic analysis. However, the structure of the sheath was based on the collision-dominated diffusion equation. The limit of validity of such a description is obtained by requiring that the electrical energy gained by a charged particle during one free flight is much less than its thermal energy. It is relatively easy to show that such a criterion implies:

$$\lambda_D \gg \ell \qquad \text{for a very negative probe,}$$

$$\left(\frac{\lambda_D}{r_p}\right)^{2/3} \gg \frac{\ell}{r_p} \qquad \text{for a moderately negative probe}^{1},$$

where λ_D is the electron Debye length based on the undisturbed charged particle density, ℓ is a typical mean free path between charged and neutral particles, and r_p is the probe radius. These inequalities place rather strong limitations on the results in Refs. 1 and 2. In general the ionization fraction has to be fairly low* (say 10^{-4}).

These continuum concepts were later extended by Lam[3] to an incompressible, isothermal flow of a weakly ionized gas for moderate surface potentials. Because of his assumptions that the gas is weakly ionized, incompressible, and isothermal, the diffusion of the charged particles to the solid surface is decoupled from the mass motion of the neutral gas. The existence of an electric field in the inviscid region was first pointed out in this work. The current collected by the body is essentially determined by the electron mobility (diffusion due to the electric field) in the inviscid region. Chung[4] has tackled the Couette flow and stagnation flow of weakly ionized gases numerically. The sheath structure he obtained checked qualitatively with that given in Refs. 1 and 2. This is not surprising since in the Couette flow there is no convective motion;

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* This was pointed out to the present writer by J. Fay and R. F. Probstein of M.I.T.

the governing diffusion equation is identical to the static case while in the stagnation flow the sheath was assumed to be thin and close to the solid surface where convective motion is entirely negligible. The analysis in the inviscid region was neglected even though he seemed to note that there was residual electric field intensity at the outer edge of the viscous boundary layer.

Previous to the work discussed above, Talbot[5] introduced the concept of a "quasi-continuum" stagnation probe analysis. The continuum equation was used to describe the diffusion of mass, momentum, and energy in the viscous boundary layer, while within the sheath, (which was assumed, a priori, to occupy a distance of one mean free path from the surface) the charged particles fall freely down the potential hill*. The change in potential in the viscous layer was neglected. It was demonstrated in Refs. 1 and 2 that within the continuum framework the potential drop for a static plasma outside the sheath can be of the same order of magnitude as (or larger than) that within the sheath. Talbot's assumption of no change in potential within the viscous layer is therefore open to question. Even though Talbot's analysis is necessarily crude, the quasi-continuum model is a more realistic one for plasmas of high ionization fraction. A complete analysis of this problem would, however, require a kinetic treatment.

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* The idea of putting a collision-dominated quasi-neutral solution and a collision-free sheath together was first suggested in 1936 by Davydov and Zmanovskaja[6]. Such an approximation can at best give gross results such as the current-voltage characteristics.

In the present paper, we shall adopt a strict continuum description. The restriction on such a continuum analysis which we have mentioned previously is the same for the present problem. It will be shown, however, that for bodies at floating potential, the structure of the sheath does not enter the calculation of the heat transfer and charged particle distribution. Therefore the limitation mentioned is not relevant to the calculation of these quantities. Our analysis will first extend Lam's results[3] to a frozen non-isothermal plasma. The electric field in the inviscid region will then be discussed for a general flow field. Since the flow characteristics in the viscous boundary layer are well known, the discussion concentrates on the diffusion of the charged particles and the accompanying electric potential distribution. It will be shown that within the viscous layer, the diffusion is ambi-polar in nature, even though the electron current is not necessarily equal to that for the ions. The potential distribution is decoupled from the system in the sense that it is determined after one has obtained the solutions for the other flow variables. The probe potential is assumed to be moderate, so that the sheath is thin and static (though with diffusion, of course). Finally the stagnation probe is discussed in more detail and an approximate analytic current-voltage characteristic is derived under the assumption of a very thin sheath.

II. Formulation.

The governing continuum equations for the physical system to be discussed are as follows:

$$\frac{\partial \rho}{\partial t} + \text{div}\,(\rho \underline{v}) = 0 \quad , \tag{1}$$

$$\rho \frac{d}{dt}\left(\frac{n_\alpha}{\rho}\right) + \text{div}\,(n_\alpha \underline{w}_\alpha) = 0 \quad ,$$

with

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \underline{v} \cdot \frac{\partial}{\partial \underline{x}}$$

and

$$\alpha = 1, 2, \ldots , \tag{2}$$

$$\rho \frac{d\underline{v}}{dt} + \frac{\partial p}{\partial \underline{x}} + \text{div}\,\underline{\underline{\sigma}}' = 0 \quad , \tag{3}$$

$$\rho \frac{d}{dt}\left(h + \frac{1}{2} v^2\right) = \frac{\partial p}{\partial t} + \text{div}\,[\underline{v} \cdot \underline{\underline{\sigma}}' - \underline{q}] + \underline{J} \cdot \underline{E} \quad , \tag{4}$$

$$\nabla^2 \phi = -4\pi e\,(n_i - n_e) \quad . \tag{5}$$

Eq. (1) is the overall continuity equation while Eq. (2) is the continuity equation for each species. We shall consider a system with three species: neutrals, ions, and electrons. The species equations which need to be considered will therefore be for the ions and electrons only. Eqs. (3), (4), and (5) are the momentum, energy, and Poisson equations respectively. The subscripts i and e stand for ions and electrons respectively and all other symbols have their usual meaning. The dissipative fluxes ($n_\alpha \underline{w}_\alpha$, $\underline{\underline{\sigma}}'$, $\underline{q}$) are given by[6]

$$\Gamma_i = n_i \underline{w}_i = -\rho D_i \left[\text{grad} \frac{n_i}{\rho} + \frac{e}{kT} \frac{n_i}{\rho} \text{grad} \phi\right] \quad , \qquad (6a)$$

$$\Gamma_e = n_e \underline{w}_e = -\rho D_e \left[\text{grad} \frac{n_e}{\rho} - \frac{e}{kT} \frac{n_e}{\rho} \text{grad} \phi\right] \quad , \qquad (6b)$$

$$\sigma'_{ik} = \mu \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3} \delta_{ik} \frac{\partial v_e}{\partial x_e}\right) + \zeta \delta_{ik} \frac{\partial v_e}{\partial x_e} \quad , \qquad (6c)$$

$$\underline{q} = -\kappa \text{ grad } T + \sum_{\alpha=1}^{3} \rho_\alpha \underline{w}_\alpha h_\alpha \qquad (\text{perfect gas})^* \quad . \qquad (6d)$$

The boundary conditions for ρ, $\underline{v}$, T are well known. In addition, we shall assume that the body is a perfect absorber of charged particles, so that on the body surface

$$\frac{n_\alpha}{\rho} = 0 \qquad \text{for } \alpha = i \text{ and } e.$$

The potential on the body is given as ϕ_p with respect to the potential far ahead of the body.

We shall assume in our subsequent analysis that the net current drawn by the body is small such that the Joule heating "$\underline{J} \cdot \underline{E}$" is negligible.

The enthalpy diffusion flux in the heat flux vector $\underline{q}$ can be simplified by writing

$$\sum \rho_\alpha \underline{w}_\alpha h_\alpha = (h_i - h_g) \rho_i \underline{w}_i + (h_e - h_g) \rho_e \underline{w}_e \quad ,$$

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* The ion and electron temperatures T are assumed to be equal.

where the subscript g stands for the neutral gas and h is the thermal enthalpy per unit mass. Since

$$h_i = h_g \quad \text{and} \quad h_e >> h_g \quad ,$$

we have, $\sum \rho_\alpha \underline{w}_\alpha h_\alpha = h_e \rho_e \underline{w}_e = h'_e n_e \underline{w}_e$, where h'_e is the thermal enthalpy per electron. We shall now assume that the flow is frozen and that electrons and ions do not recombine except on the body surface. The enthalpy h in (4) and (6d) is then taken to be thermal enthalpy only. However, in our formula for the heat transfer to the wall, h will include both the thermal enthalpy and the ionization enthalpy.

If we appropriately non-dimensionalize Eqs. (1) - (5), the order of the magnitude of each term may be expressed as follows:

1) Overall Continuity Equation

$$1 : 1 = 0 \quad .$$

2) Species Continuity Equation

$$1 : \frac{1}{Sc\ Re} = 0 \quad .$$

3) Momentum Equation

$$1 : 1 : \frac{1}{Re} = 0 \quad .$$

4) Energy Equation

$$\rho \frac{d}{dt} (h + \frac{v^2}{2}) = \frac{\partial p}{\partial t} + \text{div} \{ \underline{v} \cdot \underline{\underline{\sigma}}' - \frac{\kappa}{\bar{c}_p} \text{grad} \frac{v^2}{2} + \frac{\kappa}{\bar{c}_p} \text{grad} (h + \frac{v^2}{2})$$

$$- h_e (\frac{\kappa}{\bar{c}_p} \text{grad } C_e + \rho_\alpha \underline{w}_\alpha)\}$$

$$1 = \frac{u^2_\delta}{H_\delta} : (1 - \frac{1}{Pr}) \frac{1}{Re} \frac{u^2_\delta}{H_\delta} : \frac{1}{Pr\ Re} : (\frac{1}{Le} - 1) \frac{1}{Sc\ Re} \frac{n_\delta k T_\delta}{H_\delta \rho_\delta}$$

5) Poisson Equation

$$(\frac{\lambda_D}{r_p})^2 = 1 : 1 \quad .$$

Here

Sc = Schmidt number = $(\frac{\mu}{\rho D_a})$,

D_a = Ambipolar diffusion coefficient* ,

Re = Reynolds number = $(\frac{\rho u_\delta r_p}{\mu})$,

r_p = Typical body dimension ,

H_δ = Typical total enthalpy ,

u_δ = Typical velocity ,

n_δ = Typical charged particle number density ,

λ^2_D = Debye length = $\frac{kT_\delta}{4\pi n_\delta e^2}$,

Le = Lewis number = $\frac{\rho \bar{c}_p D_a}{\kappa}$,

Pr = Prandtl number = $\frac{\bar{c}_p \mu}{\kappa}$,

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* The reason for using the ambipolar diffusion coefficient, D_a, will be made clear in the discussion of the viscous boundary layer.

$$\bar{c}_p = \sum c_\alpha c_{p\alpha} = \sum \frac{\rho_\alpha c_{p\alpha}}{\rho} \quad ,$$

$$C_e = \frac{n_e m_e}{\rho} \quad .$$

All the quantities with subscript δ will be identified later as the values at the edge of the viscous boundary layer.

It is reasonable to assume that the non-dimensional parameters Sc, Pr, Le, u^2_δ/H_δ, $n_\delta kT_\delta/\rho_\delta H_\delta$ are all of order unity in comparison with the two important parameters Re and λ_D/r_p. In most cases of practical interest the following inequalities are satisfied, i.e.,

$$(\frac{\lambda_D}{r_p})^2 \ll \frac{1}{Re} \ll 1 \quad . \tag{7}$$

Both of the above two parameters are associated with the relevant highest order derivatives in our system of equations. It is therefore expected that there will be two singular perturbations in the problem, one for the viscous layer, associated with the Reynolds number Re (Prandtl boundary layer) and another for the sheath, associated with the Debye length parameter λ_D/r_p (Langmuir boundary layer). Because of the inequalities (7), we see that the sheath is imbedded within the viscous layer. Qualitatively, we can now say that there are three distinct regions where different physical mechanisms operate:

1) Inviscid region: Diffusion of mass, momentum, and energy are relatively unimportant compared with convection. Charge neutrality is maintained.

2) Viscous layer: Convection and diffusion operate simultaneously. Charge neutrality is also maintained in this region.

3) Sheath: Here charge separation can take place. Convection is unimportant since the region is thin and adjacent to a solid surface. Diffusion and mobility of the ions and electrons are the main features of the sheath.

Because of the assumption that the sheath region adjacent to the surface is thin, we shall automatically restrict ourselves to a moderate probe potential. If the potential is strong enough, the sheath can be thick and the problem of convection within the sheath has to be properly taken into account.

III. Inviscid Region.

In this region, the system of equations becomes doubly degenerate. First the Laplacian in the Poisson equation is neglected on the basis of the smallness of $(\lambda_D/r_p)^2$. This gives

$$n_i = n_e + 0\left(\frac{\lambda_D^2}{r_p^2}\right) \quad , \tag{8}$$

which is the well known quasi-neutral solution. Next we drop all dissipation terms in Eqs. (1) to (4), i.e.,

$$\frac{\partial \rho}{\partial t} + \text{div}\,(\rho \underline{v}) = 0 \quad ,$$

$$\rho \frac{d}{dt}\left(\frac{n_\alpha}{\rho}\right) = 0\left(\frac{1}{Re}\right) \quad ,$$

$$\rho \frac{d\underline{v}}{dt} + \frac{\partial p}{\partial \underline{x}} = 0\left(\frac{1}{Re}\right) \quad , \qquad (9)$$

$$\rho \frac{d}{dt}\left(h + \frac{1}{2} v^2\right) - \frac{\partial p}{\partial t} = 0\left(\frac{1}{Pe}\right) \quad .$$

To this order of accuracy, the density ρ, mass velocity $\underline{v}$, and fluid enthalpy h, as well as the charged particle density $n = n_i = n_e$ are determined by this degenerate set of equations. However, any information regarding the electric force is lost from the system. This lost information can be recovered by subtracting the two species continuity equations (annihilation of the dominant terms), i.e.,

$$\text{div}\,[n_e \underline{w}_e - n_i \underline{w}_i] = 0 \quad , \qquad (10a)$$

or from the flux relations (Eqs. (6a) and (6b)

$$\text{div}\,\{-\rho(D_e - D_i)\,\text{grad}\,\frac{n}{\rho} + \frac{e}{kT}(D_e + D_i)\,n\,\text{grad}\,\phi\} = 0 \quad , \qquad (10b)$$

With n, ρ, and T determined from Eq. (9), we can calculate the electric potential in the inviscid region by means of Eq. (10b). It is obvious that the validity of (10b) is independent of the large Reynolds number assumption.

It is valid as long as quasi-neutrality is maintained*. Moreover, within the sheath, even though $n_i \neq n_e$, since the mass velocity is small (of the order the ratio of the sheath thickness to the viscous layer thickness), Eq. (10b) is still approximately valid†. Note after multiplying by e, the quantity within the bracket in (10b) is the conduction current. Thus we conclude that the conduction current through a closed surface in the flow field is zero. Within the viscous and sheath layers the flux through a surface normal to the wall is negligible so that we have constancy of the conduction current density throughout the layers, i.e.,

$$- \rho D_i \left[\frac{\partial}{\partial y} \frac{n_i}{\rho} + \frac{e}{kT} \frac{n_i}{\rho} \frac{\partial \phi}{\partial y}\right] + \rho D_e \left[\frac{\partial}{\partial y} \frac{n_e}{\rho} - \frac{e}{kT} \frac{n_e}{\rho} \frac{\partial}{\partial y}\right] = - J/e \quad , \tag{11}$$

where J is an integration constant which is identified as the conduction current density collected by the probe, i.e., $J = J_i - J_e = e(\Gamma_e - \Gamma_i)$.

We have shown that the electric potential in the inviscid region is governed by Eq. (10b) with n, ρ, and T obtained from Eqs. (9). One boundary condition for Eq. (10b) is obtained by evaluating Eq. (11) at the outer edge of the viscous boundary layer, i.e.,

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* We have assumed $(\lambda_D/r_p)^2 << 1/Re$.

† Eq. (10b) is approximately valid within the sheath if the convection there is negligible. Since the convective velocity on the wall is zero, the convection within the sheath is of the order of the thickness of the sheath.

$$\frac{\partial\phi}{\partial y}\bigg|_\delta = \frac{kT_\delta}{e(D_e + D_i)} \frac{1}{n_\delta} \left[\frac{J}{e} - \rho(D_i - D_e) \frac{\partial n}{\partial y}\bigg|_\delta\right] \quad . \tag{12a}$$

In addition, we require

$$\phi \to 0 \text{ at infinity}^{*}. \tag{12b}$$

In special cases such as stagnation point flow, flow over a flat plate, and the end wall problem in a shock tube, the quantities n, ρ, and T are approximately constant in the inviscid region. In this case Eq. (10b) and the boundary condition Eq. (12a) are greatly simplified so that the equation for the potential and the boundary conditions becomes

$$\nabla^2\phi = 0 \quad ^{\dagger} \tag{13a}$$

$$\frac{\partial\phi}{\partial y}\bigg|_\delta = \frac{kT_\delta}{e(D_e + D_i)} \frac{1}{n_\delta} \frac{J}{e} \quad , \tag{13b}$$

$$\phi \to 0 \text{ at infinity} \quad . \tag{13c}$$

The conduction current density J collected by the body is still an unknown constant which must be determined by the boundary conditions specified on the body surface. In other words, J is determined only

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* Eq. (10b) is an elliptic second order linear partial differential equation. The boundary conditions we have specified uniquely define a solution.

† Eq. (13a) was first shown to be appropriate to the present problem by Lam[3].

after one solves the sheath properly. However, at the floating potential, J is by definition zero. It follows then from (13a) and (13b) that in the inviscid region

$$\phi \equiv 0$$

It is seen that the reversal of the parity of the electric field in the inviscid region occurs at the floating potential in contrast to the plasma potential in the no flow case.

IV. Viscous Layer.

Within the viscous boundary layer, the dissipative terms (with the gradient in the direction normal to the wall) are of the same order of magnitude as the convective terms. However, quasi-neutrality is still a good approximation. In this layer we have the following set of equations:

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\,(\rho \underline{v}) = 0 \quad ,$$

$$\rho \frac{d}{dt}\left(\frac{n_i}{\rho}\right) = \frac{\partial}{\partial y}\left\{\rho D_i \left[\frac{\partial}{\partial y}\left(\frac{n_i}{\rho}\right) + \frac{e}{kT}\frac{n_i}{\rho}\frac{\partial \phi}{\partial y}\right]\right\} \quad ,$$

$$\rho \frac{d}{dt}\left(\frac{n_e}{\rho}\right) = \frac{\partial}{\partial y}\left\{\rho D_e \left[\frac{\partial}{\partial y}\left(\frac{n_e}{\rho}\right) - \frac{e}{kT}\frac{n_e}{\rho}\frac{\partial \phi}{\partial y}\right]\right\} \quad , \qquad (14)$$

$$\rho \frac{du}{dt} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) \quad ,$$

$$\frac{\partial p}{\partial y} = 0 \quad ,$$

$$\rho \frac{d}{dt}\left(h + \frac{u^2}{2}\right) = \frac{\partial p}{\partial t} + \frac{\partial}{\partial y}\left\{\mu\left(1 - \frac{1}{Pr}\right)\frac{\partial}{\partial y}\left(\frac{u^2}{2}\right) + \frac{\kappa}{c_p}\frac{\partial}{\partial y}\left(h + \frac{u^2}{2}\right)\right.$$

$$\left. - h_e\left(\frac{\kappa}{c_p}\frac{\partial C_e}{\partial y} + \rho_e w_{ey}\right)\right\} \quad ,$$

$$n_i = n_e + O\left(\frac{\lambda_D^2}{r_p^2}\right)$$

where $C_e = \rho_e/\rho$ and w_{ey} is the electron diffusion velocity in the direction normal to the wall.

It was pointed out in the last section that Eq. (11) is valid within the viscous boundary layer. Such an equation gives the relation between the charged particle distribution and the electric potential within the layer. As in the inviscid region it is a great simplification that the solution for the density distribution of charged particles can be determined at first independently of the electric potential. We shall see that this is in general true, provided quasi-neutrality is valid. The second of Eqs. (14) can be integrated to give

$$\frac{1}{\rho D_i}\int^{y} \rho \frac{d}{dt}\left(\frac{n}{\rho}\right) dy = \frac{\partial}{\partial y}\left(\frac{n}{\rho}\right) + \frac{e}{kT}\frac{n}{\rho}\frac{\partial \phi}{\partial y} \quad . \qquad (14a)$$

Similarly for the electron continuity equation

$$\frac{1}{\rho D_e}\int^{y} \rho \frac{d}{dt}\left(\frac{n}{\rho}\right) dy = \frac{\partial}{\partial y}\left(\frac{n}{\rho}\right) - \frac{e}{kT}\frac{n}{\rho}\frac{\partial \phi}{\partial y} \quad . \qquad (14b)$$

From the above equations we obtain

$$\rho \frac{d}{dt}\left(\frac{n}{\rho}\right) = \frac{\partial}{\partial y}\left[\frac{2\rho}{1/D_i + 1/D_e}\frac{\partial}{\partial y}\left(\frac{n}{\rho}\right)\right] = \frac{\partial}{\partial y}\left[\rho D_a \frac{\partial}{\partial y}\left(\frac{n}{\rho}\right)\right]$$

$$= \frac{\partial}{\partial y}\left[- n_i w_{iy}\right] = \frac{\partial}{\partial y}\left[- n_e w_{ey}\right] \quad , \qquad (15)$$

where $D_a = 2/(\frac{1}{D_i} + \frac{1}{D_e})$ is the ambipolar diffusion coefficient.

We conclude from Eq. (15) that

$$n_i w_{iy} = - \rho D_a \frac{\partial}{\partial y}\left(\frac{n}{\rho}\right) + A \quad , \qquad (16)$$

$$n_e w_{ey} = - \rho D_a \frac{\partial}{\partial y}\left(\frac{n}{\rho}\right) + B^{*} \quad , \qquad (17)$$

where A and B are two arbitrary constants. It may be seen from (11) that $A - B = - J/e$. It can also be seen from Eqs. (16) and (17) that the diffusion in the viscous layer is essentially characterized by ambipolar diffusion. However, we must emphasize that the ion and electron currents are the same only when the body is at the floating potential. In view of Eq. (15), we see that the Schmidt number introduced earlier should be based on the ambipolar diffusion coefficient. Since the latter is of the same

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* Eq. (15) was first obtained by Chung for stagnation point flow. However, he did not point out the simple relations (16) and (17), which show that the ratio of the electron and ion currents cannot be a constant throughout the viscous layer.

order of magnitude as D_i, we conclude that the thickness of the diffusion layers (both for electrons and ions) are of the same order of magnitude as the viscous momentum layer, i.e., of order $Re^{-1/2}$.

The constants A and B can be determined at the outer edge of the viscous layer. We find these

$$B = -\rho D_e \left[\frac{D_e - D_i}{D_e + D_i} \frac{\partial}{\partial y} \left(\frac{n}{\rho}\right) - \frac{e}{kT} \frac{n}{\rho} \frac{\partial \phi}{\partial y}\right]\Big|_\delta \quad . \tag{18}$$

In the special cases such as stagnation point flow, flat plate flow, or the end wall of a shock tube, $\frac{\partial}{\partial y} \left(\frac{n}{\rho}\right)\Big|_\delta = 0$, then

$$B = \frac{eD_e}{kT_\delta} n_\delta \frac{\partial \phi}{\partial y}\Big|_\delta = \frac{J}{e} \frac{D_e}{D_e + D_i} \tag{19}$$

$$\approx \frac{J}{e} \quad (\text{with } D_i << D_e)$$

Similarly

$$A = -\frac{J}{e} \frac{D_i}{D_e + D_i} \approx -\frac{J}{e} \frac{D_i}{D_e} \tag{20}$$

These are essentially the currents due to the mobilities of electrons and ions. It is seen that with $D_e >> D_i$, the value of A is negligible compared with B, and thus eB is approximately the current density one draws from the plasma. At the floating potential, we see from Eqs. (19) and (20) that $A = B = 0$.

The governing equations within the viscous layer can now be simplified as follows:

$$\frac{\partial\rho}{\partial t} + \mathrm{div}\,(\rho\underline{v}) = 0 \quad ,$$

$$\rho\,\frac{d}{dt}\left(\frac{n}{\rho}\right) = \frac{\partial}{\partial y}\left[\rho D_a \frac{\partial}{\partial y}\left(\frac{n}{\rho}\right)\right] \quad ,$$

$$\rho\,\frac{du}{dt} = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left(\mu\,\frac{\partial u}{\partial y}\right) \quad , \quad \frac{\partial p}{\partial y} = 0$$

$$\rho\,\frac{d}{dt}\left(h + \frac{u^2}{2}\right) = \frac{\partial p}{\partial t} + \frac{\partial}{\partial y}\left\{\mu\left(1 - \frac{1}{Pr}\frac{\partial}{\partial y}\left(\frac{u^2}{2}\right) + \frac{u}{Pr}\frac{\partial}{\partial y}\left(h + \frac{u^2}{2}\right)\right.\right.$$

$$\left.- \left(\frac{1}{Le} - 1\right)\rho D_a h_e \frac{\partial C_e}{\partial y} - B m_e h_e\right\}$$

The electric potential is obtained from Eq. (11) after this set of equations is solved for ρ, n, $\underline{v}$, and T.

Except for the addition of the last term in the energy equations, Eqs. (21) are the usual boundary layer equations for a frozen dissociated gas[8,9]. For steady two-dimensional and axisymmetric flow (an analogous treatment applies to one-dimensional unsteady flow), we apply the usual similarity transformations used in compressible boundary layer theory[8,9].

$$\xi(x) = \int_0^x \rho_\delta u_\delta \mu_\delta r_0^{2j}\, dx$$

$$\eta = \frac{\rho_\delta u_\delta r_0^j}{\sqrt{2\xi}} \int_0^y \frac{\rho}{\rho_\delta}\, dy \quad .$$

Equations (21) then reduce to

$$(Nf'')' + ff'' = \frac{2\xi}{u_\delta}\frac{du}{d\xi}\left[f'^2 - \frac{\rho_\delta}{\rho}\right]$$

$$\left(\frac{N}{Sc} z'\right)' + fz' = \frac{2\xi f' z}{(n/\rho)_\delta}\frac{d(n/\rho)_\delta}{d\xi} \qquad (23)^*$$

$$\left(\frac{N}{Pr} g'\right)' + fg' = \frac{2\xi f' g}{H_\delta}\frac{dH_\delta}{d\xi} + \frac{u_\delta^2}{H_\delta}\left[\left(\frac{1}{Pr} - 1\right) Nf'f''\right]'$$

$$+ \left[\frac{N}{Sc}\left(\frac{1}{Le} - 1\right)\frac{(5/2)\, kT\, (n/\rho)_\delta}{H_\delta} z'\right]' - \frac{5Bk}{2}\frac{\sqrt{2\xi}}{\rho_\delta \mu_\delta u_\delta r_0^j}\frac{\partial T}{\partial \eta} \quad ,$$

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* Eqs. (23) were first used by Talbot[5] for the present type of problem. Since he restricted himself to weakly ionized gases, the last two terms in the energy equation which arise from the energy flux due to electron diffusion did not appear in his formulation.

where

$$N = \frac{\rho\mu}{\rho_\delta \mu_\delta} , \quad f' = \frac{u}{u_\delta} , \quad f = \int_0^y \frac{u}{u_\delta}\, dy ,$$

$$z = \frac{n}{\rho}/(\frac{n}{\rho})_\delta , \quad g = \frac{h + \frac{u^2}{2}}{(h + \frac{u^2}{2})_\delta} = \frac{h + \frac{u^2}{2}}{H_\delta}$$

$$Sc = \frac{\mu}{\rho D_a}\ {}^{\dagger} ,$$

$$Le = \frac{\rho \bar{c}_p D_a}{\kappa}\ {}^{*}$$

and the prime indicates differentiation with respect to η. Eqs. (23) reduce to the ordinary differential equations if all the terms on the right side of them are functions of η alone. For more detailed discussion on the conditions of similarity see Ref. 8.

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† For a weakly ionized gas only collisions between charged and neutral particles are taken into account, so that

$$Sc = \frac{\mu}{2\rho D_i} = \frac{(5\pi/32)\rho \ell c}{(3\pi/8)\rho \ell c} = 5/4 \quad [10].$$

* For monatomic atoms

$$Le = \frac{5}{2}\,\frac{\sum n_\alpha k D_a}{\kappa} = \frac{5}{2}\,\frac{\sum n_\alpha k \cdot \frac{4}{5}\frac{\mu}{\rho}}{(5/2)\ \mu C_v} = \frac{2}{3} \cdot \frac{4}{5} = \frac{8}{15} \quad [10]$$

For a fully ionized gas the charged particle collisions must be taken into account when evaluating the mixture transport coefficients, in this case one may, for example, use the mixture rules given in Ref. 11.

We recall that Eqs. (23) are obtained under the assumption of quasi-neutrality. Such an assumption will break down inside the sheath near the wall where we expect a rapid variation in potential. Since we shall later demonstrate that the sheath is thin, it is clear that the change in density, velocity, and temperature across the sheath will be very small. If we require Eqs. (23) to satisfy the boundary conditions given on the wall, the relative error introduced in the solutions for density, velocity, and temperature will be of order of the sheath thickness. The difficulty, however, arises when we try to calculate the potential distribution by means of Eq. (11) based on a charged particle density distribution obtained in the above fashion. In the first place, the current J cannot be determined. Second, and more important, the electric field becomes infinite at the edge of the sheath. This suggests that although the velocity and temperature distributions are decoupled from the system within the sheath, the charged particle density and potential distribution have to be solved simultaneously[1,2]. However, in the problem of a blunt body at floating potential ($J = 0$), one is interested only in the charged particle distribution around the body and the charged particle fluxes to the wall. As long as the sheath is thin, even though it may be collision dominated or collision free, the charged particle distribution and the heat flux can be determined with an error of order of sheath thickness by letting Eqs. (23) satisfy the boundary conditions given on the wall. We give the heat transfer to a floating wall as follows:

$$q_w = -\frac{k}{\bar{c}_p}\frac{\partial}{\partial y}\left(h + \frac{u^2}{2}\right) + h_e\left(\frac{k}{\bar{c}_p}\frac{\partial C_e}{\partial y} - \rho D_a \frac{\partial C_e}{\partial y}\right) - \rho D_a \frac{\partial}{\partial y}\left(\frac{n}{\rho}\right) h^{(0)}$$

$$= -\frac{\partial\eta}{\partial y}\frac{k}{\bar{c}_p} H_\delta \left\{g'(0) + \frac{\frac{5}{2} n_\delta kT}{\rho_\delta H_\delta} \mathrm{Le}\left[\frac{h^{(0)}}{(5/2)kT} + 1 - \frac{1}{\mathrm{Le}}\right] z'(0)\right\} \quad , \tag{24}$$

where $h^{(0)}$ is the ionization energy per electron-ion pair. For the case $h^{(0)} >> (5/2)kT$

$$q_w = -\frac{\partial\eta}{\partial y}\frac{k}{\bar{c}_p} H_\delta \left\{g'(0) + \frac{n_\delta h^{(0)}}{\rho_\delta H_\delta} \mathrm{Le}\, z'(0)\right\} \quad . \tag{24a}$$

The zero arguments of g and z in Eqs. (24) and (24a) are referred to the edge of the sheath, which in the floating case can be identified as the wall.

Near the solid surface, $f \sim \eta^2$, $f' \sim \eta$, $z \sim \eta$, so that the second equation in (23) is reduced to

$$\left(\frac{N}{Sc} z'\right)' = 0 \quad .$$

Thus

$$z' = C_1 = \text{constant} \quad , \tag{25}$$

or

$$\frac{dn}{dy} = C \quad . \tag{25a}$$

The corresponding electric potential is obtained from Eq. (11) as

$$\frac{d}{dy}\left(\frac{e\phi}{kT_w}\right) = \frac{\frac{J}{e} + (D_e - D_i)C}{D_e + D_i}\frac{1}{n} = \frac{D}{n} \tag{26}$$

It is not surprising, although interesting, that the behavior of the density and potential distributions is exactly the same as for the quasi-neutral solution in the static no flow case[1,2]. The two constants C and D are related to the number density fluxes in the following way:

$$C = -\frac{1}{2}\left(\frac{\Gamma_e}{D_e} + \frac{\Gamma_i}{D_i}\right) ,$$

$$D = \frac{1}{2}\left(\frac{\Gamma_e}{D_e} - \frac{\Gamma_i}{D_i}\right) . \tag{27}$$

These constants as we have mentioned earlier have to be determined by the boundary conditions at the wall through a careful analysis of the sheath.

V. Charge Separation Sheath.

Since we shall consider the case of moderate potential, the charge separation sheath can be assumed to be thin*. The analysis within the sheath will then be similar to that given by Cohen[2]. The coordinate y

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* For a detailed discussion on this matter, see the section on "Probes at arbitrary potentials" in Ref. 1.

in Eqs. (25a) and (26) with (27) (as is in the viscous layer) is the physical length multiplied by $Re^{1/2}$. If we choose the probe radius r_p and the free stream charged particle density n_δ to be the normalization quantities, Eqs. (25a) and (26) with (27) become

$$\frac{dn}{dy} = -\frac{1}{2}\left(\frac{\Gamma_e}{D_e} + \frac{\Gamma_i}{D_i}\right)\frac{r_p}{n_\delta} = -\frac{1}{2}(j_i + j_e) \quad , \tag{28}$$

$$\frac{d\psi}{dy} = \frac{1}{2}(j_i - j_e)\frac{1}{n} \quad , \tag{29}$$

where

$$j_{i,e} = \frac{\Gamma_{i,e}\, r_p}{D_{i,e}\, n_\delta} \quad , \quad \psi = \frac{e\phi}{kT_w} \quad .$$

The electric field outside the sheath is of order $Re^{1/2}$ in contrast to order unity in the static case. From Poisson's equation, (5), it can be seen that the stretching factor for the sheath is proportional to $(\lambda_D/r_p)^{2/3}\ Re^{1/3}$ instead of $(\lambda_D/r_p)^{2/3}$ in the static case. The electric field in the sheath is of order

$$\left(\frac{\lambda_D}{r_p}\right)^{-2/3} Re^{-1/3}\ Re^{1/2} = \left(\frac{\lambda_D}{r_p}\right)^{-2/3} Re^{1/6} \quad . \tag{30}$$

The thickness of the sheath is of the order

$$\left(\frac{\lambda_D}{r_p}\right)^{2/3} Re^{1/2}/Re^{1/2} = \left(\frac{\lambda_D}{r_p}\right)^{2/3} Re^{-1/6} \quad . \tag{31}$$

Within the sheath, one works with the electric field instead of the potential. In the numerical solutions given by Cohen[2], the body potential is not posed as the boundary condition; instead one chooses a value of $1/2(j_e + j_i)$, and the corresponding value of $1/2(j_e - j_i)$ is found by satisfying the charged particle density boundary conditions on the wall. The potential on the body is then found by integrating over the solution for the electric field. In the present problem we can also assign a value of $1/2(j_e + j_i)$, in which case $1/2(j_e - j_i)$ is determined through the sheath solution which is required to go over into Eqs. (28) and (29). With $1/2(j_e + j_i)$ and $1/2(j_e - j_i)$ known, the value of J can be calculated and the electric field through the viscous and sheath layers can then be determined. This gives the potential difference between the body and the edge of the viscous layer. The potential at the latter point, taking the potential to be zero at infinity, is obtained by solving Eq. (10b) subject to the boundary conditions (12).

The probe potential can be written as follows:

$$- \psi_p = \int_0^{\delta_s^+} \left(\frac{d\psi}{dy_0}\right)_s dy_0 + \int_{\delta_s^+}^{\delta_v^+} \left(\frac{d\psi}{dy_0}\right)_v dy_0 + \int_{\delta_v^+}^{\infty} d\underline{r} \cdot \nabla\psi_I$$

where y_0 is the physical length variable normal to the wall, and

$$1 >> \delta_s^+ >> \delta_s \qquad (\delta_s = \text{thickness of sheath})$$

$$1 >> \delta_v^+ >> \delta_v \qquad (\delta_v = \text{thickness of viscous layer})$$

Here the subscript s represents the sheath solution; v the viscous layer solution; and I the inviscid solution.

With a certain amount of manipulation, the potential formula can be put into the following form:

$$- \psi_p = \psi_\delta - \frac{1}{2} (j_e - j_i) \ln [(\frac{\lambda_D}{r_p})^{2/3} Re^{-1/6}]$$
$$+ \int_0^\infty [(\frac{d\psi}{dy})_v + (\frac{d\psi}{dy})_s - (\frac{d\psi}{dy})_{v \rightleftarrows s} - (\frac{d\psi}{dy})_\delta] \, dy \quad , \tag{32}$$

where $(\frac{d\psi}{dy})_{v \rightleftarrows s}$ is the matching between the sheath and the viscous layer as given in Eq. (29). The logarithmic term in Eq. (32) is due to a first order pole singularity of the electric field in the matching region between the sheath and viscous layer. We display this logarithmic behavior explicitly, since it is the leading term in Eq. (32) for a very thin sheath*. In Eq. (32) ψ_δ is the potential at the outer edge of the boundary as obtained from the inviscid solution.

In the next section we shall apply this general procedure to a stagnation probe.

VI. Stagnation Probe.

In the neighborhood of the stagnation point of a body, one can assume that the quantities n, ρ, and T are approximately constant in the inviscid region. The solution of the electric field is then governed by Eq. (13a) subject to the boundary conditions (13b) and (13c). Under the conventional

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* The argument of the logarithm in Eq. (32) is the thickness of the sheath (see Eq. (31)).

approximation of stagnation flow, (flow impinging on an infinite plane) there is no solution satisfied by Eq. (13a) subject to the boundary conditions unless

1) $J = 0$, i.e., for a floating body, in which case the electric field in the inviscid region is identically zero.

2) The body is floating except for a small but finite current element located at the stagnation point. Such an arrangement is of great practical interest. In what follows we shall consider this problem.

We shall make the current element mentioned above small enough such that the usual stagnation flow assumptions can be applied to such an element. However, the element is considered to be much wider than the boundary layer thickness, so that the electric potential distribution is essentially one-dimensional within the boundary layer. The solutions for the ordinary flow properties are well known. From Eqs. (13) the electric potential in the inviscid region is now given by the following equation and boundary conditions:

$$\nabla^2\phi = 0$$

$$\left.\begin{aligned} \frac{d\phi}{dy} &= \frac{kT_\delta}{e(D_e + D_i)} \frac{1}{n_\delta} \frac{J}{e} &&\text{for } r \leq a \\ &= 0 && r < a \end{aligned}\right\} \; y = 0$$

$$\phi \to 0 \text{ at infinity} \quad .$$

where y = 0 is the outer edge of the viscous layer, and where the current element is taken to be a circular disc of radius a. We are not very interested in the detailed potential distribution in the half-space $y \geq 0$. The only information required for the construction of the current-voltage characteristic is the value of the electric potential on the element

$$y = 0 \qquad r \leq a \quad .$$

The solution for the potential is given simply by the potential distribution resulting from a charged disc of surface charge density

$$\sigma = \frac{E_y}{2\pi} = - \frac{1}{2\pi} \frac{JkT_\delta}{N_\delta e^2 (D_e + D_i)} = - \frac{2\lambda_D^2}{D_e + D_i} J \quad .$$

Assuming the variation of the potential across the disc to be small, which is reasonable provided the viscous layer is thin and much smaller than the dimension of the current element, then we need only calculate the potential at the origin. The potential distribution along the y-axis is given by

$$\phi(y, r = 0) = \int_{\substack{\text{over the} \\ \text{disc}}} \frac{\sigma}{\xi} \, ds = - \frac{4\pi \, J\lambda_D^2}{D_e + D_i} \left(\sqrt{y^2 + a^2} - y\right) \quad , \tag{33}$$

and

$$\phi_\delta(y = 0, r = 0) = - \frac{4\pi \, \lambda_D^2 aJ}{D_e + D_i}$$

At the floating potential ϕ_f, $J = 0$, so that $\phi_\delta = 0$ and $\phi \equiv 0$ in the inviscid region. For $\phi < \phi_f$, $J = J_i - J_e > 0$, and by Eq. (33) we have $\phi_\delta < 0$. The potential in the inviscid region is then negative. For potentials slightly above ϕ_f, we see that $\phi_\delta > 0$. The current in the inviscid region is mainly due to the electron mobility. Since the field is zero for the floating potential, the reversal of parity of the electric field in the inviscid region occurs at the floating potential instead of the plasma potential as in the case of the static plasma. Consequently, unlike the static case, $y \equiv 0$ is not a solution of the problem, i.e., when a probe is at the plasma potential there is still an electric field within the plasma.

Within the viscous layer, the flow field is governed by Eqs. (23) with the following assumption[9]:

$$f'^2 - \frac{\rho_\delta}{\rho} \simeq 0 \quad ^{*} \quad , \quad \frac{u_\delta^2}{H_\delta} \simeq 0$$

$$\frac{d}{d\xi}\left(\frac{n}{\rho}\right)_\delta = \frac{dH_\delta}{d\xi} \simeq 0 \quad . \tag{34}$$

The last two terms in the energy equations of (23) are due to the electron diffusion flux. Under the stagnation flow assumption, they have a similarity property. The solutions for the velocity and temperature are solved in the usual fashion, i.e., one ignores the existence of the thin sheath by applying the wall conditions for the velocity and

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* For a discussion of this approximation, see Ref. 12.

temperature to the first and third equations of (23). This procedure is not possible for the second equation of (23) because of the divergence of the electric field. For stagnation flow, Eq. (25) is valid throughout the viscous layer. However as was mentioned before, the constants C and D must be determined by an analysis of the sheath. Once the charged particle distribution inside the viscous layer is solved, the electric field is then calculated by the use of Eq. (11). Together with the electric field in the sheath, the current-voltage characteristic is given by Eq. (32). A complete solution can be obtained only through a detailed numerical computation (both in the viscous and the sheath layer). However as indicated in Eq. (32), the leading behavior of the current-voltage characteristic can be obtained analytically, i.e.,

$$- \psi_p = - \frac{1}{2} (j_e - j_i) \ln [(\frac{\lambda_D}{r_p})^{2/3} \mathrm{Re}^{-16}] + O(1) \qquad (35)$$

To give an idea of the accuracy of (35), we take $\lambda_D/r_p = 10^{-4}$, $\mathrm{Re} = 10^6$. The relative error by neglecting the terms of order unity is then about 10%.

The potential ψ_p is normalized by the thermal energy at the wall. From Eqs. (26) to (29), we have

$$\frac{1}{2} (j_e - j_i) = - \frac{J}{e} \frac{r_p}{n_\delta (D_e + D_i)} - \frac{D_e - D_i}{D_e + D_i} \frac{1}{2} (j_e + j_i)$$

$$\approx - \frac{J r_p}{e n_\delta D_e} - \frac{1}{2} (j_e + j_i) \qquad \text{for } D_e \gg D_i \quad .$$

Using this relationship in Eq. (35) we obtain the current-voltage characteristic as follows:

$$\frac{J}{en_\delta D_e/r_p} = -\frac{1}{2}(j_e + j_i) + \psi_p/\ln\left[\left(\frac{r_p}{\lambda_D}\right)^{2/3} Re^{1/6}\right] \quad . \qquad (36)$$

From Eq. (28) we see that the quantity $-\frac{1}{2}(j_e + j_i)$ is the charged particle density gradient near the solid surface given by the quasi-neutral solution. To within a relative error of the order of the sheath thickness, one can obtain this slope by forcing the density equation to satisfy the boundary condition at the wall. For a complete determination of the density gradient, one has to solve Eqs. (23) simultaneously. However, if we assume that N and Sc are constant throughout the viscous layer, together with the stagnation approximations given in Eq. (34), one obtains from the first two equations in (23) that[9]

$$z'(0) = \frac{0.47\, Sc^{1/3}}{N^{1/2}} \quad , \qquad (37)$$

where N and are evaluated at the wall ($N = N_w$).

Using Eq. (28) and the transformation (22) in Eq. (37), we find that

$$-\frac{1}{2}(j_e + j_i) = 0.663\,\frac{\rho_w^2\, r_p}{\rho_\delta}\left\{\frac{(du_\delta/dx)_0}{\rho_w \mu_w}\right\}^{1/2} Sc^{1/3} \qquad (38)$$

In obtaining this, we have set $u_\delta = x(du_\delta/dx)_0$.

The current-voltage characteristic is then given explicitly as

$$\frac{J}{en_\delta D_e} = 0.663 \frac{\rho_w^2}{\rho_\delta} \frac{(du_\delta/dx)_0}{\rho_w \mu_w}^{1/2} Sc^{1/3} + \psi_p/r_p \ln [(\frac{r_p}{\lambda_D})^{2/3} Re^{1/6}] \tag{39}$$

This formula differs from the corresponding one for the static plasma first by the factor $Re^{1/6}$ in the argument of the logarithm. Furthermore, it is not difficult to show for a spherical probe in the static plasma that

$$-\frac{1}{2}(j_e + j_i) = 1 \quad .$$

In the static plasma we see then that the current collected by a probe at the plasma potential is the random current flux, while in the flowing plasma, the current is given by the first term in Eq. (38). It should be pointed out that Eqs. (35) and (38) become invalid when the probe potential is much above the plasma potential such that $j_i - j_e \not\approx 0$.

VII. Discussion.

Within the framework of the continuum assumptions, the flow of an ionized gas over a biased body is analyzed. The strongest assumption in the present work lies in the use of the continuum fluid equations for the analysis of the sheath structure. For such a description to be meaningful, one must satisfy the following condition:

$$\left(\frac{\lambda_D}{r_p}\right)^{2/3} Re^{-1/6} \gg \frac{\ell}{r_p}$$

where ℓ is the typical mean free path in the system. However, since the problem for a body at floating potential is independent of the detailed structure of the sheath, the above mentioned restriction does not apply to the results obtained for the heat flux to the surface and for the charged particle distribution around a floating body.

We have also assumed that

$$\left(\frac{\lambda_D}{r_p}\right)^2 \ll \frac{1}{Re} \ll 1 \quad .$$

Under such an assumption, which is what one generally encounters in the laboratory, the whole flow field is divided into three regions:

1) Inviscid quasi-neutral.

2) Viscous (transport of mass, momentum, and energy) quasi-neutral.

3) Charge separation sheath. The body potential is taken to be moderate (between the plasma and floating potential) such that the sheath is thin and convection can be neglected.

The conduction current is found to vanish through any closed surface. This is true in the outer two regions because of the quasi-neutrality (even for unsteady flow). The statement is true in the sheath only if the convection and the time variation inside the sheath can be neglected.

As an example, the stagnation probe problem is worked out in detail. An explicit expression for the current-voltage characteristic is given (Eq. (33)) under the assumption that

$$\ln \left[\left(\frac{r_p^2}{\lambda_D} \right)^2 Re^{1/6} \right] >> 1 \quad .$$

As we have mentioned, an error of about 10% is expected in this formula.

The potential in the inviscid region is identically zero for the probe at the floating potential. Consequently, in contrast to the static plasma, the potential distribution is not necessarily monatomic throughout the whole flow field. This fact was first pointed out by Lam[3]. Also, at the plasma potential, the current collected by the probe is, in general, not the random flux current.

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