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MASSACHUSETTS INSTITUTE OF TECHNOLOGY

LINCOLN LABORATORY

AN APPROXIMATE METHOD FOR THE CALCULATION OF SCALAR SCATTERING BY TWO-DIMENSIONAL BODIES

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ABSTRACT

A method for approximating the solution of two-dimensional scalar scattering problems has been developed and tested. Results indicate that the technique (which involves the expansion of the field in terms of cylindrical wave functions) gives good results when wave length and body size are comparable.

This technical documentary report is approved for distribution.

Franklin C. Hudson

Franklin C. Hudson, Deputy Chief
Air Force Lincoln Laboratory Office

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I. INTRODUCTION

An approximate method for calculating scalar scattering by two-dimensional bodies has been developed and used in a program written for the 7094 computer to calculate scattering from several shapes. The method consists essentially of expanding the scattered field in terms of the first N of the cylindrical wave functions. Comparison of results obtained with an exact solution and an integral equation solution shows fair agreement. The technique gives satisfactory results for small values of L/λ (of the order of one or less), where L is the perimeter of the body and λ is the wavelength of the incident radiation, but solutions become increasingly unstable as the geometric optics region is approached.

II. THE METHOD

A plane wave propagating in the direction of the positive x-axis, $E_i = e^{ikx}$, $k = 2\pi/\lambda$, is incident on an arbitrary cylindrical body, B, with axis parallel to the z axis (Fig. 1). The scattered field, E_s, may be validly expanded on and outside an imaginary circumscribed cylinder, C, in terms of the cylindrical harmonics:

$$E_s(r, \theta) = \sum_{n=0}^{\infty} C_n e^{in\theta} H_n(kr), \tag{1}$$

where $H_n(kr)$ is the Hankel function of the first kind, and r and θ are the usual polar coordinates. The scattered field, of course, is not initially known on C, so that the determination of the coefficients C_n by satisfying boundary conditions on C is not possible. If the circumscribed cylinder C is contracted to its minimum radius, say to C', it may be possible that, under certain circumstances, Eq. (1) could be used to evaluate the scattered field between B and C' with small error. In this case values of the C_n calculated by satisfying (approximately) the given boundary conditions on the surface of the scatterer may be satisfactory for determining values of the far field.

Dirichlet boundary conditions on the scatterer were employed; that is, it was specified that the total field on the surface of the scattering body be zero. It was assumed that the first N terms of the expansion Eq. (1) could be used to give a close approximation to the scattered field E_s in the neighborhood of the scatterer surface. Thus, at the surface of B the total field could be approximated by

$$E = E_i + E_s \approx e^{i\,kr\cos\theta} + \sum_{n=o}^{N} C_n e^{in\theta} H_n(kr). \qquad (2)$$

Under the given boundary conditions the left side of Eq. (2) should be equal to zero, and thus the C_n's were to be determined in such a manner as to minimize the right side of (2), in some sense, on the boundary. The obvious choice was the least squares criterion, and this was used. Thus the C_n's were determined in such a way as to minimize the mean square value of the total field on the scatterer boundary. This is given, of course, by

$$\bar{E}_N = \frac{1}{L}\int_B \left| e^{i\,kr\cos\theta} + \sum_{n=o}^{N} C_n e^{in\theta} H_n(kr)\right|^2 d\ell, \qquad (3)$$

where the line integral is over the boundary of B.

Taking the partial derivatives of $\bar{E}_n$ with respect to the C_n ($= a_n + i\,b_n$), and setting these equal to zero, a system of linear equations for the a_n and b_n is set up:

$$\left.\begin{aligned} \alpha_{pj} a_j + \beta_{pj} b_j &= \gamma_p \\ \beta_{pj} a_j - \alpha_{pj} b_j &= \delta_p \end{aligned}\right\} \quad (p = 0, 1, \ldots, N), \qquad (4)$$

where

$$\alpha_{pj} = \int_B (J_j J_p + N_j N_p) \cos j\theta \cos p\theta \, d\ell,$$

$$\beta_{pj} = \int_B (J_j N_p - N_j J_p) \cos j\theta \cos p\theta \, d\ell,$$

$$\gamma_p = -\int_B (J_p \cos(kr\cos\theta) + N_p \sin(kr\cos\theta)) \cos p\theta \, d\ell,$$

$$\delta_p = -\int_B (N_p \cos(kr\cos\theta) - J_p \sin(kr\cos\theta)) \cos p\theta \, d\ell,$$

and $J_n(kr) + i\,N_n(kr) = H_n(kr)$.

A computer program was written to evaluate these coefficients and solve Eqs. 4; results have been obtained for several scatterers.

III. RESULTS OF COMPUTATIONS

The method was first applied to a circular cylinder in which the origin of the cylindrical harmonics was offset from the cylinder axis to a point half-way between the axis and the leading edge of the cylinder, so that the surface of the cylinder was not in this case a coordinate surface in the coordinates employed. A scattering pattern was calculated for $ka = 2\pi a/\lambda = 2.0$, where a is the cylinder radius, and is shown in Fig. 2. Note the excellent agreement with the exact result. In this calculation the maximum order of the cylindrical harmonics employed was $N = 5$.

The backscattering cross section (σ) of the circular cylinder was calculated for several values of ka and for a number of values of N for each value of ka. The results are plotted in Fig. 3 along with the exact curve. At $ka = 1$ the values of σ calculated for four values of N were precisely the same and fall on the curve for the exact values. As ka increases the values of σ as calculated for different values of N begin to show a spread. At $ka = 6$, for example, the spread is about $\pm$ 20 percent of the exact value. In each case, however, there is one value of N for which the calculated cross section is quite close to the exact value. This does not occur, in general, for the largest value of N.

Scattering patterns for nose-on incidence for a 12.5° (wedge half-angle) wedge-cylinder shape (the two-dimensional analog of the cone sphere) have been calculated for values of ka (where a is the cylinder radius) of 1.93, 3.85, and 5.00. These are shown in Fig. 4. Also shown is the result of a calculation for $ka = 5.00$ done for Lincoln Laboratory by TRG, Inc. Their technique involves the numerical solution of the integral equation for the scattered field.

The agreement between the TRG solution and our solution is fair in directions near forward scattering, while it becomes poor near the backscattering aspects where the cross section is small.

IV. DISCUSSION OF THE METHOD

The method of "least squares" used above in computing the scattered scalar field from cylinders and wedge-cylinder shapes is an approximate scheme which was undertaken with the purpose of enhancing our understanding of the scattering phenomenon. In particular, we asked ourselves the following question: To what extent can the cylindrical harmonics describe the scattered field near the body when the boundary is not a level surface in polar coordinates?

We can add to our knowledge of the phenomenon of scattering and its mathematical description by explaining why our method works as well as it does. We base our argument on the following two facts, and a third fact to be added later.

(i) For the $12\ 1/2^{\circ}$ half angle wedge-cylinders being considered the number of cylindrical harmonics required to obtain a good description of the far field is $N \approx L/\lambda = k\,L/2\pi$, where L is the perimeter of the body.

(ii) The first zero of $H_n(kr)$ occurs when $kr \approx n$ (Watson), or $r \approx n/k$. Thus, the first zero of $H_N(kr)$ occurs at $r \approx L/2\pi$, since $k\,L/2\pi \approx N$. For $n < N$, the first zero of $H_n(kr)$ is within the circle of radius $L/2\pi$, for $n > N$, outside. Also, for $kr < n$, $H_n(kr) \sim 1/(kr)^n$, and for $kr > n$, $H_n(kr) \sim e^{ikr}/\sqrt{kr}$.

It will be noted that the properties of the Hankel function referred to in (ii) and the physically observed facts of (i) both make statements about the special number $N = k\,L/2\pi$. Using these two facts, we will arrive at an explanation of the degree of success of our method.

Referring to Fig. 1, we choose the center of coordinates so that as much of the scattering surface as possible is approximately $L/2\pi$ from the origin. (As a simple example, suppose the scatterer is a circular cylinder of radius $L/2\pi$. The obvious choice for the origin is at the center, since then every point on the surface is exactly $L/2\pi$ from the origin.)

Next, let s denote arc length along the surface and consider the way in which the incident wave (and therefore the scattered wave, if we have Dirichlet boundary conditions) varies on the boundary. To study this, we make the following observation.

(iii) If the angular measure for a significant change in the scattered field on the surface is less than $2\pi\lambda/L$, it is because the radius $r(\ell)$ is changing rapidly. (This is the case in the region of the tip, for example.) Furthermore, if the angular measure is less than $2\pi\lambda/L$, the linear measure cannot be greater than λ itself.

Now, we know from (i) that a good approximation for the scattered field outside a circle circumscribing the scatterer is given by

$$E_s(r, \theta) = \sum_{n=0}^{\sim\left[\frac{kL}{2\pi}\right]} C_n e^{in\theta} H_n(kr) \tag{5}$$

Our method is based on the assertion that this expression is a good approximation for the field on the surface $(r(\ell), \theta(\ell))$, even though the infinite series in general diverges.

To study this assertion and its limitations, we tentatively write

$$E_s(r(\ell), \theta(\ell)) \approx \sum_{n=0}^{\frac{kL}{2\pi}} C_n e^{in\theta(\ell)} H_n(kr(\ell)), \tag{6}$$

where ℓ is length measured on the surface. For those points on the surface where $r \geq L/2\pi$, if $n < kL/2\pi$, then $H_n(kr)$ is, according to (ii), oscillating with wave length λ. Note that this is true for each n. For those points on the surface where $r < L/2\pi$, $H_n(kr)$ has the oscillatory behavior only for $n < kr_m$, where r_m is the minimum value of r(s). For $kr_m < n < kL/2\pi$, $H_n(kr)$ behaves like $1/(kr)^n$ for $r < L/2\pi$. Thus

$$Q_n(\ell) = e^{in\theta(\ell)} H_n(kr(\ell)) \tag{7}$$

behaves qualitatively like $e^{in\theta}$ for $n < kr_m$, when $r(\ell)$ is slowly varying, and like an oscillating function with wavelength λ when $r(\ell)$ is changing rapidly. For $r_m < n < kL/2\pi$, this is true when $r(\ell)$ is slowly varying, but if it is rapidly varying, it has the $1/kr(\ell)$ factor.

Thus when r(s) is slowly varying, we are essentially expanding $E_s(r(\ell), \theta(\ell))$ in a Fourier series, and, according to FACT (iii), the number of Fourier components needed is of the order $kL/2\pi$, which agrees exactly with the assumed upper limit in Eq. (6) If $r(\ell)$ is rapidly varying the Fourier components are satisfactory up to $n = kr_m$, because the oscillatory behavior can, to a certain extent, be cancelled out by slight changes in each of the terms; or if such oscillatory behavior is needed to describe a fluctuation of wavelength λ (see iii) small adjustments in the terms will provide the required behavior. However, for $k\, r_m < n < k\, L/2\pi$, a rapidly varying $r\,(\ell)$ spoils the use of $\phi_n(\ell)$ as modified Fourier components, because for each n, the dependence on r is different (i.e. $1/(kr)^n$ instead of oscillatory, see ii), and so no cancellation can be made, nor can an oscillation with wavelength λ be achieved, if required. This last sentence describes the situation which is probably the major source of error in the method.

The above simple reasoning, admittedly a little awkward to word, explains not only why the results have value, but also (and just as important) the limitations of the method.

V. ADDITIONAL COMMENT

Any method which computes surface values must have a high degree of accuracy if k, the wave number, is large, if the results are to be used for backscatter calculations. Thus, even if the method herein described gives reasonably accurate boundary values, the backscattered field will be disproportionately increasing in error as k L increases.

REFERENCES

1. G. N. Watson, A Treatise on the Theory of Bessel Functions, (Cambridge University Press, Cambridge, 1944)

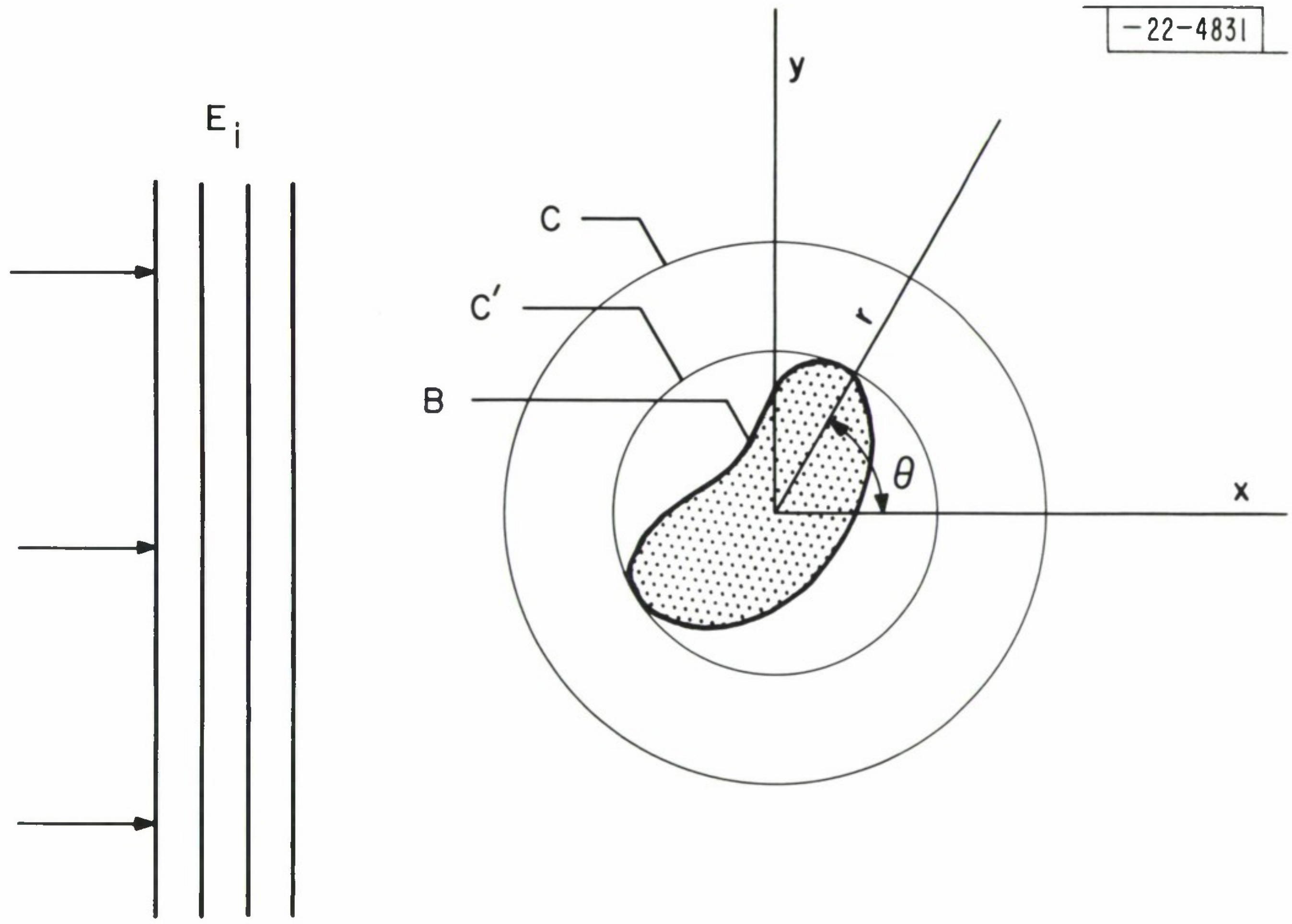

Figure 1. Scattering Body

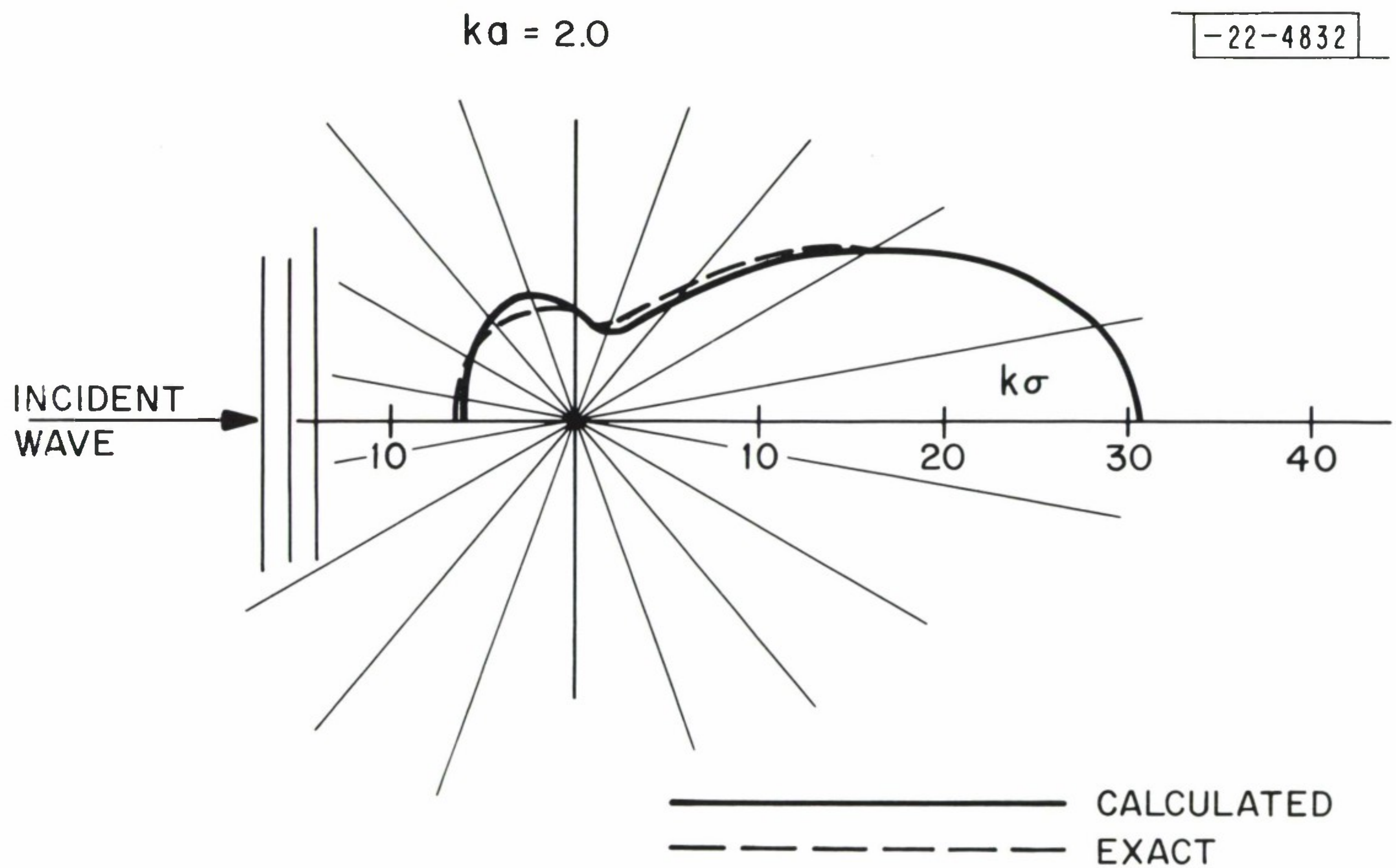

Figure 2. Scattering Pattern, Circular Cylinder, ka=2

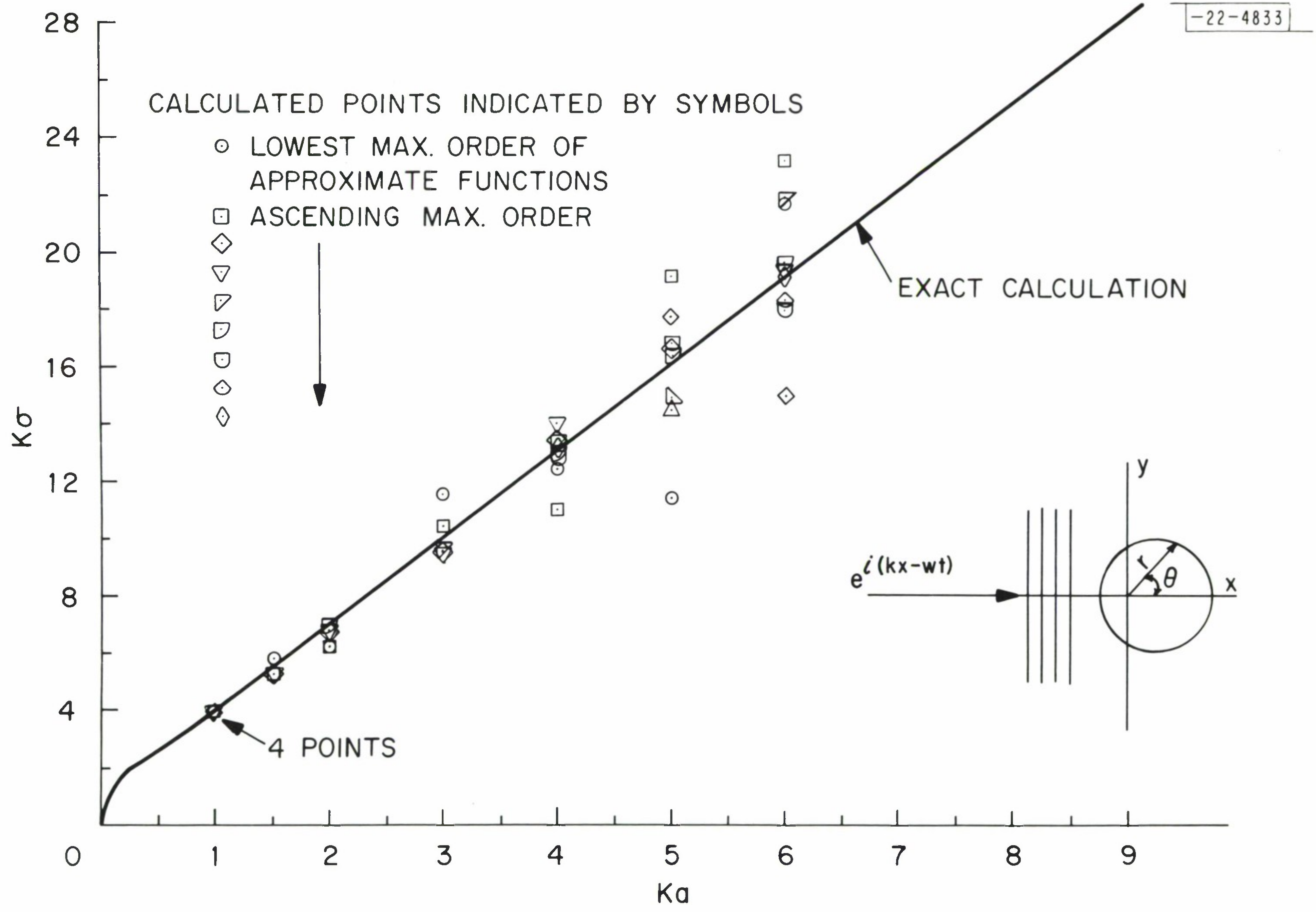

Figure 3. Backscattering from Circular Cylinder

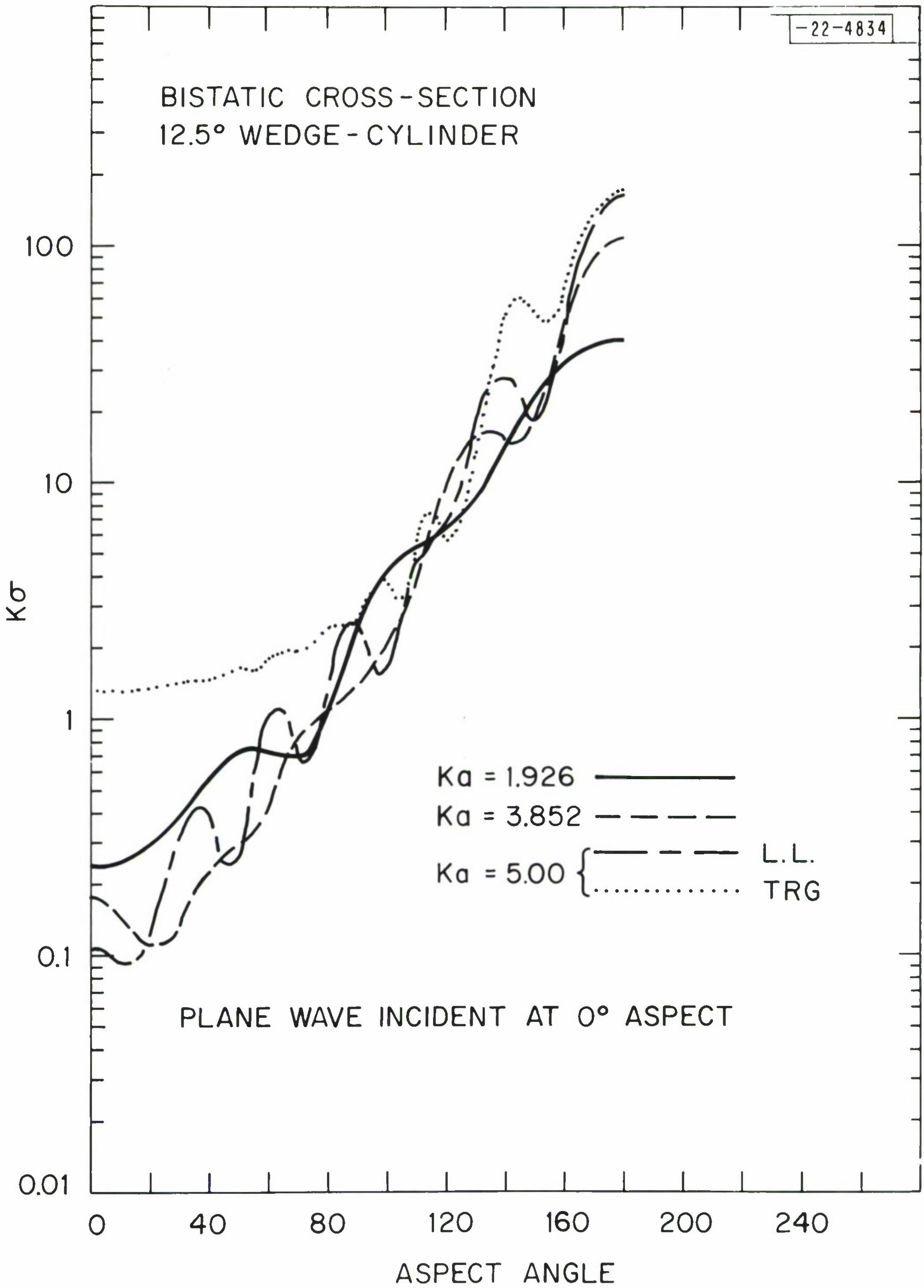

Figure 4. Scattering Patterns, 12.5° Wedge-Cylinder