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CAVITY FLOW AROUND CAMBERED HYDROFOILS

AHMED EL NIMR and BYRNE PERRY

This research was carried out under the
Bureau of Ships Fundamental Hydromechanics
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ONR Contract Nonr-225(56)

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ABSTRACT

An analytical study is made of the flow past hydrofoils with large camber at arbitrary angles of attack. The class of hydrofoils considered is restricted to those whose hodograph plane has a shape very near that bounded by two circular arcs. Under this assumption a general method is derived to calculate the parameters of the flow. In addition, a short-cut procedure is introduced for estimating the lift and drag coefficients. As a numerical example to illustrate the theory the flow past a circular-arc foil is considered. The force coefficients are in good agreement with the values computed by Rosenhead.

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1. INTRODUCTION. If a well-streamlined body is moved through water at a sufficiently high speed, cavitation will occur and become more severe as the speed increases. Because of the noise and structural damage which ordinarily ensue, elaborate precautions are often taken to avoid cavitation. Experience shows, however, that at the design speeds proposed for present-day hydrofoil craft, cavitation is, practically speaking, unavoidable. Moreover, if the zone of cavitation becomes so large that a large vapor cavity is formed and remains attached to the body (supercavitating flow), the flow regime is completely changed and damage is often avoidable. The point of view now taken is to assume that cavitation must occur and to design hydrofoils, struts, and appendages accordingly.

In regard to lifting hydrofoils and propellers, the analytical work of Tulin (1953) on supercavitating foils has been of special significance because of the comparative simplicity with which design parameters can be estimated, provided that the camber and attack angle are small. The recent extension of Tulin's theory by Chen (1962) to include second-order terms has increased the range of usefulness to intermediate values of camber and attack angle (up to 15 or 20 degrees, say). Chen's theory, which again is remarkably devoid of computational difficulties, should suffice for the great bulk of practical cases. For certain special configurations, however, it will be necessary to have in hand a theory which applies to arbitrary foil shapes having large attack angle, large camber, or a combination of both. Moreover, it is essential to have available a more inclusive theory against which to check the limits of application of the approximations of Tulin and Chen.

If the angle of attack is large but the camber is small, an excellent approximate theory might be developed by perturbing about the classical solution for a flat plate at arbitrary incidence. The structure of this theory is accordingly presented in Appendix A; however, no numerical results have as yet been worked out.

To develop a more general analytical method, one would naturally turn to the non-linear theory of Levi-Civita (presented, for example, by Milne-Thompson, 1960). The method has been applied by Brodetsky (1923), Rosenhead (1928), and Wu (1956) to a limited variety of foil sections, but the complexity of the calculations leaves something to be desired. More recently, Wu (1962) has devised a non-linear theory which overcomes most of the difficulties (Wu and Wang, 1963). However, in general, there appears to be justification for developing alternative lines of attack on the general problem for two reasons. First, each theory will have an inherent advantage in calculating a certain class of foils; for example, the method of Wu should be especially powerful in dealing with hydrofoils with flaps.

Second, the various schemes all being in some sense approximate, a chance is afforded of maintaining more control on the accuracy by checking the result of one analysis against another.

In the present work, therefore, a theory is developed to deal with a certain class of hydrofoils having rather large camber and arbitrary angle of attack. The method of analysis is such that the estimates of lift and drag are expected to be very good if the foil has a smoothly curving wetted camber line, and a hodograph plane nearly bounded by two circular arcs. The efficacy of the method is illustrated by comparison with some numerical results given by Rosenhead (1928). It almost goes without saying that a thorough review and comparison of the several methods now available would be most welcome.

2. HYDROFOILS WITH ZERO CAVITATION NUMBER. Let a curved hydrofoil abc be set at an angle α in an infinite stream with uniform velocity U_∞ and a pressure p_∞ at infinity (Figure 1a). For sufficiently high values of U_∞ the flow will separate at both edges of the foil leaving a cavity inside. If the pressure inside p_c is equal to p_∞, then the cavitation number will be equal to zero. The cavitation number σ is defined by the formula

$$\sigma = \frac{p_\infty - p_c}{\frac{1}{2}\rho \ U_\infty^2}$$

in which ρ is the fluid density. In this case the free streamlines cI and aI will extend to infinity. Take cx to be the x-axis and the perpendicular at c as the y-axis. For plane irrotational flow the complex potential W can be defined as

$$W = \varphi + i\psi \tag{1}$$

Here φ is the velocity potential and ψ is the stream function. If point b is taken as the origin for the W-plane, it will be as shown in Figure 1b. The complex velocity ω is defined to be

$$\omega = -\frac{1}{U_\infty}\frac{dW}{dz} = \frac{u}{U_\infty} - i\,\frac{v}{U_\infty} \tag{2}$$

Here u and v are the velocity components in the x and y directions respectively. The flow is assumed to be steady and the pressure in the cavity to be constant; consequently, the magnitude of the velocity on the free streamlines aI and cI is constant and equal to U_∞. Thus these streamlines are represented by a portion of the circumference of a unit circle in the ω-plane (Figure 1c). The velocity components at points c and a are given by

$$u_c = -U_\infty(1 + \alpha_c^2)^{-1/2}, \quad v_c = -\alpha_c U_\infty(1 + \alpha_c^2)^{-1/2} \tag{3}$$

$$u_a = + U_\infty(1 + \alpha_a^2)^{-1/2}, \qquad v_a = - \alpha_a U_\infty(1 + \alpha_a^2)^{-1/2} \tag{4}$$

in which α_c and α_a are the absolute values of the slope at these points. Point b is a stagnation point. For a smoothly curving foil the part abc will be represented in the ω-plane by a smooth continuous curve passing through the three known points a, b and c. The analysis will use a circular arc passing through the same points as a basis to perturb around. This choice, which is admittedly arbitrary, leads to a base flow for which the conformal mappings are given in terms of elementary functions.

3. METHOD OF ANALYSIS. If a relation between the potential W and the velocity ω can be found by conformal mapping, the physical plane z can be constructed from (2) by integration. The first step of mapping is the rotation of the ω-plane through an angle β where β is the angle made by the radius of the circular arc at point b and the u-axis. Thus

$$\omega^* = \omega e^{-i\beta} \tag{5}$$

It is convenient to map the ω^*-plane onto a new half plane $\zeta^* = \xi^* + i\eta^*$. The relation between the ω^*- and

ζ^*-planes is taken to be of the form

$$\omega^* = \frac{\omega_c^* - g(\zeta^*)\,\omega_a^*}{1 - g(\zeta^*)} \tag{6}$$

where

$$g(\zeta^*) = \frac{1}{\lambda}\left[\frac{M - N}{\zeta^*} + N\right]^{\kappa/\pi} \tag{7}$$

$$\lambda = -i\left[\frac{\omega_a^*}{\omega_a^* - \omega_c^*}\right] \tag{8}$$

$$M = \left[\lambda\,\frac{\omega_c^*}{\omega_a^*}\right]^{\pi/\kappa} \tag{9}$$

$$N = \left[\lambda\,\frac{\omega_I^* - \omega_c^*}{\omega_I^* - \omega_a^*}\right]^{\pi/\kappa} \tag{10}$$

In these expressions κ is the angle of intersection between the circular arcs cIa and abc.

The ζ^*-plane will be as shown in Figure 1d. The circular arc abc in the hodograph plane will be represented by the line segment

$$0 \le \xi^* \le T, \quad \eta = 0$$

where the constant T is defined by

$$T = \frac{N - M}{N} = 1 - \frac{M}{N}$$

The curve abc which represents the hydrofoil will deviate from that segment as shown. The free streamlines aI and cI will be represented respectively by the portions $\xi^* \le 0$ and $T \le \xi^* \le \infty$ of the ξ^*-axis.

Now define a half plane $\zeta = \xi + i\eta$ with the position of points a,b,c the same as those in the ζ^*-plane. Assume the transformation from the ζ^*-plane to the ζ-plane to be of the form

$$\zeta^* = \zeta + \epsilon(\zeta) \tag{11}$$

Here $\epsilon(\zeta)$ is an unknown complex analytic function of ζ that can be separated into real and imaginary parts as follows

$$\epsilon(\xi,\eta) = X(\xi,\eta) + iY(\xi,\eta) \tag{12}$$

Since both ζ^* and ζ are real on the lines aI and cI, then the imaginary part in equation (12) has to vanish on those lines, or

$$Y(\xi,0) = 0 \qquad \begin{array}{c} T \le \xi \le \infty \\ -\infty \le \xi \le 0 \end{array} \tag{13}$$

Thus $Y(\xi,\eta)$ is the imaginary part of an analytic function and hence satisfies the differential equation

$$Y_{\xi\xi} + Y_{\eta\eta} = 0 \qquad \eta \geq 0 \tag{14}$$

Assume that on the part abc, Y will be represented in the general Fourier series form

$$Y(\xi,0) = \sum_{n=0}^{\infty}\left[a_n \sin(nT\pi\xi) + b_n \cos(nT\pi\xi)\right]$$

$$0 \leq \xi \leq T \tag{15}$$

The boundary-value problem given by equations (13), (14) and (15) has the solution

$$Y(\xi,\eta) = \frac{1}{\pi}\int_0^{\infty}\int_0^{T}\left[\sum_{n=0}^{\infty}\{a_n \sin(nT\pi s) + b_n \cos(nT\pi s)\}\right]\cdot$$

$$\cdot\left[\cos\{t(s-\xi)\}\right]ds\ e^{-t\eta}\ dt \tag{16}$$

The analyticity of $\epsilon(\zeta)$ implies that

$$Y_\eta = X_\xi \tag{17}$$

From equations (16) and (17) the relation for $X(\xi,\eta)$ is found to be

$$X(\xi,\eta) = \frac{1}{\pi}\int_0^\infty \int_0^T \left[\sum_{n=0}^{\infty}\{a_n \sin(nT\pi s) + b_n \cos(nT\pi s)\}\right] \cdot \left[\sin\{t(s-\xi)\}\right] ds\, e^{-t\eta}\, dt \tag{18}$$

Relations (16) and (18) completely define $\epsilon(\zeta)$ in terms of the unknown constants a_n and b_n.

By the theorem of Schwarz and Christoffel, the complex potential is mapped onto the same ζ-plane through the transformation

$$W = \varphi_a(\zeta - 1)^2 \tag{19}$$

where φ_a is the velocity potential at point a. From the definition of ω given by (2) and (5) one finds that

$$\frac{dz}{d\zeta} = -\frac{e^{-i\beta}}{U_\infty} \cdot \frac{1}{\omega^*}\frac{dW}{d\zeta} \tag{20}$$

A substitution for ω^* and for $\frac{dW}{d\zeta}$ by differentiating (11) leads to the integral

$$z = -\frac{2\varphi_a\ e^{-i\beta}}{U_\infty} \int_{\zeta_c}^{\zeta} \frac{(\zeta - 1)}{F(\zeta)}\, d\zeta \tag{21}$$

Here

$$F(\zeta) = \frac{\omega_c^* - g(\zeta)\ \omega_a^*}{1 - g(\zeta)} \tag{22}$$

$$g(\zeta) = \frac{1}{\lambda}\left[\frac{(M - N)}{N\{\zeta + \epsilon(\zeta)\}} + N\right]^{\kappa/\pi} \tag{23}$$

The right-hand side of equation (21) involves the function $\epsilon(\zeta)$ which is determined in terms of the unknown constants a_n and b_n. To evaluate the first m of these constants use can be made of a method similar to that introduced by Naiman as summarized by Abbott and Von Doenhoff (1959). Divide the arc abc in both the z- and ζ-planes into $(m + 1)$ equal segments. Make a substitution for each point at the end of every segment. For example, at the end of segment number k the equation will be

$$x_k + iy_k = -\frac{2\varphi_a e^{-i\beta}}{U_\infty} \int_{\zeta_c}^{\zeta_c k/(m+1)} \frac{(\zeta - 1)}{F(\zeta)} d\zeta \qquad (24)$$

$$k = 1, 2, \ldots, m$$

Thus one obtains $2(m+1)$ equations involving the constants $a_0, a_1, \ldots, a_m, b_0, b_1, \ldots, b_m$. Solving these equations simultaneously will lead to the numerical value of each of these constants. The number m can be chosen as large as the accuracy requires.

The determination of the constants defines ω^* and hence ω completely. The net pressure at any point of the lamina is given by

$$p = \frac{1}{2}\rho U_\infty^2 \{1 - |\omega(\xi)|^2\} \qquad (25)$$

where $|\omega(\xi)|$ is the absolute value of $\omega(\xi)$ computed from (5).

The components of the total force on the lamina are found by integrating equation (25):

$$F_x = \frac{1}{2}\rho U_\infty^2 \int_0^T \{1 - |\omega(\xi)|^2\} \left|\frac{dy}{d\xi}\right| d\xi \qquad (26)$$

$$F_y = \frac{1}{2}\rho U_\infty^2 \int_0^T \{1 - |\omega(\xi)|^2\} \left|\frac{dx}{d\xi}\right| d\xi \qquad (27)$$

Thus the lift coefficient C_L will be given by

$$C_L = \frac{2(F_y \cos\alpha - F_x \sin\alpha)}{\rho U_\infty^2 B} \tag{28}$$

Similarly the drag coefficient C_D can be found from the relation

$$C_D = \frac{2(F_x \cos\alpha + F_y \sin\alpha)}{\rho U_\infty^2 B} \tag{29}$$

4. HYDROFOILS WITH FINITE CAVITATION NUMBER. In the case that the pressure p_c inside the cavity is different from p_∞ the free streamlines aI and cI seem to close at a finite distance and the length of the cavity will be no longer infinite. In this case the velocity on the free streamline V will be related to U_∞ by the relation

$$\frac{1}{2}\rho V^2 = (p_\infty - p_c) + \frac{1}{2}\rho U_\infty^2 \tag{30}$$

The mathematical model to be used here (Figure 2a) is the same as the one introduced by Wu (1962) for cavity flow around flat plates. Two curved plates dI and d'I are supposed to extend to infinity where the pressure and velocity on both plates change from p_c and V at points d and d' to p_∞ and U_∞ as they reach points I. In general the plates have a curvature that is unknown at the outset. The complex potential (Figure 2b) and the complex velocity planes are drawn with the positions of points d and d' as indicated. For convenience, however, in this case the complex velocity is defined as

$$\omega = -\frac{1}{V}\frac{dW}{dz} = \frac{u}{V} - i\,\frac{v}{V} \qquad (31)$$

Define the ζ_1-plane to be

$$\zeta_1 = \left[- i\,\frac{\omega_a^*}{\omega_a^* - \omega_c^*} \cdot \frac{\omega^* - \omega_c^*}{\omega^* - \omega_a^*}\right]^{\pi/\varkappa} \qquad (32)$$

where ω^* has the same definition given by (5). Choose the plates dI and d'I to be of such a shape to map onto a vertical slit as shown in Figure 2d. By means of the Schwarz-Christoffel theorem, the ζ_1-plane can be transformed to a half plane μ (Figure 2e). In general the shape abc will not transform into the axis as indicated, but must be mapped according to an additional transformation similar to the procedure of the foregoing section. The W-plane and μ-plane are also easily related by the method of Schwarz and Christoffel.

5. A RAPID METHOD FOR ESTIMATING LIFT AND DRAG COEFFICIENTS.

The method introduced here is a means to get a fast estimate for the lift and drag coefficients based on the assumption that the free streamline for the foil of interest is very close to that computed from the basic flow for which the hodograph boundary consists of two circular arcs. In effect a correction is made on the wetted foil surface while the proper boundary condition on the free streamlines is ignored.

First, assume that

$$\omega^* = \left[\frac{\omega_c^* - g(\zeta)\,\omega_a^*}{1 - g(\zeta)}\right] + \tau(\zeta) \tag{33}$$

on the curve abc in the ζ-plane and, second, assume that

$$\tau(\xi,0) = \sum_{-\infty}^{\infty} a_m \exp\left[\frac{2\pi m i \xi}{\xi_c}\right] \tag{34}$$

From the above two relations and the definition of W and ω the physical plane is described by the formula

$$z = -\frac{2\varphi_a e^{-i\beta}}{U_\infty}\left[\int_{\zeta_c}^{\zeta}\frac{(\zeta - 1)}{F(\zeta)}\,d\zeta - \sum_{-\infty}^{\infty} a_m \int_{\zeta_c}^{\zeta}\frac{(\zeta - 1)\ \exp\{2\pi m i\zeta/\zeta_c\}}{F^2(\zeta)}\,d\zeta + \right.$$

$$\left. + \int_{\zeta_c}^{\zeta}\frac{(\zeta - 1)\left[\sum_{-\infty}^{\infty} a_m\ \exp\{2\pi m i\zeta/\zeta_c\}\right]^2}{F^3(\zeta)}\,d\zeta + \ldots\right] \qquad ($$

By substituting for ζ and z at corresponding points the above relation yields a set of simultaneous equations for the complex constants a_m. Substitution of these constants in (34) will define ω^* along abc. The lift and drag coefficients may then be formed form (26) and (27).

As an example of the method consider the hydrofoil to be a circular arc subtending an angle $\pi/3$ set in the stream at an angle α which varies from 0 to $\pi/2$. The cavitation number is zero. The velocity components at points a and c will be

$$\omega_a^* = \frac{1}{2} - \frac{\sqrt{3}}{2}i, \qquad \omega_c^* = \frac{1}{2} + \frac{\sqrt{3}}{2}i$$

The values of the constants λ and M are

$$\lambda = \frac{1}{3\sqrt{3}} - \frac{i}{2}, \qquad M = -\frac{1}{3\sqrt{3}}$$

Table I gives the values of the constant N for selected values of α. The constants a_m have been evaluated by matching the conditions from equation (35) at ten points along the surface of the foil. The resulting values for the force coefficients are given in Table II. Figure 3 shows that the agreement with the computations of Rosenhead is very satisfactory.

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APPENDIX A

PERTURBATION OF THE FLOW PAST A FLAT PLATE

In the method of Tulin (1953) the flow about a supercavitating foil is regarded as a perturbation of a uniform flow. Thus both the attack angle and camber are restricted. An alternative scheme would use the cavity flow about a flat plate as a basis for perturbation. Such a scheme is outlined here, up to terms of the second order. The camber must be in some sense small, but there is no limitation on angle of attack.

A.1 METHOD OF ANALYSIS. Let the cambered foil a*b*c* be set such that the chord a*c* makes an angle α with the direction of the flow as shown in Figure 4. Let a*c* be considered as the x-axis, and the line perpendicular at point a* be the y-axis. Assume that the maximum deflection of the foil is h and the chord length is B. Let the smallness parameter δ be defined by

$$\delta = h/B << 1$$

It is now assumed that the stream function Ψ satisfies the Laplace equation

$$\Psi_{xx} + \Psi_{yy} = 0 \tag{A.1}$$

and moreover that it can be expressed in the form

$$\Psi(x,y) = \psi_o(x,y) + \delta\psi_1(x,y) + \delta^2\psi_2(x,y) + \dots \qquad (A.2)$$

in which $\psi_o(x,y)$ is the stream function if the foil is replaced by a flat plate ac. Also let the ordinate y for the lamina a*b*c* be given by the relation

$$y = \delta\eta(x)$$

The first objective of the analysis is to formulate and solve the appropriate boundary value problems for the determination of $\psi_o(x,y)$ and the first- and second-order perturbation terms $\psi_1(x,y)$ and $\psi_2(x,y)$. Since δ is for the moment arbitrary, it follows from (A.1) and (A.2) that ψ_o, ψ_1, ψ_2, ... satisfy the Laplace equation. Determination of the boundary conditions must now be considered. Assign for the streamlines I*b*a*I*, I*b*c*I* the values $\Psi = 0$. Then on a*b*c* we will have the condition

$$\Psi\{x,\delta\eta(x)\} = \psi_o\{x,\delta\eta(x)\} + \delta\psi_1\{x,\delta\eta(x)\}$$
$$+ \delta^2\psi_2\{x,\delta\eta(x)\} + \dots = 0$$

Expanding the terms in the right hand side by means of a Taylor series leads to the relation

$$\psi_o(x,0) + \delta\eta(x)\,\psi_{oy}(x,0) + \tfrac{1}{2}\delta^2\eta^2(x)\,\psi_{oyy}(x,0) + \dots +$$
$$+ \delta\psi_1(x,0) + \delta^2\eta(x)\,\psi_{1y}(x,0) + \dots +$$

$$+ \delta^2 \psi_2(x,0) + \dots = 0$$

By equating the terms with like powers of δ we obtain the relations

$$\psi_o(x,0) = 0 \tag{A.3}$$

$$\eta(x)\,\psi_{oy}(x,0) + \psi_1(x,0) = 0 \tag{A.4}$$

$$\tfrac{1}{2}\eta^2(x)\,\psi_{oyy}(x,0) + \eta(x)\,\psi_{1y}(x,0) + \psi_2(x,0) = 0 \tag{A.5}$$

Let x_o, y_o represent the values of x,y on the free surface for the case of a flat plate, as indicated in Figure 4. Assume that the coordinates x,y of the free streamline for the cambered foil $a^*b^*c^*$ to be given in the form

$$y = y_o,$$

$$x = x_o + \delta\xi_1(y) + \delta^2\xi_2(y) + \dots \tag{A.6}$$

Using the fact that the free surface is a streamline, we obtain the relation

$$\Psi\{x_o + \delta\xi_1(y) + \delta^2\xi_2(y) + \ldots, y_o\} = \psi_o\{x_o + \delta\xi_1(y) + \delta^2\xi_2(y) + \ldots, y_o\}$$

$$+ \delta\psi_1\{x_o + \delta\xi_1(y) + \delta^2\xi_2(y) + \ldots, y_o\} + \delta^2\psi_2\{x_o + \delta\xi_1(y) + \delta^2\xi_2(y) + \ldots, y\}$$

$$+ \ldots = 0$$

An expansion by Taylor series as before leads to the following boundary conditions

$$\psi_o(x_o, y_o) = 0 \tag{A.7}$$

$$\xi_1(y)\ \psi_{ox}(x_o, y_o) + \psi_1(x_o, y_o) = 0 \tag{A.8}$$

$$\xi_2(y)\ \psi_{ox}(x_o, y_o) + \tfrac{1}{2}\xi_1^2(y)\ \psi_{oxx}(x_o, y_o)$$

$$+ \xi_1(y)\psi_{1y}(x_o, y_o) + \psi_2(x_o, y_o) = 0 \tag{A.9}$$

One more condition on the free surface is that the resultant velocity V_c is constant, or

$$V_c^2 = \left[\frac{\partial\Psi\{x + \delta\xi_1(y) + \delta^2\xi_2(y) + \dots, y\}}{\partial x}\right]^2$$

$$+ \left[\frac{\partial\Psi\{x + \delta\xi_1(y) + \delta^2\xi_2(y) + \dots, y\}}{\partial y}\right]^2$$

A substitution for $\partial\Psi/\partial x$ and $\partial\Psi/\partial y$ from (A.2) leads to the following results:

$$V_c^2 = \Big[\psi_{oxx}(x_o,y_o) + \delta\xi_1(y)\ \psi_{oxx}(x_o,y_o) + \delta^2\xi_2(y)\ \psi_{oxx}(x_o,y_o)$$

$$+ \tfrac{1}{2}\delta^2\xi_1^2(y)\ \psi_{xxx}(x_o,y_o) + \dots + \delta\psi_{1x}(x_o,y_o) + \delta^2\xi_1(y)\ \psi_{1xx}(x_o,y_o)$$

$$+ \delta^2\psi_{2x}(x_o,y_o) + \dots\Big]^2$$

$$+ \Big[\psi_{oy}(x_o,y_o) + \delta\xi_1(y)\ \psi_{oyx}(x_o,y_o) + \delta^2\xi_2(y)\ \psi_{oyx}(x_o,y_o)$$

$$+ \tfrac{1}{2}\delta^2\xi_1(y)\ \psi_{oyxx}(x_o,y_o) + \dots + \delta\psi_{1y}(x_o,y_o) + \delta^2\xi_1(y)\ \psi_{1yx}(x_o,y_o)$$

$$+ \dots + \delta^2\psi_{2y}(x_o,y_o) + \dots\Big]^2 \qquad \text{(A.10)}$$

Equating terms with like powers of δ as before results in the relations

$$V_c^2 = (\psi_{ox})^2 + (\psi_{oy})^2 \tag{A.11}$$

$$\xi_1(y)(\psi_{ox}\psi_{oxx} + \psi_{oy}\psi_{oyx}) + \psi_{ox}\psi_{1x} + \psi_{oy}\psi_{1y} = 0 \tag{A.12}$$

$$\begin{aligned} &\xi_1^2(y)\psi_{oxx} + \psi_{1x}^2 + 2\xi_2(y)\psi_{ox}\psi_{oxx} + \xi_1^2(y)\psi_{ox}\psi_{oxxx} + \\ &+ 2\xi_1(y)\psi_{ox}\psi_{1xx} + 2\psi_{ox}\psi_{2x} + \xi_1^2(y)\psi_{oyx}^2 + \\ &+ 2\xi_2(y)\psi_{oy}\psi_{oyx} + \xi_1^2(y)\psi_{oy}\psi_{oyxx} + \\ &+ 2\xi_1(y)\psi_{oy}\psi_{1yx} + 2\psi_{oy}\psi_{2y} = 0 \end{aligned} \tag{A.13}$$

In these expressions the derivatives of the stream function terms such as ψ_{ox}, ψ_{1x}, ..., are to be evaluated on the unperturbed free streamline x_o, y_o. Conditions (A.3), (A.7) and (A.13) together with the differential equation

$$\psi_{oxx} + \psi_{oyy} = 0 \tag{A.14}$$

formulate the boundary value problem for the zero order term $\psi_o(x,y)$. The solution for this problem for zero cavitation number was given by Rayleigh (see Lamb, 1932) using the hodograph method. Since we are going to use this solution in a different form for the development of the first- and second-order terms, we will present it in the following section.

A.2 SOLUTION FOR THE ZERO ORDER TERM. The physical plane z, the complex potential plane W and the dimensionless complex velocity plane ω have the same definition as before and are shown in Figure 5.

The new plane W^{-1} is also shown in the same figure. Define the half plane t to be

$$t = -(1/\omega + \omega) \tag{A.15}$$

By using the Schwarz-Christoffel theorem the W^{-1} is mapped onto the half plane t and thus the relation between t and W is given by the formula

$$W = \frac{C}{(t - 2\cos\alpha)^2} \tag{A.16}$$

where C is a constant dependent on the dimensions of the lamina. A substitution in (A.15) with the value of ω and use of Eq. (A.16) leads to the relation

$$t = \frac{-2C}{U_\infty(t - 2\cos\alpha)^3}\frac{dt}{dz} - \frac{U_\infty(t - 2\cos\alpha)^3}{2C}\frac{dz}{dt}$$

or we get the following differential equation to relate the z and t planes:

$$\left(\frac{dz}{dt}\right)^2 + \frac{2Ct}{U_\infty(t - 2\cos\alpha)^3}\left(\frac{dz}{dt}\right) + \frac{4C^2}{U_\infty^2(t - 2\cos\alpha)^6} = 0 \tag{A.17}$$

The above has the following solution:

$$z = \frac{C}{U_\infty}\int_2^t \frac{-t + \sqrt{t^2 - 4}}{(t - 2\cos\alpha)^3}\,dt \tag{A.18}$$

where the complex constant C is determined by

$$\frac{1}{C} = \frac{-1}{BU_\infty}\int_{-2}^{2} \frac{-t + \sqrt{t^2 - 4}}{(t - 2\cos\alpha)^3}\,dt \tag{A.19}$$

Substitution for t from (A15) results in the integral

$$z = \frac{C}{U_\infty}\int_{-1}^{\omega} \frac{\omega^6 - \omega^4 - \omega^2 + 1}{\omega\{\omega^2 - (2\cos\alpha)\,\omega + 1\}^3}\,d\omega \tag{A.20}$$

Similarly by substitution for t in (A.16) we find W in terms of ω, thus

$$W = \frac{C\omega^2}{\{\omega^2 + (2\cos\alpha)\,\omega + 1\}^2} \tag{A.21}$$

Since $W = \varphi_o + i\psi_o$, one sees that

$$\psi_o(x,y) = \mathrm{Img}\left(\frac{\sqrt{C}\omega}{\omega^2 + (2\cos\alpha)\,\omega + 1}\right)^2 \tag{A.22}$$

The relations (A.20) and (A.21) give z and W in terms of the parameter ω and hence a complete solution for the zero order term is accomplished.

A.3 SOLUTION FOR HIGHER ORDER TERMS. Consider now the first order term ψ_1 which is represented by the following boundary-value problem:

$$\psi_{1xx} + \psi_{1yy} = 0 \tag{A.23a}$$

$$\eta(x)\,\psi_{1y}(x,0) + \psi_1(x,0) = 0 \tag{A.23b}$$

$$\xi_1(y)\,\psi_{ox}(x_o,y_o) + \psi_1(x_o,y_o) = 0 \tag{A.23c}$$

$$\xi_1(y)\left[\psi_{ox}(x_o,y_o)\,\psi_{oxx}(x_o,y_o) + \psi_{oy}(x_o,y_o)\,\psi_{oyx}(x_o,y_o)\right] + \psi_{ox}(x_o,y_o)\,\psi_{1x}(x_o,y_o) + \psi_{oy}(x_o,y_o)\,\psi_{1y}(x_o,y_o) = 0 \tag{A.23d}$$

The last two conditions can be combined into the condition:

$$-\psi_{1}(x_{o},y_{o})\left[\psi_{ox}(x_{o},y_{o})\,\psi_{oxx}(x_{o},y_{o}) + \psi_{y}(x_{o},y_{o})\,\psi_{oxx}(x_{o},y_{o})\right]$$

$$+\psi_{ox}(x_{o},y_{o})\left[\psi_{ox}(x_{o},y_{o})\,\psi_{1x}(x_{o},y_{o}) + \psi_{oy}(x_{o},y_{o})\,\psi_{1y}(x_{o},y_{o})\right] = 0 \qquad (A.24)$$

Due to the complexity of the z-plane, we will use Eq. (A.20) to map our domain into the semi-circular hodograph plane:

$$\omega = \omega_{1} + i\omega_{2} = u + iv$$

The last notation differs from that of the main section of the report. In what follows it is convenient to introduce the abbreviations listed below:

$$P = \psi_{ou}u_{x} + \psi_{ov}v_{x}$$

$$Q = \psi_{ou}u_{y} + \psi_{ov}v_{y}$$

$$R = \psi_{1u}u_{x} + \psi_{1v}v_{x}$$

$$S = \psi_{1u}u_{y} + \psi_{1v}v_{y}$$

The previous boundary value problem now assumes the form:

$$\psi_{1uu} + \psi_{1vv} = 0 \tag{A.25a}$$

$$\eta(u,0)\left[\psi_{1u}(u,0)\,\frac{\partial u}{\partial y} + \psi_{1v}(u,0)\,\frac{\partial v}{\partial y}\right] + \psi_1(u,0) = 0 \tag{A.25b}$$

$$\xi_1(u,v)P + \psi_1(u,v) = 0 \tag{A.25c}$$

$$\xi_1(u,v)\left\{ P\left[P_u u_x + P_v v_x\right] + Q\left[Q_u u_y + Q_v v_y\right] + PR + QR\right\} = 0 \tag{A.25d}$$

The last two boundary conditions are valid on the circumference of the semi-circle in the hodograph plane and, as before, can be combined into a single formula:

$$-\psi_1(u,v)\left\{ P\left[P_u u_x + P_v v_x\right] + Q\left[P_u u_y + P_v v_y\right]\right\} + P\left[PR + QR\right] = 0 \tag{A.26}$$

The derivatives u_x, v_x, u_y, v_y are determined from Eq. (A.20). It is obvious that this linear boundary value problem can be solved numerically in the hodograph plane by taking as small a mesh as the accuracy requires.

By a similar procedure the formulas for the second-order term $\psi_2(x,y)$ are developed and are found to be:

$$\psi_{2uu} + \psi_{2vv} = 0 \qquad \text{(A.27a)}$$

$$\frac{1}{2}\eta^2(u,v)\left[u_y Qu + v_y Q_v\right] + \eta(u,v)S + \psi(u,0) = 0 \qquad \text{(A.27b)}$$

$$\xi_2(u,v)P + \frac{1}{2}\xi_1^2(u,v)\left[u_x P_u + v_x P_v\right] + \xi_1(u,v)Q + \psi_2(u,v) = 0 \qquad \text{(A.27c)}$$

$$\begin{aligned}
&\xi_1^2(u,v)\left[u_x P_u + v_x P_v\right] + R^2 + 2\xi_2(u,v)P\left[u_x P_u + v_x P_v\right] + \\
&+ \xi_1^2(u,v)P\left[u_x(u_x P_u + v_x P_v)u + v_x(u_x P_u + v_x P_v)v\right] + \\
&+ 2\xi_1(u,v)P\left[u_x P_u + v_x P_v\right] + 2P\left[u_x\psi_{2u} + v_x\psi_{2v}\right] + \\
&+ \xi_1^2(u,v)\left[u_y P_u + v_y P_v\right]^2 + 2\xi_2(u,v)Q\left[u_y P_u + v_y P_v\right] + \\
&+ \xi_1^2(u,v)Q\left[u_y(u_x P_u + v_y P_v)u + v_y(u_x P_u + v_x P_v)v\right] + \\
&+ 2\xi_1(u,v)P\left[u_y R_u + v_y R_v\right] + 2Q\left[u_y\psi_{2u} + v_y\psi_{2v}\right] = 0 \qquad \text{(A.27d)}
\end{aligned}$$

In principle the above problem can also be done numerically in the semi-circular hodograph plane. The equations for the interior points of the mesh will be the same but those for the boundary points will have to satisfy the conditions in (A.27) instead of (A.23).

The relation for the free streamline is given in (A.6) where $\xi_1(y)$ and $\xi_2(y)$ are to be determined from Eqs. (A.8) and (A.9). The actual velocity components U and V at any point are computed from the formulas:

$$U = -\frac{\partial \Psi}{\partial y} = -\left(\frac{\partial \psi_0}{\partial y} + \delta \frac{\partial \psi_1}{\partial y} + \delta^2 \frac{\partial \psi_2}{\partial y}\right)$$

$$V = \frac{\partial \Psi}{\partial x} = \left(\frac{\partial \psi_0}{\partial x} + \delta \frac{\partial \psi_1}{\partial x} + \delta^2 \frac{\partial \psi_2}{\partial x}\right)$$

The local pressure at any point on the lamina may then be computed by use of Bernoulli's equation, and the total force obtained by an integration.

APPENDIX B

LIST OF SYMBOLS

a_n, a_m	Fourier coefficients
B	Chord length
C_D	Drag coefficient
C_L	Lift coefficient
$F(\zeta)$	Function defined by (22)
F_x, F_y	Force components
$g(\zeta)$	Function defined by (23)
H	Maximum camber of foil
M	Constant defined by (9)
N	Constant defined by (10)
p_c	Pressure inside the cavity
p_∞	Pressure at infinity
t	Complex variable
T	Constant defined by $T = 1 - M/N$
U_∞	Velocity magnitude at infinity
u, v	Velocity components
V	Velocity magnitude on the free streamline
W	Complex potential $W = \varphi + i\psi$
x, y	Cartesian coordinates
z	Physical plane, $z = x + iy$

α	Angle of attack
β	Angle through which hodograph is rotated
δ	Smallness parameter, $\delta = H/B$
$\epsilon(\zeta)$	Complex variable $\epsilon(\zeta) = X + iY$
ζ	Complex variable, $\zeta = \xi + i\eta$
ζ_1	Complex variable
κ	Angle of intersection of circle arcs in the approximate hodograph plane
λ	Constant defined by (8)
μ	Complex variable
ρ	Fluid density
σ	Cavitation number
τ	Function defined by (34)
φ	Velocity potential
ψ	Stream function
ω	Dimensionless complex velocity defined by (2)
ω^*	Complex variable defined by (5)

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TABLE I. VALUES OF N

α	N
0°	0.19253
30°	0.49074
60°	1.54024
90°	10.00725

TABLE II. LIFT AND DRAG COEFFICIENTS

Attack Angle α	Lift Coefficient C_L		Drag Coefficient C_D	
	Present Theory	Rosenhead	Present Theory	Rosenhead
0	0.8081	---	0.0406	---
30°	.7091	0.7042	.4109	0.4038
40°	---	.6620	---	.5484
50°	---	.5784	---	.6796
60°	.5132	---	.8620	---
70°	---	.3226	---	.8740
90°	0	0	.9531	.9438

TABLE III. VALUES OF THE COEFFICIENTS a_m

m	$\alpha = 0^\circ$	$\alpha = 30^\circ$	$\alpha = 60^\circ$	$\alpha = 90^\circ$
-4	.07707	.03602	-.00606	-.00102
-3	.02856	-.19926	-.05532	-.03836
-2	.13751	-.03389	.00209	.00225
-1	.17162	-.16834	-.03658	-.05085
0	-.23252	-.07860	.00392	.00428
1	.054419	-.20105	-.05209	-.03745
2	-.030412	-.08845	.00541	.00302
3	-.116138	-.22033	-.04130	-.05387
4	.04903	-.07925	.00384	.00117
5	-.07337	-.11827	-.05260	-.04211

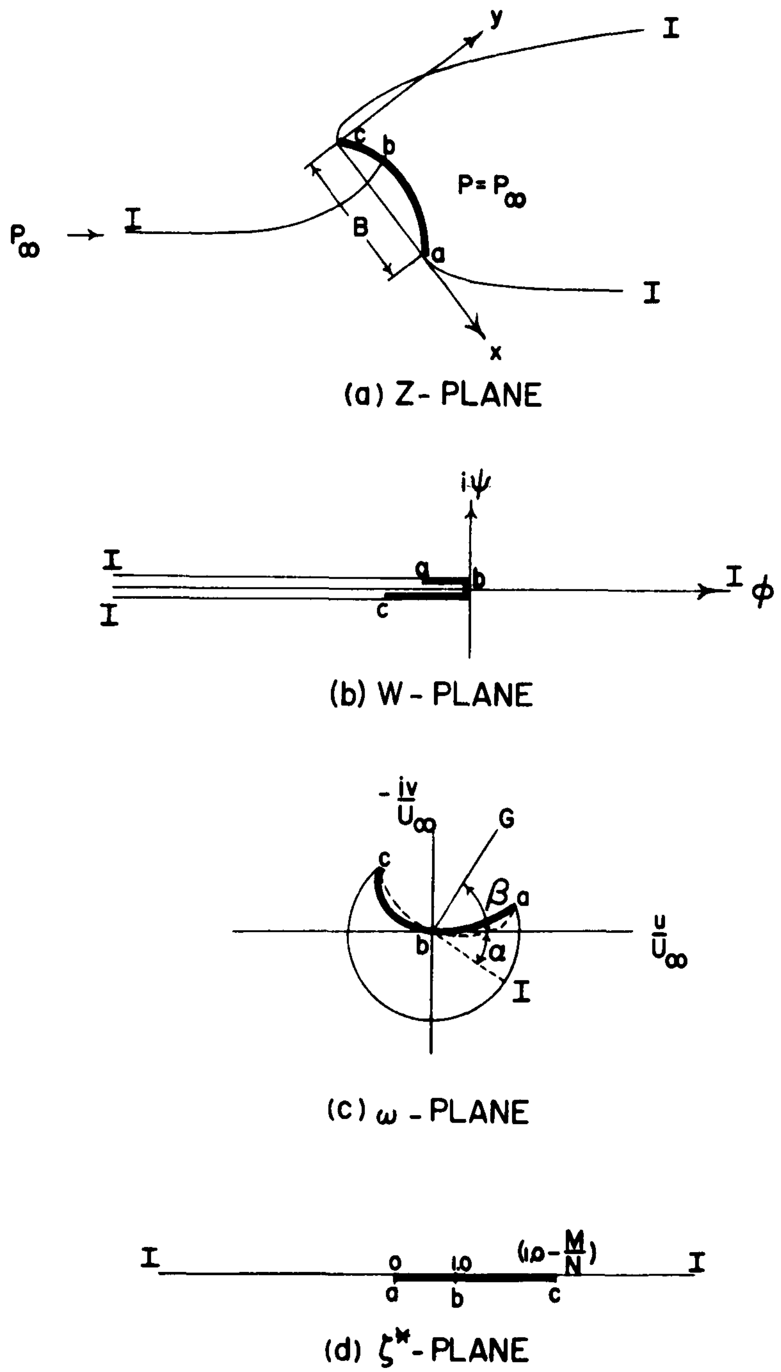

Figure 1. Conformal mapping planes for cambered hydrofoil at zero cavitation number.

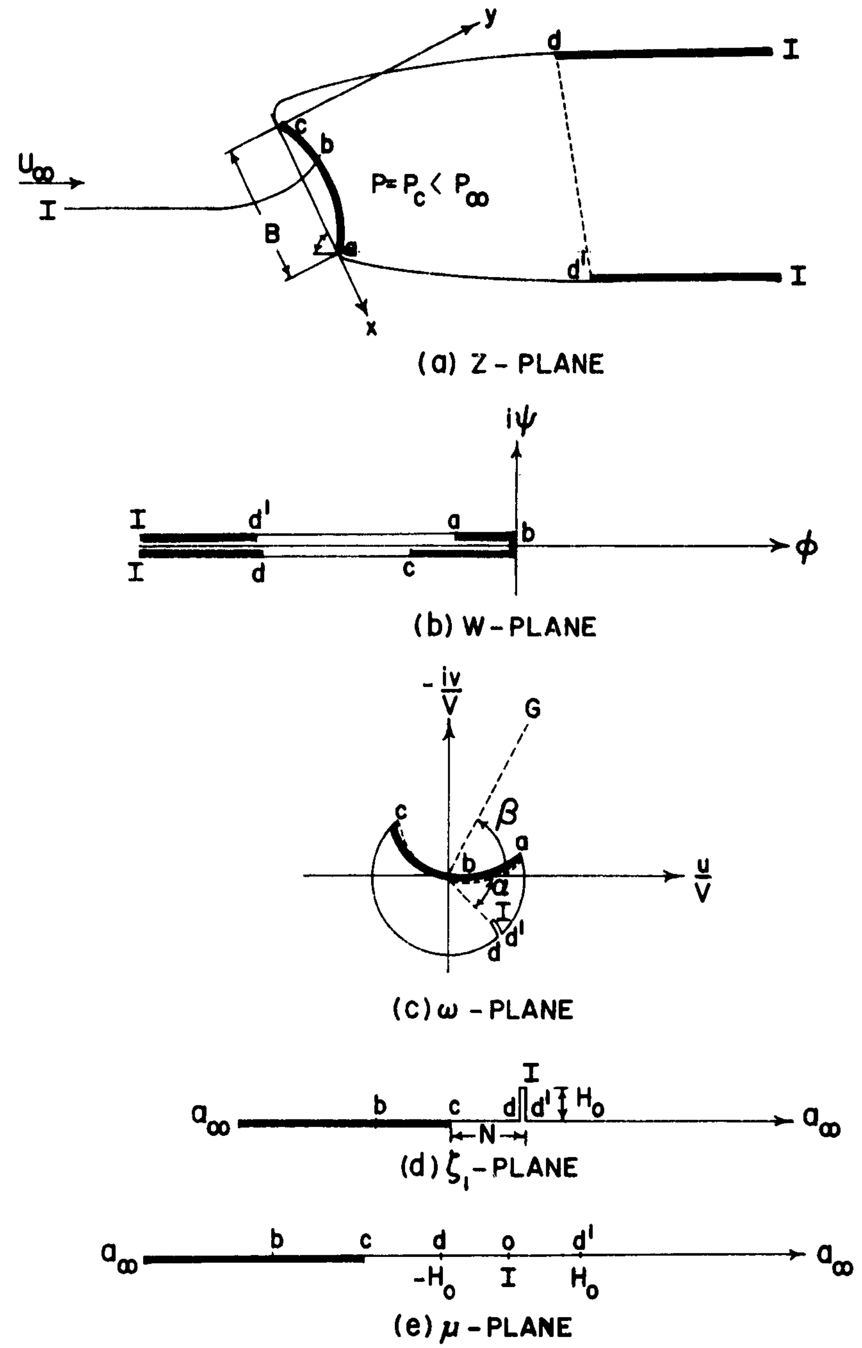

Figure 2. Conformal mapping planes for hydrofoil at finite cavitation number.

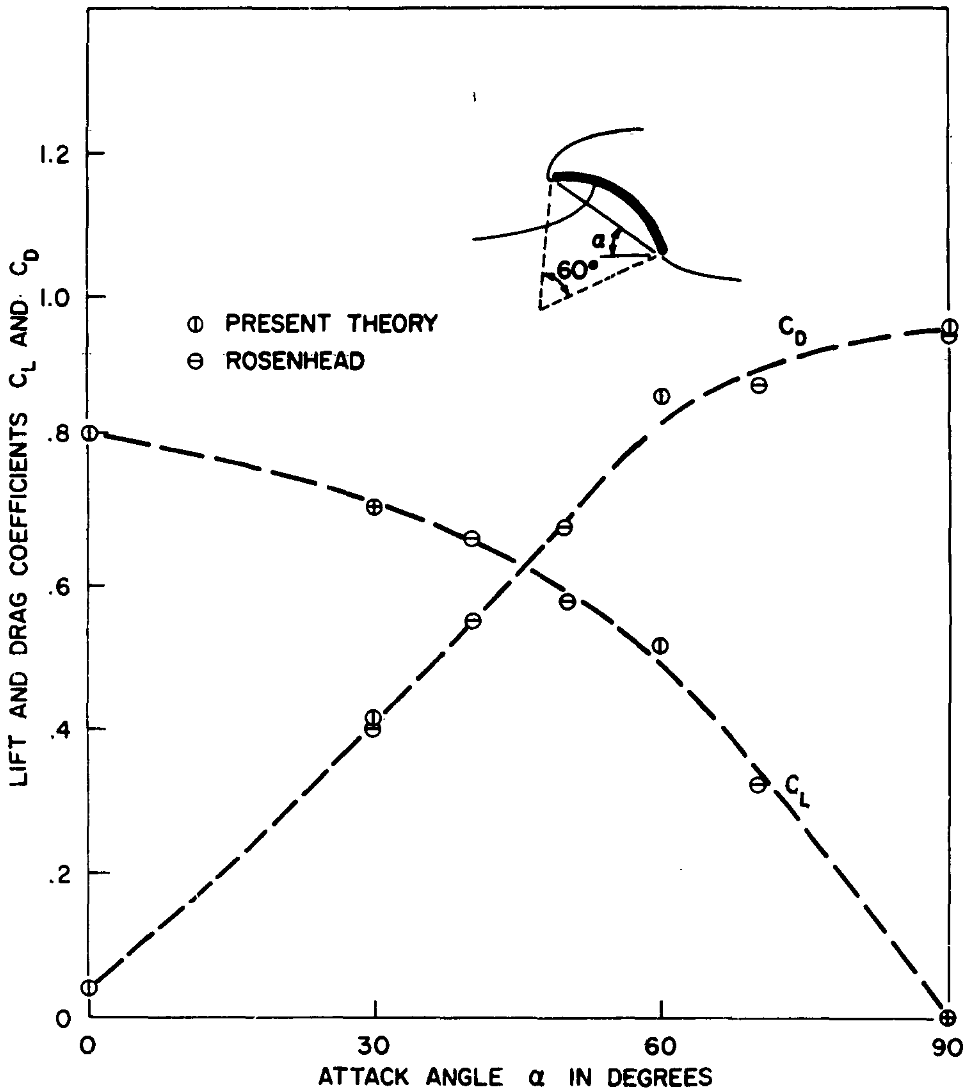

Figure 3. Comparison of present theory with that of Rosenhead (1928).

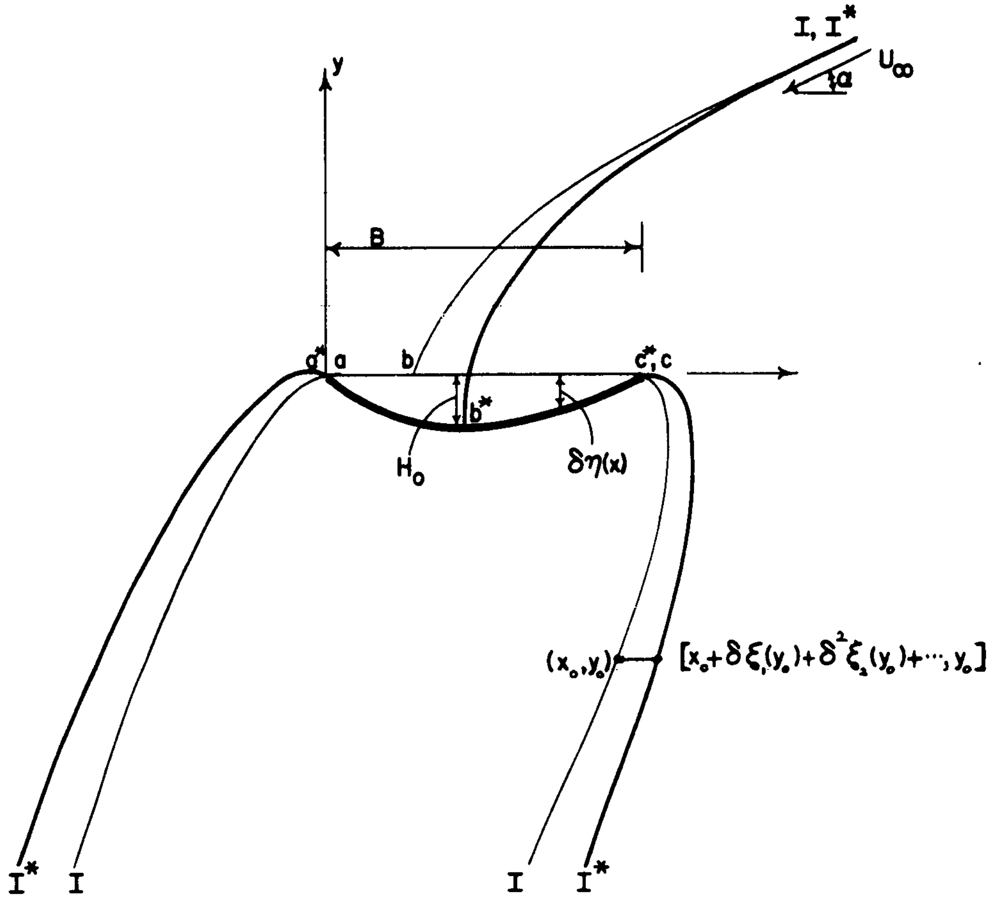

Figure 4. Cambered foil considered as a perturbation of the flow past a flat plate.

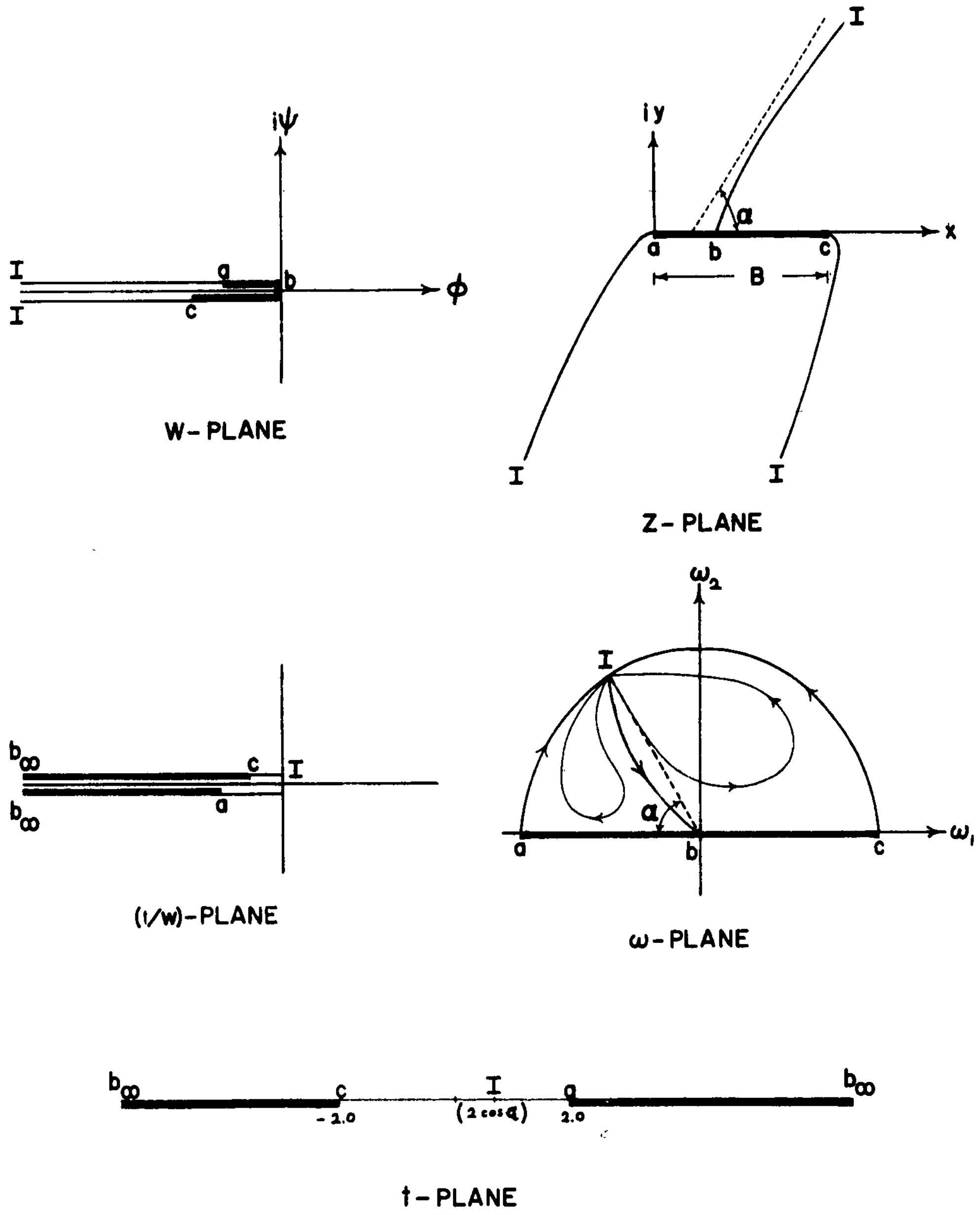

Figure 5. Conformal mapping planes for flat plate at zero cavitation number.

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