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A MODEL FOR THE POWER SPECTRUM OF RETURNED ECHOES FROM A RANDOM COLLECTION OF MOVING SCATTERERS

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A MODEL FOR THE POWER SPECTRUM OF RETURNED ECHOES FROM A RANDOM COLLECTION OF MOVING SCATTERERS

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FOREWORD

This report was prepared by the Department of Electrical Engineering of the University of Southern California under USAF Contract No. AF 04(695)-304. This contract was initiated under Project No. 3182, 730F University Program, Task No. 02, "Detection and Tracking Systems Studies and Hypervelocity Impact Problems." The work was administered under the direction of Space Systems Division, Air Force Systems Command with Captain R. D. Eaglet acting as project officer.

ABSTRACT

The importance of designing a radar receiver to operate effectively in the presence of noise which is correlated with the signal creates the necessity for developing a theoretical model of the clutter which can be used for the evaluation of various detection techniques.

Kelley and Lerner in reference 1 have developed one such model. Reed, in reference 2, has developed another model. This report presents a refinement of Reed's original model for the clutter noise process.

The clutter cloud is considered to be a collection of individual point scatterers moving about and reflecting signals independently of one another. The individual motions of the scatterers are considered to be independent of one another, but the cloud is allowed to have an overall drift velocity.

The analysis technique is the following: An RF pulse is transmitted at $t = 0$. The wavefront is reflected from the scatterers at the distances r_k. If we neglect the multiple scattering effect, then the returned echo consists of many replicas of the transmitted waveform beginning at times $t_k = \frac{2r_k}{c}$, where c is the velocity of propagation in the medium. A probability density that a scatterer will be at range r and time t is assumed. Since the scatterers are independent, the arrival of an echo at time t does not affect the probability of an echo arrival at other times; thus the echo arrivals constitute a nonstationary Poisson process with rate $\nu(t)$.

The calculations proceed from this point to determine the nonstationary expected value and covariance function for the clutter noise

process. The process is then approximated as stationary in order to determine the power spectrum.

SUMMARY

The time-varying correlation function for a radar (sonar) back-scattered noise process is determined. An approximation is made in order to obtain an expression for the power spectrum of the noise process. The correlation function and the power spectrum are both a function of the probability density that a scatter will be at range r and time t as well as the signal pulse shape. This implies that in order to minimize the noise power the signal may not be wideband but is affected by the probability distribution of the scatterers.

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I. INTRODUCTION

In order that a radar (or sonar) receiver may be properly designed to operate in an "optimum" fashion in the presence of clutter (reverberation) it is necessary to determine the statistical properties for the back-scattered noise process (which will be correlated with the signal reflection to be detected).

The statistical properties for the noise process that are analyzed in this report are the time varying correlation function and the power spectrum (for the steady state case). Since it would be difficult to determine the power spectrum for a noise-like returned echo train for a particular scattering cloud, the problem is idealized at the outset. A refinement to Reed's first order approximation (reference 2) to the power spectrum of the returned process is analyzed. This process is applicable for a wide class of possible scattering clouds.

The results of the present analysis are an extension of the results obtained by Reed. Reed assumed that the set of distributed targets is large in extent and is composed of a homogeneous random collection of individual scatterers which are possibly moving about independently of one another with possibly an overall drift. The effects of multiple scattering are neglected in order to obtain a tractable answer. In addition the scatterers are assumed to have small velocities so that an echo from a particular scatterer can be regarded as a doppler-shifted replica of the transmitted waveform. The result Reed obtained for the stationary correlation function is given in the conclusions of this report, along with the new results of this report. A comparison of the two results is also made in the conclusion.

The next section discusses the assumptions pertinent to the analysis and the refinements of the first order approximation obtained by Reed before determining the time-varying correlation function of the noise process.

II. REFINEMENT OF THE MODEL AND ITS ANALYSIS

The following assumptions are similar to Kelly and Lerner in reference 1 as is the method of calculation. The pattern of the calculations follows that of Reed in reference 2.

While most of the following assumptions and comments have been discussed in references 1 and 2 they will be repeated here for completeness.

The refinement of the model consists of

1. the inclusion of the three dimensional aspects of the problem as well as the statistical structure of the reverberation (clutter) cloud,
2. a coordinate system centered at the transmitter of the sonar (radar),
3. the inclusion of the transmitter antenna gain function, $G(\theta, \emptyset)$, normalized to unity in the direction of maximum gain (θ is the depression from the vertical, $\emptyset$ is azimuth).

The effects of scatterer rotation as well as multiple scattering will be ignored.

Following Reed (reference 2), if the transmitted signal is given by $f(t) = s(t)\, e^{2\pi i f_o t}$ where f(t) is the real transmitted signal plus i times its Hilbert transform (reference 3, p. 320), then the returned echo from a collection of scatterers at $(r, \theta, \emptyset)$ as a function of time over an observation interval (0, T) is given by

$$z(t) = \beta \sum_{t_k \varepsilon (0,\ T)} \frac{G(\theta_k, \emptyset_k)}{r_k^2}\, a_k\, s(t-t_k) \exp i \left[2\pi f_k (t-t_k) + \delta_k + \varepsilon(\theta_k, \emptyset_k) \right] \tag{1}$$

where

β includes such factors as RF gain, antenna matching, etc.

a_k is the amplitude reflection coefficient

$f_k = \left(1 + \frac{2v_o}{c} + \frac{2v_k}{c} \right) f_o$ is the doppler frequency (introduced by Reed in reference 2)

v_o is the drift radial velocity of the system of scatterers

v_k is the differential radial velocity

δ_k is the phase shift due to the effects of reflection

$\varepsilon(\theta_k, \emptyset_k)$ is a phase shift due to the transmitting antenna beam phase characteristic

c is the velocity of sound (or light) in the medium.

The subscript k refers to the k^{th} scatterer.

The following statistical assumptions concerning the random values θ_k, f_k, $\emptyset_k$, a_k, and δ_k are made:

1. The random values are taken from a joint probability distribution where it is presumed that a scatterer is present at range

$r_k = \frac{ct_k}{2}$ at the time $\frac{t_k}{2}$.

2. The scatterers are statistically independent and identically distributed.

Kelly and Lerner have determined the probability density that a scatterer will be at range r and time t as $4\pi r^2 u(r, t)$:

$$u(r, t) = \frac{1}{4\pi} \iint p_s \sin\theta \, d\theta \, d\phi$$

$$p_s(x,y,z,t) = \iiint p(x,y,z, v_x, v_y, v_z, t) \, dv_x \, dv_y \, dv_z \qquad (2)$$

$p(x, y, z, v_x, v_y, v_z, t)$ statistically describes the cloud of scatterers and is the probability density that the point (x,y,z) is occupied at time t by a scatterer with velocity (v_x, v_y, v_z). p_s is the spatial density; i.e., the probability density that (x,y,z) is occupied at time t. $u(r, t)$ is obtained by converting to polar coordinates.

Kelly and Lerner also discuss a method of handling tangential velocity components. However, in the model considered here we have considered only the radial velocity components and converted these to a doppler frequency as done by Reed.

Using the above expression for the probability density that a scatterer will be found at range r at time $\frac{t}{2}$, the probability density that an echo will arrive at a time in the interval $(t, t + dt)$ is

$$v(t)\,dt = 4\pi r^2 u(r, \tfrac{t}{2})\,dr \qquad \text{where } r = \frac{ct}{2}$$

then

$$v(t) = \pi \frac{c^3}{2} t^2 u\left(\frac{ct}{2}, \frac{t}{2}\right) \qquad (3)$$

Following Reed we next assume the scatterers are Poisson distributed in range (time). It has already been noted that they are independent. The joint probability distribution for the Poisson process for exactly n events to occur in the intervals $(t_1, t_1 + dt_1) \ldots (t_n, t_n + dt_n)$ for this nonstationary case can be expressed as:

$$\prod_{k=1}^{n} \frac{\nu(t_k)}{k} \, e^{-\int_0^T \nu(t)\, dt} \, dt_k \tag{4}$$

where it has been assumed that the signal duration Δ is short compared with T, the observation interval, and we can neglect the end effects near 0 and T.

With these assumptions we may now determine the expected value of $z(t)$ as follows.

$$E\left[z(t)\right] = \beta \sum_{n=0}^{\infty} \int_0^T \!\!\cdots\!\! \int \prod_{k=1}^{n} \frac{\nu(t_k)}{k} \, e^{-\int_0^T \nu(t)\, dt} \, dt_k$$

$$\times \int_{-\infty}^{\infty} \!\!\cdots\!\! \int p(\vec{a})\, d\vec{a} \int_{-\infty}^{\infty} \!\!\cdots\!\! \int p(\vec{\theta})\, d\vec{\theta} \tag{5}$$

$$\times \int_{-\infty}^{\infty} \!\!\cdots\!\! \int p(\vec{\phi})\, d\vec{\phi} \int_{-\infty}^{\infty} \!\!\cdots\!\! \int p(\vec{\delta})\, d\vec{\delta} \int_{-\infty}^{\infty} \!\!\cdots\!\! \int p(\vec{f})\, d\vec{f}$$

$$\sum_{k=1}^{n} \frac{G(\theta_k, \phi_k)}{t_k^2} \, \alpha_k \, s(t-t_k) \exp i \left[2\pi f_k (t-t_k) + \delta_k + \epsilon(\theta_k, \phi_k) \right]$$

where

$$p(\vec{x})\, d\vec{x} = \prod_{i=1}^{n} p(x_i)\, dx_i \, ,$$

and a $\left(\frac{c}{2}\right)^2$ has been absorbed into β.

If we assume that each of the random variables is stationary except for range (or delay t) or vary slowly in time compared to the pulse repetition frequency, then $E[z(t)]$ can be expressed as follows:

$$E[z(t)] = \beta\, E(\alpha)\, \phi_\delta(1) \sum_{n=0}^{\infty} \int_0^T \cdots \int \prod_{k=1}^{n} \frac{\nu(t_k)}{k}\, e^{-\int_0^T \nu(t)\, dt}\, dt_k$$

$$\times \int_{-\infty}^{\infty} \cdots \int p(\vec{\theta})\, d\vec{\theta} \int_{-\infty}^{\infty} \cdots \int p(\vec{\emptyset})\, d\vec{\emptyset} \int_{-\infty}^{\infty} \cdots \int p(\vec{f})\, d\vec{f}$$

$$\times \sum_{k=1}^{n} \frac{G(\theta_k, \emptyset_k)}{t_k^2}\, s(t-t_k)\, e^{i[2\pi(t-t_k) f_k + \varepsilon(\theta_k, \emptyset_k)]} \tag{6}$$

where

$$\phi_\delta(1) = \left[\int_{-\infty}^{\infty} p(\delta)\, e^{i\delta u}\, d\delta \right]_{u=1} \tag{7}$$

= characteristic function of δ with $u=1$

and the a_k's are identically distributed.

If we let $L(\theta_k, \emptyset_k) = G(\theta_k, \emptyset_k)\, e^{i\varepsilon(\theta_k, \emptyset_k)}$ and assume that the L's are identically distributed then

$$E[z(t)] = \beta\, E(a)\, E(L)\, \phi_\delta(1) \sum_{n=0}^{\infty} \int_0^T \cdots \int \prod_{k=1}^{n} \frac{\nu(t_k)}{k}\, e^{-\int_0^T \nu(t)\, dt}\, dt_k$$

$$\times \int_{-\infty}^{\infty} \cdots \int p(\vec{f})\, d\vec{f} \sum_{k=1}^{n} \frac{s(t-t_k)}{t_k^2}\, e^{2\pi i f_k (t-t_k)} \tag{8}$$

The approach used in obtaining equation (8) differs from that used by Kelly and Lerner in that they make an assumption which eliminates the doppler shift effect. Consequently, they obtain results which differ considerably from the results which are derived in the following pages.

$$E\left[z(t)\right] = \beta\ E(a)\ E(L)\ \phi_\delta(1) \sum_{n=0}^{\infty} \frac{\left[\int_0^T \nu(t')\ dt'\right]^{n-1}}{n!}\ n\ e^{-\int_0^T \nu(t)\ dt}$$

$$\times \int_0^T \frac{\nu(t')}{t'^2}\ s(t-t') \left\{ \int_{-\infty}^{\infty} p(f)\ e^{2\pi i f(t-t')}\ df \right\} dt'$$

$$= \beta\ E(a)\ E(L)\ \phi_\delta(1) \sum_{n=0}^{\infty} \frac{\left[\int_0^T \nu(t')\ dt'\right]^{n-1}}{n!}\ n\ e^{-\int_0^T \nu(t)\ dt}$$

$$\times \int_0^T \frac{\nu(t')}{(t')^2}\ s(t-t')\ \Phi(t-t')\ dt'$$

$$= \beta\ E(a)\ E(L)\ \phi_\delta(1) \int_0^T \frac{\nu(t')}{(t')^2}\ s(t-t')\ \Phi(t-t')\ dt' \tag{9}$$

where

$$\Phi(t) = \int p(f)\ e^{2\pi i f(t)}\ df \tag{10}$$

= characteristic function of $p(f)$.

If $\nu(t)$ is replaced by $\pi \frac{c^3}{2} t^2 u\left(\frac{ct}{2}, \frac{t}{2}\right)$ then

$$E\left[z(t)\right] = \beta\ E(a)\ E(L)\ \phi_\delta(1) \int_0^T \pi \frac{c^3}{2} u\left(\frac{ct'}{2}, \frac{t'}{2}\right) s(t-t')\ \Phi(t-t')\ dt' \tag{11}$$

where $u(r,t)$ is the probability density of finding a scatterer at (r, t). Let $f(t-t') = s(t-t')\ \Phi(t-t')$, $\alpha = \beta\ E(a)\ E(L)\ \phi_\delta(1)\ \pi \frac{c^3}{2}$

then

$$E\,[z(t)] = \alpha \int_0^T u\left(\frac{ct'}{2}, \frac{t'}{2}\right) f(t-t')\, dt'$$

$$= \alpha \int_0^T u\left(\frac{ct'}{2}, \frac{t'}{2}\right) f(t')\, dt' + \alpha \int_0^T u\left(\frac{ct'}{2}, \frac{t'}{2}\right)\Big[f(t-t') - f(t')\Big]\, dt'$$

$$\cong \alpha \int_0^T u\left(\frac{ct'}{2}, \frac{t'}{2}\right) f(t')\, dt' \qquad (12)$$

$$\text{provided} \quad f(t-t') - f(t') \cong 0 \qquad (13)$$

or the second integral is approximately zero.

Equation (13) establishes one functional relationship for $f(t)$ which may be useful later in determining an optimum signal pulse shape, $s(t)$.

If the interval $(0, T)$ is large, then $E[z(t)]$ can be approximated by

$$E[z(t)] = \alpha \int_{-\infty}^{\infty} u\left(\frac{ct}{2}, \frac{t}{2}\right) s(t)\, \Phi(t)\, dt \qquad (14)$$

This is a generalization of Cambell's theorem (see references 1 and 2).

In a similar manner we can show that the covariance function of $z(t)$ is

$$\operatorname{cov}\Big[\,\overline{z(t-\tau)}\; z(t)\Big] = E\Big[\,\overline{z(t-\tau)}\; z(t)\Big] - E\Big[\,\overline{z(t-\tau)}\Big]\, E\Big[\, z(t)\Big]$$

$$= |\beta|^2\, E(a^2)\, E(G^2)\, \Phi(\tau) \int_0^T \frac{\nu(t')}{(t')^4}\; \overline{s(t-\tau-t')}\;\; s(t-t')\, dt' \,. \qquad (15)$$

Again, replacing $v(t)$ by $\pi \frac{c^3}{2} \ t^2 \ u\left(\frac{ct}{2}, \frac{t}{2}\right)$ we have

$$\operatorname{cov}\left[\overline{z(t-\tau)}\ z(t)\right] = \gamma\ \phi(\tau) \int_0^T \frac{u\left(\frac{ct'}{2}, \frac{t'}{2}\right)}{(t')^2}\ \overline{s(t-\tau-t')}\ \ s(t-t')\ dt'$$

$$\gamma = |\beta|^2\ E(a^2)\ E(G^2)\ \pi\ \frac{c^3}{2} \qquad (16)$$

If we again extend the limits of integration from $-\infty$ to ∞ so that $z(t)$ is now stationary, then

$$R(\tau) = \operatorname{cov}\left[\ \overline{z(t-\tau)}\ z(t)\right] = \gamma\ \phi(\tau) \int_{-\infty}^{\infty} \frac{u\left(\frac{ct}{2}, \frac{t}{2}\right)}{t^2}\ \overline{s(t-\tau)}\ s(t)\ dt \qquad (17)$$

is the covariance function of the centered $z(t)$ complex noise (backscattering) process.

Now define the power spectrum $G(f)$ of $z(t)$ as the fourier transform of $R(\tau)$, then

$$G(f) = \int_{-\infty}^{\infty} e^{-2\pi i f\tau} R(\tau)\ d\tau \qquad (18)$$

$$S(f) = \int_{-\infty}^{\infty} s(t)\ e^{-2\pi i f t}\ dt \qquad (19)$$

as the signal spectrum. Then, by the convolution theorem,

$$G(f) = \gamma \int_{-\infty}^{\infty} p(f-f')\ H(f')\ \overline{S}(f')\ df' \qquad (20)$$

where $$h(t) = \frac{1}{t^2}\, u\left(\frac{ct}{2}, \frac{t}{2}\right) s(t)$$

$$H(f) = \int_{-\infty}^{\infty} h(t)\, e^{-2\pi i f t} dt = \int_{-\infty}^{\infty} U(f-f')\, S(f')\, df'$$

$$U(f) = \int_{-\infty}^{\infty} \frac{u\left(\frac{ct}{2}, \frac{t}{2}\right)}{t^2}\, e^{-2\pi i f t}\, dt\ .$$

It is to be noted that if $v(t)$ is proportional to t^4 then equations (15) and (20) reduce to the same result as that obtained by Reed (within a constant multiplier).

III. CONCLUSIONS

The significance of the results obtained in equations (17) or (20) can be seen if we compare with the result obtained by Reed; namely, Reed has shown that

$$R(\tau) = \nu\, E(a^2)\, \Phi(\tau) \int_{-\infty}^{\infty} s(t-\tau)\, s(t)\, dt$$

and that the noise power is

$$N_o = \frac{R(o)}{2} = \frac{\nu\, E(a^2)}{2} \int_{-\infty}^{\infty} |\, s(t)|^2\, dt\ .$$

If the pulse duration is T_s, then the noise power increases linearly with T_s or the signal energy.

However, equation (17) gives

$$N_o = \frac{R(o)}{2} = \frac{\gamma}{2} \int_{-\infty}^{\infty} \frac{u\left(\frac{ct}{2}, \frac{t}{2}\right)}{t^2} \mid s(t) \mid^2 dt .$$

This shows that the noise power may or may not increase linearly with T_s, since the probability density $u(r, t)$ now influences the noise power. Thus, the signal pulse shape is strongly determined by the probability density $u(r, t)$ if the noise power is to be minimized; i.e., for one type of probability density, $u(r, t)$, the signal spectrum might be broad, while for another $u(r, t)$ the signal spectrum might be narrow.

It is shown in the appendix, in a similar manner as that used in reference 1, that $z(t)$ approaches a Gaussian process.

APPENDIX

We show here that the probability distribution of the returned echo noise waveform approaches a Gaussian probability distribution as the mean number of scatterers per unit distance becomes large.

We start by considering Reed's original model of the returned echo noise process, which is a simplification of equation (1). If the time interval of observation (0, T) is large compared with the transmitted pulse duration, Δ, a returned echo waveform in (0, T) for a transmitted pulse is

$$z(t) = \sum_{t_k \in (0,\, T)} a_k\, s(t-t_k) \exp 2\pi i f_k (t-t_k)$$

where the various symbols have been defined in the main body of the report. Following Kelley and Lerner [reference 1] we can write immediately

$$P(z) = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty}\cdots\int p(\vec{a})\, d\vec{a} \int_{0}^{T}\cdots\int p(\vec{t})\, d\vec{t} \int_{-\infty}^{\infty}\cdots\int p(\vec{f})\, d\vec{f}\ \delta(z-z_n)$$

where

(1) $p(\vec{x})\, d(\vec{x}) = \prod_{i=1}^{n} p(x_i)\, dx_i$ = joint probability distribution of the x_i's, where $x_1, \ldots, x_n$ are assumed independent.

(2) We disregard effects within Δ of the end points of (0, T);

$z_n = \sum_{k=1}^{n} a_k\, s(t-t_k) \exp 2\pi i f_k(t-t_k)$ represents the waveform resulting from events at the times t_k $(0 < t_k < T)$;

$$z(t) = \sum_{k=-\infty}^{\infty} a_k\, s(t-t_k) \exp 2\pi i f_k (t-t_k).$$

The characteristic function of the distribution is

$$\begin{aligned}
\phi(u) &= \int_{-\infty}^{\infty} p(z)\, e^{i u z}\, dz \\
&= \sum_{n=0}^{\infty} \frac{\nu^n e^{-\nu T}}{n!} \int_{-\infty}^{\infty}\!\!\cdots\!\int p(\vec{a}_n)\, d(\vec{a}_n) \int_{-\infty}^{\infty}\!\!\cdots\!\int p(\vec{f}_n)\, d\vec{f}_n \\
&\cdot \int_0^T\!\!\cdots\!\int d\vec{t}_n \exp\left[i u \sum_{k=1}^{n} a_k\, s(t-t_k)\, e^{2\pi i f_k (t-t_k)}\right] \\
&= \sum_{n=0}^{\infty} \frac{\nu^n e^{-\nu T}}{n!} \left\{ \int_{-\infty}^{\infty} p(a)\, da \int_0^T dt' \int_0^{\infty} p(f) \exp\left[i u a\, s(t-t')\, e^{2\pi i f (t-t')}\right] df \right\}^n \\
&= \exp\left\{-\nu T + \nu \int_0^T dt' \int_{-\infty}^{\infty} p(a)\, da \int_{-\infty}^{\infty} p(f)\, df \exp\left[i u a\, s(t-t')\, e^{2\pi i f (t-t')}\right] df\right\}
\end{aligned}$$

where ν is not time varying.

If we expand the exponential involving $s(t)$ in a series and introduce the characteristic function for $p(f)$ as

$$\emptyset(t) = \int_{-\infty}^{\infty} p(f)\, e^{2\pi i f t}\, dt$$

then

$$\begin{aligned}
I(s,\emptyset) &= \int_{-\infty}^{\infty} p(f)\, df \exp\left[i u a\, s(t-t')\, e^{2\pi i f (t-t')}\right] \\
&= 1 + i u a\, s(t-t')\, \emptyset(t-t') + \frac{(i u a\, s(t-t'))^2}{2!}\, \emptyset\left[2(t-t')\right] \\
&+ \cdots
\end{aligned}$$

Next we have

$$\int_{-\infty}^{\infty} p(a)\ I(s,\emptyset)\ da = 1 + \sum_{r=1}^{\infty} \frac{(iu)^r}{r!}\ \overline{a^r}\ s^r(t-t')\ \emptyset\left[r(t-t')\right]$$

where $\overline{a^r}$ denotes the expected value of a^r.

Using the above we obtain

$$\phi(u) = \exp\left\{\nu \sum_{r=1}^{\infty} \frac{(iu)^r}{r!}\ \overline{a^r} \int_0^T s^r(t-t')\ \emptyset\left[r(t-t')\right] dt'\right\} .$$

The probability density of z is found by transforming its characteristic function as follows:

$$P(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(u)\ e^{-iuz}\ du$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left\{-iuz + \nu \sum_{r=1}^{\infty} \frac{(iu)^r}{r!}\ \overline{a^r} \int_0^T s^r(t-t')\ \emptyset\left[r(t-t')\right] dt'\right\} du$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} du\ \exp\left\{-iu(z-\bar{z})\right\} \exp\left\{\nu \sum_{r=2}^{\infty} \frac{(iu)^r}{r!}\ \overline{a^r} \int_0^T s^r(t-t')\ \emptyset\left[r(t-t')\right] dt'\right\}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left\{-iu(z-\bar{z})\right\} \exp\left\{\nu \sum_{r=2}^{\infty} \frac{(iu)^r}{r!}\ \overline{a^r}\ \sigma_r(t)\right\} du$$

where

$$\sigma_r(t) = \int_0^T s^r(t-t')\ \emptyset\left[r(t-t')\right] dt' .$$

The above gives the time varying probability distribution of z.

We assume next that the noise process is stationary in order to simplify some of the following calculations and be able to express the probability density of z in terms of its stationary variance and mean. Thus, the probability density of z is given approximately by

$$P(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left\{-iu(z-\bar{z})\right\} \exp\left\{-\frac{u^2}{2} \overline{z^2}\right\} \exp\left\{\nu \sum_{r=3}^{\infty} \frac{(iu)^r}{r!} \overline{a^r} \sigma_r\right\} du$$

where σ_r is no longer a function of time. We next make the following substitutions

$$z - \bar{z} = \sqrt{\overline{z^2}}\; x; \quad \sqrt{\overline{z^2}}\; u = \eta$$

and in addition we recall that for ν a constant then $\sqrt{\overline{z^2}} = \nu^{\frac{1}{2}} \sigma$. Then $P(z)\,dz$ becomes

$$P(x)\,dx = \frac{dx}{2\pi} \int_{-\infty}^{\infty} \exp\left\{-i\eta x - \frac{\eta^2}{2}\right\} \exp\left\{\sum_{r=3}^{\infty} \frac{(i)^r}{r!} \frac{\left(\frac{\eta}{\sigma}\right)^r}{\nu^{\frac{r}{2}-1}} \overline{a^r} \sigma_r\right\} d\eta$$

If we expand the second exponential of the above integrand in a series we obtain

$$P(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\eta \exp\left\{-i\eta x - \frac{\eta^2}{2}\right\} \left\{1 + \frac{\left(\frac{i\eta}{\sigma}\right)^3}{3!} \frac{\overline{a^3}\sigma_3}{\sqrt{\nu}} - \frac{\left(\frac{\eta}{\sigma}\right)^4}{4!} \frac{\overline{a^4}\sigma_4}{\nu} + \frac{\left(\frac{i\eta}{\sigma}\right)^5}{5!} \frac{\overline{a^5}\sigma_5}{\nu^{\frac{3}{2}}} + \dots\right\}$$

Then recalling that [see reference 4, p. 228]

$$\int_{-\infty}^{\infty} \varphi^{(n)}(x)\, e^{itx}\, dx = (-it)^n e^{-\frac{t^2}{2}}, \qquad n=0,1,2,\ldots$$

$$\varphi^{(n)}(x) = \frac{d^n}{dx^n} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$\phi(x) = \phi^{(o)}(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

we finally obtain the Edgeworth series

$$P(x) = \phi(x) - \frac{1}{3!} \frac{\overline{a^3}\, \sigma_3}{\sigma^3} \frac{1}{\sqrt{\nu}} \phi^{(3)}(x) + \frac{1}{4!} \frac{\overline{a^4}\, \sigma_4}{\sigma^4} \frac{1}{\nu} \phi^{(4)}(x) - \ldots$$

Thus $P(x)$ approaches a Gaussian distribution as ν becomes large, where ν is the mean number of events per unit time. If ρ is the mean number of scatterers per unit distance in range which are illuminated by the transmitted pulse and intercepted by the receiver, then

$$\nu = \frac{c\rho}{2}$$

Thus the returned echo noise process approaches a Gaussian process as ρ, the mean number of scatterers per unit distance, becomes large.

It is a straightforward matter to use the above formal methods to show that all joint distributions are Gaussian in the limit.

For the case when $z(t)$ is given by equation (1) and $v(t)$ is given by equation (3), then the probability distribution will again be Gaussian provided the comments made above regarding σ_r apply to the modified σ_r given by

$$\sigma_r(t) = \int_0^T u^r \left(\frac{ct'}{2} , \frac{t'}{2} \right) s^r(t-t') \, \phi\left[r(t-t') \right] dt' \; .$$

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The time-varying correlation function for a radar (sonar) back-scattered noise process is determined. An approximation is made in order to obtain an expression for the power spectrum of the noise process. The correlation function and the power spectrum are both a function of the probability density that a scatter will be at range r and time t as well as the signal pulse shape. This implies that, in order to minimize the noise power, the signal may not be wideband but is affected by the probability distribution of the scatterers.

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