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6 NETWORK FLOW THEORY,

10 by L. R. Ford, Jr.

P-923

August 14, 1956

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The RAND Corporation
1700 MAIN ST. • SANTA MONICA • CALIFORNIA

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SUMMARY

The labeling algorithm for the solution of maximal network flow problems and its application to various problems of the transportation type are discussed.

NETWORK FLOW THEORY

L. R. Ford, Jr.

1. INTRODUCTION

A network (or linear graph) is a collection of points or nodes, some of which may be joined together by arcs. We shall denote the points by P_i, $i = 0, 1, 2, \ldots, N$, and denote the arc joining P_i to P_j in that order by A_{ij}. (Note that there may also be an arc A_{ji} joining P_j to P_i.) We may also have associated with the arc A_{ij} a capacity c_{ij} and a length (or cost) ℓ_{ij}. We shall assume these to be positive integers. We shall also distinguish two points in the network, P_0, the origin, and P_N, the terminal. One may think of this system as a rail network, in which the origin represents a factory or a warehouse at which goods enter the system, and the terminal represents a consumer for these goods who removes them from the system. The c_{ij} then represent upper bounds on the shipping capacity from P_i to P_j and the ℓ_{ij} may represent variously the distance from P_i to P_j or the time required to ship from P_i to P_j or the unit cost of shipping from P_i to P_j. (We shall agree that if $c_{ij} = 0$ then $\ell_{ij} = \infty$ and that $\ell_{ij} \neq 0$.)

Evidently many different problems may be posed with this framework. We shall discuss primarily the following three problems, stated verbally below.

A. To find a maximal steady state flow of goods from the origin to the terminal, independent of cost considerations (i.e. of ℓ_{ij}).

B. To find the cheapest route from origin to terminal independent of capacity constraints (i.e. of c_{ij}).

C. To find the maximal amount of goods that can be shipped from origin to terminal in a given time, T (here interpreting ℓ_{ij} as the time required to ship from P_i to P_j).

We shall discuss A first as being by far the richest in applications. It appears as a subproblem in a very large number of transportation-type problems, and from the theory which we shall develop one may obtain a large number of combinatorial results as well.

2. FORMAL STATEMENT OF PROBLEM A

A steady state flow of goods from origin to terminal is required to have the following properties:

(a) the amount flowing into P_i must equal the amount flowing out of P_i for $i \neq 0, N$;

(b) the amount flowing along A_{ij} must be less than or equal to c_{ij}.

Subject to these restrictions, it is desired to maximize x_F, the amount flowing out of P_0, or equivalently, the amount flowing into P_N. This may be presented as a linear programming problem as follows. Denote by $x_{ij} \geq 0$ the flow from P_i to P_j. Then

$$
(1)\quad \begin{cases}
\sum_j x_{ij} - \sum_j x_{ji} = 0 & (i = 1, 2, \ldots, N-1) \\
\sum_j x_{0j} - \sum_j x_{j0} = x_F & \\
\sum_j x_{Nj} - \sum_j x_{jN} = - x_F & \\
0 \leq x_{ij} \leq c_{ij} & \\
\text{maximize } x_F \ . &
\end{cases}
$$

Thus Problem A could be solved by a direct application of the simplex method to the above system.

3. SOLUTION OF PROBLEM A BY LABELING

We here outline an iterative technique which will solve Problem A, starting with any flow whatsoever, and will also solve the dual problem to be discussed later. The labeling process attaches labels of the form $(P^{\pm}_{j}, h)$ to certain points in the following manner.

(a) Label P_0 with the label $(-, \infty)$.

(b) Take any labeled point P_j not yet scanned. Suppose labeled $(P^{\pm}_k, h)$. To all points P_i which are unlabeled and such that $c_{ji} - x_{ji} > 0$ attach the label $(P^{+}_j, \min [h, c_{ji} - x_{ji}])$. To all points P_i which are now unlabeled, and such that $x_{ij} > 0$, attach the label $(P^{-}_j, \min [h, x_{ij}])$. [In case P_i

is a candidate for a label in several ways, use any applicable label.]

If the terminal P_N is ever labeled the process ceases immediately. Otherwise the process continues until no new labels are obtained, and the terminal is left unlabeled.

In the former case we increase the flow as follows. If P_N is labeled (P_k^+, h) we replace x_{kN} by $x_{kN} + h$, and if P_N is labeled (P_k^-, h) we replace x_{Nk} by $x_{Nk} - h$. In either case we then turn our attention to P_k. In general, if P_k is labeled (P_j^+, m) replace x_{jk} by $x_{jk} + h$, and if labeled (P_j^-, m) replace x_{kj} by $x_{kj} - h$, in either case turning attention then to P_j. Eventually we arrive back at the origin and the process ceases, having increased x_F by h units. Then all labels are erased and the labeling process is begun again. We see easily that equations (1) remain satisfied.

In the latter case we are done. We shall defer proof of this momentarily until after we discuss cuts.

Definition: A cut in a network is any set of arcs whose removal disconnects the origin from the terminal. The value of a cut is the sum of the capacities of its arcs.

It is clear that, given a cut, any flow from origin to terminal must pass through it. Thus $x_F \leq$ value of any cut.

We shall prove that we have solved Problem A by producing a cut whose value is equal to x_F. Note that we have finally achieved a labeling which includes the origin and excludes the terminal. If we let I be the set of indices of labeled points and J the set of indices of unlabeled points, then the arcs A_{ij} for $i \varepsilon I$, $j \varepsilon J$ evidently form a cut. For these arcs $x_{ij} = c_{ij}$ and $x_{ji} = 0$, for otherwise P_j would be labeled. Summing equations (1) over $i \varepsilon I$ only yields

$$x_F = \sum_{\substack{i \varepsilon I \\ j \varepsilon J}} x_{ij} = \sum_{\substack{i \varepsilon I \\ j \varepsilon J}} c_{ij} = \text{value of the cut.}$$

Hence x_F is maximal; also the cut is one of minimal value.

This, in passing, is a constructive proof of the minimal cut theorem for the case of integral (or rational) c_{ij}.

The Minimal Cut Theorem. In any network the maximal steady state flow is equal to the minimal cut value.

4. A SET OF DUAL VARIABLES

Writing the last inequalities of (1) in the form $x_{ij} + y_{ij} = c_{ij}$, we find the dual problem: To determine π_i (point prices) and γ_{ij} (arc prices) satisfying

$$(2) \quad \begin{cases} \pi_i - \pi_j + \gamma_{ij} \geq 0 \\ \gamma_{ij} \geq 0 \\ \pi_N - \pi_0 \geq 1 \\ \text{minimize } \sum c_{ij}\gamma_{ij} . \end{cases}$$

Letting $\pi_i = 0$ for $i \in I$, $\pi_i = 1$ for $i \in J$ and $\gamma_{ij} = 1$ for $i \in I$, $j \in J$, $\gamma_{ij} = 0$ otherwise, yields a set of prices which satisfy (2) and which produce a dual form equal to the primal form (via the Minimal Cut Theorem). Hence these prices form an optimal solution to the dual system, from the duality theorem.

5. CERTAIN PROBLEMS SOLVABLE BY NETWORK FLOW METHODS

There are certain transportation-type problems which may be viewed as maximal flow problems, and others in which such a problem appears as an auxiliary or an associated problem.

(a) The Capacitated Hitchcock Problem. The problem may be stated as follows. To find $x_{ij} \geq 0$ such that

$$\sum_j x_{ij} = a_i \quad \sum_i x_{ij} = b_j$$

$$0 \leq x_{ij} \leq c_{ij}$$

$$\text{minimize} \sum a_{ij} x_{ij} \; .$$

Clearly, for a solution to exist, $\sum a_i$ must equal $\sum b_j$. But even if this requirement is satisfied the c_{ij} may make a solution impossible.

The question of feasibility may be settled, however, by the flow algorithm. Set up the network with points O, A_i, B_j, and $\mathcal{J}$ where O is the origin and $\mathcal{J}$ the terminal. The arcs in the network are

$\widehat{OA_i}$ with capacity a_i, $\widehat{B_j \mathcal{T}}$ with capacity b_j, and $\widehat{A_i B_j}$ with capacity c_{ij}. The maximal flow in this network is $\sum a_i$ if and only if the original problem is feasible.

(β) The Capacitated Transshipment Problem. Here the problem is in a capacitated network; certain points, O_i, have availabilities of a_i and other points, $\mathcal{T}_j$, have requirements b_j. The feasibility of this problem may be settled by adding a new origin O joined to O_i by an arc of capacity a_i, and a new terminal $\mathcal{T}$ with $\mathcal{T}_j$ joined to $\mathcal{T}$ by an arc of capacity b_j. Again, if the maximal flow is $\sum a_i = \sum b_j$, then the system is feasible.

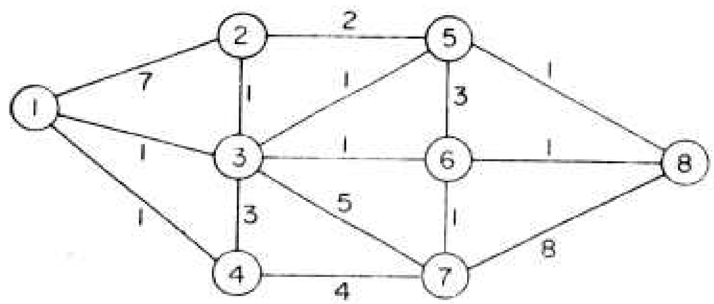
6. TWO EXAMPLES

(α) Check the feasibility of this Capacitated Hitchcock problem.

a_i \ b_j	9	10	9	12
13	3	4	2	4
10	1	3	2	4
7	5	4	3	2
10	3	3	2	1

$= c_{ij}$

(β) Check the feasibility of this Capacitated Transshipment Problem,



where the availabilities are 3 units at P_1 and 5 units at P_6; the requirements are 4 units at P_4 and 4 units at P_8.

7. A SOLUTION FOR PROBLEM B

This may be set up as a linear programming problem by starting with one unit available at the origin, one unit demanded at the terminal, and requiring the minimization of distance traveled. The equations are

$$(3)\quad \begin{cases} \sum (x_{0j} - x_{j0}) = 1 \\ \sum (x_{ij} - x_{ji}) = 0 \\ \sum (x_{Nj} - x_{jN}) = -1 \\ \text{minimize } \sum \ell_{ij} x_{ij} . \end{cases}$$

The dual problem, with which we shall work is

$$
(3') \quad \begin{cases} x_i + \ell_{ij} \geq x_j \\ x_0 = 0 \\ \text{maximize } x_n \end{cases}
$$

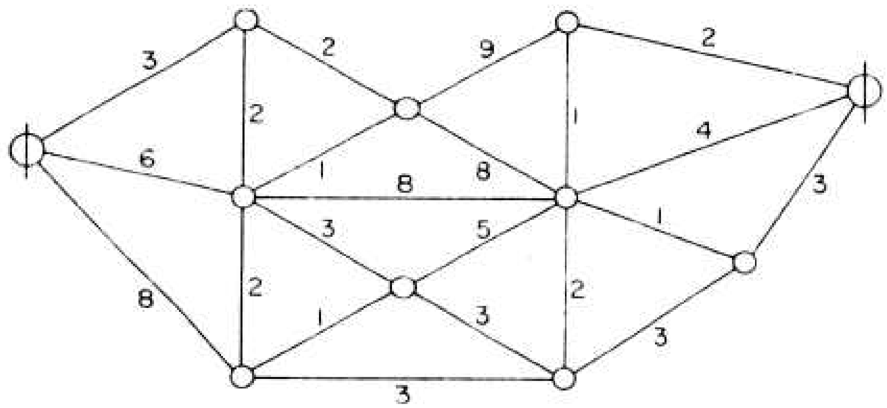
where the x_i are dual variables associated with the points of the graph and, at least in the optimal solution, x_i may be thought of as representing the shortest distance from P_0 to P_i.

A computing procedure is the following. Assign initially $x_0 = 0$ and $x_i = \infty$ for $i \neq 0$. Scan the network for a pair P_i and P_j with the property that $x_i - x_j > \ell_{ji}$. For this pair replace x_i by $x_j + \ell_{ji}$. Continue this process. Eventually no such pairs can be found, and x_N is now minimal and represents the minimal distance from P_0 to P_N. Clearly, if no such pairs can be found, the system (3) is satisfied. We shall now prove optimality.

Let P_0, P_{i_1}, ..., P_{i_m}, P_N be the shortest path. Along this path, $x_{i_{k+1}} - x_{i_k} \leq \ell_{i_k i_{k+1}}$. Summing these relations gives $x_N \leq$ (length of shortest path). On the other hand, for each P_j ($j \neq 0$) there is some P_i with $x_i + \ell_{ij} = x_j$, $\ell_{ij} \neq 0$. For if $j \neq 0$, x_j started at ∞ and has been decreasing monotonely. If x_j is still ∞, we are done; if not, at the last decrease there was such an i which must still be at the same value. Tracing out this chain backwards from P_N the x_i are monotone strictly decreasing; eventually the origin P_0 is reached. Here $x_j - x_i = \ell_{ij}$ and summing these gives x_N = length of this chain. Hence this is the shortest path.

8. AN EXAMPLE

Find the shortest path through the following network.



9. PROBLEM C VIEWED AS PROBLEM A

Problem C, the dynamic network flow problem, may be viewed as a static flow in the following larger network. Let S be the network whose nodes are points P_i^{τ}, P_i^{τ} corresponding to the old P_i but one for each basic time unit, i.e., for all τ with $0 \leq \tau \leq T$. We connect these points with directed capacitated arcs as follows.

Arc $\widehat{P_i^{\tau} P_i^{\tau+1}}$ is present with infinite capacity for $0 \leq \tau \leq T-1$ and all i.

Arc $P_i^{\tau} P_j^{\tau+\ell_{ij}}$ is present with capacity c_{ij} for $0 \leq \tau \leq T - \ell_{ij}$ and all i, j.

One sees fairly readily that a maximal steady state flow in S is a maximal dynamic flow for T time periods in the original network.

10. THE TANKER SCHEDULING PROBLEM

This problem may be stated as follows. We have given m pickup points P_i and n discharge points Q_j, and a schedule $\{t_{ij}^k\}$ where t_{ij}^k is the time at which a tanker is required to be at P_i to pick up a load destined for Q_j. (There may be several such times; these are distinguished by the index k.) We have further given two arrays of positive numbers, a_{ij} and b_{ij}, where a_{ij} represents the loading-traveling time from P_i to Q_j and b_{ij} the unloading-traveling time from Q_j to P_i. The problem is to meet the schedule with a minimal number of tankers.

This problem has been solved as a transportation problem by Dantzig and Fulkerson in [2]. Here it will be shown to be essentially of type A above.

Let T be the maximal t_{ij}^k and set up a network with points P_i^τ, Q_j^τ for $0 \leq \tau \leq T$. We join Q_j^τ to $P_i^{\tau+\ell_{ij}}$ with an arc of infinite capacity, $0 \leq \tau \leq T - \ell_{ij}$, and P_i^τ to $P_i^{\tau+1}$, Q_j^τ to $Q_j^{\tau+1}$ with an arc of infinite capacity, $0 \leq \tau \leq T-1$. The schedule may be interpreted as providing a requirement of one unit at $P_i^{t_{ij}^k}$ and an availability of one unit at $Q_j^{t_{ij}^k+a_{ij}}$ for each t_{ij}^k [such that $t_{ij}^k \leq T - a_{ij}$]. This can be accomplished by joining each such $P_i^{t_{ij}^k}$ to a terminal, $\mathcal{T}$, by an arc of capacity one, and each such $Q_j^{t_{ij}+a_{ij}}$ to an origin, O, by an arc of capacity one.

Evidently the maximal flow in this network represents the maximal reassignment potential available in the system.

Note that this problem is not, strictly speaking, of type C, since the availabilities are not the same in each time period. A new algorithm to handle type C directly, without turning to the dynamic network, has recently been developed by D. R. Fulkerson and the author.

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