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⑥ ON GAMES INVOLVING BLUFFING,

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§ 1. Introduction.

~~We wish to consider~~ a class of two-person games possessing the following general characteristics are considered

- ~~(1)~~ (a) At the beginning of the game, and at various stages of the game, a chance mechanism furnishes numbers x_1 and y_1 from the unit interval $[0,1]$ to the two players, I and II respectively. I knows x_1 but not y_1, II knows y_1 but not x_1.
- (b) Each player pays a certain amount to start the game, regardless of his subsequent moves.
- (c) The game is a many-move game of the following type. I's initial move, ~~which depends upon x_1,~~ is one of a fixed number of possible moves, which are known to II. However II does not know I's move.
- (d) After I has made the initial move, II has a choice of a finite number of counter-moves, which are known to I. However, I does not know II's reply.
- (e) After II has moved, I has again the choice of a finite number of moves, known to II, and so on. The initial maneouvering continues in this fashion for a fixed number of turns.
- (f) ~~The first phase having been concluded, the chance mechanism furnishes two new numbers x_2 and y_2 to I and II respectively. This initiates a second phase of move and counter-move.~~
- (g) The game continues in this way for a fixed number of phases, N, at the end of which there is a payoff to I of K ~~= K(x_1, x_2, ..., x_N),~~ and II receives - K.

The problem of determining the best possible mode of play for each player in the usual sense of maximizing or minimizing the expectation is one that arises in many important applications of statistics and probability theory.

The simplest case of this general problem is furnished by a class of card games which embraces some well-known diversions. Apart from the intrinsic interest and satiation of intellectual curiosity, the principal merit in considering these (relatively) simple card games lies in the fact that these games have been played over a considerable period of time and the experience of countless players has furnished a set of heuristic axioms of play which is extremely useful, as we shall see below, in unraveling the tangled skein of the mathematical problem. Once we have solved a number of particular problems, the general pattern appears to emerge. It is in the discovery of this pattern that we feel lies our principal contribution

A first question that arises, apart from the precise solution of the problem, pertains to the form of the solution. Do the players use pure or mixed strategies?

Experience would seem to indicate that a mixed strategy is required. It was with considerable surprise that we found that there existed two-person poker games where pure strategies could be used by both players, cf. [1] . Since then, we have investigated many more models of two-person card games, and developed a systematic technique which we believe will produce the solution whenever one exists in terms of pure strategies. Unfortunately, we have not been able to show that in the general game described above pure strategy solutions must exist.

We were able to show, cf. [1] , that, under mild assumptions, any mixed strategy in a game of the type described previously can be approximated by a pure strategy, in the sense that the corresponding expectations will be arbitrarily close. The reason for this lies in the fact that the chance moves may be used as substitutes for mixed strategies. For a formal proof, we refer to [1] .

To illustrate our procedure, we treat several simple models of two-person games. It is soon seen that a great mathematical simplification ensues if the games are made continuous, allowing the use of integrals rather than sums, functions rather than sequences.

Our methods are equally applicable to the problem of finding equilibrium-point solutions, of the Nash type, of N-person games in theory. In practice, the algebraic difficulty introduced by non-linear equations causes a bit of grief. However, we feel that interesting and important as the equilibrium point theory is, it omits many important features of an actual game involving more than two players and for that reason we do not apply it.

The plan of the paper is as follows. In the second section we discuss a game which can be thought of as half of a simplified poker game. The experience gained here is useful in discussing the bilateral game. We then treat in detail three types of two-person games, each with its particular feature of interest. Sandwiched between is a short discussion of a game which occupies an intermediate position in the hierarchy.

§ 2. First Example.

We consider first a deceptively innocent game played according to the following rules. There are N players, where the exact number is immaterial. If $N=1$, gambling is not possible, although a solitaire game remains. If N is very large, several decks of cards may be required. In general, $2 \leq N \leq 10$. In turn, each player deals himself and the other players four cards each. Before play begins, each player antes one, so that there is a sum of N in the pot. Upon looking at his cards, the first player to the left of the dealer has the option of not betting, whereupon he automatically leaves his ante in the pot, or of betting an amount f, $1 \leq f \leq N$, that he can beat, *in its suit*, the next card the dealer turns up from the deck. If he does, he wins f from the pot, if not, he contributes f to the pot. The next player to the left has a similar choice, with the difference that the new upper limit will be $N+f$ if the first player has lost. Whenever the pot falls beneath N in size, each player antes one.

Each player sees at most one card from any other player's hand, for the rules demand that the winning player show only the necessary majorizing card, and that the losing player throw his hand away face down. We shall in the discussion below ignore the additional information, in most cases of negligible effect, which may be gained that way, and the fact that the upper limit for individual wagers may increase considerably above N, which in actual play occurs alarmingly often.

We begin by considering the following simple version.

There are two players, B, for bettor, and D for dealer. D deals B a card, x_1, $0 \leq x_1 \leq 1$, and one to himself, y_1, $0 \leq y_1 \leq 1$. Before the betting begins, both players ante one. Upon receiving his card, B has the choice of betting an amount, $f(x_1)$, $1 \leq f(x_1) \leq M$, or of folding, in which case D automatically wins the ante. If B bets $f(x_1)$, D has no choice but to cover. If $x \geq y$, B wins $f(x) + 1$, if $x < y$, B loses $f(x) + 1$.

Given the distributions of x and y, the problem is to determine the best possible mode of play for B.

Let x have the distribution function $F(x)$, and y, the distribution function $G(y)$ where $F(0) = G(0) = 1$, $\int_0^1 dF(x) = \int_0^1 dG(y) = 1$.

It is intuitively clear, and may easily be shown rigorously, that since D cannot be bluffed out, $f(x)$ must be a non-decreasing function of x, and hence if B does not bet on x_1, he does not bet on x_2, if $x_1 \geq x_2$.

A consequence of this is that B folds, i.e. drops out and allows D to win the ante, if $x \leq a_1$, where a_1 is some as yet undetermined number in the unit interval, and bets $f(x)$ if $x \geq a_1$.

B's expectation is given by

$$(1) \qquad E_B = - \int_{x \leq a_1} dF + \int_{\substack{a_1 \leq x \leq 1 \\ 0 \leq y \leq 1}} (1+f(x))k(x,y)dF(x)dG(y),$$

where $K(x,y)$ is defined by

$$(2) \qquad \begin{aligned} K(x,y) &= 1 \text{ if } x \geq y \\ &= -1 \text{ otherwise.} \end{aligned}$$

E_B may be simplified to

$$(3) \qquad \int_{x \geq a_1} (1+f(x))\ (2G(x) - 1)dF(x) - \int_{x \leq a_1} dF(x).$$

B must now choose a_1 and $f(x)$ so that this expectation is a maximum.

Let x_o be a point where

$$(4) \qquad 2G(x) - 1 = 0.$$

Since $G(0) = 0$, $\int_o^1 dG = 1$, there is one such point. The point is unique if $dG > 0$. It is clear that $a_1 \leq x_o$, since if $a_1 > x_o$, we may always increase E_B by decreasing a_1.

Therefore, regardless of the value of a_1, $f(x)$ is chosen as follows

$$(5) \qquad \begin{aligned} f(x) &= 1, & a_1 &\leq x \leq x_o \\ &= M, & x_o &\leq x \leq 1. \end{aligned}$$

The expectation now takes the form

$$(6) \qquad 2 \int_{a_1 \leq x \leq x_o} (2G(x)-1)dF(x) + (1+M)\int_{x_o}^1 \left[2G(x)-1\right] dF(x) - \int_{x \leq a_1} dF(x),$$

$$= \int_{a_1 \leq x \leq x_o} (4G(x)-1)dF + (M+1)\int_{x_o}^1 (2G(x)-1)dF - 1 + \int_{x \geq x_o} dF.$$

From this it follows that a_1 is chosen by the condition

$$a_1 = \inf\left[x;\ 4G(x) - 1 \geq 0\right] . \tag{7}$$

Collecting the previous results, B's best strategy is determined by

$$\begin{aligned} f(x) &= 1, & a_1 \leq x \leq x_o, \\ &= M, & x_o \leq x \leq 1, \end{aligned} \tag{8}$$

where

(9) (a) $2G(x_o) - 1 = 0,$

(b) a_1 is determined by ( ),

and B folds if $0 \leq x \leq a_1$.

In particular, if we assume that x and y are uniformly distributed over the unit interval, we have

$$\begin{aligned} f(x) &= 1, & \tfrac{1}{4} \leq x \leq \tfrac{1}{2} \\ &= M, & \tfrac{1}{2} \leq x \leq 1, \end{aligned} \tag{10}$$

and B folds for $x \leq \frac{1}{4}$.

Having seen the pattern of a solution consider the more complicated game:

"Given two players, B and D, let B be dealt a card $C_1 = (x_1, x_2, \ldots, x_n)$, a point in the n-dimensional unit region $0 \leq x_i \leq 1$, where the distribution function of C is known, and D be given a card $C_2 = z$, $0 \leq z \leq 1$, which has probability p_i of being compared to x_i in order to determine the outcome. If B and D both ante one, and B is allowed to bet an amount $f = f(C)$, $1 \leq f \leq M$, winning $(1+f)$ if $z \leq x_i$, losing $(1+f)$ if $z \geq x_i$, what is the best possible mode of play for B?"

B's strategy, as before, will be to fold if $C = (x_1, x_2, \ldots, x_n)$ is within a certain region B, and to bet if C is outside this region. B's expectation is easily written,

$$(11) \qquad E_B = -\int_{C\in R} \prod_{i=1}^{n} dF_i + \sum_{i=1}^{n} p_i \int_{C\in I-R} \left[1+f(C)\right] K(x_i, z) \prod_{i=1}^{n} dF_i dG(z)$$

$$= -\int_{C\in R} \prod_{i=1}^{n} dF_i + \int_{C\in I-R} \left[1+f(C)\right] \left\{ \sum_{i=1}^{n} p_i(2G(x_i)-1) \right\} \prod dF_i(x_i) .$$

The notation is as follows: I denotes the unit cube, $C\in R$ means that C is within the region R, and in what follows, $A \cap B$ denotes the intersection of the two regions A and B.

To maximize E_B, it is clear that we should choose $f(C) = M$ in the region S, belonging to I-R, defined by

$$(12) \qquad \prod dF_i \left\{ \sum_{i=1}^{n} p_i(2G(x_i) - 1) \right\} \geq 0,$$

and f(C) = 1 in the complementary part of I-R. It is clear that it is a matter of indifference what the value of f(C) is whenever the expression in (12) is zero.

Hence

$$(13) \qquad E_B = -\int_{C\in R} \prod_{i=1}^{n} dF_i + (M+1)\int_{C\in S\cap(I-R)} \left[\sum_{i=1}^{n} p_i(2G(x_i)-1)\right] \prod dF_i(x_i)$$

$$+ 2\int_{C\in(I-S)\cap(I-R)} \left[\sum_{i=1}^{n} p_i(2G(x_i)-1)\right] \prod dF_i(x_i).$$

It remains to determine R. Write

$$(14) \qquad \int_{C\in R} \prod_{i=1}^{n} dF_i = \int_{C\in I} - \int_{C\in I-R}$$

$$= 1 - \int_{C\in(I-R)\cap S} - \int_{C\in(I-R)\cap(I-S)}.$$

Thus

$$(15) \qquad E_B = -1 + {}_{C\in(I-S)\cap(I-R)} \left[2\sum_{i=1}^{n} p_i(2G(x_i)-1\right] \prod dF_i$$

$$+ \text{ terms independent of } R.$$

Decreasing the region R will increase B until

(16) $$\left[2\sum_{i=1}^{n} p_i(2G(x_i) -1) - 1\right] \leq 0.$$

Hence R is defined by

(17) $$2\sum_{i=1}^{n} p_i(2G(x_i) -1) - 1 \geq 0.$$

Let us now consider the application to Red Dog. Let us assume that the x_i are uniformly distributed, z is uniformly distributed, and $p_i = \frac{1}{4}$, corresponding to four suits. The region where the maximum should be bet is determined by

(18) $$2\sum_{i=1}^{4} p_i G(x_i) - 1 = \frac{\sum_{i=1}^{4} x_i}{2} - 1 \geq 0,$$

and the folding region by

(19) $$1 + 2\sum_{i=1}^{4} p_i(2G(x_i) -1) = \sum_{i=1}^{4} x_i - 1 \leq 0.$$

In the actual game of Red Dog, F_i is a step function with jump at $\frac{k}{13}$, k = 1, 2, ..., 12. Furthermore in order to complete the discussion we should consider the case where a hand is void of one or more suits. The second point may easily be taken care of in the continuous case. The first point requires only a good deal of arithmetic.

It is interesting to note that the general structure of the game may be determined without knowledge of the fine structure. This observation will be very valuable in what follows.

§3. Discussion of an Intermediate Game.

In the previous section, we have discussed the strategy to be employed by the first player whenever the second player is forced to cover all bets. A first extension of this situation is a game where the dealer's strategy is partly fixed, partly free. An example of a game of this type is the game of blackjack, or twenty-one. A well-known variation is the game of seven-and-a-half.

The game of blackjack, stripped of inessential minor features, is played as follows. A bridge deck is used with the picture cards counted as ten, an ace as one or eleven at the choice of the player, and the other cards retaining their numerical values. There are two players, a dealer and a bettor. Each player is dealt a closed card. The first player has the choice of folding immediately, in which case he loses a token amount, the ante, or of betting an amount which depends upon the card he receives and the allowable bets. The object of the game is for each player to get a total of twenty-one by drawing open cards from the deck. If neither player attains a total of twenty-one, the hands are compared, and the hand with higher point total wins, with ties going in favor of the dealer. What prevents repeated drawing is the rule that a player automatically loses if his point total ever exceeds twenty-one.

After B has drawn one card, he has the choice of continuing to draw, "pulling", or of not pulling, "sticking". Once B has concluded his moves, D has the same choices. As currently played, however, D is compelled to pull if he has fifteen or under, and must stick if he has sixteen or over.[1] This may be regarded as an answer to possible bluffing on B's part, and we shall see that in the two-person poker games we discuss, exactly this method is used to counter bluffing.

B bluffs by sticking with a low point total, such as twelve or thirteen, hoping that D will pull over twenty-one.

It is rather interesting to observe that in this case experience has dictated the use of a pure strategy on the part of the dealer. It is easy to concoct various simplified models of blackjack, and in each of these models, pure strategies will be found to exist.

§ 4. A Simple Poker Game.

We begin by considering the game where there are two players A and B, each of whom receives a card, x and y, respectively, $0 \leq x, y \leq 1$, where for simplicity we assume that x and y are equidistributed. Each antes one before play begins. A has the choice of folding or of betting an amount $a > 0$. B has a choice of folding or of seeing A's bet.

Let us use the following notation:

(1) A_F = set of x values where A folds.

A_B = set of x values where A bets.

B_F = set of y values where B folds.

B_B = set of y values where B bets.

Let ϕ_F, ϕ_B be the characteristic functions of A_F and A_B, ψ_F, ψ_B, be the characteristic functions of B_F and B_B.

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[1] It is commonly believed that this policy is pursued to prevent the dealer from using marked cards.

Then

$$(2) \qquad E_A = \int_{A_F} (-1)\,dx + \int_{A_B} \left[\int_{B_F} dy + (a+1) \int_{B_B} K(x,y)\,dy \right] dx$$

$$= \int_{B_F} \left[\int_{A_B} dx \right] dy + \int_{B_B} \left[(a+1) \int_{A_B} K(x,y)\,dx \right] dy - \int_{A_F} dx.$$

Writing E_A in terms of characteristic functions, this becomes

$$(3) \qquad E_A = - \int_0^1 \phi_F dx + \int_0^1 \phi_B(x) \left[\int_0^1 \psi_F dy + (a+1) \int_0^1 \psi_B(y) D(x,y)\,dy \right] dx.$$

Viewing E_A from A's point of view, with the aim of maximizing E_A we see that for fixed ψ_F, ψ_B, A chooses ϕ_F or ϕ_B, for a particular value of x, depending upon the relation between

$$(4) \qquad I_1 = -1 \quad \text{and} \quad I_2 = \int_0^1 \psi_F dy + (a+1) \int_0^1 \psi_B(y) K(x,y)\,dy.$$

Let us now make the fundamental assumption that a solution in terms of pure strategies exists. Under this assumption we shall find the form of the solution. It is then easy to show that what we have actually is a solution.

At $x = 0$,

(5) $$I_2 = \int_0^1 \psi_F dy - (a+1) \int_0^1 \psi_B(y) dy$$

$$= \int_0^1 \psi_F dy - (a+1) \left[1 - \int_0^1 \psi_F dy \right] = - (a+1) + (a+2) \int_0^1 \psi_F dy.$$

Let us now, assume first that $I_1 > I_2(0)$, so that A always folds at $x = 0$, and hence in some neighborhood of $x = 0$.

Turning now to E_A, viewed from B's vantage, we must compare

(6) $$J_1 = \int_{A_B} dx \text{ and } J_2 = (a+1) \int_{A_B} K(x,y) dy,$$

or

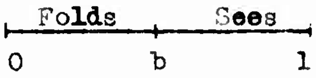
$$J_1 = \int_0^1 \phi_B dx \text{ and } J_2 = (a+1) \int_0^1 \phi_B(x) K(x,y) dx.$$

At $y = 0$, these are

(7) $$J_1 = \int_0^1 \phi_B dx, . \quad J_2(0) = (a+1) \int_0^1 \phi_B(x) dx.$$

Consequently, it is always true that B folds in some neighborhood of the origin. Furthermore, since J_1 is a constant and $J_2(y)$ is a monotone decreasing function of y, it is clear that if B starts seeing A's bet with $y = y_1$, he sees whenever he has a $y_2 > y_1$.

(8) B:



The question arises as to the determination of b. Referring to (6), we have

$$(9) \qquad J_2(y) = -(a+1)\int_0^y \phi_B(x)dx + (a+1)\int_y^1 \phi_B(x)dx$$

$$= (a+1)\left[1 - 2\int_0^y \phi_B(x)dx\right].$$

Therefore B only changes over from folding to betting at y if the measure of the set upon which A bets to the left of y is large enough to make $J_2(y) = J_1$.

Let us now return to A and see what it is that will make him bet in the interval [0,b] . We have, as in (9),

$$(10) \qquad I_2 = \int_0^1 {}_F dy + (a+1)\left[2\int_0^x {}_B(y)dy - 1\right].$$

Hence, A only changes from folding to betting at x if the measure of the set upon which B bets in [0,x] is large enough to make $I_2(x) = I_1$. This is clearly impossible in [0,b] , so that if A ever starts by folding at 0, he continues folding up to b, at least. But this implies that B has no motivation for changing over at b. Continuing in this way, we see that the only solution would be for both to fold regardless of the card each receives.

This seems rather far-fetched, and we consequently investigate the other two possibilities; viz.

(11) a. A always sees.

b. At $x = 0$, it makes no difference to A whether he sees or folds.

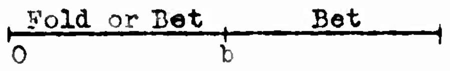
The first alternative is again improbable, and it is sensible to consider the second alternative. Using (5), 11b implies the equation

(12) $$-1 = -(a+1) + (a+2) \int_0^1 \psi_F dy,$$

$$\frac{a}{a+2} = \int_0^1 \psi_F dy .$$

A's strategy must now take the form

(13) A:



The reason why A must bet in $[b,1]$ is that B is betting in $[b,1]$, so that although $I_1 = -1 = I_2(x)$ for $0 \leq x \leq b$, $I_2(x) > -1$ in $[b,1]$. The measure of the set upon which A bets in $[0,b]$ is determined by the condition that at $y = b$

(14) $$J_1(b) = J_2(b)$$

$$\int_0^1 \phi_B dx = (a+1) \int_0^1 \phi_B(x) K(x,b) dx$$

$$\int_0^b \phi_B dx + 1-b = \int_0^1 \phi_B dx = -(a+1) \int_0^b \phi_B(x) dx + (a+1) \int_b^1 dx$$

$$(a+2) \int_0^b \phi_B(x) dx = a(1-b)$$

$$\int_{o}^{b} \phi_B(x)\,dx = \frac{a(1-b)}{a+2} .$$

Unfortunately b is still undetermined. Taking A's strategy to be of the form

(15) 

$, \quad C = (2(a+1)b-a)/)a+2),$

E_A is easily computed, referring to (2) and (13), namely

$$(16) \qquad E_A = - C + b(1-C) - (a+1)(b-C)(1-b).$$

Maximizing over b, we find

$$(17) \qquad b = a/a+2, \quad C = (a/a+2)^2.$$

This solution is valid for all $a \geq 0$, and we see that b has the correct behavior as $a \to 0$ or ∞ .

§ 4. Another Simple Model.

Let us now consider the following game where we increase the complexity by introducing 2 bets z_1, z_2, $z_2 > z_1$. This is the game whose solution was given in our original note, [1] . We now have the following three sets for A:

(1) A_F = set where A folds ,

A_L = set where A bets low, z_1 ,

A_H = set where B bets high, z_2,

and the sets for B:

(2) B_{FL} = set where B folds if A makes a low bet,

B_{FH} = set where B folds if A makes a high bet,

B_{SL} = set B sees if A makes a low bet,

B_{SH} = set B sees if A makes a high bet.

Then

(3)
$$\begin{aligned} E_A = & - \int_{A_F} dx \\ & + \int_{A_L} \left[\int_{B_{FL}} dy + (z_1+1) \int_{B_{SL}} K(x,y)\,dy \right] dx \\ & + \int_{A_H} \left[\int_{B_{FH}} dy + (z_2+1) \int_{B_{SH}} K(x,y)\,dy \right] dx \\ = & - \int_{A_F} dx \\ & + \int_{B_{FL}} \left[\int_{A_L} dx \right] dy + \int_{B_{SL}} \left[(z_1+1) \int_{A_L} K(x,y)\,dx \right] dy \\ & + \int_{B_{FH}} \left[\int_{A_H} dx \right] dy + \int_{B_{SH}} \left[(z_2+1) \int_{A_H} K(x,y)\,dx \right] dy . \end{aligned}$$

Let us begin by looking at E_A from B's point of view. Since B's decisions to see low bets or high bets are independent, we compare first

(4) $$I_1 = \int_{A_L} dx \quad \text{and} \quad (z_1+1) \int_{A_L} K(x,y)\,dx = I_2$$

and then

(5) $$I_3 = \int_{A_H} \quad \text{and} \quad (z_2+1) \int_{A_H} K(x,y)\,dx = I_4 .$$

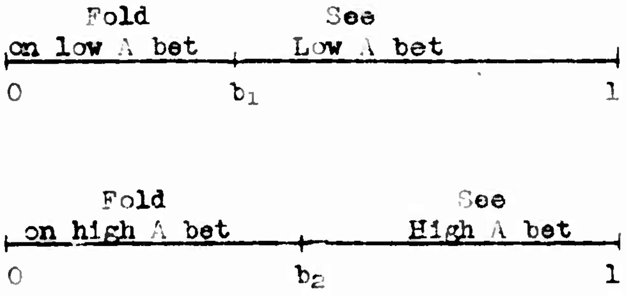
At $y = 0$,

(6) $$I_1 = \int_{A_L} dx \, , \qquad I_2 = \int_{A_H} dx$$

$$I_2(0) = (z_1+1) \int_{A_L} dx, \qquad I_4(0) = (z_2+1) \int_{A_H} dx.$$

Hence, if there is a solution, B's strategy is as follows:

(7)



where we will discuss the determination of b_1 and b_2 below.

Now let us return to E_A viewed from A's eyes. We must compare

$$(8) \qquad J_1 = -1$$

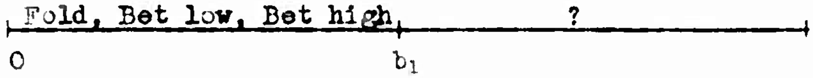
$$J_2 = \int_{B_{FL}} dy + (z_1+1) \int_{B_{SL}} K(x,y)\,dy$$

$$J_3 = \int_{B_{FH}} dy + (z_2+1) \int_{B_{SH}} K(x,y)\,dy \,.$$

From what has preceded, we suspect that at $x = 0$, $J_1 = J_2(0) = J_3(0)$. To avoid tiresome repetition, we shall not go through the mathematical argument which shows this, since we will go into detail in the next example, but assume it.

Thus A's strategy is, so far,

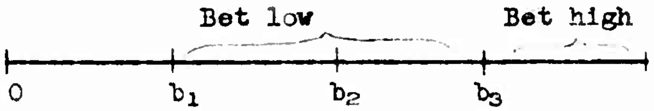
(9)



Referring to B's diagram, we see that at b_1, betting low must become preferable to folding for A, since B starts seeing at b_1. It is reasonable to assume that $b_2 > b_1$, at first. We continually use our experience with the actual game to reduce the number of possible cases.

We see that from b_1 to b_2, A must bet low

(10)



From b_2 on J_3 increases, but it takes some set of seeing high bets on B's part to catch up to J_2, so that at only at b_3, $J_2 = J_3$, and from then on A bets high. J_2 and J_3 can intersect in only one point, since B's strategy makes the two curves straight lines.

We now turn to the determination of the constants. We have the following constraints:

(11) a. At $x = 0$, $J_1 = J_2 = J_3$

b. At $y = b_1$, $I_1 = I_2$

c. At $y = b_2$, $I_3 = I_4$

d. At $x = b_3$, $J_2 = J_3$.

There are five equations for the five unknowns b_1, b_2, b_3, and the measures of the bet-low and bet-high sets in $[0,b_1]$.

We have then, using 11a:

(12)
$$-1 = b_1 - (z_1+1)(1-b_1),$$
$$b_1 = z_1/z_1+2,$$
$$-1 = b_2 - (z_1+1)(1-b_2),$$
$$b_2 = z_2/z_2+2.$$

Using 11b, we have at $y = b_1$

(13) $$\int_{A_L} dx = (z_1+1) \int_{A_L} K(x,b_1)dx$$

$$m_L = -(z_1+1)m_L + (b_3-b_1()z_1+1).$$

Using 11c, at $y = b_2$

(14) $$m_H = -(z_1+1)m_H + (1-b_3(z_2+1),$$

where m_L is the measure of the set where A bets low, and m_H is the measure of the set where A bets high.

Using 11d at $x = b_3$

(15) $$\int_{B_{FL}} dy + (z_1+1) \int_{B_{SL}} K(b_3,y)dy = \int_{B_{FH}} dy + (z_2+1) \int_{B_{SH}} K(b_3,y)dy$$

$$b_1 + (z_1+1)\left[\int_{b_1}^{b_3} K + \int_{b_3}^{1} K\right] = b_2 + (z_2+1)\left[\int_{b_2}^{b_3} K + \int_{b_3}^{1} K\right],$$

which reduces to

$$b_1 + (z_1+1)(2b_3-b_1-1) = b_2 + (z_2+1)(2b_3-b_2-1),$$

whence

(16) $$\frac{1}{2} + \frac{b_2z_2 - b_1z_1}{2(z_2-z_1)} = b_3.$$

From (16), we have b_3, and from (13) and (14) m_L and m_H. We must now examine the values for consistency, that is, we must check

$$(17) \qquad 0 \leq m_L + m_H \leq b_1, \qquad m_L \geq 0, \qquad m_H \geq 0,$$

$$0 \leq b_1 \leq b_2 \leq b_3.$$

Clearly, from (12), $1 > b_2 > b_1 > 0$.

From (13)

$$(18) \qquad m_L = \frac{(b_3-b_1)(z_1+1)}{z_1+2}$$

$$m_H = \frac{(1-b_3)(z_2+1)}{z_2+2} \quad .$$

Without difficulty we find that our solution is valid for $z_2 \geq z_1 \geq .62$, where .62 is an approximation to $2/c - 2$, and c is the smallest root of $\frac{c^3}{4} - 2c^2 + 4c-2 = 0$. We have not investigated the problem for $z_1 < .62$. The value of the game is easily found to be

$$(19) \qquad E_A = \left[-c_1 + (2-c_1)(c_2-c_1)^2 + 2(2-c_1)(c_2-c_1)c_2 + (2-c_2)c_2c_1\right] \Big/ c_1,$$

where $c_1 = 2/2+z_1$, $c_2 = 2/2+z_2$.

§5. The Poker Game with a Raise.

We now introduce one of the characteristic features of actual poker, the raise. We consider the following model. A and B are each dealt cards, x and y respectively. Before play begins, both players ante one. After the cards are dealt, A has the choice of folding, in which case B wins the ante, or of betting an amount $a \geq 1$. After A has bet, B has the choice of folding, of seeing A's bet, in which case the hands are compared, or of raising an amount a, in which case A may either see the raise or fold.

We shall use the following notation:

(1) A_F = set on which A folds.

A_B = set on which A bets, but does not see a raise.

A_S = set on which A bets and sees a raise.

B_F = set on which B folds.

B_S = set on which B sees A's bet, but does not raise.

B_R = set on which B raises A's bet.

Let E_A be A's expectation. Then

(2)
$$E_A = \int_{A_F} (-1)dx$$
$$+ \int_{A_B} \left[\int_{B_F} dy + (1+a) \int_{B_S} K(x,y)dy - (a+1) \int_{B_R} dy \right] dx$$
$$+ \int_{A_S} \left[\int_{B_F} dy + (1+a) \int_{B_S} K(x,y)dy + (2a+1) \int_{B_R} K(x,y)dy \right] dx$$

$$= \int_{B_F} \left[\int_{A_B} dx + \int_{A_S} dx \right] dy - \int_{A_F} dx$$

$$+ \int_{B_S} \left[(1+a) \int_{A_S} K(x,y)dx + (1+a) \int_{A_S} K(x,y)dx \right] dy$$

$$+ \int_{B_R} \left[(2a+1) \int_{A_S} K(x,y)dx - (a+1) \int_{A_B} dx \right] dy.$$

Let us write

$$(3) \qquad E_A = \int_{A_F} I_1(x)dx + \int_{A_B} I_2(x)dx + \int_{A_S} I_3(x)dx$$

$$= \int_{B_F} J_1(y)dy + \int_{B_S} I_2(y)dy + \int_{B_R} I_3(y)dy$$

We note the following immediate properties of the I's and J's

(4) (a) $I_1(x)$ is a monotone increasing function of x.

(b) $J_1(y)$ is a monotone decreasing function of y.

(c) $I_1(x)$, $J_1(y)$ are constants -

(d) $I_2(x)$ is constant over any subset of $B_F + B_R$.

(e) $I_3(x)$ is constant over any subset of B_F.

(f) $I_3(x) - I_2(x)$ is a monotone increasing function of x.

(g) $J_2(y)$ is constant over any subset of A_F.

(h) $J_3(y)$ is constant over any subset of $A_F + A_B$.

As before, we now assume the existence of a solution and derive its properties. It is first of all clear that A never sees a raise in some interval $[0,c]$. For, at $x = 0$ we have

$$(5) \qquad I_1(0) = -1$$

$$I_2(0) = \int_{B_F} dy - (1+a) \int_{B_S} dy - (a+1) \int_{B_R} dy$$

$$I_3(0) = \int_{B_F} dy - (1+a) \int_{B_S} dy - (2a+1) \int_{B_R} dy.$$

Consequently, if B raises a non-null set, $I_3(0) < I_2(0)$, and A must choose only between folding and betting without seeing at $x = 0$ and consequently in some neighborhood of 0. It is certainly plausible that B raises if he gets a card close to 1.

Similarly, in some interval $[0,d]$, B never sees. At $y = 0$, we have

$$(6) \qquad J_1(0) = \int_{A_B} dx + \int_{A_S} dx$$

$$J_2(0) = (1+a) \int_{A_B} dx +)1+a) \int_{A_S} dx$$

$$J_3(0) = (2a+1) \int_{A_S} dx - (1+a) \int_{A_B} dx!$$

Since B wishes to minimize E_A and $J_2(0) > J_1(0)$, he never sees A in some neighborhood of $y = 0$.

There are now three alternatives for A in some initial interval $[0,c]$.

(7) (a) A always folds in $[0,c]$.

(b) A always bets in $[0,c]$.

(c) It is immaterial whether A folds or bets in $[0,c]$, and he combines folding and betting in some (as yet unknown) proportions.

Let us begin by assuming that (a) is valid. B has the alternatives

(8) (a) B always folds in $[0,d]$.

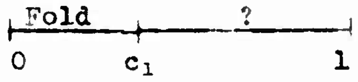
(b) B always raises in $[0,d]$.

(c) It is immaterial whether B folds or raises in $[0,d]$, and he combines folding and raising in some (as yet unknown) proportions.

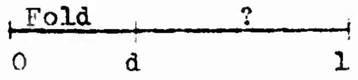
Since $J_1(y)$ is constant and $J_3(y)$ is monotone decreasing, if B begins by raising in $[0,d]$, he never folds. Consequently, we regard (b) as least possible, and consider first (a) and then (c).

We have then as a first possibility

(9) A:



B:



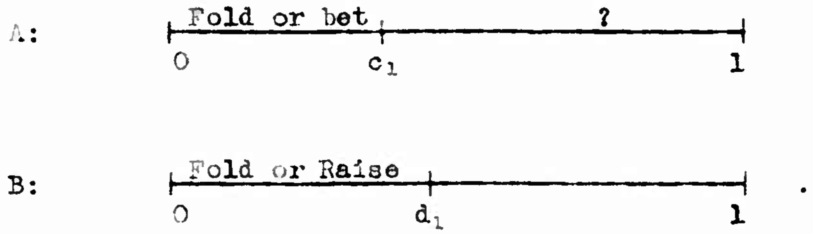
.

Let us show that $d > c$ is impossible. Referring to (2), it is clear that the only thing that will force A to change from folding to seeing or raising at c is B seeing or raising in $0 \leq y \leq c$, on a set of positive-measure. If $d > c$, this is not so, and hence $c \leq d$. Exactly the same type of reasoning shows that B only changes from folding to seeing or raising if A bets or sees in $0 \leq x \leq c$, which is not true.

Therefore, if a solution exists, it cannot have the form of (9). Referring to (7), we have two remaining alternatives for A. However, from the monotone character of $I_2(x)$ it follows that if A begins by betting, he never folds -- which seems improbable.

Hence, it is reasonable to try a solution of the form

(10)



Let us continue our discussion with the observation that in some interval $[c_1, c_2]$, $c_1 < c_2 < 1$, there must be betting on A's part. For if seeing a raise is preferable to betting without seeing at x_1, then because of the monotone behavior of $I_3(x) - I_2(x)$ it is preferable for $x \geq x_1$, and it is not reasonable to suppose that A never just bets, but always sees a raise.

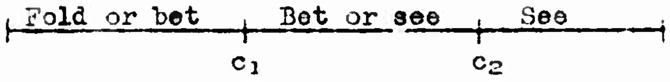
On the basis of this remark, we can now show that $c_1 = d_1$. As we have already pointed out, the only thing that forces A to change his pattern of play at c_1 from folding or betting to always betting is betting on B's part in $[0,c_1]$. This does not occur. There is still the possibility that the amount of raising on B's part can force A to see raises at c_1. In this case, there would be nothing to force B to change his strategy at d_1, and hence he would always fold or raise, which is implausible. Consequently $d_1 > c_1$ is not possible.

Let us now examine $c_1 > d_1$. Precisely the same type of argument shows that this case is highly unlikely. Hence, we take $c_1 = d_1$, and proceed to the next step.

If A is ever to change over betting, or seeing and betting, B must begin seeing in (c_1,c_2). Suppose that it were true that in some interval $[c_1,c_2]$, A always bets and B always sees. If $c_2 < 1$, it is clear that it ends at c_2 as far as A is concerned only if B raises in $[c_1,c_2]$, which is not so.

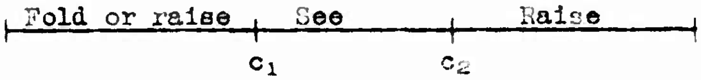
Therefore in $[c_1,c_2]$, it must be a matter of indifference to A whether he bets or sees a raise. From the monotonic behavior of $I_3 - I_2$, A"s strategy must be

(11) A:



and B's

(12) B:



,

since A only changes over to seeing raises at c_2 because B raises in $[c_2,1]$.

As yet unknown are the amounts of folding and betting in $[0,c_1]$, betting and seeing in $[c_1,c_2]$ on A's part, and folding or raising in $[0,c_1]$ on B's part. These unknowns are determined by the following considerations.

(13) (a) The amount of betting on A's part in $[0,c_1]$ must be sufficient to make B start seeing at c_1.

(b) The amount of seeing raises on A's part in $[c_1,c_2]$ must be sufficient to make B start raising at c_2.

(c) The amount of raising on B's part in $[0,c_1]$ must be sufficient to make A start betting or seeing raises in $[c_1,c_2]$.

Add to these the facts that

(14) (d) At $x = 0$, folding or betting are equivalent for A.

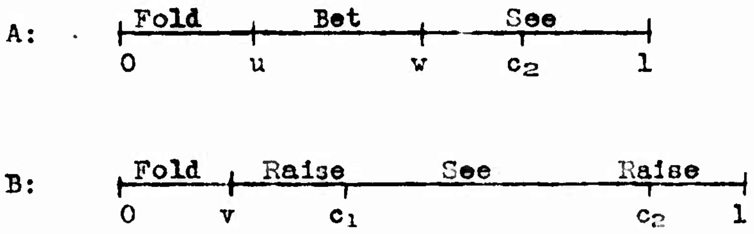
(e) At $y = 0$, folding or raising are equivalent for B,

and we have five conditions to determine the five unknowns c_1, c_2, A_B, A_S, B_F.

If these five conditions are consistent, we shall have a solution to our game, which will be unique, apart from the location of A_F, B_F in $[0,c_1]$, which is of no importance, and A_B in $[c_1,c_2]$, which is subject to some constraints. It is simplest to put A_B at the end of $[c_1,c_2]$, as we shall do.

We take A's and B's strategies to be, for the purpose of calculation,

(15)



From (c), we have the equation,

$$-1 = v - (a+1)(1-v) \tag{16}$$

$$v = a/a+2,$$

where a is the size of the bet.

Similarly, the other conditions, after some slight simplification, yield the conditions

$$\begin{aligned} (a+2)\ u - (3a+2)\ w &= -2a \\ (a+2)\ u - 2(1+a)\ c_1 &= -a \\ w - 2c_2 &= -1 \\ (3a+2)c_1 + ac_2 &= 2a + 2av\ . \end{aligned} \tag{17}$$

From the equations, it follows that, for all values of u, we have the consistency condition

$$c_2 \geq w \geq c_1 \geq u \tag{18}$$

satisfied. Solving for c_1 we have

$$c_1 = \frac{8a^2 + 6a + (12a^2 + 8a)v}{20a^2 + 26a + 8}, \tag{19}$$

and a slight calculation shows $a_1 > v$ for all $a > 0$. Knowing c_1, we easily determine c_2, w, u from the other equations of (18). To give some idea of the solutions, let us take a = 1 and 5.

(20)

	c_1	c_2	u	v	w
a = 1	$\frac{31}{81}$	$\frac{61}{81}$	$\frac{43}{243}$	$\frac{1}{3}$	$\frac{41}{81}$
a = 5	$\frac{3}{4}$	$\frac{31}{34}$	$\frac{4}{7}$	$\frac{2}{7}$	$\frac{14}{17}$

The values for a = 5 are approximate, e.g., $\frac{3}{4}$ is an approximation to 3310/4466.

The expectation itself may be calculated and is given by

$$E_A = -1 + (a+1)(c_2-c_1)^2 + 2(a+1)(c_2-c_1)(1-c_2) + (2a+1)(1-c_2)^2 \tag{21}$$

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[1] R. Bellman and D. Blackwell, On Games Involving Bluffing, Proc. Nat. Acad. Sci., 35(1949), pp. 600-5.

cf. also,

[2] A. Dvoretsky, A. Wald and J. Wolfowitz, Elimination of Randomization in Certain Problems of Statistics and of the Theory of Games, Proc. Nat. Acad. Sci., 36(1950), pp. 256-60.

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